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# PROJECTION ESTIMATORS OF THE STATIONARY DENSITY OF A DIFFERENTIAL EQUATION DRIVEN BY THE FRACTIONAL BROWNIAN MOTION

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ABSTRACT. The paper deals with projection estimators of the density of the stationary solution  $X$  to a differential equation driven by the fractional Brownian motion under a dissipativity condition on the drift function. A model selection method is provided and, thanks to the concentration inequality for Lipschitz functionals of discrete samples of  $X$  proved in Bertin et al. (2020), an oracle inequality is established for the adaptive estimator.

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## 1. INTRODUCTION

Consider the differential equation

$$(1) \quad X_t = X_0 + \int_0^t b(X_s) ds + \sigma B_t ; t \in \mathbb{R}_+,$$

where  $B = (B_t)_{t \in \mathbb{R}_+}$  is a fractional Brownian motion of Hurst index  $H \in (0, 1)$ ,  $b : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous map and  $\sigma \in \mathbb{R}^*$ .

For over two decades, many authors have investigated statistical questions related to differential equations driven by the fractional Brownian motion (fDE).

A large part of the papers published on statistical inference for fDEs deals with parametric estimators of the drift function  $b$  when  $H > 1/2$ . In Kleptsyna and Le Breton [10] and Hu and Nualart [8], continuous-time estimators of the drift parameter in Langevin's equation are studied. Kleptsyna and Le Breton [10] provides a maximum likelihood estimator, where the stochastic integral with respect to the solution to Equation (1) returns to an It o integral. In [17], Tudor and Viens extend this estimator to equations with a drift function depending linearly on the unknown parameter. On the maximum likelihood estimator in fDEs with multiplicative noise, see Mishura and Ralchenko [12]. Hu and Nualart [8] provides a least squares estimator, where the stochastic integral with respect to the solution of Equation (1) is taken in the sense of Skorokhod. In [9], Hu, Nualart and Zhou extend this estimator to equations with a drift function depending linearly on the unknown parameter. Tindel and Neuenkirch [13] provides a discrete-time least squares type estimator defined by an objective function tailor-made to make use of the main result of Tudor and Viens [18] on the rate of convergence of the quadratic variation of the fractional Brownian motion. In [15], Panloup, Tindel and Varvenne extend the results of [13] under much more flexible conditions.

More recently, nonparametric methods were investigated to estimate the drift function  $b$  in Equation (1). For instance, Sausseureau [16] and Comte and Marie [5] study the consistency of continuous-time Nadaraya-Watson type estimators of  $b$ .

The common point of all the references mentioned above is that the existence and uniqueness of the stationary solution to Equation (1) is required. Even if to estimate the distribution of the stationary solution is not necessary to study estimators of  $b$ , this is a very important question already investigated via kernel based methods in Bertin et al. [1]. Precisely, when the stationary solution has a density  $f$  with respect to Lebesgue's measure, the authors establish a risk bound on Parzen's estimator of  $f$  and provide an oracle inequality for an adaptive estimator obtained via a Goldenshluger-Lepski type method. This is a nice application of a powerful concentration inequality for Lipschitz functionals of discrete samples of  $X$  also established by Bertin et al. in [1].

Let  $\mathcal{S}_m$  be the vector space generated by an orthonormal family  $\mathcal{B}_m = \{\varphi_1, \dots, \varphi_m\}$  of  $\mathbb{L}^2(I, dr)$ , where  $I \subset \mathbb{R}$  is an interval, and consider  $t_0, \dots, t_n > 0$  such that  $t_i := i\Delta_n$  for every  $i \in \{1, \dots, n\}$ ,  $\Delta_n > 0$ ,

$$\lim_{n \rightarrow \infty} \Delta_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} n\Delta_n = \infty.$$

Our paper deals with the following projection estimator of  $f$ :

$$\hat{f}_m = \hat{f}_{m,n} := \sum_{j=1}^m [\hat{\theta}_{m,n}]_j \varphi_j \quad \text{with} \quad \hat{\theta}_{m,n} := \left( \frac{1}{n} \sum_{i=1}^n \varphi_j(X_{t_i}) \right)_{j \in \{1, \dots, m\}}.$$

Precisely, a concentration inequality on the supremum of the empirical process is derived from Bertin et al. [1], Theorem 1 in Section 2. Then, a risk bound on  $\hat{f}_m$  is established in Section 3. Finally, Section 4 deals with a model selection method and an oracle inequality for the adaptive estimator.

As in the *i.i.d. context*, the main advantage of the projection based approach for one-dimensional fDEs is that the optimization problem defining the model selection method is numerically easier to solve than the one defining the Goldenshluger-Lepski method because it involves only one variable (see Problem (2)). Note also that in the kernel based approach, even in dimension 1 in the *i.i.d. context*, there is no simple model selection method as (2). This is the main advantage of the adaptive estimator studied in this paper with respect to the kernel-based adaptive estimator of Bertin et al. [1]. In fact, the same advantage than in the *i.i.d. context* (see the remarks at the end of Section 4 for details).

#### Notations:

- (1) Throughout the paper,  $\mathbb{R}^n$  is equipped with the distance  $d_1$  defined by

$$d_1(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^n |\mathbf{y}_i - \mathbf{x}_i|; \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

- (2) For any metric space  $E$ ,  $\text{Lip}(E; \mathbb{R})$  is the space of Lipschitz continuous maps from  $E$  into  $\mathbb{R}$ , equipped with its usual semi-norm, always denoted by  $\|\cdot\|_{\text{Lip}}$  for the sake of simplicity.  
(3) The space  $C^0(E; \mathbb{R})$  is equipped with the uniform norm, always denoted by  $\|\cdot\|_{\infty}$  for the sake of simplicity.  
(4) The space  $\mathbb{L}^2(I, dr)$  is equipped with its usual scalar product  $\langle \cdot, \cdot \rangle$ . The associated norm is denoted by  $\|\cdot\|$ .  
(5) Throughout the paper,  $\mathbb{R}^I$  is identified to  $\{\varphi : \mathbb{R} \rightarrow \mathbb{R} : \text{supp}(\varphi) = I\}$ .

## 2. PRELIMINARIES: STATIONARY SOLUTIONS OF THE FDE

This section deals with existing results on the existence and uniqueness of the stationary solution to Equation (1) under a dissipativity condition on the drift function  $b$ , and then with some consequences of a concentration inequality for Lipschitz functionals of  $(X_{t_1}, \dots, X_{t_n})$  due to Bertin et al. [1].

**Assumption 2.1.** *The function  $b$  belongs to  $C^1(\mathbb{R})$ ,  $b'$  is bounded and there exists  $\mathfrak{m}_b > 0$  such that*

$$b'(x) \leq -\mathfrak{m}_b; \quad \forall x \in \mathbb{R}.$$

Under Assumption 2.1, Equation (1) has a unique stationary solution  $X = (X_t)_{t \in \mathbb{R}_+}$  (see Hairer [6]) and  $X_0$  has a density  $f$  with respect to Lebesgue's measure (see Bertin et al. [1], Proposition 1).

**Example 2.2.** *A famous example of fDE satisfying Assumption 2.1 is the fractional Langevin equation ( $b = -\theta \text{Id}_{\mathbb{R}}$  with  $\theta > 0$ ). Its solution is the fractional Ornstein-Uhlenbeck process and, in this case, the stationary density  $f$  is Gaussian (see Cheridito et al. [3]).*

The following theorem provides a concentration inequality for Lipschitz functionals of  $(X_{t_1}, \dots, X_{t_n})$  (see Bertin et al. [1], Theorem 1).

**Theorem 2.3.** *Under Assumption 2.1, there exists a constant  $\mathfrak{c}_{2.3} > 0$ , not depending on  $n$ , such that for every  $F \in \text{Lip}(\mathbb{R}^n; \mathbb{R})$  and  $r > 0$ ,*

$$\mathbb{P}(F(X_{t_1}, \dots, X_{t_n}) - \mathbb{E}(F(X_{t_1}, \dots, X_{t_n})) > r) \leq \exp\left(-\frac{r^2}{\mathfrak{c}_{2.3} \|F\|_{\text{Lip}}^2 n^{\mathfrak{a}_H} \Delta_n^{-\mathfrak{b}_H}}\right)$$

with  $\mathfrak{a}_H = (2H) \vee 1$  and  $\mathfrak{b}_H = 1 \wedge (2 - 2H)$ .

Now, let us state a consequence of Theorem 2.3 on the (centered) empirical process, already established in Bertin et al. [1] (see Corollary 1).

**Corollary 2.4.** *Under Assumption 2.1, for every  $\varphi \in \text{Lip}(\mathbb{R})$  and  $r > 0$ ,*

$$\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n [\varphi(X_{t_i}) - \mathbb{E}(\varphi(X_{t_i}))] > r\right) \leq \exp\left(-\frac{r^2}{\mathfrak{c}_{2.3} \|\varphi\|_{\text{Lip}}^2 (n\Delta_n)^{\mathfrak{b}_H}}\right).$$

Finally, because Corollary 2.4 is not sufficient to establish a risk bound on the adaptive estimator at Section 4, let us derive a concentration inequality on the supremum of the empirical process from Theorem 2.3.

**Corollary 2.5.** *Consider  $\mathcal{F} \subset \text{Lip}(\mathbb{R})$  such that*

$$\mathfrak{c}_{\mathcal{F}} := \sup_{\varphi \in \mathcal{F}} \|\varphi\|_{\text{Lip}} < \infty.$$

For every  $\varphi \in \mathcal{F}$ , consider the empirical process

$$\nu_n(\varphi) := \frac{1}{n} \sum_{i=1}^n [\varphi(X_{t_i}) - \mathfrak{m}(\varphi)] \quad \text{with} \quad \mathfrak{m}(\varphi) := \langle \varphi, f \rangle.$$

Under Assumption 2.1, for every

$$\mathfrak{h} \geq \mathbb{E}\left(\sup_{\varphi \in \mathcal{F}} |\nu_n(\varphi)|\right)$$

and every  $r > 0$ ,

$$\mathbb{P}\left(\sup_{\varphi \in \mathcal{F}} |\nu_n(\varphi)| - \mathfrak{h} > r\right) \leq \exp\left(-\frac{r^2}{\mathfrak{c}_{2.3} \mathfrak{c}_{\mathcal{F}}^2 (n\Delta_n)^{\mathfrak{b}_H}}\right).$$

*Proof.* Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be the map defined by

$$F(\mathbf{x}) := \sup_{\varphi \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n [\varphi(\mathbf{x}_i) - \mathfrak{m}(\varphi)] \right|; \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

For any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , by the triangle inequality for the uniform norm,

$$\begin{aligned} |F(\mathbf{y}) - F(\mathbf{x})| &\leq \frac{1}{n} \sup_{\varphi \in \mathcal{F}} \left| \sum_{i=1}^n [\varphi(\mathbf{y}_i) - \mathfrak{m}(\varphi)] - \sum_{i=1}^n [\varphi(\mathbf{x}_i) - \mathfrak{m}(\varphi)] \right| \\ &\leq \frac{1}{n} \sup_{\varphi \in \mathcal{F}} \left\{ \sum_{i=1}^n |\varphi(\mathbf{y}_i) - \varphi(\mathbf{x}_i)| \right\} \leq \frac{1}{n} d_1(\mathbf{x}, \mathbf{y}) \sup_{\varphi \in \mathcal{F}} \|\varphi\|_{\text{Lip}}. \end{aligned}$$

Then,  $F \in \text{Lip}(\mathbb{R}^n; \mathbb{R})$  and

$$\|F\|_{\text{Lip}} \leq \frac{\mathfrak{c}_{\mathcal{F}}}{n}.$$

Therefore, by Theorem 2.3 and since  $2 - \mathfrak{a}_H = \mathfrak{b}_H$ , for every  $r > 0$ ,

$$\begin{aligned} \mathbb{P}\left(\sup_{\varphi \in \mathcal{F}} |\nu_n(\varphi)| - \mathfrak{h} > r\right) &\leq \mathbb{P}(F(X_{t_1}, \dots, X_{t_n}) - \mathbb{E}(F(X_{t_1}, \dots, X_{t_n})) > r) \\ &\leq \exp\left(-\frac{r^2}{\mathfrak{c}_{2.3}^2 \mathfrak{c}_{\mathcal{F}}^2} (n\Delta_n)^{\mathfrak{b}_H}\right). \end{aligned}$$

□

### 3. RISK BOUND ON THE PROJECTION ESTIMATORS

This section deals with a risk bound on  $\widehat{f}_m$  obtained via the concentration inequality on the empirical process stated in Corollary 2.4.

In the sequel,  $f_m$  is the orthogonal projection of  $f$  on  $\mathcal{S}_m$  (in  $\mathbb{L}^2(I, dr)$ ), and  $\mathcal{B}$  fulfills the following assumption.

**Assumption 3.1.** *The  $\varphi_j$ 's are bounded and Lipschitz continuous functions.*

Now, let us establish the main result of this section: a risk bound on  $\widehat{f}_m$ .

**Proposition 3.2.** *Under Assumptions 2.1 and 3.1,*

$$\mathbb{E}(\|\widehat{f}_m - f\|^2) \leq \min_{t \in \mathcal{S}_m} \|t - f\|^2 + 2\mathfrak{c}_{2.3} \frac{mL(m)}{(n\Delta_n)^{\mathfrak{b}_H}} \quad \text{with} \quad L(m) := \sum_{j=1}^m \|\varphi_j\|_{\text{Lip}}^2.$$

*Proof.* First of all, for any  $x \in I$ ,

$$\mathbb{E}(\|\widehat{f}_m(x) - f(x)\|^2) = (\mathbb{E}(\widehat{f}_m(x)) - f(x))^2 + \text{var}(\widehat{f}_m(x)).$$

Let us express well the bias term, and then give a suitable bound for the variance term. On the one hand,

$$\mathbb{E}(\widehat{f}_m(x)) = \frac{1}{n} \sum_{j=1}^m \sum_{i=1}^n \mathbb{E}(\varphi_j(X_{t_i})) \varphi_j(x) = \sum_{j=1}^m \langle f, \varphi_j \rangle \varphi_j(x) = f_m(x).$$

So,

$$\int_I (\mathbb{E}(\widehat{f}_m(x)) - f(x))^2 dx = \|f_m - f\|^2 = \min_{t \in \mathcal{S}_m} \|t - f\|^2.$$

On the other hand, consider

$$\varphi(x, \cdot) := \sum_{j=1}^m \varphi_j(\cdot) \varphi_j(x).$$

By Corollary 2.4 applied to  $\varphi(x, \cdot)$ ,

$$\begin{aligned} \text{var}(\widehat{f}_m(x)) &= 2 \int_0^\infty r \mathbb{P}\left(\frac{1}{n} \left| \sum_{i=1}^n [\varphi(x, X_{t_i}) - \mathbb{E}(\varphi(x, X_{t_i}))] \right| > r\right) dr \\ &\leq 4 \int_0^\infty r \exp\left(-\frac{r^2}{\mathfrak{c}_{2.3} \|\varphi(x, \cdot)\|_{\text{Lip}}^2} (n\Delta_n)^{\mathfrak{b}_H}\right) dr = 2\mathfrak{c}_{2.3} \Gamma(1) \frac{\|\varphi(x, \cdot)\|_{\text{Lip}}^2}{(n\Delta_n)^{\mathfrak{b}_H}}. \end{aligned}$$

Moreover, by Jensen's inequality and since  $\mathcal{B}_m$  is an orthonormal family of  $\mathbb{L}^2(I, dx)$ ,

$$\int_I \|\varphi(x, \cdot)\|_{\text{Lip}}^2 dx \leq \int_I \left[ \sum_{j=1}^m \|\varphi_j\|_{\text{Lip}} |\varphi_j(x)| \right]^2 dx \leq m \sum_{j=1}^m \|\varphi_j\|_{\text{Lip}}^2 = mL(m).$$

Therefore,

$$\mathbb{E}(\|\widehat{f}_m - f\|^2) \leq \min_{t \in \mathcal{S}_m} \|t - f\|^2 + 2\mathfrak{c}_{2.3} \frac{mL(m)}{(n\Delta_n)^{\mathfrak{b}_H}}.$$

□

**Example 3.3.** Assume that  $I = [0, 1]$  and that  $\mathcal{B}_m$  is the trigonometric basis. Precisely, for every  $x \in I$ ,  $\varphi_1(x) = 1$ , and for every  $j \in \mathbb{N}$  such that  $2j + 1 \leq m$ ,  $\varphi_{2j}(x) = \sqrt{2} \cos(2\pi jx)$  and  $\varphi_{2j+1}(x) = \sqrt{2} \sin(2\pi jx)$ . Then, there exists a constant  $\mathfrak{c}_{3.3} > 0$ , not depending on  $m$  and  $n$ , such that

$$L(m) = \sum_{j=1}^m \|\varphi'_j\|_\infty^2 \leq \mathfrak{c}_{3.3} m^3.$$

So, the variance term in the risk bound on  $\hat{f}_m$  stated in Proposition 3.2 is of order  $m^4(n\Delta_n)^{-b_H}$ . Note that the variance term in the risk bound on Parzen's estimator of bandwidth  $h > 0$  obtained in Bertin et al. [1], Proposition 3, is of same order  $h^{-4}(n\Delta_n)^{-b_H}$ .

#### 4. MODEL SELECTION

As in the *classic* projection density estimation framework, note that

$$\hat{f}_m = \arg \min_{\varphi \in \mathcal{S}_m} \hat{\gamma}_n(\varphi) \quad \text{with} \quad \hat{\gamma}_n(\varphi) := \|\varphi\|^2 - \frac{2}{n} \sum_{i=1}^n \varphi(X_{t_i}); \quad \forall \varphi \in \mathcal{S}_m.$$

So, for the proposal set

$$\mathcal{M} := \{m \in \mathbb{N}^* : mL(m) \leq (n\Delta_n)^{b_H}\}$$

and a hyper-parameter  $\mathfrak{R} > 0$ , it is natural to consider the adaptive estimator  $\hat{f}_{\hat{m}}$  of  $f$ , where

$$(2) \quad \hat{m} = \arg \min_{m \in \mathcal{M}} \{\hat{\gamma}_n(\hat{f}_m) + \text{pen}(m)\}$$

with

$$\text{pen}(m) := \mathfrak{R} \frac{(m+1)L(m)}{(n\Delta_n)^{b_H}}; \quad \forall m \in \mathcal{M}.$$

In order to provide an oracle inequality for  $\hat{f}_{\hat{m}}$  at Theorem 4.5, let us first establish the following technical lemma.

**Lemma 4.1.** Under Assumption 3.1, with the notations of Corollary 2.5, if

$$\mathcal{F} = \mathcal{F}_m := \{\varphi \in \mathcal{S}_m : \|\varphi\| = 1\},$$

then

$$\mathfrak{c}_{\mathcal{F}_m} \leq L(m)^{1/2}.$$

*Proof.* Consider  $\varphi \in \mathcal{S}_m$  such that  $\|\varphi\| = 1$ . Then, there exist  $a_1, \dots, a_m \in \mathbb{R}$  such that

$$\varphi = \sum_{j=1}^m a_j \varphi_j \quad \text{and} \quad \sum_{j=1}^m a_j^2 = 1.$$

So,

$$\|\varphi\|_{\text{Lip}} \leq \sum_{j=1}^m |a_j| \cdot \|\varphi_j\|_{\text{Lip}} \leq \left( \sum_{j=1}^m a_j^2 \right)^{1/2} \left( \sum_{j=1}^m \|\varphi_j\|_{\text{Lip}}^2 \right)^{1/2} = L(m)^{1/2}.$$

Therefore,

$$\mathfrak{c}_{\mathcal{F}_m} = \sup_{\varphi \in \mathcal{F}_m} \|\varphi\|_{\text{Lip}} \leq L(m)^{1/2}.$$

□

In the sequel, the  $\mathcal{S}_m$ 's are nested:

**Assumption 4.2.** For every  $m, m' \in \mathcal{M}$ , if  $m \geq m'$ , then  $\mathcal{S}_{m'} \subset \mathcal{S}_m$ .

Moreover,  $L(\cdot)$  fulfills the following assumption.

**Assumption 4.3.** The map  $m \mapsto L(m)$  has polynomial growth.

**Example 4.4.** *On the one hand, note that Assumption 4.2 is fulfilled by several usual bases: the trigonometric basis, Hermite's basis, Laguerre's basis, etc. On the other hand, for instance, the trigonometric basis fulfills Assumption 4.3 (see Example 3.3).*

Now, let us establish the main result of this section: an oracle inequality for  $\widehat{f}_{\widehat{m}}$ .

**Theorem 4.5.** *Under Assumptions 2.1, 3.1, 4.2 and 4.3, if  $\mathfrak{R} \geq 16\mathfrak{c}_{2,3}$ , then there exist two positive constants  $\mathfrak{c}_{4.5,1}$  and  $\mathfrak{c}_{4.5,2}$ , not depending on  $n$ , such that*

$$\mathbb{E}(\|\widehat{f}_{\widehat{m}} - f\|^2) \leq \mathfrak{c}_{4.5,1} \min_{m \in \mathcal{M}} \{\|f_m - f\|^2 + \text{pen}(m)\} + \frac{\mathfrak{c}_{4.5,2}}{(n\Delta_n)^{\mathfrak{b}_H}}.$$

*Proof.* The proof is dissected in two steps. In the first one, with the same arguments than in the *classic* projection density estimation framework (see Comte [4], Theorem 5.2 or Massart [14], Chapter 7), it is established that for any  $m \in \mathcal{M}$ ,

$$\|\widehat{f}_{\widehat{m}} - f\|^2 \leq 3\|f_m - f\|^2 + 4\text{pen}(m) + R_{m,n},$$

where  $R_{m,n}$  is a remainder term. In the second step, it is established that  $\mathbb{E}(R_{m,n})$  is of order  $(n\Delta_n)^{-\mathfrak{b}_H}$  thanks to Corollary 2.5 and Lemma 4.1.

**Step 1.** Note that

$$\widehat{\gamma}_n(\widehat{f}_{\widehat{m}}) + \text{pen}(\widehat{m}) \leq \widehat{\gamma}_n(f_m) + \text{pen}(m); \forall m \in \mathcal{M}$$

and

$$\widehat{\gamma}_n(\varphi) - \widehat{\gamma}_n(\psi) = \|\varphi - f\|^2 - \|\psi - f\|^2 - 2\nu_n(\varphi - \psi); \forall \varphi, \psi \in \text{Lip}(I; \mathbb{R}).$$

Then, for any  $m \in \mathcal{M}$ , since  $2uv \leq u^2 + v^2$  for every  $u, v \in \mathbb{R}_+$ , and since  $\mathcal{S}_m + \mathcal{S}_{\widehat{m}} \subset \mathcal{S}_{m \vee \widehat{m}}$  by Assumption 4.2,

$$\begin{aligned} \|\widehat{f}_{\widehat{m}} - f\|^2 &\leq \|f_m - f\|^2 + \text{pen}(m) + 2 \cdot \frac{1}{2} \|\widehat{f}_{\widehat{m}} - f_m\| \cdot 2\nu_n \left( \frac{\widehat{f}_{\widehat{m}} - f_m}{\|\widehat{f}_{\widehat{m}} - f_m\|} \right) - \text{pen}(\widehat{m}) \\ &\leq \|f_m - f\|^2 + \text{pen}(m) + \frac{1}{4} \|\widehat{f}_{\widehat{m}} - f_m\|^2 + 4 \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \widehat{m}}} |\nu_n(\varphi)| \right]^2 - \text{pen}(\widehat{m}). \end{aligned}$$

For every  $\overline{m} \in \mathcal{M}$ , consider

$$\begin{aligned} p(m, \overline{m}) &:= \frac{\mathfrak{R}}{4} \cdot \frac{((m \vee \overline{m}) + 1)L(m \vee \overline{m})}{(n\Delta_n)^{\mathfrak{b}_H}} \\ &\leq \frac{1}{4} (\text{pen}(m) + \text{pen}(\overline{m})). \end{aligned}$$

So,

$$\begin{aligned} \|\widehat{f}_{\widehat{m}} - f\|^2 &\leq \|f_m - f\|^2 + \text{pen}(m) + \frac{1}{4} \|\widehat{f}_{\widehat{m}} - f_m\|^2 \\ &\quad + 4 \left( \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \widehat{m}}} |\nu_n(\varphi)| \right]^2 - p(m, \widehat{m}) \right)_+ + 4p(m, \widehat{m}) - \text{pen}(\widehat{m}) \\ &\leq \|f_m - f\|^2 + 2\text{pen}(m) + \frac{1}{2} (\|\widehat{f}_{\widehat{m}} - f\|^2 + \|f_m - f\|^2) \\ &\quad + 4 \sum_{\overline{m} \in \mathcal{M}} \left( \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \overline{m}}} |\nu_n(\varphi)| \right]^2 - p(m, \overline{m}) \right)_+ \end{aligned}$$

and then,

$$\|\widehat{f}_{\widehat{m}} - f\|^2 \leq 3\|f_m - f\|^2 + 4\text{pen}(m) + 8 \sum_{\overline{m} \in \mathcal{M}} \left( \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \overline{m}}} |\nu_n(\varphi)| \right]^2 - p(m, \overline{m}) \right)_+.$$

**Step 2.** For any  $\bar{m} \in \mathcal{M}$ , by Corollary 2.4 (as in the proof of Proposition 3.2), and since  $\mathfrak{K} \geq 16\mathfrak{c}_{2.3}$ ,

$$\begin{aligned} \mathbb{E} \left( \sup_{\varphi \in \mathcal{F}_{m \vee \bar{m}}} |\nu_n(\varphi)| \right)^2 &\leq \sum_{j=1}^{m \vee \bar{m}} \mathbb{E}(\nu_n(\varphi_j)^2) = \sum_{j=1}^{m \vee \bar{m}} \text{var}(\nu_n(\varphi_j)) \\ &\leq 2\mathfrak{c}_{2.3} \frac{1}{(n\Delta_n)^{\mathfrak{b}_H}} \sum_{j=1}^{m \vee \bar{m}} \|\varphi_j\|_{\text{Lip}}^2 \leq \frac{\mathfrak{K}}{8} \cdot \frac{L(m \vee \bar{m})}{(n\Delta_n)^{\mathfrak{b}_H}} =: \mathfrak{h}(m, \bar{m}). \end{aligned}$$

Now, consider

$$\mathfrak{p}(m, \bar{m}) := \frac{1}{2}p(m, \bar{m}) - \mathfrak{h}(m, \bar{m}) = \frac{\mathfrak{K}}{8} \cdot \frac{(m \vee \bar{m})L(m \vee \bar{m})}{(n\Delta_n)^{\mathfrak{b}_H}} > 0.$$

So,

$$\begin{aligned} \left( \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \bar{m}}} |\nu_n(\varphi)| \right]^2 - p(m, \bar{m}) \right)_+ &\leq \left( 2 \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \bar{m}}} |\nu_n(\varphi)| - \mathfrak{h}(m, \bar{m})^{1/2} \right]^2 + 2\mathfrak{h}(m, \bar{m}) - p(m, \bar{m}) \right)_+ \\ &= 2 \left( \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \bar{m}}} |\nu_n(\varphi)| - \mathfrak{h}(m, \bar{m})^{1/2} \right]^2 - \mathfrak{p}(m, \bar{m}) \right)_+ \end{aligned}$$

and then,

$$\begin{aligned} \mathbb{E} \left[ \left( \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \bar{m}}} |\nu_n(\varphi)| \right]^2 - p(m, \bar{m}) \right)_+ \right] &\leq 2 \int_0^\infty \mathbb{P} \left[ \left( \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \bar{m}}} |\nu_n(\varphi)| - \mathfrak{h}(m, \bar{m})^{1/2} \right]^2 - \mathfrak{p}(m, \bar{m}) \right)_+ > r \right] dr \\ &= 2 \int_0^\infty \mathbb{P} \left( \sup_{\varphi \in \mathcal{F}_{m \vee \bar{m}}} |\nu_n(\varphi)| - \mathfrak{h}(m, \bar{m})^{1/2} > (r + \mathfrak{p}(m, \bar{m}))^{1/2} \right) dr. \end{aligned}$$

Thus, by Corollary 2.5,

$$\begin{aligned} \mathbb{E} \left[ \left( \left[ \sup_{\varphi \in \mathcal{F}_{m \vee \bar{m}}} |\nu_n(\varphi)| \right]^2 - p(m, \bar{m}) \right)_+ \right] &\leq 2 \exp \left( -\frac{\mathfrak{p}(m, \bar{m})(n\Delta_n)^{\mathfrak{b}_H}}{\mathfrak{c}_{2.3}\mathfrak{c}_{\mathcal{F}_{m \vee \bar{m}}}^2} \right) \int_0^\infty \exp \left( -\frac{r(n\Delta_n)^{\mathfrak{b}_H}}{\mathfrak{c}_{2.3}\mathfrak{c}_{\mathcal{F}_{m \vee \bar{m}}}^2} \right) dr \\ &= 2\Gamma(1)\mathfrak{c}_{2.3} \frac{1}{(n\Delta_n)^{\mathfrak{b}_H}} \mathfrak{c}_{\mathcal{F}_{m \vee \bar{m}}}^2 \exp \left( -\frac{\mathfrak{p}(m, \bar{m})(n\Delta_n)^{\mathfrak{b}_H}}{\mathfrak{c}_{2.3}\mathfrak{c}_{\mathcal{F}_{m \vee \bar{m}}}^2} \right). \end{aligned}$$

Moreover, by Assumption 4.3, there exist three constants  $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3 > 0$ , not depending on  $m$  and  $n$ , such that

$$\begin{aligned} \sum_{\bar{m} \in \mathcal{M}} \mathfrak{c}_{\mathcal{F}_{m \vee \bar{m}}}^2 \exp \left( -\frac{\mathfrak{p}(m, \bar{m})(n\Delta_n)^{\mathfrak{b}_H}}{\mathfrak{c}_{2.3}\mathfrak{c}_{\mathcal{F}_{m \vee \bar{m}}}^2} \right) &\leq \sum_{\bar{m} \in \mathcal{M}} L(m \vee \bar{m}) \exp \left( -\frac{\mathfrak{K}}{8\mathfrak{c}_{2.3}}(m \vee \bar{m}) \right) \\ &= mL(m) \exp \left( -\frac{\mathfrak{K}}{8\mathfrak{c}_{2.3}}m \right) \\ &\quad + \sum_{\bar{m} \in \mathcal{M} : \bar{m} > m} L(\bar{m}) \exp \left( -\frac{\mathfrak{K}}{16\mathfrak{c}_{2.3}}\bar{m} \right) \exp \left( -\frac{\mathfrak{K}}{16\mathfrak{c}_{2.3}}\bar{m} \right) \\ &\leq \mathfrak{c}_1 + \mathfrak{c}_2 \sum_{\bar{m} \in \mathcal{M}} \exp \left( -\frac{\mathfrak{K}}{16\mathfrak{c}_{2.3}}\bar{m} \right) \leq \mathfrak{c}_3. \end{aligned}$$

Therefore,

$$\mathbb{E}(\|\widehat{f}_{\bar{m}} - f\|^2) \leq \min_{m \in \mathcal{M}} \{3\|f_m - f\|^2 + 4\text{pen}(m)\} + \frac{2\mathfrak{c}_{2.3}\mathfrak{c}_3}{(n\Delta_n)^{\mathfrak{b}_H}}.$$

□

**Remarks:**



- (1) Note that the penalty and the remainder term in the risk bound in Theorem 4.5 are of same order than in Bertin et al. [1], Theorem 3.
- (2) As in the *i.i.d. context*, since it involves only one variable, the main advantage of the projection based approach for one-dimensional fDEs is that Problem (2) is numerically easier to solve than the optimization problem defining the Goldenshluger-Lepski method. Note also that in the kernel based approach, even in dimension 1 in the *i.i.d. context*, there is no simple model selection method as (2). Recently, in [11], Lacour, Massart and Rivoirard have provided a bandwidth selection method, called PCO method, bypassing the mentioned drawbacks of Goldenshluger-Lepski's method, but to extend the PCO method to the fDE framework requires more than Bertin et al. [1], Theorem 1. An extension of the concentration inequality for  $U$ -statistics of order 2 of Houdré and Reynaud-Bouret [7] from the *i.i.d. context* to  $(X_{t_1}, \dots, X_{t_n})$  is required.
- (3) Note that without additional arguments, one can extend the result of our paper to multidimensional fDEs for isotropic projection estimators. However, to extend it to anisotropic projection estimators requires a Goldenshluger-Lepski type procedure again (see Chagny [2] for this type of method).

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