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1     **OUTPUT FEEDBACK CONTROL OF A CASCADE SYSTEM OF**  
2     **LINEAR KORTEWEG-DE VRIES EQUATIONS\***

3     CONSTANTINOS KITSOS <sup>§†</sup>, EDUARDO CERPA <sup>‡</sup>, GILDAS BESANÇON <sup>§</sup>, AND  
4     CHRISTOPHE PRIEUR<sup>§</sup>

5     **Abstract.** This paper is about the stabilization of a cascade system of  $n$  linear Korteweg-de  
6     Vries equations in a bounded interval. It considers an output feedback control placed at the left  
7     endpoint of the last equation, while the output involves only the solution to the first equation.  
8     The boundary control problems investigated include two cases: a classical control on the Dirichlet  
9     boundary condition and a less standard one on its second-order derivative. The feedback control law  
10    utilizes the estimated solutions of a high-gain observer system and the output feedback control leads  
11    to stabilization for any  $n$  for the first boundary conditions case and for  $n = 2$  for the second one.

12    **Key words.** Korteweg-de Vries equation, cascade systems, output feedback control

13    **AMS subject classifications.** 68Q25, 68R10, 68U05

14    **1. Introduction.** In this paper, we study the following cascade system of  $n$   
15    linear Korteweg-de Vries (KdV for short) equations posed in a bounded interval of  
16    length  $L$

17    (1.1)                  $v_t + v_x + v_{xxx} = (A_n - B)v$ , in  $(0, \infty) \times (0, L)$ ,

19    where  $v = (v_1 \ \cdots \ v_n)^\top$  is the state and

20                  $A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}$ ,                  $B = \text{diag}(1, 1, \dots, 1, -1)$ .

22    Let us consider two different types of boundary conditions, where the input control  $u$   
23    in both of them is placed on the left side and only acts on the  $n$ -th coordinate of the  
24    state.

25    *Boundary conditions A (BC-A):*

26    (1.2a)                  $v_i(t, 0) = 0, i = 1, \dots, n - 1$ , for all  $t > 0$ ,  
                                $v_n(t, 0) = u(t)$ , for all  $t > 0$ ,  
27                                  $v(t, L) = 0, v_x(t, L) = 0$ , for all  $t > 0$ .

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28 *Boundary conditions B (BC-B):*

$$\begin{aligned}
 &v_{i,xx}(t, 0) = 0, i = 1, \dots, n - 1, \text{ for all } t > 0, \\
 29 \quad (1.2b) \quad &v_{n,xx}(t, 0) = u(t), \text{ for all } t > 0, \\
 30 \quad &v(t, L) = 0, v_x(t, L) = 0, \text{ for all } t > 0.
 \end{aligned}$$

31 In order to complete our control system, we add an initial condition given by

$$32 \quad (1.3) \quad v(0, x) = v^0(x), x \in (0, L)$$

34 and a distributed measurement given by the following output

$$\begin{aligned}
 35 \quad (1.4) \quad &y(t, x) = Cv(t, x); \\
 36 \quad &C = (1 \quad 0 \quad \dots \quad 0).
 \end{aligned}$$

38 The nonlinear version of a single KdV equation describes propagation of waters  
 39 with small amplitude in closed channels. It was introduced in 1895 and since then  
 40 its properties have gained much consideration, see for instance [6]. Surveys on recent  
 41 progresses and open problems on control and stabilization of such models can be found  
 42 in [26] and [3].

43 The aim of the present work is to stabilize the cascade system (1.1) considering  
 44 any of the boundary control problems (BC-A) and (BC-B) and by utilizing the knowl-  
 45 edge of the first state only, while the other states are estimated via an observer. Notice  
 46 that this system is unstable due to the instability of the trajectory corresponding to  
 47 its last equation, as it can be seen by following classical energy arguments. In the  
 48 recent decades, stabilization of single KdV equations has gained significant interest,  
 49 see for instance [4], where backstepping method is used for feedback controls placed  
 50 on the left boundary, see also [29, 7, 27, 13]. Output feedback laws for single linearized  
 51 and nonlinear KdV equations have been already established via boundary observers in  
 52 [21, 23] (see also [28, 11, 14, 2]), by means of backstepping and Lyapunov techniques.  
 53 In these two works, the measurement injected in the observer involves the right end-  
 54 point of the domain, more precisely, the second derivative of the boundary or the  
 55 Dirichlet condition, depending on the boundary conditions. Output feedback control  
 56 laws for systems written in the cascade form considered here have not yet appeared  
 57 in the literature, while controllability of coupled KdV equations but with couplings,  
 58 different from the ones studied here (particularly, internal couplings in one-order de-  
 59 rivatives), describing strong interactions of weakly nonlinear long waves, have been  
 60 investigated for instance in [5]. However, coupling in zero-order internal terms, rep-  
 61 resented by matrix  $A_n$ , might result from the linearization of coupled nonlinear KdV  
 62 equations of some forms appearing in [20], describing oceanic and atmospheric phe-  
 63 nomena, such as the atmospheric blockings, the interactions between the atmosphere  
 64 and ocean, the oceanic circulations, and hurricanes (see system (27)-(28) and model  
 65 5, therein, according to the well-known Painlevé classification), see also the Hirota-  
 66 Satsuma model [12] and [10] for multicomponent KdV equations (related to the weak  
 67 nonlinear dispersion). For these systems, it is often difficult to control and observe di-  
 68 rectly all the equations. Also, general settings of coupled infinite-dimensional systems  
 69 with couplings in zero-order terms, as the ones considered here, have been studied  
 70 with respect to their controllability and observability properties, when considering  
 71 reduced numbers of controls and observations, see [1], [19]. In those works, the au-  
 72 thors have shown that the problem of control of undeactuated systems with reduced

73 number of observations is quite challenging. Furthermore, placing the control on the  
 74 second derivative of the left boundary, as in the considered second boundary control  
 75 problem is even more original and its investigation exhibits some technical difficulties,  
 76 for which solutions are proposed in the present work. To the best of our knowledge,  
 77 boundary control problems of this second type have not appeared in the literature.

78 Here, we aim at observing the full state of a system of KdV equations written in a  
 79 cascade form and finally controlling it, by considering a single observation. Observer  
 80 design for nonlinear systems of partial differential equations written in such a form,  
 81 based on the well-known high-gain methodology, have been considered, for instance  
 82 in [17, 16, 18], in the framework of first-order hyperbolic systems, extending results  
 83 for finite-dimensional systems [15]. A similar form considered here, in its linearized  
 84 version, allows an observer design, which relies on a choice of a sufficiently large  
 85 parameter in its equations, while appropriate choice of the latter leads simultaneously  
 86 to the closed-loop output feedback stabilization. In summary, the contribution of the  
 87 present work first lies in stabilizing the trajectory of the last equation by means of an  
 88 observer relying on the measurement of the first state only. The control placed on the  
 89 left boundary, combined with the observer gain, brings this trajectory asymptotically  
 90 to zero in an arbitrarily fast manner (first part of [Theorem 3.2](#)). Subsequently, it  
 91 is proven that 1) the whole cascade system becomes asymptotically stable for any  
 92  $L > 0$ , when boundary condition (BC-A) is considered, and 2) this result holds for  
 93 boundary condition (BC-B), only when  $n = 2$  (number of equations), noting that for  
 94  $n > 2$ , stabilization is achieved for quite small  $L$  (last part of [Theorem 3.2](#)). The  
 95 methodology relies on backstepping techniques and appropriate Lyapunov analysis.  
 96 Exponential stabilization for (BC-B) is proven here to be linked to the solvability of an  
 97 ordinary differential equations problem, similar to the differential equation satisfied  
 98 by the eigenvectors of the associated differential operator to these KdV equations,  
 99 and being subject to some constraints.

100 In [Section 2](#) we prove a preliminary result on the stability of a single damped  
 101 KdV equation and then we prove the full state stabilization of the cascade system for  
 102 both boundary condition problems. In [Section 3](#), we first present the observer design  
 103 for the coupled system and finally the main output feedback stabilization result. In  
 104 [Section 4](#), we provide conclusions and some perspectives.

105 **2. Full State Feedback Stabilization.** In this section, we study the full state  
 106 feedback stabilization of system (1.1) for boundary control problems (BC-A) and  
 107 (BC-B).

108 **2.1. Stability of a single KdV equation.** Prior to the stabilization of the  
 109 cascade system, we present a preliminary result about the stability of a single damped  
 110 linear KdV equation, which will be invoked in the sequel. Consider a single KdV  
 111 equation in the domain  $(0, L)$

$$112 \quad (2.1) \quad w_t + w_x + w_{xxx} + \lambda w = 0, \text{ in } (0, \infty) \times (0, L),$$

114 satisfying one of the following distinct cases of boundary conditions

$$115 \quad (2.2a) \quad w(t, 0) = w(t, L) = w_x(t, L) = 0, t > 0,$$

$$116 \quad (2.2b) \quad w_{xx}(t, 0) = w(t, L) = w_x(t, L) = 0, t > 0,$$

118 and initial condition of the form

$$119 \quad (2.3) \quad w(0, x) = w^0(x), x \in (0, L).$$

121 The stability result for solutions  $w$  to the above problem is presented in the following  
 122 propositions. Although asymptotic stability assuming boundary conditions (2.2a) is  
 123 ensured for every  $\lambda > 0$ , for (2.2b) asymptotic stability is guaranteed only when  $\lambda \geq \lambda_0$   
 124 for some  $\lambda_0 > 0$ . These results are stated precisely in the next two propositions and  
 125 will be used throughout this work.

126 PROPOSITION 2.1. Consider system (2.1) with boundary conditions (2.2a) and  
 127 initial condition  $w^0 \in L^2(0, L)$ . Then for all  $\lambda > 0$ , we have

$$128 \quad (2.4) \quad \|w(t, \cdot)\|_{L^2(0, L)} \leq e^{-\lambda t} \|w^0(\cdot)\|_{L^2(0, L)}, t \geq 0,$$

130 for every  $L > 0$ .

131 Proposition 2.1 concerning boundary conditions (2.2a) is a standard result and can  
 132 be derived from energy estimates. Well-posedness of this equation is presented in  
 133 Appendix B.1. Let us note here, that asymptotic stability for this case can be proven  
 134 even when the damping is not constant in the domain but localized to a part of it,  
 135 see for instance [24], and even when the damping is saturated, see [22].

136 To proceed to the stability result for boundary conditions (2.2b), we utilize the  
 137 following lemma.

138 LEMMA 2.2. There exists  $\lambda_0 > 0$ , such that the following assertions hold true.

139 Assertion 1: For every  $\lambda \geq \lambda_0$ , there exist  $\pi(\cdot)$  in  $C^3(0, \infty)$  and  $b > 0$ , such that  
 140 the following holds for all  $x \geq 0$

$$141 \quad (2.5) \quad \begin{cases} \pi'''(x) + \pi'(x) - 2\lambda\pi(x) = -2b\pi(x), \\ \pi''(0)\pi(0) + (\pi'(0))^2 + \pi^2(0) \leq 0, \\ \pi(x) > 0, \\ \pi'(x) \geq 0. \end{cases}$$

143 Assertion 2: For every  $\lambda \in (0, \lambda_0)$ , there exist  $\bar{L}, b > 0$  and  $\pi(\cdot)$  in  $C^3(0, \infty)$   
 144 satisfying (2.5) for all  $x \in [0, \bar{L}]$ .

145 Proof. See Appendix A □

146 The following proposition concerns the second case of boundary conditions.

147 PROPOSITION 2.3. Consider system (2.1) with boundary conditions (2.2b) and  
 148 initial condition  $w^0 \in L^2(0, L)$ . Then, there exists  $\lambda_0 > 0$ , such that:

149 1) For all  $\lambda \geq \lambda_0$ , there exist  $a, b > 0$ , such that the solution to (2.1)-(2.3)-(2.2b)  
 150 satisfies the following:

$$151 \quad (2.6) \quad \|w(t, \cdot)\|_{L^2(0, L)} \leq ae^{-bt} \|w^0(\cdot)\|_{L^2(0, L)}, t \geq 0,$$

153 for every  $L > 0$ .

154 2) For all  $\lambda \in (0, \lambda_0)$ , there exist  $\bar{L}, a, b > 0$  such that (2.6) is satisfied for all  
 155  $L \in (0, \bar{L}]$ .

156 Proof. In this context, we are interested by unique solutions  $w$  belonging to  
 157  $C(0, \infty; L^2(0, L))$ . Well-posedness of the initial boundary value problem (2.1)-(2.3)-  
 158 (2.2b) can be easily proven by invoking classical arguments, although these boundary  
 159 conditions are less common in the literature. More details about the well-posedness  
 160 of such systems are presented in Appendix B.1.

161 To prove the stability result, let us consider the following weighted  $L^2$ -norm

$$162 \quad E(t) := \int_0^L \pi(x)w^2(x)dx$$

163

164 along the  $L^2$  solutions to (2.1)-(2.3)-(2.2b), for some appropriate choice of positive  
 165  $\pi(\cdot) \in C^3[0, L]$ . Calculating its time-derivative and applying integrations by parts,  
 166 we obtain

$$\begin{aligned} 167 \quad \dot{E}(t) &= \int_0^L (\pi'''(x) + \pi'(x) - 2\lambda) w^2(x) dx - 3 \int_0^L \pi'(x) w_x^2(x) dx \\ 168 \quad &+ [(-\pi''(x) - \pi(x)) w^2(x) - 2\pi(x) w_{xx}(x) w(x) + \pi(x) w_x^2(x) \\ 169 \quad &+ 2\pi'(x) w_x(x) w(x)]_0^L. \end{aligned}$$

171 Substituting boundary conditions (2.2b) we get

$$\begin{aligned} 172 \quad \dot{E}(t) &= \int_0^L (\pi'''(x) + \pi'(x) - 2\lambda) w^2(x) dx - 3 \int_0^L \pi'(x) w_x^2(x) dx \\ 173 \quad &- (w_x(0) \quad w(0)) \begin{pmatrix} \pi(0) & -\pi'(0) \\ -\pi'(0) & -\pi(0) - \pi''(0) \end{pmatrix} \begin{pmatrix} w_x(0) \\ w(0) \end{pmatrix}. \end{aligned}$$

174 To ensure the exponential decay of  $E(t)$ , we invoke assertions of Lemma 2.2 for  $\pi(\cdot)$ ,  
 175 for which we assume that it satisfies (2.5). By Assertions 1 and 2, the second integral  
 176 and the third boundary term of the above equation become nonpositive and we obtain  
 177 the existence of a constant  $b > 0$ , such that

$$178 \quad \dot{E}(t) \leq -2bE(t)$$

180 and, therefore, (2.6) holds with  $a = \sqrt{\frac{\pi(L)}{\pi(0)}}$ . This completes the proof of Proposi-  
 181 tion 2.3.  $\square$

182 **2.2. Full state stabilization.** Following the previous results, we are in a po-  
 183 sition to study the closed-loop stabilization. Here, the considered state feedback  
 184 controls, which are placed in the last equation, will be proven to be of the following  
 185 form for each of the problems (BC-A) and (BC-B)

$$186 \quad (2.7a) \quad (\text{BC-A}): u(t) = \int_0^L p(0, y) v_n(t, y) dy,$$

$$187 \quad (2.7b) \quad (\text{BC-B}): u(t) = -\frac{\omega + 1}{3} L v_n(t, 0) + \int_0^L p_{xx}(0, y) v_n(t, y) dy,$$

189 with  $\omega > 0$  to be chosen appropriately and kernel function  $p : \Pi \rightarrow \mathbb{R}$  depending on  
 190  $\omega$ , where  $\Pi := \{(x, y); x \in [0, L], y \in [x, L]\}$ .

191 We now present the exponential decay result of the solution  $v$  to the cascade  
 192 system (1.1) via the control (2.7), which utilizes the full state. The proof uses back-  
 193 stepping tools appearing in [4], [3] for single KdV equations.

194 **THEOREM 2.4.** Consider system (1.1) with boundary conditions (BC-A) or (BC-  
 195 B), feedback control laws of the form (2.7a) or (2.7b), respectively, and initial condi-  
 196 tion  $v^0 \in L^2(0, L)^n$ .

197 a) If (BC-A) holds and  $n \geq 2$ , then for every  $L > 0$ , there exist constants  $c, d > 0$ ,  
 198 such that the solution  $v$  to (1.1) satisfies the following

$$199 \quad (2.8) \quad \|v\|_{L^2(0, L)^n} \leq ce^{-dt} \|v^0\|_{L^2(0, L)^n}, \forall t \geq 0.$$

201 b) If (BC-B) holds and  $n = 2$ , then for every  $L > 0$ , there exist constants  $c, d > 0$ ,  
 202 such that solution  $v$  to (1.1) satisfies (2.8).

203 c) If (BC-B) holds and  $n > 2$ , then there exists  $\bar{L} > 0$ , such that (2.8) is guaran-  
 204 teed for all  $L \in (0, \bar{L}]$ .

205 *Proof.* The well-posedness of controlled system (1.1) with boundary conditions  
 206 (BC-A) or (BC-B) is shown in Appendix B.1.

207 We first prove a preliminary result concerning the exponential stability of  $v_n$ . We  
 208 apply a Volterra transformation  $\mathcal{T} : L^2(0, L) \rightarrow L^2(0, L)$  of the form

$$209 \quad (2.9) \quad z(x) = \mathcal{T}[v_n](x) := v_n(x) - \int_x^L p(x, y)v_n(y)dy$$

211 to the solution to the last equation of the cascade system, with  $p$  defined on  $\Pi$ . Under  
 212 appropriate choice of  $p(\cdot, \cdot)$ , we prove that this transformation maps solution  $v_n$  to  
 213 the trajectory  $z$  satisfying the following target equation in  $[0, \infty) \times [0, L]$

$$214 \quad (2.10) \quad z_t + z_x + z_{xxx} + \omega z = 0,$$

$$215 \quad (\text{BC-A}, z) : z(t, 0) = z(t, L) = z_x(t, L) = 0,$$

$$216 \quad (\text{BC-B}, z) : z_{xx}(t, 0) = z(t, L) = z_x(t, L) = 0.$$

218 with control given by (2.7). Indeed, performing standard differentiations and integra-  
 219 tions by parts (for more intuition about such operations, the reader can refer to [4]),  
 220 we derive the following equations

$$221 \quad z_t(t, x) + z_x(t, x) + z_{xxx}(t, x) + \omega z(t, x) =$$

$$222 \quad - \int_x^L (p_{xxx}(x, y) + p_{yyy}(x, y) + p_y(x, y) + (\omega + 1)p(x, y)) v_n(t, y)dy$$

$$223 \quad + p(x, L)v_{n,xx}(t, L) + p(x, L)v_n(t, L) + p_{yy}(x, L)v_n(t, L) - p_y(x, L)v_{n,x}(t, L)$$

$$224 \quad + \left( \omega + 1 + \frac{d^2}{dx^2}p(x, x) + \frac{d}{dx}p_x(x, x) + p_{xx}(x, x) - p_{yy}(x, x) \right) v_n(t, x)$$

$$225 \quad + \left( p_x(x, x) + p_y(x, x) + 2\frac{d}{dx}p(x, x) \right) v_{n,x}(t, x).$$

227 By choosing  $p(\cdot, \cdot)$  satisfying the following equations

$$228 \quad (2.11) \quad \begin{cases} p_{xxx} + p_{yyy} + p_x + p_y + (\omega + 1)p = 0, & (x, y) \in \Pi \\ p(x, x) = p(x, L) = 0, & x \in [0, L] \\ p_x(x, x) = \frac{\omega+1}{3}(L-x), & x \in [0, L] \end{cases}$$

230 we achieve to obtain target system (2.10) for both boundary problems (BC-A,  $z$ ) and  
 231 (BC -B,  $z$ ). Solutions to (2.11) are proven to be unique in  $C^3(\Pi)$  in [4], by following  
 232 successive approximation methods. The feedback control  $u$  is easily checked to satisfy  
 233 (2.7), if we use (2.9) and also calculate the value of the second derivative, viz.

$$234 \quad z_{xx}(x) = v_{n,xx}(x) + \frac{d}{dx}p(x, x)v_n(x) + p(x, x)v_{n,x}(x) + p_x(x, x)v_n(x)$$

$$235 \quad - \int_0^L p_{xx}(x, y)v_n(y)dy$$

237 for  $x = 0$ .

238 Now, as we saw in Propositions 2.1 and 2.3 of the previous subsection, solution  
 239  $z$  to target system (2.10) is asymptotically stable for every  $L > 0$ , if  $\omega > 0$  when  
 240 (BC A,  $z$ ) holds and if  $\omega \geq 1$  when (BC B,  $z$ ) holds. This implies the asymptotic  
 241 stability of  $v_n$ , solution to (1.1), with control given by (2.7) for each of the boundary

242 problems (BC-A) and (BC-B). The latter follows from the fact that, as proven in  
 243 [4], transformation (2.9), mapping solution  $v_n$  to  $z$ , is bounded and invertible with  
 244 bounded inverse. So, for every  $\bar{d} > 0$ , there exist  $\omega_0, \bar{c} > 0$ , such that for all  $\omega \geq \omega_0$ ,  
 245 we have

$$246 \quad (2.12) \quad \|v_n\|_{L^2(0,L)} \leq \bar{c}e^{-\bar{d}t} \|v_n^0\|_{L^2(0,L)}, \forall t \geq 0.$$

248 To prove the asymptotic stability of the full state, consider vector  $v_{[n-1]} :=$   
 249  $(v_1 \ \cdots \ v_{n-1})^\top$ . Then,  $v_{[n-1]}$  satisfies the following equations

$$250 \quad (2.13) \quad \begin{cases} v_{[n-1],t} + v_{[n-1],x} + v_{[n-1],xxx} = (A_{n-1} - I_{n-1})v_{[n-1]} + \ell v_n, \\ \text{(BC-A, } v_{[n-1]}) : v_{[n-1]}(t, 0) = v_{[n-1]}(t, L) = v_{[n-1],x}(t, L) = 0, \\ \text{(BC-B, } v_{[n-1]}) : v_{[n-1],xx}(t, 0) = v_{[n-1]}(t, L) = v_{[n-1],x}(t, L) = 0, \end{cases}$$

251 where

$$\ell := (0 \ \cdots \ 0 \ 1)^\top.$$

252 To prove stability of this system, consider a Lyapunov functional of the following form

$$253 \quad W(t) = \int_0^L \pi(x) |v_{[n-1]}(x)|^2 dx$$

254 along the  $L^2(0, L)^{n-1}$  solutions  $v_{[n-1]}$  to the last equations, where  $\pi(\cdot)$  is a positive  
 255 increasing  $C^3$  function to be chosen. After substituting the above equations satisfied  
 256 by  $v_{[n-1]}$  and applying integrations by parts, we obtain for the time-derivative of  $W$

$$258 \quad \begin{aligned} \dot{W}(t) &= \int_0^L (\pi'''(x) + \pi'(x)) |v_{[n-1]}(x)|^2 dx - 3 \int_0^L \pi'(x) |v_{[n-1],x}(x)|^2 dx \\ &\quad - \int_0^L \pi(x) v_{[n-1]}^\top(x) (2I_{n-1} - A_{n-1}^\top - A_{n-1}) v_{[n-1]}(x) dx \\ &\quad + 2 \int_0^L \pi(x) v_{n-1}(x) v_n(x) dx + W_0, \end{aligned}$$

260 with

$$263 \quad W_0 := \left[ -(\pi''(x) + \pi(x)) |v_{[n-1]}(x)|^2 + \pi(x) \left( |v_{[n-1],x}(x)|^2 - 2v_{[n-1],xx}^\top(x) v_{[n-1]}(x) \right) \right. \\ \left. + 2\pi'(x) v_{[n-1],x}^\top(x) v_{[n-1]}(x) \right]_0^L.$$

264 Matrix  $2I_{n-1} - A_{n-1}^\top - A_{n-1}$  is positive definite and its eigenvalues are

$$\rho := 2 - 2 \cos \frac{\pi j}{n}, j = 1, \dots, n-1.$$

265 Hence, its minimal eigenvalue is given by

$$267 \quad (2.15) \quad \rho_n := \lambda_{\min}(2I_{n-1} - A_{n-1}^\top - A_{n-1}) = 2 - 2 \cos \frac{\pi}{n}, \mathbb{N} \ni n \geq 2.$$

268 Since  $\pi'(x) \geq 0$ , by use of Young's inequality we obtain

$$270 \quad \begin{aligned} \dot{W}(t) &\leq \int_0^L (\pi'''(x) + \pi'(x) - \rho_n \pi(x)) |v_{[n-1]}(x)|^2 dx \\ &\quad + 2\delta \int_0^L \pi(x) |v_{[n-1]}(x)|^2 dx + \frac{1}{2\delta} \int_0^L \pi(x) v_n^2 dx + W_0, \end{aligned}$$

271  
272



273 and  $\delta > 0$  is chosen sufficiently small as in (A.4) in the proof of Lemma 2.2 of the  
274 previous subsection.

275 Now, we choose  $\pi(\cdot)$  for each of the two boundary problems as follows.  
For (BC-A,  $v_{[n-1]}$ ) we choose

$$\pi(\cdot) = 1.$$

276 From this, taking also into account the exponential stability of  $v_n$  (2.12), we get  
277 for the case (BC-A,  $v_{[n-1]}$ ) the following estimate

$$278 \quad (2.16) \quad \dot{W}(t) \leq -2dW(t) + \frac{1}{2\delta}\pi(L)\bar{c}^2 e^{-2\bar{d}t} \|v_n^0\|_{L^2(0,L)}^2$$

280 with  $d = \rho_n/2 - \delta$ .

281 For (BC-B,  $v_{[n-1]}$ ) we choose a positive and increasing  $\pi(\cdot)$  satisfying (2.5) (see  
282 Assertion 1 in Lemma 2.2) with  $\lambda = \frac{\rho_n}{2} - \delta$  and  $b > 0$ . It turns out by Assertion 1  
283 that there is  $\pi(\cdot)$  and  $b > 0$  satisfying this equation for any  $L > 0$ , when  $\lambda = 1 - \delta$ ,  
284 corresponding to  $\rho_2 = 2$  (for  $n = 2$ ). Then, the exponential decay of the Lyapunov  
285 functional is ensured similarly as in Proposition 2.3. More precisely, there exists  $d > 0$ ,  
286 such that for all  $L > 0$ , (2.16) is satisfied for (BC-B,  $v_{[n-1]}$ ) as well. Also, as shown  
287 in Proposition 2.3, for  $n > 2$ , which renders  $\rho_n < 2$ , (2.16) is satisfied for some  $\pi(\cdot)$ ,  
288  $b > 0$ , only when  $0 < L \leq \bar{L}$ , with  $\bar{L}$  depending on  $n$ .

289 Combining the above results, from (2.16), which holds for both (BC-A,  $v_{[n-1]}$ )  
290 and (BC-B,  $v_{[n-1]}$ ), we derive by Gronwall's inequality

$$291 \quad (2.17) \quad W(t) \leq e^{-2dt}W(0) + \frac{\pi(L)\bar{c}^2}{4\delta(d-\bar{d})} \left( e^{-2\bar{d}t} - e^{-2dt} \right) \|v_n^0\|_{L^2(0,L)}^2,$$

293 recalling also, that  $\bar{d}$  depending on the parameter  $\omega$  of the control laws, can be chosen,  
294 such that  $\bar{d} > d$ . Combining (2.17) and (2.12), we get

$$295 \quad \|v\|_{L^2(0,L)^n} \leq \|v_{[n-1]}\|_{L^2(0,L)^{n-1}} + \|v_n\|_{L^2(0,L)} \leq \sqrt{\frac{\pi(L)}{\pi(0)}} e^{-dt} \|v_{[n-1]}(0, \cdot)\|_{L^2(0,L)^{n-1}} \\ 296 \quad + \frac{\bar{c}\sqrt{\pi(L)}}{2\sqrt{\pi(0)\delta(\bar{d}-d)}} \sqrt{e^{-2dt} - e^{-2\bar{d}t}} \|v_n^0\|_{L^2(0,L)} + \bar{c}e^{-\bar{d}t} \|v_n^0\|_{L^2(0,L)}.$$

298 The last inequality leads to (2.8) for a suitable choice of  $c$ .

299 This concludes the proof and shows, also, that although the exponential conver-  
300 gence to zero of  $v_n$  can become arbitrarily fast by the choice of parameter  $\omega$  inside  
301 the controls, solution  $v$  to the whole cascade system has a fixed convergence rate.  $\square$

302 *Remark 2.5.* Note that in the above proof, parameter  $\rho_n$  in (2.15), depending  
303 on  $n$ , does not permit the stabilization of the closed-loop system to be achieved for  
304 any number of equations  $n$ , when the length of the domain  $L$  is arbitrary. As it  
305 was shown in Proposition 2.3, the damped KdV equation in the case of boundary  
306 conditions of the type (BC-B) requires a damping with coefficient  $\lambda$  larger than a  
307 critical damping coefficient  $\lambda_0$ . The parameter  $\rho_n$ , which appears in the stabilization  
308 of the closed-loop system corresponding to the damping coefficient, is decreasing with  
309  $n$ . For  $n > 2$  the stabilization cannot be ensured for any  $L > 0$ , since, because of  
310  $\rho_n$ , the damping coefficient becomes lower than the critical one, while for  $n = 2$ , the  
311 damping coefficient of the coupled equation is exactly equal to the critical one.

**3. Observer Design and Output Feedback Stabilization.** In this section, we first present the proposed observer, along with its convergence proof for each of the boundary control problems (BC-A) and (BC-B). Then, we study the output feedback stabilization of system (1.1) with controls placed on the left boundaries as described in each of problems (BC-A) and (BC-B). We note here that, even though the considered system is linear, the use of the high-gain observer design is instrumental in the output feedback control in the two following manners and is based on the methodology [17], introduced for quasilinear hyperbolic systems. 1) For (BC-B), the choice of the high-gain parameter is needed to establish convergence of the observer, contrarily to a simpler Luenberger observer design, which would be sufficient for (BC-A); 2) The high-gain parameter is used in the stabilization of the closed-loop system for both boundary control problems (BC-A) and (BC-B).

In the following subsection we present the observer for the cascade system, whose exponential stability relies on the result presented in Proposition 2.3 of Section 2.

**3.1. Observer.** Define, first, diagonal matrix  $\Theta_n$  by

$$\Theta_n := \text{diag}(\theta, \theta^2, \dots, \theta^n),$$

where  $\theta > 0$  represents a gain, which will be selected later. Consider a vector gain  $K_n = (k_1 \ \dots \ k_n)^\top$  and let  $P \in \mathbb{R}^{n \times n}$  be a symmetric and positive definite matrix satisfying a quadratic Lyapunov equation of the form

$$(3.1) \quad P(A_n + K_n C) + (A_n + K_n C)^\top P = -I_n.$$

The previous equation is always feasible, due to the observability of the pair  $(A_n, C)$ .

Then, our observer is defined to satisfy the following equations in  $(0, \infty) \times (0, L)$

$$(3.2) \quad \hat{v}_t(t, x) + \hat{v}_x(t, x) + \hat{v}_{xxx}(t, x) = (A_n - B)\hat{v}(t, x) - \Theta_n K_n (y(t, x) - C\hat{v}(t, x))$$

with boundary conditions for each of (BC-A) and (BC-B) as follows

$$(3.3a) \quad \begin{aligned} & \hat{v}_i(t, 0) = 0, i = 1, \dots, n-1, \text{ for all } t > 0, \\ \text{(BC-A): } & \hat{v}_n(t, 0) = u(t), \text{ for all } t > 0, \\ & \hat{v}(t, L) = \hat{v}_x(t, L) = 0, \text{ for all } t > 0, \end{aligned}$$

$$(3.3b) \quad \begin{aligned} & \hat{v}_{i,xx}(t, 0) = 0, i = 1, \dots, n-1, \text{ for all } t > 0, \\ \text{(BC-B): } & \hat{v}_{n,xx}(t, 0) = u(t), \text{ for all } t > 0, \\ & \hat{v}(t, L) = \hat{v}_x(t, L) = 0, \text{ for all } t > 0. \end{aligned}$$

and initial condition

$$\hat{v}(0, x) = \hat{v}^0(x), x \in (0, L).$$

The main observer result is stated in the following theorem.

**THEOREM 3.1.** *Consider system (1.1) with output (1.4) and boundary conditions satisfying (1.2) ((BC-A) or (BC-B)) and  $v^0 \in L^2(0, L)^n, u \in L^2_{loc}(0, \infty)$ . Consider, also,  $P$  and  $K_n$  satisfying a Lyapunov equation as in (3.1). Then, (3.2), with boundary conditions (3.3) and initial condition  $\hat{v}^0 \in L^2(0, L)^n$  is an observer for solution of (1.1), in the sense that for  $\theta$  large it estimates the state  $v$  arbitrarily fast. More precisely, for every  $\kappa > 0$ , there exist  $\theta_0$ , such that for every  $\theta > \theta_0$ , the following holds for all  $v^0, \hat{v}^0 \in L^2(0, L)^n, t \geq 0$ :*

$$(3.4) \quad \|\hat{v}(t, \cdot) - v(t, \cdot)\|_{L^2(0, L)^n} \leq \nu \theta^{n-1} e^{-\kappa t} \|\hat{v}^0(\cdot) - v^0(\cdot)\|_{L^2(0, L)^n},$$

with  $\nu > 0$ , depending on  $n$  and  $L$ .

351 *Proof.* First, we prove in [Appendix B.2](#) that observer system [\(3.2\)](#)-[\(3.3\)](#) is well-  
 352 posed. Then, to prove its asymptotic convergence to the state  $v$ , let us define a scaled  
 353 observer error  $\varepsilon$  by

$$354 \quad (3.5) \quad \varepsilon = \Theta_n^{-1}(\hat{v} - v).$$

356 Then,  $\varepsilon$  satisfies the following equations

$$357 \quad (3.6) \quad \varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = \theta(A_n + K_n C)\varepsilon - B\varepsilon$$

359 and boundary conditions for each of the cases (BC-A) and (BC-B) as follows

$$360 \quad (3.7a) \quad \varepsilon(t, 0) = \varepsilon(t, L) = \varepsilon_x(t, L) = 0,$$

$$361 \quad (3.7b) \quad \varepsilon_{xx}(t, 0) = \varepsilon(t, L) = \varepsilon_x(t, L) = 0.$$

363 We expect that solutions to the previous coupled equations can approach zero  
 364 exponentially fast, since  $A_n + K_n C$  being Hurwitz will exhibit a damping effect (as  
 365 in the single KdV equation), with its magnitude being controlled by  $\theta$ . Indeed, to  
 366 prove exponential stability, we choose the following Lyapunov functional defined on  
 367 the  $L^2(0, L)^n$  solutions to the observer error equations

$$368 \quad (3.8) \quad V(t) := \int_0^L \mu(x) \varepsilon^\top(x) P \varepsilon(x) dx,$$

370 with positive  $\mu(\cdot) \in C^3[0, L]$  to be chosen suitably for each of the boundary conditions  
 371 cases. Taking its time-derivative and substituting [\(3.6\)](#) and Lyapunov equation [\(3.1\)](#),  
 372 yields

$$373 \quad \dot{V}(t) = \int_0^L \mu(x) [-\partial_x^3 (\varepsilon^\top(x) P \varepsilon(x)) - \partial_x (\varepsilon^\top(x) P \varepsilon(x)) + 3\partial_x (\varepsilon_x^\top(x) P \varepsilon_x(x)) \\ 374 \quad - \theta \varepsilon^\top(x) \varepsilon(x) - 2\varepsilon^\top(x) P B \varepsilon(x)] dx.$$

376 Performing successive integrations by parts, we obtain

$$377 \quad \dot{V}(t) \leq \int_0^L \left( \mu'''(x) + \mu'(x) + \left( -\theta \frac{1}{|P|} + 2 \frac{|P|}{\lambda_{\min}(P)} \right) \mu(x) \right) \varepsilon^\top(x) P \varepsilon(x) dx \\ 378 \quad - 3 \int_0^L \mu'(x) \varepsilon_x^\top(x) P \varepsilon_x(x) dx + V_0,$$

380 where

$$381 \quad V_0 := [(-\mu''(x) - \mu(x)) \varepsilon^\top(x) P \varepsilon(x) - \mu(x) (\varepsilon_{xx}^\top(x) P \varepsilon(x) + \varepsilon^\top(x) P \varepsilon_{xx}(x)) \\ 382 \quad + \mu(x) \varepsilon_x^\top(x) P \varepsilon_x(x) + \mu'(x) (\varepsilon_x^\top(x) P \varepsilon(x) + \varepsilon^\top(x) P \varepsilon_x(x))]_0^L$$

384 and  $\lambda_{\min}(P)$  is the minimal eigenvalue of  $P$ .

385 Let us now choose for boundary conditions case [\(3.7a\)](#)

$$386 \quad (3.10) \quad \mu(\cdot) := 1,$$

388 for which we obtain

$$389 \quad V_0 = -\varepsilon_x^\top(0) P \varepsilon_x(0) \leq 0.$$

391 Note that given (3.10) for the boundary conditions case (3.7a), for every  $\theta > \theta_{0,A}$ ,  
 392 with

$$393 \quad \theta_{0,A} := 2 \frac{|P|^2}{\lambda_{\min}(P)},$$

394 we get

$$395 \quad (3.11) \quad \dot{V}(t) \leq -2\kappa_A V(t), t \geq 0,$$

396 for some  $\kappa_A > 0$ .

397 Considering boundary conditions of case (3.7b), (3.9) is written as

$$398 \quad V_0 := - \begin{pmatrix} \varepsilon_x^\top(0) & \varepsilon^\top(0) \end{pmatrix} \begin{pmatrix} P\mu(0) & P\mu'(0) \\ P\mu'(0) & -P(\mu''(0) + \mu(0)) \end{pmatrix} \begin{pmatrix} \varepsilon_x(0) \\ \varepsilon(0) \end{pmatrix}$$

399 For this case (3.7b), we see here that for all

$$400 \quad \theta \geq \theta_{0,B} := 2 \frac{|P|^2}{\lambda_{\min}(P)} + 2|P|,$$

401 Assertion 1 ((2.5) in Lemma 2.2) is satisfied with  $\mu(\cdot)$  in the place of  $\pi(\cdot)$ ,  $\lambda =$   
 402  $\theta \frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)}$ ,  $\lambda_0 = 1$  and  $b = \kappa_B$  for some  $\kappa_B > 0$ . For all  $\theta \geq \theta_0$ , we choose,  
 403 therefore,  $\pi(\cdot) = \mu(\cdot)$  satisfying (2.5) and we derive again (3.11) satisfied for every  
 404  $\theta \geq \theta_{0,B}$ , with  $\kappa_A$  substituted by  $\kappa_B$ .

Combining the previous estimates, we directly obtain (3.4) with

$$\nu := \sqrt{\frac{\mu(L)}{\mu(0)}} \sqrt{\frac{|P|}{\lambda_{\min}(P)}}$$

405 and this concludes the proof of Theorem 3.1.  $\square$

406 **3.2. Output feedback stabilization.** Next, it is proven that plugging the ob-  
 407 server's state considered in Theorem 3.1 in the feedback laws (2.7) of the previous  
 408 section, the closed-loop system is stabilized. This is done in two steps. First, it is  
 409 proven that the considered output feedback law stabilizes arbitrarily fast the solution  
 410 of the last KdV equation and second, the stabilization of the whole cascade system  
 411 of KdV equations follows. However, for system with boundary conditions (BC-B),  
 412 stabilization for any  $L$  is only achieved when  $n = 2$ , corresponding to a cascade sys-  
 413 tem of two equations, while for  $n > 2$ , stabilization is achieved for small  $L$ , following  
 414 the result of Proposition 2.3 of the previous section. Even if this case is restrictive,  
 415 we find several physical applications, where only two coupled equations appear in the  
 416 model, see [20]. These statements are presented in the following theorem.

417 **THEOREM 3.2.** *Consider the closed-loop system (1.1)-(3.2), output (1.4), and*  
 418 *boundary conditions being of the form (BC-A) or (BC-B). Then, for any  $\bar{d} > 0$ ,*  
 419 *there exist an output feedback law  $u(t)$  of the form (2.7), where  $v$  is substituted by*  
 420 *the observer state  $\hat{v}$ , and constants  $\theta_0, \omega_0, \bar{c} > 0$ , such that for any design parameters*  
 421  *$\theta > \theta_0, \omega > \omega_0$  (with  $\theta$  involved in the observer and  $\omega$  involved in the control laws),*  
 422 *the closed-loop system solution with  $v^0, \hat{v}^0 \in L^2(0, L)^n$  satisfies the following stability*  
 423 *inequality (on the estimation error and last observer state)*

$$424 \quad (3.12) \quad \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}_n\|_{L^2(0,L)} \\ 425 \leq \bar{c} e^{-\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)}), \forall t \geq 0.$$

432 Moreover, whenever the previous assertion holds, we get the following (full state  
433 convergence)

434 a) When boundary conditions (BC-A) hold with  $n \geq 2$ , for every  $L > 0$ , there  
435 exist constants  $c, d > 0$ , such that solutions  $v, \hat{v}$  satisfy the following  
436

$$437 \quad (3.13) \quad \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} \\ 438 \quad \leq ce^{-dt} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}), \forall t \geq 0,$$

440 with  $d$  depending on  $n$ .

441 b) When boundary conditions (BC-B) hold with  $n = 2$ , for every  $L > 0$ , there  
442 exist constants  $c, d > 0$ , such that solutions  $v, \hat{v}$  satisfy (3.13).

443 c) When (BC-B) holds, with  $n > 2$  there exists  $\bar{L} > 0$  small, such that asymptotic  
444 stability (3.13) is guaranteed for all  $L \in (0, \bar{L}]$ .

445 *Proof.* To address the closed-loop control problem, let us rewrite observer error  
446 and observer coupled equations, viz. (see (3.6), (3.2))

$$447 \quad (3.14) \quad \begin{cases} \varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = \theta(A_n + K_n C)\varepsilon - B\varepsilon, \\ \hat{v}_t + \hat{v}_x + \hat{v}_{xxx} = (A_n - B)\hat{v} + \theta\Theta_n K_n \varepsilon_1, \end{cases}$$

448 with boundary conditions (3.7), (3.3).

450 Let us perform a Volterra transformation to the solution of the  $n$ -th equation of  
451 the observer, which by (3.2) is written as

$$452 \quad (3.15) \quad \hat{v}_{n,t} + \hat{v}_{n,x} + \hat{v}_{n,xxx} = \hat{v}_n + k_n \theta^{n+1} \varepsilon_1.$$

454 The Volterra transformation

$$455 \quad (3.16) \quad q(x) = \mathcal{T}[\hat{v}_n](x) := (k_n \theta^{n+1})^{-1} \hat{v}_n(x) - (k_n \theta^{n+1})^{-1} \int_x^L p(x, y) \hat{v}_n(y) dy,$$

457 under appropriate choice of  $p(\cdot, \cdot)$  maps (3.15) into target system

$$458 \quad (3.17) \quad q_t + q_x + q_{xxx} = -\omega q + \varepsilon_1 - \int_x^L p(x, y) \varepsilon_1(t, y) dy,$$

460 with  $\omega$  a constant involved in the controller, and boundary conditions for each of the  
461 two considered cases as follows

$$462 \quad (3.18a) \quad q(t, 0) = q(t, L) = q_x(t, L) = 0,$$

$$463 \quad (3.18b) \quad q_{xx}(t, 0) = q(t, L) = q_x(t, L) = 0.$$

465 Then, the kernel functions  $p(\cdot, \cdot)$  satisfy (2.11) for both problems (3.18a), (3.18b). It  
466 is easy to check this if we apply successive differentiations of (3.16) as in Theorem 2.4,  
467 we obtain the above target system, by choosing  $p(\cdot, \cdot)$  satisfying (2.11). Subsequently,  
468 the output feedback control  $u(\cdot)$  for (BC-A) is given by

$$469 \quad (3.19a) \quad u(t) = \int_0^L p(0, y) \hat{v}_n(t, y) dy,$$

470 and for (BC-B),

$$472 \quad (3.19b) \quad u(t) = -\frac{\omega + 1}{3} L \hat{v}_n(t, 0) + \int_0^L p_{xx}(0, y) \hat{v}_n(t, y) dy.$$

473

474 As noticed in proof of [Theorem 2.4](#), it has been proven that the kernel equations  
 475 [\(2.11\)](#) are solvable in  $\Pi$  and the corresponding Volterra transformation is bounded  
 476 and injective with bounded inverse.

477 Consider now the Lyapunov function

$$478 \quad U_1(t) = U_{1,1}(t) + U_{1,2}(t);$$

$$479 \quad U_{1,1}(t) := \int_0^L \mu(x) \varepsilon^\top(x) P \varepsilon(x) dx, U_{1,2}(t) := \int_0^L \sigma(x) q^2(x) dx,$$

481 along the solutions to [\(3.6\)](#)-[\(3.7\)](#) and [\(3.17\)](#)-[\(3.18\)](#), where  $U_{1,1}$  is the same as [\(3.8\)](#) of  
 482 [Theorem 3.1](#) and  $\sigma(\cdot)$  is a positive  $C^3$  increasing function in  $[0, L]$  to be chosen later.

483 Taking the time-derivative of  $U_{1,2}$  and substituting [\(3.17\)](#), we infer

$$484 \quad \dot{U}_{1,2}(t) = \int_0^L (\sigma'''(x) + \sigma'(x) - 2\omega) q^2(x) dx - 3 \int_0^L \sigma'(x) q_x^2(x) dx$$

$$485 \quad + 2 \int_0^L \sigma(x) q(x) \varepsilon_1(x) dx - 2 \int_0^L \sigma(x) q(x) \int_x^L p(x, y) \varepsilon_1(y) dy dx$$

$$486 \quad + [(-\sigma''(x) - \sigma(x)) q^2(x) - 2\sigma(x) q_{xx}(x) q(x) + \sigma(x) q_x^2(x)$$

$$487 \quad + 2\sigma'(x) q_x(x) q(x)]_0^L.$$

489 By using

$$490 \quad 2 \int_0^L \sigma(x) q(x) \int_x^L p(x, y) \varepsilon_1(y) dy dx \leq U_{1,2}(t) + \sigma(L) \int_0^L \left( \int_x^L p(x, y) \varepsilon_1(y) dy \right)^2 dx$$

$$491 \quad \leq U_{1,2}(t) + L^2 \sigma(L) \max_{x, y \in [0, L]} p^2(x, y) \int_0^L \varepsilon_1^2(y) dy$$

$$492 \quad \leq U_{1,2}(t) + L^2 \frac{\sigma(L)}{\mu(0) \lambda_{\min}(P)} \max_{x, y \in [0, L]} p^2(x, y) U_{1,1}(t),$$

494 we get

$$495 \quad \dot{U}_{1,2}(t) \leq \int_0^L (\sigma'''(x) + \sigma'(x) - 2(\omega - 1)) q^2(x) dx - 3 \int_0^L \sigma'(x) q_x^2(x) dx + h U_{1,1}(t)$$

$$496 \quad + [(-\sigma''(x) - \sigma(x)) q^2(x) - 2\sigma(x) q_{xx}(x) q(x) + \sigma(x) q_x^2(x)$$

$$497 \quad + 2\sigma'(x) q_x(x) q(x)]_0^L,$$

499 where  $h := (L^2 \max_{x, y \in [0, L]} p^2(x, y) + 1) \frac{\sigma(L)}{\mu(0) \lambda_{\min}(P)}$ .

500 We can prove that for each of the two cases of boundary conditions we get

$$501 \quad (3.21) \quad \dot{U}_1(t) \leq -2\bar{d} U_1(t).$$

503 **Case (BC-A):**

504 We choose  $\mu(\cdot) = \sigma(\cdot) = 1$  and we obtain:

$$505 \quad \dot{U}_{1,2}(t) \leq -2(\omega - 1) U_{1,2}(t) + h U_{1,1}(t).$$

507 As seen in [Theorem 3.1](#), for  $\mu(\cdot) = 1$ , we have:

$$508 \quad \dot{U}_{1,1}(t) \leq \left( -\theta \frac{1}{|P|} + 2 \frac{|P|}{\lambda_{\min}(P)} \right) U_{1,1}(t).$$

509

510 Combining the last two equations, if we choose  $\theta, \omega$  as follows:

$$511 \quad (3.22) \quad \theta > h|P| + 2\frac{|P|^2}{\lambda_{\min}(P)}, \omega > 1,$$

512 we get a  $\bar{d} > 0$ , such that (3.21) holds.

514 **Case (BC-B):**

515 We see for this case of boundary conditions that for all

$$516 \quad (3.23) \quad \theta \geq \theta_0 := 2\frac{|P|^2}{\lambda_{\min}(P)} + h|P| + 2|P|,$$

518 Assertion 1 ((2.5) in Lemma 2.2) is satisfied with  $\mu(\cdot)$  in the place of  $\pi(\cdot)$ ,  $\lambda =$   
 519  $\theta\frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)} - \frac{h}{2}$ ,  $\lambda_0 = 1$ . For all  $\theta \geq \theta_0$ , we choose, therefore,  $\mu(\cdot)$  satisfying (2.5)  
 520 and we get that the first term of the right hand side of

$$521 \quad \dot{U}_1(t) \leq \int_0^L \left( \mu'''(x) + \mu'(x) - 2 \left( \theta\frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)} - \frac{h}{2} \right) \mu(x) \right) \varepsilon^\top(x) P \varepsilon(x) dx$$

$$522 \quad + \dot{U}_{1,2}(t), t \geq 0.$$

524 becomes negative.

Similarly, for every

$$\omega \geq 2,$$

525 we can find  $\sigma(\cdot) = \pi(\cdot)$  satisfying (2.5), with  $\lambda = \omega - 1$  and  $\lambda_0 = 1$  and by virtue  
 526 of Proposition 2.3, right hand side of (3.20) becomes negative. Hence, returning to  
 527  $\dot{U}_1$  and choosing  $\theta \geq \theta_0$  and  $\omega \geq 2$ , we can always find  $\mu(\cdot), \sigma(\cdot)$  as in Assertion 1 of  
 528 Lemma 2.2, in a such way that we always get a  $c_2 > 0$ , satisfying again (3.21).

529 Consequently, for each of the two problems (BC-A) and (BC-B), for each  $\bar{d} > 0$   
 530 we can find  $\theta, \omega$ , chosen as before in such a way that there exists constant  $\gamma > 0$   
 531 depending polynomially on  $\theta$ , such that

$$532 \quad \|\hat{v} - v\|_{L^2(0,L)^n} + \|q\|_{L^2(0,L)} \leq \gamma e^{-\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|q(0, \cdot)\|_{L^2(0,L)}), \forall t \geq 0.$$

534 Transformation  $\mathcal{T}$  is bounded with bounded inverse (see the comments in Theo-  
 535 rem 2.4) and, therefore, we obtain an inequality as (3.12).

536 *Remark 3.3.* The previous calculations indicate that gain  $\theta$  appearing in observer  
 537 system (3.2)-(3.3) is crucial in the stabilization of the closed-loop system. Indeed, in  
 538 (3.23), we see that choice of  $\theta$  compensates for some terms appearing therein. The  
 539 dependence of the terms on the eigenvalues of matrix  $P$  indicates that a simpler  
 540 Luenberger observer with pole placement would not suffice for the stabilization of  
 541 the closed-loop system. These terms play the role of the nonlinearities, appearing in  
 542 the Lyapunov derivative used for the observer error in high-gain observer designs for  
 543 finite-dimensional systems. Although in finite dimension, a pole-placement observer  
 544 is enough for linear systems, in the present framework of infinite dimension, a design  
 545 similar to high-gain observers in finite-dimension is required.

546 We are now in a position to prove the closed-loop stability for the whole system  
 547 following the methodology of Theorem 2.4. Let  $\hat{v}_{[n-1]} := (\hat{v}_1 \cdots \hat{v}_{n-1})^\top$ . Then,  $\hat{v}_{[n-1]}$   
 548 satisfies the following equations

$$549 \quad \hat{v}_{[n-1],t} + \hat{v}_{[n-1],x} + \hat{v}_{[n-1],xxx} = (A_{n-1} - I_{n-1})\hat{v}_{[n-1]} + \ell\hat{v}_n + \Theta_{n-1}K_{n-1}(\hat{v}_1 - v_1),$$

$$550 \quad (\text{BC-A}, \hat{v}_{[n-1]}) : \hat{v}_{[n-1]}(t, 0) = \hat{v}_{[n-1]}(t, L) = \hat{v}_{[n-1],x}(t, L) = 0$$

$$551 \quad (\text{BC-B}, \hat{v}_{[n-1]}) : \hat{v}_{[n-1],xx}(t, 0) = \hat{v}_{[n-1]}(t, L) = \hat{v}_{[n-1],x}(t, L) = 0$$

551 where  $\ell := (0 \ \cdots \ 0 \ 1)^\top$  and  $\Theta_{n-1}, K_{n-1}$  are involved in observer (3.2).

552 By choosing

$$553 \quad U_2(t) = \int_0^L \pi(x) |\hat{v}_{[n-1]}(x)|^2 dx$$

554 as a Lyapunov functional along the  $L^2(0, L)^{n-1}$  solutions to the last equations, with  
555  $\pi(\cdot)$  a positive increasing  $C^3$  function, we obtain

$$557 \quad \dot{U}_2(t) = \int_0^L (\pi'''(x) + \pi'(x)) |\hat{v}_{[n-1]}(x)|^2 dx - 3 \int_0^L \pi'(x) |\hat{v}_{[n-1],x}(x)|^2 dx$$

$$558 \quad - 2 \int_0^L \pi(x) \hat{v}_{[n-1]}^\top(x) \text{Sym}(I_{n-1} - A_{n-1}) \hat{v}_{[n-1]}(x) dx$$

$$559 \quad + 2 \int_0^L \pi(x) \hat{v}_{n-1}(x) \hat{v}_n(x) dx + 2 \int_0^L \pi(x) \hat{v}_{[n-1]}^\top \Theta_{n-1} K_{n-1} (\hat{v}_1 - v_1) dx + U_{2,0},$$

561 where  $U_{2,0}$  is as  $W_0$  in (2.14) (see the proof of Theorem 2.4), while  $v_{[n-1]}$  is substituted  
562 by  $\hat{v}_{[n-1]}$ . Applying Young's inequality, we get

$$563 \quad \dot{U}_2(t) \leq \int_0^L (\pi'''(x) + \pi'(x) - (\rho_n - 2\delta)\pi(x)) |\hat{v}_{[n-1]}(x)|^2 dx$$

$$564 \quad (3.24) \quad + \frac{1}{\delta} \int_0^L \pi(x) \hat{v}_n^2(x) dx + \frac{1}{\delta} \theta^{2n-2} |K_{n-1}|^2 \int_0^L \pi(x) |\hat{v}_1(x) - v_1(x)|^2 dx + U_{2,0},$$

566 with  $\delta > 0$  chosen sufficiently small, as in (A.4), determined in the proof of Lemma 2.2  
567 of previous section, and  $\rho_n$  defined in (2.15).

568 Now, to ensure negativity of the Lyapunov derivative, we choose  $\pi(\cdot)$  for each of  
569 the two boundary problems as follows.

570 **Case (BC A,  $\hat{v}_{[n-1]}$ ):**

$$571 \quad \pi(\cdot) = 1.$$

573 Then, in conjunction with the previously proven equation (3.12), we get from (3.24)

$$574 \quad (3.25) \quad \dot{U}_2(t) \leq -2dU_2(t) + me^{-2\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)})^2$$

576 where  $d := \rho_n - 2\delta > 0$  and

$$577 \quad (3.26) \quad m := \frac{1}{\delta} \pi(L) \bar{c}^2 \max(1, \theta^{2n-2} |K_{n-1}|^2).$$

579 **Case (BC B,  $\hat{v}_{[n-1]}$ ):**

580 For boundary conditions (BC B,  $\hat{v}_{[n-1]}$ ), to obtain an asymptotic stability result,  
581 we first check that for  $n = 2$ , we have  $\rho_n = 2$ . For this  $\rho_2$ , proof of Lemma 2.2  
582 suggests that there exists  $\pi(\cdot)$  satisfying (2.5) for some  $b > 0$ , with the same  $\pi(\cdot)$ ,  
583  $\lambda = \frac{\rho_2}{2} - \delta$ . Then, a similar inequality as (3.25) is satisfied for all  $L > 0$ ,  $d = b$  and  
584  $m$  as in (3.26). Additionally, following Assertion 2 in the proof of Lemma 2.2, we see  
585 that for any  $n > 2$ , implying  $\rho_n < 2$ , there exist again  $\pi(\cdot)$ ,  $d = b > 0$ , such that (2.5)  
586 holds for  $L \in (0, \bar{L}]$ .

587 Now, we see that for both boundary problems (BC-A) and (BC-B), (3.25) gives

$$588 \quad U_2(t) \leq e^{-2dt} W(0) + \frac{m}{2d - 2\bar{d}} (e^{-2\bar{d}t} - e^{-2dt}) (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)})^2.$$

589



590 The latter implies

(3.27)

$$591 \quad U_2(t) \leq \left( e^{-2dt} + \frac{m}{2d-2\bar{d}}(e^{-2\bar{d}t} - e^{-2dt}) \right) (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n})^2.$$

592

593 Recalling that  $\bar{d}$  depends on the adjustable observer parameter  $\theta$ , we suppose, without  
 594 loss of generality, that can be chosen such that  $\bar{d} > d$ , so that the previous inequality  
 595 has meaning.

596 Now, using trivial inequalities and by virtue of (3.27) and (3.12), we easily get:

$$597 \quad \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} \leq \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}_n\|_{L^2(0,L)} + \|\hat{v}_{[n-1]}\|_{L^2(0,L)^{n-1}}$$

$$598 \quad \leq \left[ \bar{c}e^{-\bar{d}t} + \frac{1}{\sqrt{\pi(0)}} \sqrt{e^{-2dt} + \frac{m}{2d-2\bar{d}}(e^{-2\bar{d}t} - e^{-2dt})} \right]$$

$$599 \quad \times (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}).$$

600

601 The latter completes the proof of [Theorem 3.2](#), suggesting also that the asymptotic  
 602 rate of the whole closed-loop cascade system is no larger than  $d$ , which is decreasing  
 603 with  $n$ , contrary to the asymptotic rate for the last state  $v_n$ , which is adjusted by the  
 604 observer and control parameters.  $\square$

605 *Remark 3.4.* The considered stabilization problem of under-actuated and under-  
 606 observed cascade systems of KdV equations was here limited to the linear case, and  
 607 special forms of couplings. Even though stabilization results of the original nonlinear  
 608 KdV equation can be found (see survey [3]), or observer results for some infinite-  
 609 dimensional systems with nonlinearities (satisfying some “triangular structure”), as  
 610 in [16, 18], extensions of our output feedback stabilization to more general couplings  
 611 and/or nonlinearities are quite challenging, and are thus left for future studies: a  
 612 strong difficulty comes from the coefficients of system’s differential operator, where  
 613 the presence of distinct elements raises problems related to the notion of algebraic  
 614 solvability, which has been given attention in [1, 19] and other works of the same  
 615 authors. In the Lyapunov-based approach we have considered, this problem translates  
 616 into the lack of a commutative property between a Lyapunov matrix and coefficients of  
 617 system’s differential operator. Handling nonlinearities in one-order term (for instance  
 618 terms  $v_i v_{i,x}$  or even couplings of this type between the equations) and zero-order  
 619 term at the same time is also part of the challenge, as this commutative property  
 620 would not be fulfilled. We also refer to [9], where some open problems concerning  
 621 such coupling are presented, while the reader can understand the difficulties in the  
 622 controllability analysis of under-actuated systems with nontrivial coefficients of the  
 623 differential operators and the presence of nonlinearities.

624 Notice yet that it could be possible to adopt an indirect approach, based on our  
 625 previous approaches [18], to deal with the case where the one-order and dispersion  
 626 terms would be of the form  $A_1 v_x + A_2 v_{xxx}$ , for some  $A_1, A_2 \in \mathbb{R}^{n \times n}$ . Consideration of  
 627 linear lower triangular couplings of one-order and third-order terms would be feasible  
 628 as well, but more general cases remain open.

629 Notice also that this under-observed problem being already challenging, the case  
 630 when only a boundary measurement is available (instead of an internal one, at least  
 631 localized to a part of the domain) is even more difficult: even though a solution does  
 632 exist for a single equation and boundary measurement [23], it does not easily extends  
 633 to the case of coupled KdV equations, via a backstepping and a single measurement

634 instead of a distributed one. For the case of  $n$  coupled equations, a backstepping  
 635 approach that would lead to an exponentially stable observer error system would fail,  
 636 even for  $n = 2$ , if the observations were fewer than the states. In addition, the control  
 637 problem of under-actuated systems by itself is a hard problem, and if  $2 \times 2$  systems  
 638 have local solutions via backstepping, see [8], they concern hyperbolic systems where a  
 639 dissipative target system is feasible. For the coupled KdV equations, the exponentially  
 640 stable target system for the observer error would be needed to be a damped system,  
 641 which cannot be achieved by a single observation.

642 Some possible generalizations of the present framework, as the ones described  
 643 before, will be a subject of our future works.

644 **4. Conclusion.** In this work, output feedback stabilization for a class of cascade  
 645 system of linear KdV equations was achieved. Two boundary control problems, with  
 646 controls placed on the left side of the last equation, were investigated. Distributed  
 647 measurement of the first state was considered, which provided an estimation (using a  
 648 high-gain observer) of the states fed in the control laws. The cascade system is stabi-  
 649 lized for both boundary problems, but with a limitation on the number of equations  
 650 and length of the domain for the second one.

651 Future developments might include the same stabilization framework, but with  
 652 more general couplings (in zero-order and one-order derivative terms), including lo-  
 653 calized terms and nonlinearities.

654 **Appendix A. Proof of Lemma 2.2.** To prove Assertions 1 and 2 of  
 655 Lemma 2.2, it is more convenient to write the characteristic equation of the differential  
 656 equation in (2.5) as

$$657 \quad (A.1) \quad r^3 + r - s^3 - s = 0$$

659 (as in [25], a technique used to solve the characteristic equation of the KdV operator),  
 660 where

$$661 \quad (A.2) \quad s^3 + s = 2\lambda - 2b$$

and considering  $s$  being the real root of the latter equation. Then, solutions to (A.1)  
 are given by

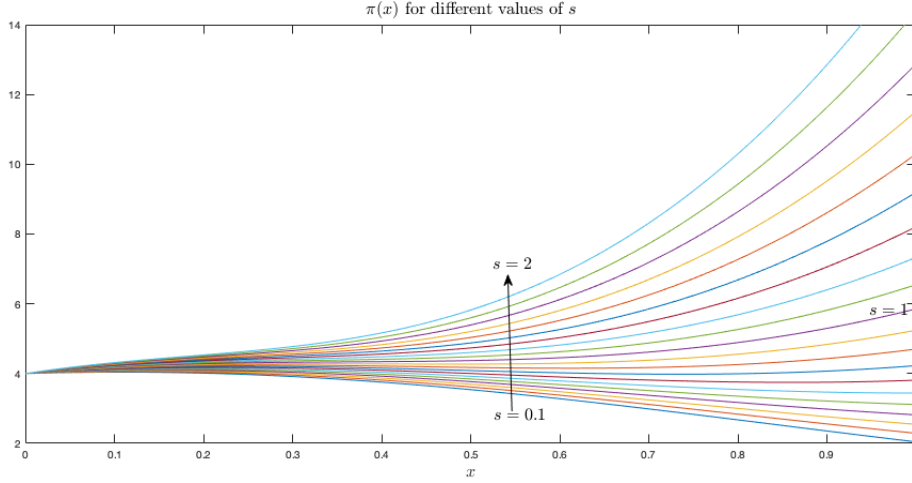
$$r_1 = s, r_2 = -\frac{s}{2} + i\frac{\sqrt{3s^2 + 4}}{2}, r_3 = -\frac{s}{2} - i\frac{\sqrt{3s^2 + 4}}{2}$$

663 and, therefore, unique solutions to the differential equation in (2.5) are given by

$$664 \quad (A.3) \quad \pi(x) = \alpha e^{sx} + \beta e^{-\frac{s}{2}x} \cos \frac{\sqrt{3s^2 + 4}}{2}x + \gamma e^{-\frac{s}{2}x} \sin \frac{\sqrt{3s^2 + 4}}{2}x$$

666 with  $\alpha, \beta, \gamma \in \mathbb{R}$  chosen, such that restriction on initial conditions in (2.5) is satisfied.  
 667 We can check numerically that there exists a number  $\epsilon > 0$  near zero, such that for  
 668  $s \geq 1 - \epsilon$ ,  $\pi(\cdot)$  given by (A.3) with initial conditions  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$   
 669 (corresponding to  $\alpha = 56/25, \beta = 44/25, \gamma = 8/25$ ) is positive and increasing and,  
 670 therefore, it satisfies (2.5). Defining a small constant  $\delta > 0$  by

$$671 \quad (A.4) \quad \delta := \frac{\epsilon^3}{5} - \frac{3\epsilon^2}{5} + \frac{7\epsilon}{5},$$

FIG. 1. Solutions to (2.5) for different  $s$ 

673 we see from (A.2) that for  $s \geq 1 - \epsilon$  we have  $\lambda \geq \lambda_0 := 1 - \delta$  for choice  $b = \frac{11}{16}\lambda$ .  
 674 Thus, for all  $\lambda \geq \lambda_0$ , there exists  $b > 0$ , such that conditions (2.5) are satisfied.  
 675 Hence, Assertion 1 is proven to hold for  $\lambda_0 = 1 - \delta$ , where  $\delta$  is defined above. Now,  
 676 notice that for  $s < 1 - \epsilon$ , corresponding to  $\lambda < \lambda_0$ , and for any initial condition of  
 677  $\pi(\cdot)$ , satisfying second equation of (2.5), there is a  $\bar{L} > 0$ , such that for  $x > \bar{L}$ ,  $\pi(\cdot)$   
 678 becomes decreasing and, thus, fails to satisfy all conditions (2.5). This implies that  
 679 for  $0 < \lambda < \lambda_0$ , Assertion 2 is satisfied for some small  $\bar{L} > 0$ . Letting  $s \rightarrow 0^+$ , and  
 680 choosing initial conditions  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$  as before,  $\pi$  approaches the  
 681 trajectory of  $\pi(x) = -1 + 5 \cos(x) + 2 \sin(x)$ , for which  $\pi'(x) < 0$  for  $L > \arctan(2/5)$ .  
 682 By this, for  $\lambda \rightarrow 0^+, b \rightarrow 0^+$ , Assertion 2 is satisfied with  $\bar{L} = \arctan(2/5)$ .

683 In Figure 1, we represent the evolution of  $\pi(x)$  for choice of initial condition  
 684  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$  and various values of  $s$ , corresponding to various  
 685 values of  $\lambda$ . For small values of  $s$ , corresponding to small values  $\lambda$ ,  $\pi(\cdot)$  is increasing  
 686 until some point  $x = \bar{L}$  quite small, but for  $x > \bar{L}$ , it is decreasing and, thus, fails to  
 687 satisfy fourth equation of (2.5) after this point, in accordance with Assertion 2. We  
 688 also see that for all  $s \geq 1 - \epsilon$ , for  $\epsilon > 0$  small, given as before,  $\pi(\cdot)$  is always increasing,  
 689 verifying Assertion 1. The proof is complete.

## 690 Appendix B. Well-posedness of system and observer.

691 We show here the well-posedness of both controlled system (1.1)-(1.3) and ob-  
 692 server system (3.2)-(3.3).

693 **B.1. Well-posedness of (1.1)-(1.3).** First, for system (1.1), with boundary  
 694 conditions (BC-A) or (BC-B), feedback control laws of the form (2.7a) or (2.7b),  
 695 respectively, and initial condition  $v^0 \in L^2(0, L)^n$ , it is sufficient to prove the well-  
 696 posedness of target system, which results after applying the isomorphic transformation  
 697  $\mathcal{T}$ , see (2.9), in conjunction with (2.10):

$$\begin{cases} v_{[n-1],t} + v_{[n-1],x} + v_{[n-1],xxx} = (A_{n-1} - I_{n-1})v_{[n-1]} + \ell \mathcal{T}^{-1}[z], \\ z_t + z_x + z_{xxx} + \omega z = 0 \\ \text{(BC-A) : } \eta(t, 0) = \eta(t, L) = \eta_x(t, L) = 0, \\ \text{(BC-B) : } \eta_{xx}(t, 0) = \eta(t, L) = \eta_x(t, L) = 0, \end{cases}$$

700 where  $\eta = (v_{[n-1]} \ z)$  is the target state and we adopt the same notation as in (2.13).  
 701 We rewrite the above system as an abstract evolution system in  $L^2(0, L)^n$  as

$$702 \quad \dot{\zeta} = \mathcal{A}\zeta,$$

704 where  $\mathcal{A} := (\mathcal{A}_1 \ \cdots \ \mathcal{A}_n) : D(\mathcal{A}) \rightarrow L^2(0, L)^n$  is an linear unbounded operator  
 705 defined as

$$706 \quad \begin{aligned} \mathcal{A}_i \zeta_i &= -\zeta_{i,x} - \zeta_{i,xxx} - \zeta_i + \zeta_{i+1}, \quad i = 1, \dots, n-2 \\ 707 \quad \mathcal{A}_{n-1} \zeta_{n-1} &= -\zeta_{n-1,x} - \zeta_{n-1,xxx} - \zeta_{n-1} + \mathcal{T}^{-1}[\zeta_n] \\ 708 \quad \mathcal{A}_n \zeta_n &= -\zeta_{n,x} - \zeta_{n,xxx} - \omega \zeta_n, \end{aligned}$$

710 with domain

$$711 \quad D(\mathcal{A}) = \{ \zeta \in H^3(0, L)^n; \zeta(0) = \zeta(L) = \zeta'(L) = 0, \text{ for (BC-A)}, \\ 712 \quad \text{or } \zeta''(0) = \zeta(L) = \zeta'(L) = 0, \text{ for (BC-B)} \}.$$

714 Its adjoint operator satisfies

$$715 \quad \begin{aligned} \mathcal{A}_i^* \zeta_i &= \zeta_{i,x} + \zeta_{i,xxx} - \zeta_i + \zeta_{i+1}, \quad i = 1, \dots, n-2 \\ 716 \quad \mathcal{A}_{n-1}^* \zeta_{n-1} &= \zeta_{n-1,x} + \zeta_{n-1,xxx} - \zeta_{n-1} + \mathcal{T}^{-1}[\zeta_n] \\ 717 \quad \mathcal{A}_n^* \zeta_n &= \zeta_{n,x} + \zeta_{n,xxx} - \omega \zeta_n, \end{aligned}$$

719 with domain

$$720 \quad D(\mathcal{A}^*) = \{ \zeta \in H^3(0, L)^n; \zeta(0) = \zeta'(0) = \zeta(L) = 0, \text{ for (BC-A)}, \\ 721 \quad \text{or } \zeta''(0) = -\zeta(0), \zeta(L) = \zeta'(0) = \zeta(L) = 0, \text{ for (BC-B)} \}.$$

723 Operator  $\mathcal{A}$  and its adjoint  $\mathcal{A}^*$  are closed with domains dense in  $L^2(0, L)^n$ . Further-  
 724 more, they are both dissipative. Indeed, from the stability proof of Theorem 2.4, we  
 725 first see that  $\mathcal{A}_n$  is dissipative. Then, the exponential stability of  $\mathcal{T}[\zeta_n]$  in (2.12), in  
 726 conjunction with (2.17) implies that

$$727 \quad (\text{B.1}) \quad \langle \mathcal{A}_{[n-1]} \zeta_{[n-1]}, \pi \zeta_{[n-1]} \rangle_{L^2(0, L)^{n-1}} \leq 0,$$

729 where  $\pi(\cdot)$  satisfies (2.5). Inequality (B.1) is satisfied for every  $L > 0$ , if  $n \geq 2$  under  
 730 boundary conditions (BC-A) and if  $n = 2$  under boundary conditions (BC-B). For  
 731 the latter case, the same inequality holds for  $n > 2$ , when  $L \in (0, \bar{L}]$ , for some  $\bar{L} > 0$ .  
 732 This was shown in the stability proof of Theorem 2.4 and it implies that operator  $\mathcal{A}$   
 733 is dissipative, namely

$$734 \quad \langle \mathcal{A}\zeta, \zeta \rangle_{L^2(0, L)^n} \leq 0.$$

736 To show dissipativity of the adjoint operator  $\mathcal{A}^*$ , i.e., that  $\langle \mathcal{A}^*\zeta, \zeta \rangle_{L^2(0, L)^n} \leq 0$ ,  
 737 we can easily show first that  $\mathcal{A}_n^*$  is dissipative, by applying integrations by parts.  
 738 Then, we show that  $\langle \mathcal{A}_{[n-1]}^* \zeta_{[n-1]}, \zeta_{[n-1]} \rangle_{L^2(0, L)^{n-1}} \leq 0$ . This implies that  $\mathcal{A}^*$  is  
 739 dissipative.

Consequently, we can apply the Lumer-Phillips theorem and we conclude that  $\mathcal{A}$   
 generates a  $C^0$ -semigroup of contractions and, thus, returning to the original coordi-  
 nates via  $\mathcal{T}^{-1}$ , we have that for any initial condition  $v^0 \in L^2(0, L)^n$ , there exists a  
 unique mild solution

$$v \in C^0(0, \infty; L^2(0, L)^n)$$

740 for system (1.1)-(1.3), noting also, that for (BC-A), the above holds for all  $L > 0$   
 741 and  $n \geq 2$ , while for (BC-B), the above holds for all  $L > 0$ , when  $n = 2$  and for all  
 742  $L \in (0, \bar{L}]$ , when  $n > 2$ , where  $\bar{L}$  is given in Lemma 2.2.

743 The above well-posedness result for the  $n$  coupled equations holds also for the  
 744 single damped KdV equation, see (2.1)-(2.3), as this system's operator is equal to  $\mathcal{A}_n$ ,  
 745 as defined above.

746 **B.2. Well-posedness of (3.2)-(3.3).** Observer system given by (3.2) with  
 747 boundary conditions (3.3) is well posed. To see this, it suffices to show the well-  
 748 posedness of the error system (3.6)-(3.7), invoking also the well-posedness of initial  
 749 system (1.1)-(1.3) that we showed before. The differential operator for error system  
 750 (3.6)-(3.7) is given by

$$751 \quad \mathcal{A}\zeta = -\zeta_x - \zeta_{xxx} + \theta(A_n + K_n C)\zeta - B\zeta,$$

753 with domain

$$754 \quad D(\mathcal{A}) = \{\zeta \in H^3(0, L)^n; \zeta(0) = \zeta(L) = \zeta'(L) = 0, \text{ for (BC-A)},$$

$$755 \quad \text{or } \zeta''(0) = \zeta(L) = \zeta'(L) = 0, \text{ for (BC-B)}\}.$$

757 and its adjoint operator is given by

$$758 \quad \mathcal{A}^*\zeta = \zeta_x + \zeta_{xxx} + \theta(A_n + K_n C)\zeta - B\zeta,$$

760 with domain

$$761 \quad D(\mathcal{A}^*) = \{\zeta \in H^3(0, L)^n; \zeta(0) = \zeta'(0) = \zeta(L) = 0, \text{ for (BC-A)},$$

$$762 \quad \text{or } \zeta''(0) = -\zeta(0), \zeta(L) = \zeta'(0) = \zeta(L) = 0, \text{ for (BC-B)}\}.$$

764 By the stability proof in Theorem 3.1, we see that

$$765 \quad (B.2) \quad \langle \mathcal{A}\zeta, \mu P\zeta \rangle_{L^2(0, L)^n} \leq 0,$$

whenever  $\theta > \theta_0$ , where  $\theta_0$  is defined in the proof of Theorem 3.1 and where function  
 $\mu(\cdot)$  and matrix  $P$  are defined in the proof of Theorem 3.1. Inequality (B.2) implies  
 that  $\mathcal{A}$  is dissipative. The adjoint  $\mathcal{A}^*$  is also dissipative and it can be shown by  
 proving that  $\langle \mathcal{A}^*\zeta, P\zeta \rangle_{L^2(0, L)^n} \leq 0$ , by applying successive integrations by parts.  
 This, similarly as in Appendix B.1, proves the well-posedness of the error equations,  
 which along with the well-posedness of the initial system results in the well-posedness  
 of the observer system (3.2)-(3.3), namely for any initial condition  $\hat{v}^0 \in L^2(0, L)^n$ ,  
 there exists a unique mild solution

$$\hat{v} \in C^0(0, \infty; L^2(0, L)^n),$$

767 for all  $\theta > \theta_0$ .

768

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