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1     **OUTPUT FEEDBACK CONTROL OF A CASCADE SYSTEM OF**  
2     **LINEAR KORTEWEG–DE VRIES EQUATIONS\***

3     CONSTANTINOS KITSOS <sup>§†</sup>, EDUARDO CERPA <sup>‡</sup>, GILDAS BESANÇON <sup>§</sup>, AND  
4     CHRISTOPHE PRIEUR<sup>§</sup>

5     **Abstract.** This paper is about the stabilization of a cascade system of  $n$  linear Korteweg–de  
6     Vries equations in a bounded interval. It considers an output feedback control placed at the left  
7     endpoint of the last equation, while the output involves only the solution to the first equation.  
8     The boundary control problems investigated include two cases: a classical control on the Dirichlet  
9     boundary condition and a less standard one on its second-order derivative. The feedback control law  
10    utilizes the estimated solutions of a high-gain observer system, and the output feedback control leads  
11    to stabilization for any  $n$  for the first boundary conditions case and for  $n = 2$  for the second one.

12    **Key words.** Korteweg–de Vries equation, cascade systems, output feedback control

13    **AMS subject classifications.** 68Q25, 68R10, 68U05

14    **1. Introduction.** In this paper, we study the following cascade system of  $n$   
15    linear Korteweg–de Vries (KdV for short) equations posed in a bounded interval of  
16    length  $L$ :

17 (1.1)                    $v_t + v_x + v_{xxx} = (A_n - B)v$ , in  $(0, \infty) \times (0, L)$ ,

19 where  $v = (v_1 \ \cdots \ v_n)^\top$  is the state and

20                    $A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix},$                     $B = \text{diag}(1, 1, \dots, 1, -1).$

22 Let us consider two different types of boundary conditions, where the input control  $u$   
23 in both of them is placed on the left side and only acts on the  $n$ th coordinate of the  
24 state.

25     *Boundary conditions A (BC-A):*

26 (1.2a)                    $v_i(t, 0) = 0, \quad i = 1, \dots, n - 1, \quad \forall t > 0,$   
                              $v_n(t, 0) = u(t), \quad \forall t > 0,$   
27                            $v(t, L) = 0, v_x(t, L) = 0, \quad \forall t > 0.$

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28 *Boundary conditions B (BC-B):*

$$\begin{aligned}
 & v_{i,xx}(t, 0) = 0, \quad i = 1, \dots, n-1, \quad \forall t > 0, \\
 29 \quad (1.2b) \quad & v_{n,xx}(t, 0) = u(t), \quad \forall t > 0, \\
 30 \quad & v(t, L) = 0, v_x(t, L) = 0, \quad \forall t > 0.
 \end{aligned}$$

31 In order to complete our control system, we add an initial condition given by

$$32 \quad (1.3) \quad v(0, x) = v^0(x), \quad x \in (0, L)$$

34 and a distributed measurement given by the following output:

$$\begin{aligned}
 35 \quad (1.4) \quad & y(t, x) = Cv(t, x); \\
 36 \quad & C = (1 \quad 0 \quad \cdots \quad 0).
 \end{aligned}$$

38 The nonlinear version of a single KdV equation describes propagation of waters  
 39 with small amplitude in closed channels. It was introduced in 1895, and since then its  
 40 properties have gained much consideration; see, for instance, [6]. Surveys on recent  
 41 progress and open problems on control and stabilization of such models can be found  
 42 in [26] and [3].

43 The aim of the present work is to stabilize the cascade system (1.1) considering any  
 44 of the boundary control problems (BC-A) and (BC-B) and by utilizing the knowledge  
 45 of the first state only, while the other states are estimated via an observer. Notice  
 46 that this system is unstable due to the instability of the trajectory corresponding to  
 47 its last equation, as can be seen by following classical energy arguments. In recent  
 48 decades, stabilization of single KdV equations has gained significant interest. See, for  
 49 instance, [4], where backstepping method is used for feedback controls placed on the  
 50 left boundary, see also [29, 7, 27, 13]. Output feedback laws for single linearized and  
 51 nonlinear KdV equations have been already established via boundary observers in [21,  
 52 23] (see also [28, 11, 14, 2]), by means of backstepping and Lyapunov techniques. In  
 53 these two works, the measurement injected in the observer involves the right endpoint  
 54 of the domain, more precisely, the second derivative of the boundary or the Dirichlet  
 55 condition, depending on the boundary conditions. Output feedback control laws for  
 56 systems written in the cascade form considered here have not yet appeared in the  
 57 literature, while controllability of coupled KdV equations but with couplings, different  
 58 from the ones studied here (particularly, internal couplings in first-order derivatives),  
 59 describing strong interactions of weakly nonlinear long waves, has been investigated  
 60 for instance in [5]. However, coupling in zero-order internal terms, with coupling  
 61 coefficient  $A_n$ , might result from the linearization of coupled nonlinear KdV equations  
 62 of some forms appearing in [20], describing oceanic and atmospheric phenomena, such  
 63 as the atmospheric blockings, the interactions between the atmosphere and ocean,  
 64 the oceanic circulations, and hurricanes (see system (27)–(28) and model 5 in [20],  
 65 according to the well-known Painlevé classification), see also the Hirota–Satsuma  
 66 model [12] and [10] for multicomponent KdV equations (related to the weak nonlinear  
 67 dispersion). For these systems, it is often difficult to control and observe directly all  
 68 the equations. Also, general settings of coupled infinite-dimensional systems with  
 69 couplings in zero-order terms, as the ones considered here, have been studied with  
 70 respect to their controllability and observability properties, when considering reduced  
 71 numbers of controls and observations, see [1, 19]. In those works, the authors have  
 72 shown that the problem of control of underactuated systems with a reduced number

73 of observations is quite challenging. Furthermore, placing the control on the second  
 74 derivative of the left boundary, as in the considered second boundary control problem,  
 75 is even more original, and its investigation exhibits some technical difficulties, for  
 76 which solutions are proposed in the present work. To the best of our knowledge,  
 77 boundary control problems of this second type have not appeared in the literature.

78 Here, we aim at observing the full state of a system of KdV equations written in a  
 79 cascade form and finally controlling it, by considering a single observation. Observer  
 80 design for nonlinear systems of partial differential equations written in such a form,  
 81 based on the well-known high-gain methodology, has been considered, for instance,  
 82 in [17, 16, 18], in the framework of first-order hyperbolic systems, extending results  
 83 for finite-dimensional systems [15]. A similar form considered here, in its linearized  
 84 version, allows an observer design, which relies on a choice of a sufficiently large  
 85 parameter in its equations, while appropriate choice of the latter leads simultaneously  
 86 to the closed-loop output feedback stabilization. In summary, the contribution of the  
 87 present work first lies in stabilizing the trajectory of the last equation by means of an  
 88 observer relying on the measurement of the first state only. The control placed on the  
 89 left boundary, combined with the observer gain, brings this trajectory asymptotically  
 90 to zero in an arbitrarily fast manner (first part of [Theorem 3.2](#)). Subsequently, it  
 91 is proven that (1) the whole cascade system becomes asymptotically stable for any  
 92  $L > 0$  when boundary condition (BC-A) is considered and (2) this result holds for  
 93 boundary condition (BC-B) only when  $n = 2$  (number of equations), noting that for  
 94  $n > 2$ , stabilization is achieved for quite small  $L$  (last part of [Theorem 3.2](#)). The  
 95 methodology relies on backstepping techniques and appropriate Lyapunov analysis.  
 96 Exponential stabilization for (BC-B) is proven here to be linked to the solvability of an  
 97 ordinary differential equations problem, similar to the differential equation satisfied  
 98 by the eigenvectors of the associated differential operator to these KdV equations,  
 99 and being subject to some constraints.

100 In [Section 2](#) we prove a preliminary result on the stability of a single damped  
 101 KdV equation, and then we prove the full state stabilization of the cascade system for  
 102 both boundary condition problems. In [Section 3](#), we first present the observer design  
 103 for the coupled system and finally the main output feedback stabilization result. In  
 104 [Section 4](#), we provide conclusions and some perspectives.

105 **2. Full State Feedback Stabilization.** In this section, we study the full state  
 106 feedback stabilization of system (1.1) for boundary control problems (BC-A) and  
 107 (BC-B).

108 **2.1. Stability of a single KdV equation.** Prior to the stabilization of the  
 109 cascade system, we present a preliminary result about the stability of a single damped  
 110 linear KdV equation, which will be invoked in what follows. Consider a single KdV  
 111 equation in the domain  $(0, L)$ ,

$$112 \quad (2.1) \quad w_t + w_x + w_{xxx} + \lambda w = 0, \quad \text{in } (0, \infty) \times (0, L),$$

114 satisfying one of the following distinct cases of boundary conditions,

$$115 \quad (2.2a) \quad w(t, 0) = w(t, L) = w_x(t, L) = 0, \quad t > 0,$$

$$116 \quad (2.2b) \quad w_{xx}(t, 0) = w(t, L) = w_x(t, L) = 0, \quad t > 0,$$

118 and initial condition of the form

$$119 \quad (2.3) \quad w(0, x) = w^0(x), \quad x \in (0, L).$$

121 The stability result for solutions  $w$  to the above problem is presented in the following  
 122 propositions. Although asymptotic stability assuming boundary conditions (2.2a) is  
 123 ensured for every  $\lambda > 0$ , for (2.2b) asymptotic stability is guaranteed only when  $\lambda \geq \lambda_0$   
 124 for some  $\lambda_0 > 0$ . These results are stated precisely in the next two propositions and  
 125 will be used throughout this work.

126 PROPOSITION 2.1. Consider system (2.1) with boundary conditions (2.2a) and  
 127 initial condition  $w^0 \in L^2(0, L)$ . Then for all  $\lambda > 0$ , we have

$$128 \quad (2.4) \quad \|w(t, \cdot)\|_{L^2(0, L)} \leq e^{-\lambda t} \|w^0(\cdot)\|_{L^2(0, L)}, \quad t \geq 0,$$

130 for every  $L > 0$ .

131 Proposition 2.1 concerning boundary conditions (2.2a) is a standard result and can  
 132 be derived from energy estimates. Well-posedness of this equation is presented in  
 133 Appendix B.1. Let us note here, that asymptotic stability for this case can be proven  
 134 even when the damping is not constant in the domain but localized to a part of it  
 135 (see, for instance, [24]) and even when the damping is saturated (see [22]).

136 To proceed to the stability result for boundary conditions (2.2b), we utilize the  
 137 following lemma.

138 LEMMA 2.2. There exists  $\lambda_0 > 0$  such that the following assertions hold true.

139 Assertion 1: For every  $\lambda \geq \lambda_0$ , there exist  $\pi(\cdot)$  in  $C^3(0, \infty)$  and  $b > 0$  such that  
 140 the following holds for all  $x \geq 0$ :

$$141 \quad (2.5) \quad \begin{cases} \pi'''(x) + \pi'(x) - 2\lambda\pi(x) = -2b\pi(x), \\ \pi''(0)\pi(0) + (\pi'(0))^2 + \pi^2(0) \leq 0, \\ \pi(x) > 0, \\ \pi'(x) \geq 0. \end{cases}$$

143 Assertion 2: For every  $\lambda \in (0, \lambda_0)$ , there exist  $\bar{L}, b > 0$  and  $\pi(\cdot)$  in  $C^3(0, \infty)$   
 144 satisfying (2.5) for all  $x \in [0, \bar{L}]$ .

145 Proof. See Appendix A □

146 The following proposition concerns the second case of boundary conditions.

147 PROPOSITION 2.3. Consider system (2.1) with boundary conditions (2.2b) and  
 148 initial condition  $w^0 \in L^2(0, L)$ . Then there exists  $\lambda_0 > 0$  such that the following hold:

149 (1) For all  $\lambda \geq \lambda_0$ , there exist  $a, b > 0$  such that the solution to (2.1)–(2.3)–(2.2b)  
 150 satisfies the following:

$$151 \quad (2.6) \quad \|w(t, \cdot)\|_{L^2(0, L)} \leq ae^{-bt} \|w^0(\cdot)\|_{L^2(0, L)}, \quad t \geq 0$$

153 for every  $L > 0$ .

154 (2) For all  $\lambda \in (0, \lambda_0)$ , there exist  $\bar{L}, a, b > 0$  such that (2.6) is satisfied for all  
 155  $L \in (0, \bar{L}]$ .

156 Proof. In this context, we are interested by unique solutions  $w$  belonging to  
 157  $C(0, \infty; L^2(0, L))$ . Well-posedness of the initial boundary value problem (2.1)–(2.3)–  
 158 (2.2b) can be easily proven by invoking classical arguments, although these boundary  
 159 conditions are less common in the literature. More details about the well-posedness  
 160 of such systems are presented in Appendix B.1.

161 To prove the stability result, let us consider the following weighted  $L^2$ -norm:

$$162 \quad E(t) := \int_0^L \pi(x)w^2(x)dx$$

163

164 along the  $L^2$  solutions to (2.1)–(2.3)–(2.2b), for some appropriate choice of positive  
 165  $\pi(\cdot) \in C^3[0, L]$ . Calculating its time-derivative and applying integrations by parts,  
 166 we obtain

$$\begin{aligned} 167 \quad \dot{E}(t) &= \int_0^L (\pi'''(x) + \pi'(x) - 2\lambda) w^2(x) dx - 3 \int_0^L \pi'(x) w_x^2(x) dx \\ 168 \quad &+ [(-\pi''(x) - \pi(x)) w^2(x) - 2\pi(x) w_{xx}(x) w(x) + \pi(x) w_x^2(x) \\ 169 \quad &+ 2\pi'(x) w_x(x) w(x)]_0^L. \end{aligned}$$

171 Substituting boundary conditions (2.2b) we get

$$\begin{aligned} 172 \quad \dot{E}(t) &= \int_0^L (\pi'''(x) + \pi'(x) - 2\lambda) w^2(x) dx - 3 \int_0^L \pi'(x) w_x^2(x) dx \\ 173 \quad &- (w_x(0) \quad w(0)) \begin{pmatrix} \pi(0) & -\pi'(0) \\ -\pi'(0) & -\pi(0) - \pi''(0) \end{pmatrix} \begin{pmatrix} w_x(0) \\ w(0) \end{pmatrix}. \end{aligned}$$

174 To ensure the exponential decay of  $E(t)$ , we invoke assertions of Lemma 2.2 for  $\pi(\cdot)$ ,  
 175 for which we assume that it satisfies (2.5). By Assertions 1 and 2, the second integral  
 176 and the third boundary term of the above equation become nonpositive, and we obtain  
 177 the existence of a constant  $b > 0$  such that

$$178 \quad \dot{E}(t) \leq -2bE(t)$$

and, therefore, (2.6) holds with

$$a = \sqrt{\frac{\pi(L)}{\pi(0)}}.$$

180 This completes the proof of Proposition 2.3.  $\square$

181 **2.2. Full state stabilization.** Following the previous results, we are in a po-  
 182 sition to study the closed-loop stabilization. Here, the considered state feedback  
 183 controls, which are placed in the last equation, will be proven to be of the following  
 184 form for each of the problems (BC-A) and (BC-B):

$$185 \quad (2.7a) \quad (\text{BC-A}): u(t) = \int_0^L p(0, y) v_n(t, y) dy,$$

$$186 \quad (2.7b) \quad (\text{BC-B}): u(t) = -\frac{\omega + 1}{3} L v_n(t, 0) + \int_0^L p_{xx}(0, y) v_n(t, y) dy$$

188 with  $\omega > 0$  to be chosen appropriately and kernel function  $p : \Pi \rightarrow \mathbb{R}$  depending on  
 189  $\omega$ , where  $\Pi := \{(x, y); x \in [0, L], y \in [x, L]\}$ .

190 We now present the exponential decay result of the solution  $v$  to the cascade  
 191 system (1.1) via the control (2.7), which utilizes the full state. The proof uses back-  
 192 stepping tools appearing in [4, 3] for single KdV equations.

193 **THEOREM 2.4.** Consider system (1.1) with boundary conditions (BC-A) or (BC-  
 194 B), feedback control laws of the form (2.7a) or (2.7b), respectively, and initial condi-  
 195 tion  $v^0 \in L^2(0, L)^n$ .

196 (a) If (BC-A) holds and  $n \geq 2$ , then for every  $L > 0$  there exist constants  $c, d > 0$   
 197 such that the solution  $v$  to (1.1) satisfies the following:

$$198 \quad (2.8) \quad \|v\|_{L^2(0,L)^n} \leq ce^{-dt} \|v^0\|_{L^2(0,L)^n}, \quad \forall t \geq 0.$$

200 (b) If (BC-B) holds and  $n = 2$ , then for every  $L > 0$ , there exist constants  $c, d > 0$   
 201 such that solution  $v$  to (1.1) satisfies (2.8).

202 (c) If (BC-B) holds and  $n > 2$ , then there exists  $\bar{L} > 0$  such that (2.8) is guaran-  
 203 teed for all  $L \in (0, \bar{L}]$ .

204 *Proof.* The well-posedness for controlled system (1.1) with boundary conditions  
 205 (BC-A) or (BC-B) is shown in Appendix B.1.

206 We first prove a preliminary result concerning the exponential stability of  $v_n$ . Let  
 207 us apply a Volterra transformation  $\mathcal{T} : L^2(0, L) \rightarrow L^2(0, L)$  of the form

$$208 \quad (2.9) \quad z(x) = \mathcal{T}[v_n](x) := v_n(x) - \int_x^L p(x, y)v_n(y)dy$$

210 to the solution to the last equation of the cascade system, with  $p$  defined on  $\Pi$ . Under  
 211 appropriate choice of  $p(\cdot, \cdot)$ , we prove that this transformation maps solution  $v_n$  to  
 212 the trajectory  $z$  satisfying the following target equation in  $[0, \infty) \times [0, L]$ :

$$213 \quad (2.10) \quad z_t + z_x + z_{xxx} + \omega z = 0,$$

$$214 \quad (\text{BC-A}, z) : z(t, 0) = z(t, L) = z_x(t, L) = 0,$$

$$215 \quad (\text{BC-B}, z) : z_{xx}(t, 0) = z(t, L) = z_x(t, L) = 0$$

217 with control given by (2.7). Indeed, performing standard differentiations and integra-  
 218 tions by parts (for more intuition about such operations, the reader can refer to [4]),  
 219 we derive the following equations:

$$220 \quad z_t(t, x) + z_x(t, x) + z_{xxx}(t, x) + \omega z(t, x) =$$

$$221 \quad - \int_x^L (p_{xxx}(x, y) + p_{yyy}(x, y) + p_y(x, y) + (\omega + 1)p(x, y)) v_n(t, y)dy$$

$$222 \quad + p(x, L)v_{n,xx}(t, L) + p(x, L)v_n(t, L) + p_{yy}(x, L)v_n(t, L) - p_y(x, L)v_{n,x}(t, L)$$

$$223 \quad + \left( \omega + 1 + \frac{d^2}{dx^2}p(x, x) + \frac{d}{dx}p_x(x, x) + p_{xx}(x, x) - p_{yy}(x, x) \right) v_n(t, x)$$

$$224 \quad + \left( p_x(x, x) + p_y(x, x) + 2\frac{d}{dx}p(x, x) \right) v_{n,x}(t, x).$$

226 By choosing  $p(\cdot, \cdot)$  satisfying the equations

$$227 \quad (2.11) \quad \begin{cases} p_{xxx} + p_{yyy} + p_x + p_y + (\omega + 1)p = 0, & (x, y) \in \Pi, \\ p(x, x) = p(x, L) = 0, & x \in [0, L], \\ p_x(x, x) = \frac{\omega+1}{3}(L-x), & x \in [0, L], \end{cases}$$

229 we obtain target system (2.10) for both boundary problems (BC-A,  $z$ ) and (BC-B,  
 230  $z$ ). Solutions to (2.11) are proven in [4] to be unique in the space  $C^3(\Pi)$  by following  
 231 successive approximation methods. The feedback control  $u$  is easily checked to satisfy  
 232 (2.7) if we use (2.9) and also calculate the value of the second derivative, viz.,

$$233 \quad z_{xx}(x) = v_{n,xx}(x) + \frac{d}{dx}p(x, x)v_n(x) + p(x, x)v_{n,x}(x) + p_x(x, x)v_n(x)$$

$$234 \quad - \int_0^L p_{xx}(x, y)v_n(y)dy$$

236 for  $x = 0$ .

237 Now, as we saw in [Propositions 2.1](#) and [2.3](#) of the previous subsection, solution  
 238  $z$  to target system [\(2.10\)](#) is asymptotically stable for every length  $L > 0$  if  $\omega > 0$   
 239 under boundary conditions (BC-A,  $z$ ) and if  $\omega \geq 1$  under boundary conditions (BC-  
 240 B,  $z$ ). This implies the asymptotic stability of  $v_n$ , solution to [\(1.1\)](#), with control given  
 241 by [\(2.7\)](#) for each of the boundary problems (BC-A) and (BC-B). The latter follows  
 242 from the fact that, as proven in [\[4\]](#), transformation [\(2.9\)](#), mapping solution  $v_n$  to  
 243  $z$ , is bounded and invertible with bounded inverse. So, for every  $\bar{d} > 0$ , there exist  
 244  $\omega_0, \bar{c} > 0$  such that for all  $\omega \geq \omega_0$ , we have

$$245 \quad (2.12) \quad \|v_n\|_{L^2(0,L)} \leq \bar{c}e^{-\bar{d}t} \|v_n^0\|_{L^2(0,L)}, \quad \forall t \geq 0.$$

247 To prove the asymptotic stability of the full state, consider vector  $v_{[n-1]} :=$   
 248  $(v_1 \ \cdots \ v_{n-1})^\top$ . Then,  $v_{[n-1]}$  satisfies the following equations:

$$249 \quad (2.13) \quad \begin{cases} v_{[n-1],t} + v_{[n-1],x} + v_{[n-1],xxx} = (A_{n-1} - I_{n-1})v_{[n-1]} + \ell v_n, \\ \text{(BC-A, } v_{[n-1]}) : v_{[n-1]}(t, 0) = v_{[n-1]}(t, L) = v_{[n-1],x}(t, L) = 0, \\ \text{(BC-B, } v_{[n-1]}) : v_{[n-1],xx}(t, 0) = v_{[n-1]}(t, L) = v_{[n-1],x}(t, L) = 0, \end{cases}$$

250 where

$$\ell := (0 \ \cdots \ 0 \ 1)^\top.$$

251 To prove stability of this system, consider a Lyapunov functional of the form

$$252 \quad W(t) = \int_0^L \pi(x) |v_{[n-1]}(x)|^2 dx$$

253  
 254 along the  $L^2(0, L)^{n-1}$  solutions  $v_{[n-1]}$  to the last equations, where  $\pi(\cdot)$  is a positive  
 255 increasing  $C^3$  function to be chosen. After substituting the above equations satisfied  
 256 by  $v_{[n-1]}$  and applying integrations by parts, we obtain, for the time-derivative of  $W$ ,

$$257 \quad \begin{aligned} \dot{W}(t) = & \int_0^L (\pi'''(x) + \pi'(x)) |v_{[n-1]}(x)|^2 dx - 3 \int_0^L \pi'(x) |v_{[n-1],x}(x)|^2 dx \\ & - \int_0^L \pi(x) v_{[n-1]}^\top(x) (2I_{n-1} - A_{n-1}^\top - A_{n-1}) v_{[n-1]}(x) dx \\ & + 2 \int_0^L \pi(x) v_{n-1}(x) v_n(x) dx + W_0 \end{aligned}$$

258  
 259 with

$$260 \quad W_0 := \left[ -(\pi''(x) + \pi(x)) |v_{[n-1]}(x)|^2 + \pi(x) \left( |v_{[n-1],x}(x)|^2 - 2v_{[n-1],xx}^\top(x) v_{[n-1]}(x) \right) \right. \\ \left. + 2\pi'(x) v_{[n-1],x}^\top(x) v_{[n-1]}(x) \right]_0^L.$$

261 Matrix  $2I_{n-1} - A_{n-1}^\top - A_{n-1}$  is positive definite and its eigenvalues are given by

$$\rho := 2 - 2 \cos \frac{\pi j}{n}, \quad j = 1, \dots, n-1.$$

262 Consequently, its minimal eigenvalue is given by

$$263 \quad (2.15) \quad \rho_n := \lambda_{\min}(2I_{n-1} - A_{n-1}^\top - A_{n-1}) = 2 - 2 \cos \frac{\pi}{n}, \quad \mathbb{N} \ni n \geq 2.$$



268 Since  $\pi'(x) \geq 0$ , by use of Young's inequality we obtain

$$269 \quad \dot{W}(t) \leq \int_0^L (\pi'''(x) + \pi'(x) - \rho_n \pi(x)) |v_{[n-1]}(x)|^2 dx$$

$$270 \quad + 2\delta \int_0^L \pi(x) |v_{[n-1]}(x)|^2 dx + \frac{1}{2\delta} \int_0^L \pi(x) v_n^2 dx + W_0,$$

271

272 and  $\delta > 0$  is chosen sufficiently small as in (A.4) in the proof of Lemma 2.2 of the  
273 previous subsection.

274 Now, we choose  $\pi(\cdot)$  for each of the two boundary problems as follows.

For (BC-A,  $v_{[n-1]}$ ) we choose

$$\pi(\cdot) = 1.$$

275 From this, taking also into account the exponential stability of  $v_n$  (2.12), we get  
276 for the case (BC-A,  $v_{[n-1]}$ ) the following estimate:

$$277 \quad (2.16) \quad \dot{W}(t) \leq -2dW(t) + \frac{1}{2\delta} \pi(L) \bar{c}^2 e^{-2\bar{d}t} \|v_n^0\|_{L^2(0,L)}^2$$

278

279 with  $d = \rho_n/2 - \delta$ .

280 For (BC-B,  $v_{[n-1]}$ ) we choose a positive and increasing  $\pi(\cdot)$  satisfying (2.5) (see  
281 Assertion 1 in Lemma 2.2) with  $\lambda = \frac{\rho_n}{2} - \delta$  and  $b > 0$ . It turns out by Assertion 1  
282 that there are  $\pi(\cdot)$  and  $b > 0$  satisfying this equation for any  $L > 0$ , when  $\lambda = 1 - \delta$ ,  
283 corresponding to  $\rho_2 = 2$  (for  $n = 2$ ). Then, the exponential decay of the Lyapunov  
284 functional is ensured similarly as in Proposition 2.3. More precisely, there exists  $d > 0$   
285 such that for all  $L > 0$ , (2.16) is satisfied for (BC-B,  $v_{[n-1]}$ ) as well. Also, as shown  
286 in Proposition 2.3, for  $n > 2$ , which renders  $\rho_n < 2$ , (2.16) is satisfied for some  $\pi(\cdot)$ ,  
287  $b > 0$ , only when  $0 < L \leq \bar{L}$ , with  $\bar{L}$  depending on  $n$ .

288 Combining the above results, from (2.16), which holds for both (BC-A,  $v_{[n-1]}$ )  
289 and (BC-B,  $v_{[n-1]}$ ), we derive by Gronwall's inequality

$$290 \quad (2.17) \quad W(t) \leq e^{-2dt} W(0) + \frac{\pi(L) \bar{c}^2}{4\delta(d - \bar{d})} \left( e^{-2\bar{d}t} - e^{-2dt} \right) \|v_n^0\|_{L^2(0,L)}^2,$$

291

292 recalling also that  $\bar{d}$ , depending on the parameter  $\omega$  of the control laws, can be chosen  
293 such that  $\bar{d} > d$ . Combining (2.17) and (2.12), we get

$$294 \quad \|v\|_{L^2(0,L)^n} \leq \|v_{[n-1]}\|_{L^2(0,L)^{n-1}} + \|v_n\|_{L^2(0,L)} \leq \sqrt{\frac{\pi(L)}{\pi(0)}} e^{-dt} \|v_{[n-1]}(0, \cdot)\|_{L^2(0,L)^{n-1}}$$

$$295 \quad + \frac{\bar{c} \sqrt{\pi(L)}}{2\sqrt{\pi(0)\delta(\bar{d} - d)}} \sqrt{e^{-2dt} - e^{-2\bar{d}t}} \|v_n^0\|_{L^2(0,L)} + \bar{c} e^{-\bar{d}t} \|v_n^0\|_{L^2(0,L)}.$$

296

297 The last inequality leads to (2.8) for a suitable choice of  $c$ .

298 This concludes the proof and shows also that although the exponential conver-  
299 gence to zero of  $v_n$  can become arbitrarily fast by the choice of parameter  $\omega$  inside  
300 the controls, solution  $v$  to the whole cascade system has a fixed convergence rate.  $\square$

301 *Remark 2.5.* Note that in the above proof, parameter  $\rho_n$  in (2.15), depending  
302 on  $n$ , does not permit the stabilization of the closed-loop system for any number  
303 of equations  $n$  when the length of the domain  $L$  is arbitrary. As was shown in

304 **Proposition 2.3**, the damped KdV equation in the case of boundary conditions of  
 305 the type (BC-B) requires a damping with coefficient  $\lambda$  larger than a critical damping  
 306 coefficient  $\lambda_0$ . The parameter  $\rho_n$ , which appears in the stabilization of the closed-  
 307 loop system corresponding to the damping coefficient, is decreasing with  $n$ . For  $n > 2$   
 308 the stabilization cannot be ensured for any  $L > 0$ , since, because of  $\rho_n$ , the damping  
 309 coefficient becomes lower than the critical one, while for  $n = 2$ , the damping coefficient  
 310 of the coupled equation is exactly equal to the critical one.

311 **3. Observer Design and Output Feedback Stabilization.** In this section,  
 312 we first present the proposed observer, along with its convergence proof for each of the  
 313 boundary control problems (BC-A) and (BC-B). Then, we study the output feedback  
 314 stabilization of system (1.1) with controls placed on the left boundaries as described in  
 315 each of problems (BC-A) and (BC-B). We note here that, even though the considered  
 316 system is linear, the use of the high-gain observer design is instrumental in the output  
 317 feedback control in the two following manners and is based on the methodology [17],  
 318 introduced for quasilinear hyperbolic systems: (1) For (BC-B), the choice of the  
 319 high-gain parameter is needed to establish convergence of the observer, contrary to  
 320 a simpler Luenberger observer design, which would be sufficient for (BC-A). (2) The  
 321 high-gain parameter is used in the stabilization of the closed-loop system for both  
 322 boundary control problems (BC-A) and (BC-B).

323 In the following subsection we present the observer for the cascade system, whose  
 324 exponential stability relies on the result presented in **Proposition 2.3** of **Section 2**.

325 **3.1. Observer.** Define, first, diagonal matrix  $\Theta_n$  by

$$326 \quad \Theta_n := \text{diag}(\theta, \theta^2, \dots, \theta^n),$$

327 where  $\theta > 0$  represents a gain, which will be selected later. Consider a vector gain  
 328  $K_n = (k_1 \ \dots \ k_n)^\top$ , and let  $P \in \mathbb{R}^{n \times n}$  be a symmetric and positive definite matrix  
 329 satisfying a quadratic Lyapunov equation of the form

$$330 \quad (3.1) \quad P(A_n + K_n C) + (A_n + K_n C)^\top P = -I_n.$$

331 The previous equation is always feasible due to the observability of the pair  $(A_n, C)$ .

332 Then, our observer is defined to satisfy the following equations in  $(0, \infty) \times (0, L)$ :

$$333 \quad (3.2) \quad \hat{v}_t(t, x) + \hat{v}_x(t, x) + \hat{v}_{xxx}(t, x) = (A_n - B)\hat{v}(t, x) - \Theta_n K_n (y(t, x) - C\hat{v}(t, x))$$

335 with boundary conditions for each of (BC-A) and (BC-B) as follows:

$$336 \quad (3.3a) \quad \begin{aligned} & \hat{v}_i(t, 0) = 0, \quad i = 1, \dots, n-1, \quad \forall t > 0, \\ & \text{(BC-A): } \hat{v}_n(t, 0) = u(t), \quad \forall t > 0, \\ & \hat{v}(t, L) = \hat{v}_x(t, L) = 0, \quad \forall t > 0, \end{aligned}$$

$$337 \quad (3.3b) \quad \begin{aligned} & \hat{v}_{i,xx}(t, 0) = 0, \quad i = 1, \dots, n-1, \quad \forall t > 0, \\ & \text{(BC-B): } \hat{v}_{n,xx}(t, 0) = u(t), \quad \forall t > 0, \\ & \hat{v}(t, L) = \hat{v}_x(t, L) = 0, \quad \forall t > 0. \end{aligned}$$

338 and initial condition

$$\hat{v}(0, x) = \hat{v}^0(x), \quad x \in (0, L).$$

339 The main observer result is stated in the following theorem.

340 **THEOREM 3.1.** *Consider system (1.1) with output (1.4) and boundary conditions*  
 341 *satisfying (1.2) ((BC-A) or (BC-B)) and  $v^0 \in L^2(0, L)^n, u \in L^2_{loc}(0, \infty)$ . Consider,*  
 342 *also,  $P$  and  $K_n$  satisfying a Lyapunov equation as in (3.1). Then (3.2), with boundary*  
 343 *conditions (3.3) and initial condition  $\hat{v}^0 \in L^2(0, L)^n$ , is an observer for solution of*  
 344 *(1.1), in the sense that for  $\theta$  large it estimates the state  $v$  arbitrarily fast. More*  
 345 *precisely, for every  $\kappa > 0$ , there exists  $\theta_0$  such that for every  $\theta > \theta_0$ , the following*  
 346 *holds for all  $v^0, \hat{v}^0 \in L^2(0, L)^n, t \geq 0$ :*

$$347 \quad (3.4) \quad \|\hat{v}(t, \cdot) - v(t, \cdot)\|_{L^2(0, L)^n} \leq \nu \theta^{n-1} e^{-\kappa t} \|\hat{v}^0(\cdot) - v^0(\cdot)\|_{L^2(0, L)^n}$$

349 *with  $\nu > 0$ , depending on  $n$  and  $L$ .*

350 *Proof.* First, we prove in [Appendix B.2](#) that observer system (3.2)–(3.3) is well-  
 351 posed. Then, to prove its asymptotic convergence to the state  $v$ , let us define a scaled  
 352 observer error  $\varepsilon$  by

$$353 \quad (3.5) \quad \varepsilon = \Theta_n^{-1}(\hat{v} - v).$$

355 Then,  $\varepsilon$  satisfies the following equations:

$$356 \quad (3.6) \quad \varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = \theta(A_n + K_n C)\varepsilon - B\varepsilon$$

358 and boundary conditions for each of the cases (BC-A) and (BC-B) as follows:

$$359 \quad (3.7a) \quad \varepsilon(t, 0) = \varepsilon(t, L) = \varepsilon_x(t, L) = 0,$$

$$360 \quad (3.7b) \quad \varepsilon_{xx}(t, 0) = \varepsilon(t, L) = \varepsilon_x(t, L) = 0.$$

362 We expect that solutions to the previous coupled equations can approach zero  
 363 exponentially fast, since  $A_n + K_n C$  being Hurwitz will exhibit a damping effect (as  
 364 in the single KdV equation), with its magnitude being controlled by  $\theta$ . Indeed, to  
 365 prove exponential stability, we choose the following Lyapunov functional defined on  
 366 the  $L^2(0, L)^n$  solutions to the observer error equations:

$$367 \quad (3.8) \quad V(t) := \int_0^L \mu(x) \varepsilon^\top(x) P \varepsilon(x) dx$$

369 with positive  $\mu(\cdot) \in C^3[0, L]$  to be chosen suitably for each of the boundary conditions  
 370 cases. Taking its time-derivative and substituting (3.6) and Lyapunov equation (3.1)  
 371 yields

$$372 \quad \dot{V}(t) = \int_0^L \mu(x) [-\partial_x^3(\varepsilon^\top(x) P \varepsilon(x)) - \partial_x(\varepsilon^\top(x) P \varepsilon(x)) + 3\partial_x(\varepsilon_x^\top(x) P \varepsilon_x(x)) \\ 373 \quad - \theta \varepsilon^\top(x) \varepsilon(x) - 2\varepsilon^\top(x) P B \varepsilon(x)] dx.$$

375 Performing successive integrations by parts, we obtain

$$376 \quad \dot{V}(t) \leq \int_0^L \left( \mu'''(x) + \mu'(x) + \left( -\theta \frac{1}{|P|} + 2 \frac{|P|}{\lambda_{\min}(P)} \right) \mu(x) \right) \varepsilon^\top(x) P \varepsilon(x) dx \\ 377 \quad - 3 \int_0^L \mu'(x) \varepsilon_x^\top(x) P \varepsilon_x(x) dx + V_0,$$

378

379 where

$$380 \quad V_0 := [(-\mu''(x) - \mu(x))\varepsilon^\top(x)P\varepsilon(x) - \mu(x)(\varepsilon_{xx}^\top(x)P\varepsilon(x) + \varepsilon^\top(x)P\varepsilon_{xx}(x)) \\ 381 \quad (3.9) \quad + \mu(x)\varepsilon_x^\top(x)P\varepsilon_x(x) + \mu'(x)(\varepsilon_x^\top(x)P\varepsilon(x) + \varepsilon^\top(x)P\varepsilon_x(x))]_0^L$$

383 and  $\lambda_{\min}(P)$  is the minimal eigenvalue of  $P$ .

384 Let us now choose  $\mu$  for boundary conditions case (3.7a) as follows:

$$385 \quad (3.10) \quad \mu(\cdot) := 1,$$

387 for which we obtain

$$388 \quad V_0 = -\varepsilon_x^\top(0)P\varepsilon_x(0) \leq 0.$$

390 Note that given (3.10) for the boundary conditions case (3.7a), for every  $\theta > \theta_{0,A}$   
391 with

$$392 \quad \theta_{0,A} := 2 \frac{|P|^2}{\lambda_{\min}(P)},$$

394 we get

$$395 \quad (3.11) \quad \dot{V}(t) \leq -2\kappa_A V(t), \quad t \geq 0$$

397 for some  $\kappa_A > 0$ .

398 Considering boundary conditions of case (3.7b), (3.9) is written as

$$399 \quad V_0 := -(\varepsilon_x^\top(0) \quad \varepsilon^\top(0)) \begin{pmatrix} P\mu(0) & P\mu'(0) \\ P\mu'(0) & -P(\mu''(0) + \mu(0)) \end{pmatrix} \begin{pmatrix} \varepsilon_x(0) \\ \varepsilon(0) \end{pmatrix}.$$

401 For these boundary conditions (3.7b), we see that for all

$$402 \quad \theta \geq \theta_{0,B} := 2 \frac{|P|^2}{\lambda_{\min}(P)} + 2|P|,$$

404 Assertion 1 ((2.5) in Lemma 2.2) is satisfied with  $\mu(\cdot)$  in the place of  $\pi(\cdot)$ ,  $\lambda =$   
405  $\theta \frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)}$ ,  $\lambda_0 = 1$ , and  $b = \kappa_B$  for some  $\kappa_B > 0$  depending on  $\theta$ . For all  
406  $\theta \geq \theta_{0,B}$ , we choose, therefore,  $\pi(\cdot) = \mu(\cdot)$  satisfying (2.5), and we derive again (3.11)  
407 with  $\kappa_A$  substituted by  $\kappa_B$ .

Combining the previous estimates, we directly obtain (3.4) with

$$\nu := \sqrt{\frac{\mu(L)}{\mu(0)}} \sqrt{\frac{|P|}{\lambda_{\min}(P)}},$$

408 and this concludes the proof of Theorem 3.1.  $\square$

409 **3.2. Output feedback stabilization.** Next, it is proven that by plugging the  
410 observer's state considered in Theorem 3.1 into the feedback laws (2.7) of the previous  
411 section, the closed-loop system is stabilized. This is done in two steps. First, it is  
412 proven that the considered output feedback law stabilizes arbitrarily fast the solution  
413 of the last KdV equation and second, the stabilization of the whole cascade system  
414 of KdV equations follows. However, for system with boundary conditions (BC-B),  
415 stabilization for any  $L$  is only achieved when  $n = 2$ , corresponding to a cascade system  
416 of two equations, while for  $n > 2$ , stabilization is achieved for small  $L$ , following the  
417 result of Proposition 2.3 of the previous section. Even if this requirement is restrictive,  
418 we find several physical applications, where only two coupled equations appear in the  
419 model; see [20]. These statements are presented in the following theorem.

420 **THEOREM 3.2.** Consider the closed-loop system (1.1)–(3.2), output (1.4), and  
 421 boundary conditions being of the form (BC-A) or (BC-B). Then, for any  $\bar{d} > 0$ ,  
 422 there exist an output feedback law  $u(t)$  of the form (2.7), where  $v$  is substituted by  
 423 the observer state  $\hat{v}$ , and constants  $\theta_0, \omega_0, \bar{c} > 0$  such that for any design parameters  
 424  $\theta > \theta_0, \omega > \omega_0$  (with  $\theta$  involved in the observer and  $\omega$  involved in the control laws),  
 425 the closed-loop system solution with  $v^0, \hat{v}^0 \in L^2(0, L)^n$  satisfies the following stability  
 426 inequality (on the estimation error and last observer state):

$$427 \quad (3.12) \quad \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}_n\|_{L^2(0,L)} \\ 428 \quad \leq \bar{c}e^{-\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)}), \quad \forall t \geq 0.$$

431 Moreover, whenever the previous assertion holds, we get the following (full state  
 432 convergence):

433 (a) When boundary conditions (BC-A) hold with  $n \geq 2$ , then for every  $L > 0$ ,  
 434 there exist constants  $c, d > 0$ , such that solutions  $v, \hat{v}$  satisfy the following:

$$435 \quad (3.13) \quad \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} \\ 436 \quad \leq ce^{-dt} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}), \quad \forall t \geq 0$$

437 with  $d$  depending on  $n$ .

438 (b) When boundary conditions (BC-B) hold with  $n = 2$ , then for every  $L > 0$ ,  
 439 there exist constants  $c, d > 0$  such that solutions  $v, \hat{v}$  satisfy (3.13).

440 (c) When (BC-B) holds, with  $n > 2$  there exists  $\bar{L} > 0$  small such that asymptotic  
 441 stability (3.13) is guaranteed for all  $L \in (0, \bar{L}]$ .

442 *Proof.* To address the closed-loop control problem, let us rewrite observer error  
 443 and observer coupled equations, viz. (see (3.6), (3.2))

$$444 \quad (3.14) \quad \begin{cases} \varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = \theta(A_n + K_n C)\varepsilon - B\varepsilon, \\ \hat{v}_t + \hat{v}_x + \hat{v}_{xxx} = (A_n - B)\hat{v} + \theta\Theta_n K_n \varepsilon_1, \end{cases}$$

445 with boundary conditions (3.7), (3.3).

446 Let us perform a Volterra transformation to the solution of the  $n$ th equation of  
 447 the observer, which by (3.2) is written as

$$448 \quad (3.15) \quad \hat{v}_{n,t} + \hat{v}_{n,x} + \hat{v}_{n,xxx} = \hat{v}_n + k_n \theta^{n+1} \varepsilon_1.$$

449 The Volterra transformation

$$450 \quad (3.16) \quad q(x) = \mathcal{T}[\hat{v}_n](x) := (k_n \theta^{n+1})^{-1} \hat{v}_n(x) - (k_n \theta^{n+1})^{-1} \int_x^L p(x, y) \hat{v}_n(y) dy,$$

451 under appropriate choice of  $p(\cdot, \cdot)$  maps (3.15) into target system

$$452 \quad (3.17) \quad q_t + q_x + q_{xxx} = -\omega q + \varepsilon_1 - \int_x^L p(x, y) \varepsilon_1(t, y) dy$$

453 with  $\omega$  a constant involved in the controller, and boundary conditions for each of the  
 454 two considered cases as follows:

$$455 \quad (3.18a) \quad q(t, 0) = q(t, L) = q_x(t, L) = 0,$$

$$456 \quad (3.18b) \quad q_{xx}(t, 0) = q(t, L) = q_x(t, L) = 0.$$

464 Then, the kernel functions  $p(\cdot, \cdot)$  satisfy (2.11) for both problems (3.18a), (3.18b). It  
 465 is easy to check this if we apply successive differentiations of (3.16) as in Theorem 2.4,  
 466 we obtain the above target system by choosing  $p(\cdot, \cdot)$  satisfying (2.11). Subsequently,  
 467 the output feedback control  $u(\cdot)$  for (BC-A) is given by

$$468 \quad (3.19a) \quad u(t) = \int_0^L p(0, y) \hat{v}_n(t, y) dy,$$

469  
 470 and for (BC-B),

$$471 \quad (3.19b) \quad u(t) = -\frac{\omega + 1}{3} L \hat{v}_n(t, 0) + \int_0^L p_{xx}(0, y) \hat{v}_n(t, y) dy.$$

473 As noticed in proof of Theorem 2.4, it has been proven that the kernel equations  
 474 (2.11) are solvable in  $\Pi$  and the corresponding Volterra transformation is bounded  
 475 and injective with bounded inverse.

476 Consider now the Lyapunov function

$$477 \quad U_1(t) = U_{1,1}(t) + U_{1,2}(t);$$

$$478 \quad U_{1,1}(t) := \int_0^L \mu(x) \varepsilon^\top(x) P \varepsilon(x) dx, U_{1,2}(t) := \int_0^L \sigma(x) q^2(x) dx$$

480 along the solutions to (3.6)–(3.7) and (3.17)–(3.18), where  $U_{1,1}$  is the same as (3.8) of  
 481 Theorem 3.1 and  $\sigma(\cdot)$  is a positive  $C^3$  increasing function in  $[0, L]$  to be chosen later.

482 Taking the time-derivative of  $U_{1,2}$  and substituting (3.17), we infer

$$483 \quad \dot{U}_{1,2}(t) = \int_0^L (\sigma'''(x) + \sigma'(x) - 2\omega) q^2(x) dx - 3 \int_0^L \sigma'(x) q_x^2(x) dx$$

$$484 \quad + 2 \int_0^L \sigma(x) q(x) \varepsilon_1(x) dx - 2 \int_0^L \sigma(x) q(x) \int_x^L p(x, y) \varepsilon_1(y) dy dx$$

$$485 \quad + [(-\sigma''(x) - \sigma(x)) q^2(x) - 2\sigma(x) q_{xx}(x) q(x) + \sigma(x) q_x^2(x)$$

$$486 \quad + 2\sigma'(x) q_x(x) q(x)]_0^L.$$

488 By using

$$489 \quad 2 \int_0^L \sigma(x) q(x) \int_x^L p(x, y) \varepsilon_1(y) dy dx \leq U_{1,2}(t) + \sigma(L) \int_0^L \left( \int_x^L p(x, y) \varepsilon_1(y) dy \right)^2 dx$$

$$490 \quad \leq U_{1,2}(t) + L^2 \sigma(L) \max_{x, y \in [0, L]} p^2(x, y) \int_0^L \varepsilon_1^2(y) dy$$

$$491 \quad \leq U_{1,2}(t) + L^2 \frac{\sigma(L)}{\mu(0) \lambda_{\min}(P)} \max_{x, y \in [0, L]} p^2(x, y) U_{1,1}(t),$$

493 we get

$$494 \quad \dot{U}_{1,2}(t) \leq \int_0^L (\sigma'''(x) + \sigma'(x) - 2(\omega - 1)) q^2(x) dx - 3 \int_0^L \sigma'(x) q_x^2(x) dx + h U_{1,1}(t)$$

$$495 \quad + [(-\sigma''(x) - \sigma(x)) q^2(x) - 2\sigma(x) q_{xx}(x) q(x) + \sigma(x) q_x^2(x)$$

$$496 \quad + 2\sigma'(x) q_x(x) q(x)]_0^L,$$

498 where  $h := (L^2 \max_{x,y \in [0,L]} p^2(x,y) + 1) \frac{\sigma(L)}{\mu(0)\lambda_{\min}(P)}$ .

499 We can prove that for each of the two cases of boundary conditions we get

$$500 \quad (3.21) \quad \dot{U}_1(t) \leq -2\bar{d}U_1(t).$$

502 **Case (BC-A):**

503 We choose  $\mu(\cdot) = \sigma(\cdot) = 1$ , and we obtain

$$504 \quad \dot{U}_{1,2}(t) \leq -2(\omega - 1)U_{1,2}(t) + hU_{1,1}(t).$$

506 As seen in [Theorem 3.1](#), for  $\mu(\cdot) = 1$ , we have

$$507 \quad \dot{U}_{1,1}(t) \leq \left( -\theta \frac{1}{|P|} + 2 \frac{|P|}{\lambda_{\min}(P)} \right) U_{1,1}(t).$$

509 Combining the last two equations, if we choose  $\theta, \omega$  as

$$510 \quad (3.22) \quad \theta > h|P| + 2 \frac{|P|^2}{\lambda_{\min}(P)}, \quad \omega > 1,$$

512 we get a  $\bar{d} > 0$  such that [\(3.21\)](#) holds.

513 **Case (BC-B):**

514 We see for this case of boundary conditions that for all

$$515 \quad (3.23) \quad \theta \geq \theta_0 := 2 \frac{|P|^2}{\lambda_{\min}(P)} + h|P| + 2|P|,$$

517 Assertion 1 ([\(2.5\)](#) in [Lemma 2.2](#)) is satisfied with  $\mu(\cdot)$  in the place of  $\pi(\cdot)$ ,  $\lambda =$   
 518  $\theta \frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)} - \frac{h}{2}$ ,  $\lambda_0 = 1$ . For all  $\theta \geq \theta_0$ , we choose, therefore,  $\mu(\cdot)$  satisfying [\(2.5\)](#),  
 519 and we get that the first term of the right-hand side of

$$520 \quad \dot{U}_1(t) \leq \int_0^L \left( \mu'''(x) + \mu'(x) - 2 \left( \theta \frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)} - \frac{h}{2} \right) \mu(x) \right) \varepsilon^\top(x) P \varepsilon(x) dx$$

$$521 \quad + \dot{U}_{1,2}(t), \quad t \geq 0$$

523 becomes negative.

Similarly, for every

$$\omega \geq 2,$$

524 we can find  $\sigma(\cdot) = \pi(\cdot)$  satisfying [\(2.5\)](#) with  $\lambda = \omega - 1$  and  $\lambda_0 = 1$ , and by virtue of  
 525 [Proposition 2.3](#), the right-hand side of [\(3.20\)](#) becomes negative. Hence, returning to  
 526  $\dot{U}_1$  and choosing  $\theta \geq \theta_0$  and  $\omega \geq 2$ , we can always find  $\mu(\cdot), \sigma(\cdot)$  as in Assertion 1 of  
 527 [Lemma 2.2](#) in a such way that we always get a  $c_2 > 0$ , satisfying again [\(3.21\)](#).

528 Consequently, for each of the two problems (BC-A) and (BC-B), for each  $\bar{d} > 0$   
 529 we can find  $\theta, \omega$ , chosen as before in such a way that there exists constant  $\gamma > 0$   
 530 depending polynomially on  $\theta$  such that

$$531 \quad \|\hat{v} - v\|_{L^2(0,L)^n} + \|q\|_{L^2(0,L)} \leq \gamma e^{-\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|q(0, \cdot)\|_{L^2(0,L)}), \quad \forall t \geq 0.$$

533 Transformation  $\mathcal{T}$  is bounded with bounded inverse (see the comments in [Theo-](#)  
 534 [rem 2.4](#)), and, therefore, we obtain an inequality as [\(3.12\)](#).

535 *Remark 3.3.* The previous calculations indicate that gain  $\theta$  appearing in observer  
 536 system (3.2)–(3.3) is crucial in the stabilization of the closed-loop system. Indeed,  
 537 in (3.23), we see that choice of  $\theta$  compensates for some terms appearing therein.  
 538 The dependence of the terms on the eigenvalues of matrix  $P$  indicates that a simpler  
 539 Luenberger observer with pole placement would not suffice for the stabilization of  
 540 the closed-loop system. These terms play the role of the nonlinearities, appearing in  
 541 the Lyapunov derivative used for the observer error in high-gain observer designs for  
 542 finite-dimensional systems. Although in finite dimensions, a pole-placement observer  
 543 is enough for linear systems, in the present framework of infinite dimensions, a design  
 544 similar to high-gain observers in finite dimensions is required.

545 We are now in a position to prove the closed-loop stability for the whole system  
 546 following the methodology of Theorem 2.4. Let  $\hat{v}_{[n-1]} := (\hat{v}_1 \cdots \hat{v}_{n-1})^\top$ . Then,  $\hat{v}_{[n-1]}$   
 547 satisfies the following equations

$$\begin{aligned} & \hat{v}_{[n-1],t} + \hat{v}_{[n-1],x} + \hat{v}_{[n-1],xxx} = (A_{n-1} - I_{n-1})\hat{v}_{[n-1]} + \ell\hat{v}_n + \Theta_{n-1}K_{n-1}(\hat{v}_1 - v_1), \\ 548 \text{ (BC-A, } \hat{v}_{[n-1]}) : \hat{v}_{[n-1]}(t, 0) = \hat{v}_{[n-1]}(t, L) = \hat{v}_{[n-1],x}(t, L) = 0, \\ 549 \text{ (BC-B, } \hat{v}_{[n-1]}) : \hat{v}_{[n-1],xx}(t, 0) = \hat{v}_{[n-1]}(t, L) = \hat{v}_{[n-1],x}(t, L) = 0, \end{aligned}$$

550 where  $\ell := (0 \ \cdots \ 0 \ 1)^\top$  and  $\Theta_{n-1}, K_{n-1}$  are involved in observer (3.2).

551 By choosing

$$552 \quad U_2(t) = \int_0^L \pi(x) |\hat{v}_{[n-1]}(x)|^2 dx$$

554 as a Lyapunov functional along the  $L^2(0, L)^{n-1}$  solutions to the last equations, with  
 555  $\pi(\cdot)$  a positive increasing  $C^3$  function, we obtain

$$\begin{aligned} 556 \quad \dot{U}_2(t) &= \int_0^L (\pi'''(x) + \pi'(x)) |\hat{v}_{[n-1]}(x)|^2 dx - 3 \int_0^L \pi'(x) |\hat{v}_{[n-1],x}(x)|^2 dx \\ 557 \quad & - 2 \int_0^L \pi(x) \hat{v}_{[n-1]}^\top(x) \text{Sym}(I_{n-1} - A_{n-1}) \hat{v}_{[n-1]}(x) dx \\ 558 \quad & + 2 \int_0^L \pi(x) \hat{v}_{n-1}(x) \hat{v}_n(x) dx + 2 \int_0^L \pi(x) \hat{v}_{[n-1]}^\top \Theta_{n-1} K_{n-1} (\hat{v}_1 - v_1) dx + U_{2,0}, \end{aligned}$$

560 where  $U_{2,0}$  is as  $W_0$  in (2.14) (see the proof of Theorem 2.4), while  $v_{[n-1]}$  is substituted  
 561 by  $\hat{v}_{[n-1]}$ . Applying Young's inequality, we get

$$\begin{aligned} 562 \quad \dot{U}_2(t) &\leq \int_0^L (\pi'''(x) + \pi'(x) - (\rho_n - 2\delta)\pi(x)) |\hat{v}_{[n-1]}(x)|^2 dx \\ 563 \quad (3.24) \quad & + \frac{1}{\delta} \int_0^L \pi(x) \hat{v}_n^2(x) dx + \frac{1}{\delta} \theta^{2n-2} |K_{n-1}|^2 \int_0^L \pi(x) |\hat{v}_1(x) - v_1(x)|^2 dx + U_{2,0} \end{aligned}$$

565 with  $\delta > 0$  chosen sufficiently small, as in (A.4), determined in the proof of Lemma 2.2  
 566 of previous section, and  $\rho_n$  defined in (2.15).

567 Now, to ensure negativity of the Lyapunov derivative, we choose  $\pi(\cdot)$  for each of  
 568 the two boundary problems as follows.

569 **Case (BC-A,  $\hat{v}_{[n-1]}$ ):**

$$570 \quad \pi(\cdot) = 1.$$



572 Then, in conjunction with the previously proven equation (3.12), we get, from (3.24),

$$573 \quad (3.25) \quad \dot{U}_2(t) \leq -2dU_2(t) + me^{-2\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)})^2,$$

575 where  $d := \rho_n - 2\delta > 0$  and

$$576 \quad (3.26) \quad m := \frac{1}{\delta} \pi(L) \bar{c}^2 \max(1, \theta^{2n-2} |K_{n-1}|^2).$$

578 **Case (BC-B,  $\hat{v}_{[n-1]}$ ):**

579 For boundary conditions (BC-B,  $\hat{v}_{[n-1]}$ ), to obtain an asymptotic stability result,  
580 we first check that for  $n = 2$ , we have  $\rho_n = 2$ . For this  $\rho_2$ , the proof of Lemma 2.2  
581 suggests that there exists  $\pi(\cdot)$  satisfying (2.5) for some  $b > 0$ , with the same  $\pi(\cdot)$ ,  
582  $\lambda = \frac{\rho_2}{2} - \delta$ . Then, a similar inequality as (3.25) is satisfied for all  $L > 0$ ,  $d = b$ , and  
583  $m$  as in (3.26). Additionally, following Assertion 2 in the proof of Lemma 2.2, we see  
584 that for any  $n > 2$ , implying  $\rho_n < 2$ , there exist again  $\pi(\cdot)$ ,  $d = b > 0$  such that (2.5)  
585 holds for  $L \in (0, \bar{L}]$ .

586 Now, we see that for both boundary problems (BC-A) and (BC-B), (3.25) gives

$$587 \quad U_2(t) \leq e^{-2dt} W(0) + \frac{m}{2d - 2\bar{d}} (e^{-2\bar{d}t} - e^{-2dt}) (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)})^2.$$

589 The latter implies

$$(3.27) \quad U_2(t) \leq \left( e^{-2dt} + \frac{m}{2d - 2\bar{d}} (e^{-2\bar{d}t} - e^{-2dt}) \right) (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n})^2.$$

592 Recalling that  $\bar{d}$  depends on the adjustable observer parameter  $\theta$ , we suppose, without  
593 loss of generality, that it can be chosen such that  $\bar{d} > d$ , so that the previous inequality  
594 has meaning.

595 Now, using trivial inequalities and by virtue of (3.27) and (3.12), we easily get

$$596 \quad \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} \leq \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}_n\|_{L^2(0,L)} + \|\hat{v}_{[n-1]}\|_{L^2(0,L)^{n-1}} \\ 597 \quad \leq \left[ \bar{c}e^{-\bar{d}t} + \frac{1}{\sqrt{\pi(0)}} \sqrt{e^{-2dt} + \frac{m}{2d - 2\bar{d}} (e^{-2\bar{d}t} - e^{-2dt})} \right] \\ 598 \quad \times (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}).$$

600 The latter completes the proof of Theorem 3.2, suggesting also that the asymptotic  
601 rate of the whole closed-loop cascade system is no larger than  $d$ , which is decreasing  
602 with  $n$ , contrary to the asymptotic rate for the last state  $v_n$ , which is adjusted by the  
603 observer and control parameters.  $\square$

604 *Remark 3.4.* The considered stabilization problem of underactuated and under-  
605 observed cascade systems of KdV equations was here limited to the linear case, and  
606 special forms of couplings. Even though stabilization results of the original nonlinear  
607 KdV equation can be found (see survey [3]), or observer results for some infinite-  
608 dimensional systems with nonlinearities (satisfying some “triangular structure”), as  
609 in [16, 18], extensions of our output feedback stabilization to more general couplings  
610 and/or nonlinearities are quite challenging and are thus left for future studies: a strong  
611 difficulty comes from the coefficients of the system’s differential operator, where the  
612 presence of distinct elements raises problems related to the notion of algebraic solv-  
613 ability, which has been given attention in [1, 19] and other works of the same authors.

614 In the Lyapunov-based approach we have considered, this problem translates into the  
 615 lack of a commutative property between a Lyapunov matrix and coefficients of sys-  
 616 tem's differential operator. Handling nonlinearities in a first-order term (for instance,  
 617 terms  $v_i v_{i,x}$  or even couplings of this type between the equations) and zero-order  
 618 term at the same time is also part of the challenge, as this commutative property  
 619 would not be fulfilled. We also refer to [9], where some open problems concerning  
 620 such coupling are presented, while the reader can understand the difficulties in the  
 621 controllability analysis of underactuated systems with nontrivial coefficients of the  
 622 differential operators and the presence of nonlinearities.

623 Notice yet that it could be possible to adopt an indirect approach, based on our  
 624 previous approaches [18], to deal with the case where the first-order and dispersion  
 625 terms would be of the form  $A_1 v_x + A_2 v_{xxx}$  for some  $A_1, A_2 \in \mathbb{R}^{n \times n}$ . Consideration of  
 626 linear lower triangular couplings of first-order and third-order terms would be feasible  
 627 as well, but more general cases remain open.

628 Notice also that while this underobserved problem is already challenging, the case  
 629 when only a boundary measurement is available (instead of an internal one, at least  
 630 localized to a part of the domain) is even more difficult: even though a solution does  
 631 exist for a single equation and boundary measurement [23], it does not easily extend  
 632 to the case of coupled KdV equations, via a backstepping and a single measurement  
 633 instead of a distributed one. For the case of  $n$  coupled equations, a backstepping  
 634 approach that would lead to an exponentially stable observer error system would fail,  
 635 even for  $n = 2$ , if the observations were fewer than the states. In addition, the control  
 636 problem of underactuated systems by itself is a hard problem, and if  $2 \times 2$  systems have  
 637 local solutions via backstepping (see [8]), they concern hyperbolic systems where a  
 638 dissipative target system is feasible. For the coupled KdV equations, the exponentially  
 639 stable target system for the observer error would be needed to be a damped system,  
 640 which cannot be achieved by a single observation.

641 Some possible generalizations of the present framework, as the ones described  
 642 before, will be a subject of our future work.

643 **4. Conclusion.** In this work, output feedback stabilization for a class of cascade  
 644 system of linear KdV equations was achieved. Two boundary control problems, with  
 645 controls placed on the left side of the last equation, were investigated. Distributed  
 646 measurement of the first state was considered, which provided an estimation (using a  
 647 high-gain observer) of the states fed in the control laws. The cascade system is stabi-  
 648 lized for both boundary problems but with a limitation on the number of equations  
 649 and length of the domain for the second one.

650 Future developments might include the same stabilization framework, but with  
 651 more general couplings (in zero-order and first-order derivative terms), including lo-  
 652 calized terms and nonlinearities.

653 **Appendix A. Proof of Lemma 2.2.** To prove Assertions 1 and 2 of  
 654 Lemma 2.2, it is more convenient to write the characteristic equation of the differential  
 655 equation in (2.5) as

$$656 \quad (A.1) \quad r^3 + r - s^3 - s = 0$$

658 (as in [25], a technique used to solve the characteristic equation of the KdV operator),  
 659 where

$$660 \quad (A.2) \quad s^3 + s = 2\lambda - 2b$$

and considering  $s$  being the real root of the latter equation. Then, solutions to (A.1) are given by

$$r_1 = s, \quad r_2 = -\frac{s}{2} + i\frac{\sqrt{3s^2 + 4}}{2}, \quad r_3 = -\frac{s}{2} - i\frac{\sqrt{3s^2 + 4}}{2},$$

and, therefore, unique solutions to the differential equation in (2.5) are given by

$$(A.3) \quad \pi(x) = \alpha e^{sx} + \beta e^{-\frac{s}{2}x} \cos \frac{\sqrt{3s^2 + 4}}{2}x + \gamma e^{-\frac{s}{2}x} \sin \frac{\sqrt{3s^2 + 4}}{2}x$$

with  $\alpha, \beta, \gamma \in \mathbb{R}$  chosen such that restriction on initial conditions in (2.5) is satisfied. We can check numerically that there exists a number  $\epsilon > 0$  near zero such that for  $s \geq 1 - \epsilon$ ,  $\pi(\cdot)$  given by (A.3) with initial conditions  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$  (corresponding to  $\alpha = 56/25, \beta = 44/25, \gamma = 8/25$ ) is positive and increasing and, therefore, satisfies (2.5). Defining a small constant  $\delta > 0$  by

$$(A.4) \quad \delta := \frac{\epsilon^3}{5} - \frac{3\epsilon^2}{5} + \frac{7\epsilon}{5},$$

we see from (A.2) that for  $s \geq 1 - \epsilon$  we have  $\lambda \geq \lambda_0 := 1 - \delta$  for choice  $b = \frac{11}{16}\lambda$ . Thus, for all  $\lambda \geq \lambda_0$ , there exists  $b > 0$ , such that conditions (2.5) are satisfied. Hence, Assertion 1 is proven to hold for  $\lambda_0 = 1 - \delta$ , where  $\delta$  is defined above. Now, notice that for  $s < 1 - \epsilon$ , corresponding to  $\lambda < \lambda_0$ , and for any initial condition of  $\pi(\cdot)$  satisfying the second equation of (2.5), there is a  $\bar{L} > 0$  such that for  $x > \bar{L}$ ,  $\pi(\cdot)$  becomes decreasing and, thus, fails to satisfy all conditions (2.5). This implies that for  $0 < \lambda < \lambda_0$ , Assertion 2 is satisfied for some small  $\bar{L} > 0$ . Letting  $s \rightarrow 0^+$  and choosing initial conditions  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$  as before,  $\pi$  approaches the trajectory of  $\pi(x) = -1 + 5 \cos(x) + 2 \sin(x)$ , for which  $\pi'(x) < 0$  for  $L > \arctan(2/5)$ . By this, for  $\lambda \rightarrow 0^+, b \rightarrow 0^+$ , Assertion 2 is satisfied with  $\bar{L} = \arctan(2/5)$ .

In Figure 1, we represent the evolution of  $\pi(x)$  for choice of initial condition  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$  and various values of  $s$  corresponding to various values of  $\lambda$ . For small values of  $s$  corresponding to small values of  $\lambda$ ,  $\pi(\cdot)$  is increasing until some point  $x = \bar{L}$  quite small, but for  $x > \bar{L}$ , it is decreasing and, thus, fails to satisfy the fourth equation of (2.5) after this point, in accordance with Assertion 2. We also see that for all  $s \geq 1 - \epsilon$ , for  $\epsilon > 0$  small, given as before,  $\pi(\cdot)$  is always increasing, verifying Assertion 1. The proof is complete.

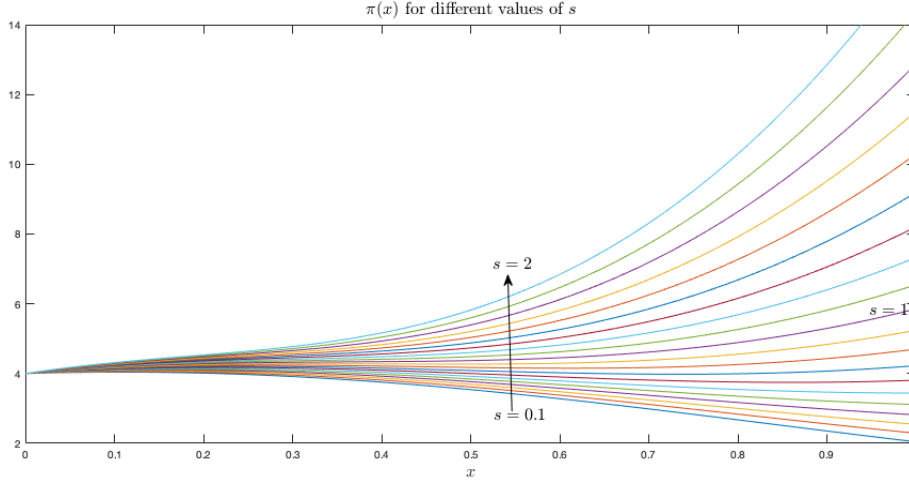
## 689 Appendix B. Well-posedness of system and observer.

690 We show here the well-posedness of both controlled system (1.1)–(1.3) and ob-  
691 server system (3.2)–(3.3).

692 **B.1. Well-posedness of (1.1)–(1.3).** First, for system (1.1), with boundary  
693 conditions (BC-A) or (BC-B), feedback control laws of the form (2.7a) or (2.7b),  
694 respectively, and initial condition  $v^0 \in L^2(0, L)^n$ , it is sufficient to prove the well-  
695 posedness of target system, which results after applying the isomorphic transformation  
696  $\mathcal{T}$  (see (2.9)) in conjunction with (2.10):

$$697 \quad \begin{cases} v_{[n-1],t} + v_{[n-1],x} + v_{[n-1],xxx} = (A_{n-1} - I_{n-1})v_{[n-1]} + \ell \mathcal{T}^{-1}[z], \\ z_t + z_x + z_{xxx} + \omega z = 0, \\ \text{(BC-A)} : \eta(t, 0) = \eta(t, L) = \eta_x(t, L) = 0, \\ \text{(BC-B)} : \eta_{xx}(t, 0) = \eta(t, L) = \eta_x(t, L) = 0, \end{cases}$$

698

FIG. 1. Solutions to (2.5) for different  $s$ 

699 where  $\eta := (v_{[n-1]} \ z)^\top$  is the target state and we adopt the same notation as in  
 700 (2.13). We rewrite the above system as an abstract evolution system in  $L^2(0, L)^n$  as

$$701 \quad \dot{\zeta} = \mathcal{A}\zeta,$$

703 where  $\mathcal{A} := \text{diag}(\mathcal{A}_1, \dots, \mathcal{A}_n) : D(\mathcal{A}) \rightarrow L^2(0, L)^n$  is an linear unbounded operator  
 704 defined as

$$705 \quad \begin{aligned} \mathcal{A}_i \zeta_i &= -\zeta_{i,x} - \zeta_{i,xxx} - \zeta_i + \zeta_{i+1}, \quad i = 1, \dots, n-2, \\ 706 \quad \mathcal{A}_{n-1} \zeta_{n-1} &= -\zeta_{n-1,x} - \zeta_{n-1,xxx} - \zeta_{n-1} + \mathcal{T}^{-1}[\zeta_n], \\ 707 \quad \mathcal{A}_n \zeta_n &= -\zeta_{n,x} - \zeta_{n,xxx} - \omega \zeta_n \end{aligned}$$

709 with domain

$$710 \quad D(\mathcal{A}) = \{ \zeta \in H^3(0, L)^n; \zeta(0) = \zeta(L) = \zeta'(L) = 0 \text{ for (BC-A)} \\ 711 \quad \text{or } \zeta''(0) = \zeta(L) = \zeta'(L) = 0 \text{ for (BC-B)} \}.$$

713 Its adjoint operator satisfies

$$714 \quad \begin{aligned} \mathcal{A}_i^* \zeta_i &= \zeta_{i,x} + \zeta_{i,xxx} - \zeta_i + \zeta_{i+1}, \quad i = 1, \dots, n-2, \\ 715 \quad \mathcal{A}_{n-1}^* \zeta_{n-1} &= \zeta_{n-1,x} + \zeta_{n-1,xxx} - \zeta_{n-1} + \mathcal{T}^{-1}[\zeta_n], \\ 716 \quad \mathcal{A}_n^* \zeta_n &= \zeta_{n,x} + \zeta_{n,xxx} - \omega \zeta_n \end{aligned}$$

718 with domain

$$719 \quad D(\mathcal{A}^*) = \{ \zeta \in H^3(0, L)^n; \zeta(0) = \zeta'(0) = \zeta(L) = 0 \text{ for (BC-A)} \\ 720 \quad \text{or } \zeta''(0) = -\zeta(0), \zeta(L) = \zeta'(0) = \zeta(L) = 0 \text{ for (BC-B)} \}.$$

722 Operator  $\mathcal{A}$  and its adjoint  $\mathcal{A}^*$  are closed with domains dense in  $L^2(0, L)^n$ . Further-  
 723 more, they are both dissipative. Indeed, from the stability proof of [Theorem 2.4](#), we  
 724 first see that  $\mathcal{A}_n$  is dissipative. Then, the exponential stability of  $\mathcal{T}[\zeta_n]$  in (2.12), in  
 725 conjunction with (2.17), implies that

$$726 \quad (B.1) \quad \langle \mathcal{A}_{[n-1]} \zeta_{[n-1]}, \pi \zeta_{[n-1]} \rangle_{L^2(0, L)^{n-1}} \leq 0,$$

728 where  $\pi(\cdot)$  satisfies (2.5). Inequality (B.1) is satisfied for every  $L > 0$  if  $n \geq 2$  under  
 729 boundary conditions (BC-A) and if  $n = 2$  under boundary conditions (BC-B). For  
 730 the latter case, the same inequality holds for  $n > 2$ , when  $L \in (0, \bar{L}]$ , for some  $\bar{L} > 0$ .  
 731 This was shown in the stability proof of Theorem 2.4, and it implies that operator  $\mathcal{A}$   
 732 is dissipative, namely,

$$733 \quad \langle \mathcal{A}\zeta, \zeta \rangle_{L^2(0,L)^n} \leq 0.$$

735 To show dissipativity of the adjoint operator  $\mathcal{A}^*$ , i.e., that  $\langle \mathcal{A}^*\zeta, \zeta \rangle_{L^2(0,L)^n} \leq 0$ ,  
 736 we can easily show first that  $\mathcal{A}_n^*$  is dissipative, by applying integrations by parts.  
 737 Then, we show that  $\langle \mathcal{A}_{[n-1]}^* \zeta_{[n-1]}, \zeta_{[n-1]} \rangle_{L^2(0,L)^{n-1}} \leq 0$ . This implies that  $\mathcal{A}^*$  is  
 738 dissipative.

Consequently, we can apply the Lumer–Phillips theorem, and we conclude that  
 $\mathcal{A}$  generates a  $C^0$ -semigroup of contractions, and, thus, returning to the original  
 coordinates via  $\mathcal{T}^{-1}$ , we have that for any initial condition  $v^0 \in L^2(0, L)^n$ , there  
 exists a unique mild solution

$$v \in C^0(0, \infty; L^2(0, L)^n)$$

739 for system (1.1)–(1.3), noting also that for (BC-A), the above holds for all  $L > 0$   
 740 and  $n \geq 2$ , while for (BC-B), the above holds for all  $L > 0$ , when  $n = 2$  and for all  
 741  $L \in (0, \bar{L}]$ , when  $n > 2$ , where  $\bar{L}$  is given in Lemma 2.2.

742 The above well-posedness result for the  $n$  coupled equations holds also for the  
 743 single damped KdV equation (see (2.1)–(2.3)), as this system’s operator is equal to  
 744  $\mathcal{A}_n$ , as defined above.

745 **B.2. Well-posedness of (3.2)–(3.3).** The observer system given by (3.2) with  
 746 boundary conditions (3.3) is well-posed. To see this, it suffices to show the well-  
 747 posedness of the error system (3.6)–(3.7), invoking also the well-posedness of initial  
 748 system (1.1)–(1.3) that we showed before. The differential operator for error system  
 749 (3.6)–(3.7) is given by

$$750 \quad \mathcal{A}\zeta = -\zeta_x - \zeta_{xxx} + \theta(A_n + K_n C)\zeta - B\zeta$$

752 with domain

$$753 \quad D(\mathcal{A}) = \{ \zeta \in H^3(0, L)^n; \zeta(0) = \zeta(L) = \zeta'(L) = 0 \text{ for (BC-A)} \\ 754 \quad \text{or } \zeta''(0) = \zeta(L) = \zeta'(L) = 0 \text{ for (BC-B)} \},$$

756 and its adjoint operator is given by

$$757 \quad \mathcal{A}^*\zeta = \zeta_x + \zeta_{xxx} + \theta(A_n + K_n C)\zeta - B\zeta$$

759 with domain

$$760 \quad D(\mathcal{A}^*) = \{ \zeta \in H^3(0, L)^n; \zeta(0) = \zeta'(0) = \zeta(L) = 0 \text{ for (BC-A)} \\ 761 \quad \text{or } \zeta''(0) = -\zeta(0), \zeta(L) = \zeta'(0) = \zeta(L) = 0 \text{ for (BC-B)} \}.$$

763 By the stability proof in Theorem 3.1, we see that

$$764 \quad (B.2) \quad \langle \mathcal{A}\zeta, \mu P\zeta \rangle_{L^2(0,L)^n} \leq 0$$

whenever  $\theta > \theta_0$ , where  $\theta_0$  is defined in the proof of [Theorem 3.1](#) and where function  $\mu(\cdot)$  and matrix  $P$  are defined in the proof of [Theorem 3.1](#). Inequality (B.2) implies that  $\mathcal{A}$  is dissipative. The adjoint  $\mathcal{A}^*$  is also dissipative, and it can be shown by proving that  $\langle \mathcal{A}^* \zeta, P \zeta \rangle_{L^2(0,L)^n} \leq 0$  by applying successive integrations by parts. This, similarly as in [Appendix B.1](#), proves the well-posedness of the error equations, which along with the well-posedness of the initial system results in the well-posedness of the observer system (3.2)–(3.3), namely, for any initial condition  $\hat{v}^0 \in L^2(0, L)^n$ , there exists a unique mild solution

$$\hat{v} \in C^0(0, \infty; L^2(0, L)^n)$$

766 for all  $\theta > \theta_0$ .

767

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