



HAL
open science

Computing the Characteristic Polynomial of Generic Toeplitz-like and Hankel-like Matrices

Clément Pernet, Hippolyte Signargout, Pierre Karpman, Gilles Villard

► **To cite this version:**

Clément Pernet, Hippolyte Signargout, Pierre Karpman, Gilles Villard. Computing the Characteristic Polynomial of Generic Toeplitz-like and Hankel-like Matrices. ISSAC 2021 - International Symposium on Symbolic and Algebraic Computation, Jul 2021, Saint Petersburg, Russia. pp.249-256, 10.1145/3452143.3465542 . hal-03189115

HAL Id: hal-03189115

<https://hal.science/hal-03189115v1>

Submitted on 2 Apr 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Computing the Characteristic Polynomial of Generic Toeplitz-like and Hankel-like Matrices

Pierre Karpman

Université Grenoble Alpes
Laboratoire Jean Kuntzmann, CNRS UMR 5224
Grenoble, France

Hippolyte Signargout

Univ. Lyon, ENS de Lyon, CNRS, Inria, UCBL
LIP UMR 5668 and LJK UMR 5224
Lyon, France

Clément Pernet

Université Grenoble Alpes
Laboratoire Jean Kuntzmann, CNRS UMR 5224
Grenoble, France

Gilles Villard

Univ. Lyon, CNRS, ENS de Lyon, Inria, UCBL
LIP UMR 5668
Lyon, France

ABSTRACT

New algorithms are presented for computing annihilating polynomials of Toeplitz, Hankel, and more generally Toeplitz+Hankel-like matrices over a field. Our approach follows works on Copersmith’s block Wiedemann method with structured projections, which have been recently successfully applied for computing the bivariate resultant. A first baby-step/giant step approach—directly derived using known techniques on structured matrices—gives a randomized Monte Carlo algorithm for the minimal polynomial of an $n \times n$ Toeplitz or Hankel-like matrix of displacement rank α using $\tilde{O}(n^{\omega-c(\omega)}\alpha^{c(\omega)})$ arithmetic operations, where ω is the exponent of matrix multiplication and $c(2.373) \approx 0.523$ for the best known value of ω . For generic Toeplitz+Hankel-like matrices a second algorithm computes the characteristic polynomial in $\tilde{O}(n^{2-1/\omega})$ operations when the displacement rank is considered constant. Previous algorithms required $O(n^2)$ operations while the exponents presented here are respectively less than 1.86 and 1.58 with the best known estimate for ω .

KEYWORDS

Characteristic polynomial, minimal polynomial, Toeplitz matrix, Hankel matrix, Toeplitz+Hankel-like matrix.

1 INTRODUCTION

We consider the problem of computing the minimal or the characteristic polynomial of Toeplitz-like and Hankel-like matrices, which include Toeplitz and Hankel ones. The necessary definitions about those structures are given in Section 2.

Throughout the paper $T \in \mathbb{K}^{n \times n}$ is non-singular and either Toeplitz-like or Hankel-like, where \mathbb{K} is a commutative field. The structure is parameterized by the displacement rank $1 \leq \alpha \leq n$ of T [12, 19]. In particular a Toeplitz or a Hankel matrix has displacement rank $\alpha = 2$.

The determinant of T can be computed in $\tilde{O}(\alpha^{\omega-1}n)$ operations in \mathbb{K} , where $\omega \leq 3$ is a feasible exponent for square $n \times n$ matrix multiplication. For the best known value of ω one can take $\omega \approx 2.373$ [1, 18]. When T has generic rank profile (the leading principal submatrices are non singular) a complexity bound $\tilde{O}(\alpha^2 n)$ for the determinant is derived from [19, Cor. 5.3.3, p. 161]. In the general case, for ensuring the rank profile one uses rank-regularization

techniques initially developed in [13, 15] that lead to randomized Las Vegas algorithms assuming that the cardinality of \mathbb{K} is large enough; see [19, Sec. 5.6-5.7] and [3] for detailed studies in our context. Taking advantage of fast matrix multiplication is possible using the results in [3], where fundamental matrix operations, including the determinant, are performed in time $\tilde{O}(\alpha^{\omega-1}n)$ for a wide spectrum of displacement structures. In this approach the determinant is revealed by the recursive factorization of the inverse.

The characteristic polynomial $\det(xI_n - T)$ of T is a polynomial of degree n . Using an evaluation-interpolation scheme it follows that it can be computed in $\tilde{O}(\alpha^{\omega-1}n^2)$ operations in \mathbb{K} . We also refer to [19, Ch. 7] for a Newton-Structured iteration scheme in time $\tilde{O}(\alpha^2 n^2)$.

For a Toeplitz or Hankel matrix these complexity bounds for computing the characteristic polynomial were quadratic; our contribution establishes an improved bound $\tilde{O}(n^{2-1/\omega})$ for generic matrices (given in compressed form), which is sub-quadratic including when using $\omega = 3$. We build on the results of [23] where only the case of a Sylvester matrix was treated, and show that the approach can be generalized to larger displacement rank families. In particular, the Hankel-(like) case requires the use of sophisticated techniques in order to handle the Toeplitz+Hankel structure [7, 9] and its generalizations [19].

The algorithms we propose fit into the broad family of Copersmith’s block Wiedemann algorithms; we refer to [16] for the necessary material and detailed considerations on the approach. Another interpretation in terms of structured lifting and matrix fraction reconstruction is given in [23].

From $T \in \mathbb{K}^{n \times n}$, the problem is to compute the determinant (or a divisor) of the characteristic matrix $M(x) = xI_n - T$. For $1 \leq m \leq n$ and well chosen projection matrices V and W in $\mathbb{K}^{n \times m}$, the principle is to reconstruct an irreducible fraction description $P(x)Q^{-1}(x)$ of $V^T M(x)^{-1} W \in \mathbb{K}(x)^{m \times m}$, where $P, Q \in \mathbb{K}[x]^{m \times m}$, from a truncated series expansion of the fraction. The denominator matrix Q carries information on the Smith normal form of $M(x)$ [16, Thm. 2.12]. Using random V and W allows to recover the minimal polynomial of T from the largest invariant factor of $M(x)$, and for a generic matrix T the characteristic polynomial is obtained [16, 23].

The matrix Q is computed from a truncation $S^{(m)} \in \mathbb{K}[x]^{m \times m}$ of the series expansion of $V^T M(x)^{-1} W$,

$$S^{(m)}(x) = - \sum_{k \geq 0}^{2 \lceil n/m \rceil} V^T (T^{-1})^k W x^k \quad (1)$$

using for example matrix fraction reconstruction [2, 5]. We will not detail these latter aspects in this paper since they can be found elsewhere in the literature: see [16, 23] for the general techniques involved; [22, Cor. 6.4] for the power series truncation; and [17] for alternative reconstruction possibilities. The results we need on matrix polynomials are recalled in Section 3.

We focus on the computation of the power series terms $H_k = V^T (T^{-1})^k W$ in Eq. (1). The idea for improving the complexity bounds is to use structured projections V and W in order to speed up the computation of the expansion such as in [4, 23]. A typical choice is such that the matrix product by V and W is reduced. The central difficulty is to show that the algorithm remains correct; special choices for V and W could prevent a fraction reconstruction with appropriate cost, or give a denominator matrix Q with too little information on the invariant structure of T .

For a generic input matrix and our best exponent, in Section 5 we follow the choice of [23] and work with $V = W = X$ where $X = (I_m \ 0)^T \in \mathbb{K}^{n \times m}$. An $n \times n$ Toeplitz or a Hankel matrix is defined by $2n-1$ elements of \mathbb{K} , and our algorithm is correct except on a certain hypersurface of \mathbb{K}^{2n-1} . The same way, a Toeplitz-like or Hankel-like matrix of displacement rank α is defined by the $2n\alpha$ coefficients of its generators, and our algorithm is correct for all values of $\mathbb{K}^{2n\alpha}$ except for a hypersurface. If T is Hankel, the matrix $M(x) = xI_n - T$ is Toeplitz+Hankel and the algorithm involves a compressed form that generalizes the use of generators associated to displacement operators [9, 19]. The algorithm computes a compressed representation of $M(x)^{-1}$ modulo $x^{2 \lceil n/m \rceil + 1}$, and exploits its structure to truncate it into a compressed representation of $S^{(m)}(x) = X^T M(x)^{-1} X \pmod{x^{2 \lceil n/m \rceil + 1}}$ at no cost. The parameter m can be optimised to get an algorithm using $\tilde{O}(n^{2-1/\omega})$ operations when the displacement rank is considered constant.

Before considering the fast algorithm for the generic case, in Section 4 we consider the baby steps/giant steps algorithm of [16]. Indeed, thanks to the incorporation of fast matrix multiplication in basis structured matrix operations [3], the overall approach with dense projections V and W already allows a slight exponent improvement. Taking into account that the input matrix T is structured, a direct cost analysis of the algorithm of [16] improves on the quadratic cost for Toeplitz and Hankel matrices as soon as one takes $\omega < 3$. However it is unclear to us how to compute the characteristic polynomial in this case (see the related Open Problem 3 in [14]). The algorithm we propose is randomized Monte Carlo and we compute the minimal polynomial in $\tilde{O}(n^{\omega-c(\omega)})$ operations with $c(\omega) = \frac{\omega-1}{5-\omega}$. For Toeplitz-like and Hankel-like matrices with displacement rank α , the cost is multiplied by $\tilde{O}(\alpha^{c(\omega)})$.

Notation. Indices of matrix and vectors start from zero. The vectors of the n -dimensional canonical basis are denoted by e_0^n, \dots, e_{n-1}^n . For a matrix M , $M_{i,j}$ denotes the coefficient (i, j) of this matrix, $M_{i,*}$, its row of index i and $M_{*,j}$ its column of index j .

2 MATERIAL FOR RANK DISPLACEMENT STRUCTURES

A wide range of structured matrices are efficiently described by the action of a displacement operator [12]. There are two types of such operators: the Sylvester operators of the form

$$\nabla_{M,N} : A \mapsto MA - AN,$$

and the Stein operators of the form

$$\Delta_{M,N} : A \mapsto A - MAN;$$

where M and N are fixed matrices. A Toeplitz matrix T is defined by $2n-1$ coefficients $t_{-n+1}, \dots, t_{n-1} \in \mathbb{K}$ such that $T = (t_{i-j})_{i,j}$. Its image through Δ_{Z_n, Z_n^T} , where $Z_n = (\delta_{i,j+1})_{0 \leq i, j \leq n-1}$ has rank 2. Similarly, a Hankel matrix H is defined by $2n-1$ coefficients h_0, \dots, h_{2n-2} such that $H = (h_{i+j})_{i,j}$ and its image through ∇_{Z_n, Z_n^T} has rank 2.

As a generalization, the class of Toeplitz-like (resp. Hankel-like) matrices is defined [8, 19] as those matrices which image through Δ_{Z_n, Z_n^T} (resp. ∇_{Z_n, Z_n^T}) has a bounded rank α , called the displacement rank. Lastly, any sum of a Toeplitz and a Hankel matrix, (forming the class of Toeplitz+Hankel matrices) has an image of rank 4 through the displacement operator ∇_{U_n, U_n} where $U_n = Z_n + Z_n^T$. However, contrarily to the previous instances, this operator is no longer regular, and the low rank image does not suffice to uniquely reconstruct the initial matrix: additional data (usually a first or a last column) is required for a unique reconstruction. The class of Toeplitz+Hankel-like matrices is formed by those matrices whose image through ∇_{U_n, U_n} has a bounded rank.

2.1 Product of Structured Matrices

PROPOSITION 2.1 ([3, THEOREM 1.2]). *Let $A \in \mathbb{K}^{n \times n}$ be a Toeplitz-like or Hankel-like matrix with displacement rank α given by its generators and $B \in \mathbb{K}^{n \times m}$ be a dense matrix. The multiplication of A by B can be computed in $\tilde{O}(n \max(\alpha, m) \min(\alpha, m)^{\omega-2})$ operations in \mathbb{K} .*

PROPOSITION 2.2. *Let $A, B \in \mathbb{K}^{n \times n}$ be two Toeplitz-like matrices of displacement rank α and β respectively, then their product AB is a Toeplitz-like matrix of displacement rank at most $\alpha + \beta + 1$. Furthermore, given generators for A and B w.r.t. Δ_{Z_n, Z_n^T} , one can compute generators for AB w.r.t. the same operator in $\tilde{O}(n(\alpha + \beta)^{\omega-1})$ field operations.*

PROOF. Let G_A, H_A and G_B, H_B be the generators of A and B respectively. They satisfy $A - Z_n A Z_n^T = G_A H_A^T$ and $B - Z_n B Z_n^T = G_B H_B^T$. Consequently

$$\begin{aligned} AB &= (Z_n A Z_n^T + G_A H_A^T)(Z_n B Z_n^T + G_B H_B^T) \\ &= Z_n A B Z_n^T - Z_n A_{*,n} B_{n,*} Z_n^T + (Z_n A Z_n^T G_B) H_B^T \\ &\quad + G_A (H_A^T Z_n B Z_n^T + H_A^T G_B H_B^T), \end{aligned}$$

and therefore $AB - Z_n A B Z_n^T = G_{AB} H_{AB}^T$ for

$$\begin{aligned} G_{AB} &= \left(G_A \mid Z_n A Z_n^T G_B \mid -Z_n A_{*,n} \right) \\ H_{AB} &= \left(Z_n B^T Z_n^T H_A + H_B G_B^T H_A \mid H_B \mid Z_n B_{n,*}^T \right), \end{aligned}$$

thus showing that AB has displacement rank at most $\alpha + \beta + 1$.

Computing these generators involves applying A on a dense $n \times \beta$ matrix and B on a dense $\alpha \times n$ matrix, and computing the product of an $\alpha \times n$ by an $n \times \beta$ matrix and the product of an $\alpha \times \beta$ by a $\beta \times n$ matrix. Using [3, Theorem 1.2], these cost $\tilde{O}(n(\alpha + \beta)^{\omega-1})$ field operations. \square

PROPOSITION 2.3. *Let $A, B \in \mathbb{K}^{n \times n}$ be two Hankel-like matrices of displacement rank α and β respectively, then their product AB is a Toeplitz-like matrix of displacement rank at most $\alpha + \beta + 1$. Furthermore, given generators for A and B w.r.t. ∇_{Z_n, Z_n^\top} , generators for AB w.r.t. Δ_{Z_n, Z_n^\top} can be computed in $\tilde{O}(n(\alpha + \beta)^{\omega-1})$.*

PROOF. Let G_A, H_A and G_B, H_B be the generators of A and B respectively, satisfying $Z_n A - A Z_n^\top = G_A H_A^\top$ and $Z_n B - B Z_n^\top = G_B H_B^\top$. Using a similar reasoning as for Proposition 2.2 we can deduce that $AB - Z_n A B Z_n^\top = G_{AB} H_{AB}^\top$ for

$$\begin{aligned} G_{AB} &= \left(G_A \mid A Z_n^\top G_B \mid A_{*,n} \right) \\ H_{AB} &= \left(H_B G_B^\top H_A - B^\top Z_n^\top H_A \mid H_B \mid B_{n,*}^\top \right), \end{aligned}$$

thus showing that AB has displacement rank at most $\alpha + \beta + 1$. Computing these generators again costs $\tilde{O}(n(\alpha + \beta)^{\omega-1})$ field operations. \square

PROPOSITION 2.4. *Let $A \in \mathbb{K}^{n \times n}$ be a Toeplitz-like (resp. Hankel-like) matrix of displacement rank α , then for an arbitrary (resp. even) r , A^r is a Toeplitz-like matrix of displacement rank at most $(\alpha + 1)r$ and its generators can be computed in $\tilde{O}(n(\alpha r)^{\omega-1})$ field operations.*

PROOF. Using fast exponentiation one computes A^r as:

$$A^r = \prod_{k=0}^{\lfloor \log r \rfloor} \left(A^{2^k} \right)^{l_k} \quad \text{where } r = \sum_{k=0}^{\log r} l_k 2^k,$$

which only requires squarings and products between matrices of the form A^{2^k} . When A is Toeplitz-like the result is a straightforward consequence of Proposition 2.2; when it is Hankel-like the product A^2 is computed using Proposition 2.3, the remaining products are between Toeplitz-like matrices, and the result again follows from Proposition 2.2. \square

2.2 Reconstruction of a Toeplitz+Hankel-like Matrix from its Generators

The operator ∇_{U_n, U_n} is defined in [19, Section 4.5] as partly-regular, which means that a Toeplitz+Hankel-like matrix is completely defined by its generators and its irregularity set that contains all the entries in either its first row, its last row, its first column or its last column.

A formula to recover a dense representation of the matrix from its generators and its first column is given in [19, Theorem 4.5.1].

THEOREM 2.5 ([19]). *Let $M \in \mathbb{K}^{n \times n}$ be a Toeplitz+Hankel-like matrix, $G, H \in \mathbb{K}^{n \times \alpha}$ its generators and $c_0 = M e_0^n$ its first column, then*

$$M = \tau_{U_n}(c_0) - \sum_{j=0}^{\alpha-1} \tau_{U_n}(G_{*,j}) \tau_{Z_n}(Z_n H_{*,j})^\top \quad (2)$$

where for an $n \times n$ matrix A and a vector v of length n $\tau_A(v)$ denotes the matrix of the algebra generated by A which has v as its first column.

We show that one can derive a fast reconstruction algorithm for a Toeplitz+Hankel-like matrix from Eq. (2) and first detail the structure of the various $\tau_A(v)$ matrices.

LEMMA 2.6. $\tau_{Z_n}(v)^\top$ is the Toeplitz upper-triangular matrix with v^\top as its first row.

LEMMA 2.7. $\tau_{U_n}(v) = \sum_{i=0}^{n-1} v_i Q_i(U_n)$ where $Q_0(x) = 1$, $Q_1(x) = x$ and $Q_{i+1}(x) = xQ_i(x) - Q_{i-1}(x)$.

PROOF. The first column of $Q_i(U_n)$ is e_i^n . \square

COROLLARY 2.8. *Column j of $\tau_{U_n}(v)$ is $Q_j(U_n)v$.*

PROOF. With Lemma 2.7 and after checking the property for $j \in \{0, 1\}$, it suffices to prove $Q_i(U_n)_{*,j+1} = U_n Q_i(U_n)_{*,j} - Q_{i-1}(U_n)_{*,j-1}$. This is true for $i \in \{0, 1\}$ and if it is for i and $i-1$, then

$$\begin{aligned} Q_{i+1}(U_n)_{*,j+1} &= U_n^2 Q_i(U_n)_{*,j} - U_n Q_i(U_n)_{*,j-1} \\ &\quad - U_n Q_{i-1}(U_n)_{*,j} + Q_{i-1}(U_n)_{*,i-1} \end{aligned}$$

\square

From these we can write the following proposition, inspired by [7, Proposition 4.2], and which enables fast recursive reconstruction of the columns of a Toeplitz+Hankel-like matrix.

PROPOSITION 2.9. *Let $M \in \mathbb{K}^{n \times n}$ be a Toeplitz+Hankel-like matrix, $G, H \in \mathbb{K}^{n \times \alpha}$ its generators for ∇_{U_n, U_n} and $c_0 = M e_0^n$ its first column. With the notation $c_{-1} = 0$, the columns $(c_k)_{0 \leq k \leq n-1}$ of M follow the recursion:*

$$c_{k+1} = U_n c_k - c_{k-1} - \sum_{j=0}^{\alpha-1} H_{k,j} G_{*,j}. \quad (3)$$

PROOF. Let C be the matrix defined by the recursion formula and initial conditions of Proposition 2.9, we will prove $C = M$.

By definition c_0 is the first column of M ; assume now that for $j \leq k$, c_j is column j of M , then Eq. (3) can be detailed as

$$\begin{aligned} c_{k+1} &= Q_{k+1}(U_n) c_0 - \sum_{j=0}^{\alpha-1} \sum_{i=1}^{k-1} Q_{k+1}(U_n) G_{*,j} H_{i,j} \\ &\quad - U_n \sum_{j=1}^{\alpha} Q_k(U_n) G_{*,j} H_{k,j} - \sum_{j=1}^{\alpha} H_{k,j} G_{*,j} \\ &= M_{*,k+1} \text{ by Eq. (2)} \end{aligned}$$

\square

3 MATERIAL FOR MATRIX POLYNOMIALS

We rely on the material from [16, 23]. For matrix polynomials and fractions the reader may refer to [11]. The rational matrix $H(x) = V^\top M(x)^{-1} W$ over $\mathbb{K}(x)$ can be written as a fraction of two polynomial matrices. A right fraction description is given by square polynomial matrices $P(x)$ and $Q(x)$ such that $H(x) = P(x)Q(x)^{-1} \in \mathbb{K}(x)^{m \times m}$, and a left description by $P_l(x)$ and $Q_l(x)$ such that $H(x) = Q_l(x)^{-1} P_l(x) \in \mathbb{K}(x)^{m \times m}$. Degrees of denominator matrices are

minimized using column-reduced forms. A non-singular polynomial matrix is said to be column-reduced if its leading column coefficient matrix is non-singular [11, Sec. 6.3]. We also have the notion of irreducible and minimal fraction descriptions. If P and Q (resp. P_l and Q_l) have unimodular right (resp. left) matrix gcd's [11, Sec. 6.3] then the description is called irreducible. If Q (resp. Q_l) is column-reduced then the description is called minimal.

For a given m , define $1 \leq v \leq n$ to be the sum of the degrees of the first m largest invariant factors of $M(x)$ (equivalently, the first m diagonal elements of its Smith normal form). The following will ensure that the minimal polynomial of T , which is the largest invariant factor of $M(x)$ can be computed from the Smith normal form of an appropriate denominator $Q(x)$; see Corollary 4.2.

THEOREM 3.1. ([16, Thm. 2.12] and [22]) *Let V and W be block vectors over a sufficiently large field \mathbb{K} whose entries are sampled uniformly and independently from a finite subset $S \subseteq \mathbb{K}$. Then with probability at least $1 - 2n/|S|$, $H(x) = V^T M(x)^{-1} W$ has left and right irreducible descriptions with denominators of degree $\lceil v/m \rceil$, of determinantal degree v , and whose i^{th} invariant factor (starting from the largest degree) is the i^{th} invariant factor of $M(x)$.*

The next result we need is concerned with the computation of an appropriate denominator Q as soon as the truncated power series in Eq. (1) is known. We notice that $H(x) = V^T M(x)^{-1} W$ is strictly proper in that it tends to zero when x tends to infinity. For fraction reconstruction we use the computation of minimal approximant bases (or σ -bases) [2, 21], and the algorithm with complexity bound $\tilde{O}(m^{\omega-1}n)$ in [5, 10].

THEOREM 3.2 ([5, LEMMA 3.7]). *Let $H \in \mathbb{K}(x)^{m \times m}$ be a strictly proper power series, with left and right matrix fractions descriptions of degree at most d . A denominator Q of a right irreducible description $H(x) = P(x)Q(x)^{-1}$ can be computed in $\tilde{O}(m^{\omega-1}n)$ arithmetic operations from the first $2d + 1$ terms of the expansion of H .*

In our case, from Theorem 3.1 we will obtain the existence of appropriate fractions of degree less than $\lceil n/m \rceil$, and use Theorem 3.2 for bounding the cost of the computation of Q .

4 A BABY-STEP GIANT STEP ALGORITHM

In this section, we propose a direct adaptation of the baby steps/giant steps variant of Coppersmith's block-Wiedemann algorithm from [16, Sec. 4] to the case of structured matrices. In order to compute the terms of the series (1), we will assume that the input matrix T has been inverted, using [3, Theorem 6.6]. In this section we will therefore denote by T this inverse and compute the projections of its powers.

4.1 Description of the Algorithm

Let $V, W \in \mathbb{K}^{n \times m}$ be the block vectors used for the projection. Algorithm 1 performs r baby steps and s giant steps to compute the first terms of the sequence $H_k = V^T T^k W = V^T (T^r)^j T^i W$ for $0 \leq k \leq 2\lceil n/m \rceil$, $0 \leq i < r$, $0 \leq j < s$ and $rs \geq k + 1$.

This algorithm relies on three main operations:

- (1) the product of a structured matrix to dense rectangular matrix, supported by Proposition 2.1 for Lines 3 and 7;
- (2) the exponentiation of a structured matrix, supported by Proposition 2.4 for Line 4;

Algorithm 1 Compute $H_k = V^T T^k W$ for $0 \leq k \leq 2\lceil n/m \rceil$

Input: Generators of $T \in \mathbb{K}^{n \times n}$, Toeplitz-like or Hankel-like

Input: $m, r, s \in \mathbb{N}$ s.t. $rs \geq 2\lceil n/m \rceil + 1$, r even if T is Hankel-like

Input: $V, W \in \mathbb{K}^{n \times m}$

Output: $H = (H_{rj+i})_{j < s, i < r}$ where $H_k = V^T T^k W$

- 1: $W_0 \leftarrow W$
 - 2: **for** $1 \leq i \leq r - 1$ **do**
 - 3: $W_i \leftarrow T W_{i-1}$
 - 4: $R \leftarrow T^r$
 - 5: $V_0 \leftarrow V$
 - 6: **for** $1 \leq j \leq s - 1$ **do**
 - 7: $V_j^T \leftarrow V_{j-1}^T R$
 - 8: $H \leftarrow (V_0 \ \dots \ V_{s-1})^T (W_0 \ \dots \ W_{r-1})$
-

(3) the product of two dense rectangular matrices for Line 8.

4.2 Cost Analysis

THEOREM 4.1. *Algorithm 1 runs in $\tilde{O}(n^{\omega - \frac{\omega-1}{5-\omega}} \alpha^{\frac{\omega-1}{5-\omega}})$ operations in \mathbb{K} for well chosen m, r and s .*

For instance, when the displacement rank α is constant, and with the best known estimate $\omega = 2.373$ [1] the cost becomes $\tilde{O}(n^{1.851})$ while it is $\tilde{O}(n^2)$ for $\omega = 3$.

PROOF. From Proposition 2.1, applying an $n \times m$ block to T can be done in $\tilde{O}(n \max(m, \alpha) \min(m, \alpha)^{\omega-2})$ field operations. Hence the r baby-steps, Line 3, computing the $(T^i W)_{0 \leq i < r}$ cost overall

$$\tilde{O}(nr \max(m, \alpha) \min(m, \alpha)^{\omega-2}) \quad (4)$$

field operations.

By Proposition 2.4, the initialization of the giant steps, Line 4 computing a structured representation for T^r , can be done in

$$\tilde{O}(nr^{\omega-1} \alpha^{\omega-1}) \quad (5)$$

operations in \mathbb{K} .

Then each of the giant steps, Line 7, is a product of an $m \times n$ dense matrix with an $n \times n$ matrix of displacement rank αr . From Proposition 2.1, these s steps cost

$$\tilde{O}(ns \max(m, \alpha r) \min(m, \alpha r)^{\omega-2}) \quad (6)$$

Lastly, the computation of the product resulting in H , Line 8, uses $\tilde{O}(n \max(mr, ms) \min(mr, ms)^{\omega-2})$ or equivalently

$$\tilde{O}(nm^{\omega-1} \max(r, s) \min(r, s)^{\omega-2}) \quad (7)$$

field operations.

Let $m = \left\lceil n^{\frac{\omega-3}{\omega-5}} \alpha^{\frac{2}{5-\omega}} \right\rceil$ and set $r = s = \left\lceil \sqrt{2n/m} \right\rceil$. Note that $\alpha \leq m \leq \alpha r$. Therefore (4) is dominated by (7). Moreover (6) writes $\tilde{O}(n^2 m^{\omega-3} \alpha)$, (7) writes $\tilde{O}(n^{\frac{\omega+1}{2}} m^{\frac{\omega-1}{2}})$ and both terms equal

$$\tilde{O}(n^{\omega - \frac{\omega-1}{5-\omega}} \alpha^{\frac{\omega-1}{5-\omega}}).$$

Finally, (5) writes $\tilde{O}(n^{\frac{\omega+1}{2}} (\frac{\alpha^2}{m})^{\frac{\omega-1}{2}})$ and is thus dominated by (7). \square

Let us now suppose that the entries of V and W are sampled uniformly and independently from a finite subset $S \subseteq \mathbb{K}$, we then have the following:

COROLLARY 4.2. *The minimal polynomial of an $n \times n$ Toeplitz-like or Hankel-like matrix with displacement rank α can be computed by a Monte Carlo algorithm in*

$$\tilde{O}\left(n^{\omega - \frac{\omega-1}{5-\omega}} \alpha^{\frac{\omega-1}{5-\omega}}\right)$$

field operations with a probability of success of at least $1 - (n^2 + 3n)/|S|$.

PROOF. The first step is to compute the inverse of T , using [3, Theorem 6.6] in $\tilde{O}(n\alpha^{\omega-1})$ operations in \mathbb{K} . Then running Algorithm 1 on T^{-1} costs $\tilde{O}\left(n^{\omega - \frac{\omega-1}{5-\omega}} \alpha^{\frac{\omega-1}{5-\omega}}\right)$ which dominates $\tilde{O}(n\alpha^{\omega-1})$ since $\alpha \leq n$. From the sequence of matrices $(H_k)_{0 \leq k \leq 2n/m}$, one can compute a minimal denominator Q for $H(x) = V^T(xI_n - T)^{-1}W \in \mathbb{K}[x]^{m \times m}$ in $\tilde{O}(nm^{\omega-1})$ field operations, by Theorem 3.2.

Using Theorem 3.1, the minimal polynomial is then obtained as the first invariant factor in the Smith form of Q , computed by [20, Proposition 41]. This step also costs $\tilde{O}(nm^{\omega-1})$ field operations and since $m \leq n$ we have

$$nm^{\omega-1} \leq n^{\frac{\omega+1}{2}} m^{\frac{\omega-1}{2}}$$

which shows that the cost of these last two computations will always be dominated by the cost of the product (7). The probability of failure for the computation of T^{-1} is $n(n+1)/|S|$ by [3, Lemma 6.2]. A union bound combining this probability and the failure probability of Theorem 3.1 yields a probability of failure of $(n^2 + 3n)/|S|$. \square

Note that this result carries over to the computation of the characteristic polynomial of any Toeplitz-like or Hankel-like matrix T having fewer than m invariant factors in its Frobenius normal form.

5 AN ALGORITHM BASED ON STRUCTURED INVERSION

In this section we propose an algorithm computing the determinant of a generic structured polynomial matrix $M \in \mathbb{K}[x]^{n \times n}$ with displacement rank α based on the structure of the \mathcal{SLU} representation of Toeplitz-like matrix, or a generalization thereof for Hankel-like matrices, as presented in (2).

Principle of the algorithm. Here, the sequence $(H_k = V^T T^{-k} W)_k$ is obtained as the matrix coefficients of the series expansion $V^T M^{-1} W$. As $2\lceil n/m \rceil + 1$ terms are required, and with the special choice $V = W = X = (I_m \mid 0)^T$, this boils down to computing a dense representation of the $m \times m$ leading principal submatrix of $M^{-1} \bmod x^{2\lceil n/m \rceil + 1}$. The outline of the algorithm is as follows.

- (1) Compute the inverse $M^{-1} \bmod x^{2\lceil n/m \rceil + 1}$ in a compressed representation
- (2) Crop this representation to form a representation of the $m \times m$ leading principal submatrix;
- (3) Extract the dense representation from this representation.

We will now present the algorithm specialized for the two classes of interest.

5.1 The Algorithm for Toeplitz-like Matrices

A Toeplitz-like matrix T is represented by a pair of generators $G, H \in \mathbb{K}^{n \times \alpha}$ satisfying $T = \sum_{i=0}^{\alpha-1} L(G_{*,i})L(H_{*,i})^T$, where $L(v)$ is the lower triangular Toeplitz matrix with v as its first column [12, 13]. The $m \times m$ leading principal submatrix of any product $L(v)L(w)^T$ is the product of the $m \times m$ leading principal submatrix of these factors, which in turn is $L(v_{1..m})L(w_{1..m})^T$. Algorithm 2 relies on this property to produce $S^{(m)}$ from the m first rows of the generators of T^{-1} .

Algorithm 2 Compute $S^{(m)}$: Toeplitz-like case

Input: (G, H) generators of $M \in \mathbb{K}[x]^{n \times n}$, a Toeplitz-like matrix of displacement rank α

Output: Dense representation of $S^{(m)} = X^T M^{-1} X \bmod x^{2\lceil n/m \rceil + 1}$

- 1: $(E, F) \leftarrow$ generators for $M^{-1} \bmod x^{2\lceil n/m \rceil + 1}$
 - 2: $E' \leftarrow X^T E; F' \leftarrow FX$
 - 3: $S^{(m)} \leftarrow \sum_{i=0}^{\alpha-1} L(E'_{*,i})L(F'_{*,i})^T \bmod x^{2\lceil n/m \rceil + 1}$
-

THEOREM 5.1. *Algorithm 2 is correct for $M = xI_n - T$ and T generic and uses*

$$\tilde{O}\left(\frac{n^2}{m} \alpha^{\omega-1} + n\alpha\right)$$

operations in \mathbb{K} .

PROOF. From the above remark, $E' = E_{1..m,*}$ and $F' = F_{1..m,*}$ are generators for $S^{(m)} = X^T M^{-1} X$. Note that no division by x in the ring $\mathbb{K}[x]/\langle x^{2\lceil n/m \rceil + 1} \rangle$ will occur in Line 1 as T has generic rank profile, and consequently all leading principal minors of $M(x)$ are not divisible by x which shows the correctness.

By [3, Theorem 34], Line 1, computing the generators of M^{-1} , can be computed in $\tilde{O}(n\alpha^{\omega-1})$ operations over $\mathbb{K}[x]/\langle x^{2\lceil n/m \rceil + 1} \rangle$ which in turn is

$$\tilde{O}\left(\frac{n^2}{m} \alpha^{\omega-1}\right) \quad (8)$$

operations in \mathbb{K} .

The dense reconstruction of $S^{(m)}$ in Line 3 is achieved by α products of an $m \times m$ Toeplitz matrix $L(E'_{*,i})$ by an $m \times m$ dense matrix $L(F'_{*,i})^T$ for a total cost of

$$\tilde{O}(n\alpha) \quad (9)$$

operations in \mathbb{K} . \square

COROLLARY 5.2. *The characteristic polynomial of a generic $n \times n$ Toeplitz-like matrix with displacement rank α can be computed in $\tilde{O}\left(n^{2-\frac{1}{\omega}} \alpha^{\frac{(\omega-1)^2}{\omega}}\right)$ operations in \mathbb{K} when $\alpha = O\left(n^{\frac{\omega-2}{-\omega^2+4\omega-2}}\right)$, and $\tilde{O}\left(n^{\frac{3}{2}} \alpha^{\frac{\omega}{2}}\right)$ otherwise.*

Note that this is $O(n^{1.579})$ (resp. $O(n^{1.667})$) for α constant and $\omega = 2.373$ (resp. $\omega = 3$). When $\alpha = \Theta\left(n^{\frac{\omega-2}{-\omega^2+4\omega-2}}\right)$ and taking $\omega = 2.373$ (resp. $\omega = 3$), both expressions become $\tilde{O}(n^{1.74})$ (resp. $\tilde{O}(n^3)$).

The complexity when α is low can also be written as

$$\tilde{O}\left(n^{\omega-f(\omega)}\alpha^{f(\omega)}\right),$$

similarly as in Theorem 4.1, which can be interpreted as a transfer of part of the exponent from n to α by using the structure of the matrix.

PROOF. The family of Toeplitz matrices presented in Section 6.1 proves that for a generic Toeplitz-like matrix T , the matrix $\mathcal{H}^{(n)} = \mathcal{H}_{1..n,1..n}$ is non-singular, where

$$\mathcal{H} = \left(V^T T^{i+j} W\right)_{0 \leq i, j \leq \lceil n/m \rceil - 1}.$$

Then [23, Lemma 2.4] implies that the irreducible left and right fractions descriptions of $X^T M^{-1} X$ have degree at most $\lceil n/m \rceil$. Thus Theorem 3.2 ensures that an appropriate denominator Q of a right fraction description of $X^T M^{-1} X$ can be computed from $S^{(m)} = X^T M^{-1} X \pmod{x^{2\lceil n/m \rceil + 1}}$.

Besides the computation of $S^{(m)}$ by Theorem 5.1, the computation of the denominator Q of its irreducible right fraction description costs

$$\tilde{O}\left(nm^{\omega-1}\right) \quad (10)$$

operations by Theorem 3.2. Computing the determinant of Q has same cost by [6, 20]. The total cost depends on α .

Case 1: $\alpha = O\left(n^{\frac{\omega-2}{-\omega^2+4\omega-2}}\right)$. We set $m = n^{\frac{1}{\omega}}\alpha^{\frac{\omega-1}{\omega}}$ so that $\alpha = O(m^{\omega-2})$ and the term (9) is dominated by (10). For the chosen value of m the terms (8) (decreasing in m) and (10) (increasing in m) are equal, leading to a full cost of $\tilde{O}\left(n^{2-\frac{1}{\omega}}\alpha^{\frac{(\omega-1)^2}{\omega}}\right)$ operations in \mathbb{K} .

Case 2: $\alpha = \Omega\left(n^{\frac{\omega-2}{-\omega^2+4\omega-2}}\right)$. We set $m = n^{\frac{1}{2}}\alpha^{\frac{\omega-2}{2}}$ so that $\alpha = \Omega(m^{\omega-2})$. In this case the term (10) is dominated by (9) and for this value of m we have equality between the terms (8) and (9), leading to a full cost of $\tilde{O}\left(n^{\frac{3}{2}}\alpha^{\frac{3}{2}}\right)$ operations in \mathbb{K} . \square

5.2 The Algorithm for Hankel-like Matrices

In this section we are interested in adapting the previous algorithm to Hankel-like matrices. If T is Hankel-like then $M(x) = xI_n - T$ is Toeplitz+Hankel-like.

We will thus generalize and consider that T is a Toeplitz+Hankel-like matrix. We are interested in computing the first $2\lceil n/m \rceil + 1$ terms of the series $X^T M(x)^{-1} X$. We are going to adapt the Toeplitz algorithm and use Pan's Divide-and-Conquer algorithm for inversion [19, Chapter 5]. Computing the characteristic polynomial from there does not depend on the structure of M or T .

The strategy consists in computing generators for the truncated matrix from which we can recover a dense representation. Algorithm 3 details the steps. The generators and irregularity set of the inverse in Line 1 are computed with Pan's Divide and Conquer algorithm [19], as well as the solution to the linear system. The following lines are dedicated to the reconstruction of the dense representation of $S^{(m)}(x)$ from the generators. The correctness of Algorithm 3 is proved by Proposition 2.9.

Algorithm 3 Compute $S^{(m)}$: Toeplitz+Hankel-like case

Input: (G, H, v) generators and irregularity set of $M \in \mathbb{K}[x]^{n \times n}$, a Toeplitz+Hankel-like matrix of displacement rank α .

Output: Dense representation of $S^{(m)}(x) = X^T M^{-1}(x) X \pmod{x^{2\lceil n/m \rceil + 1}}$

- 1: $(E, F, u), c \leftarrow$ generators and irregularity set of the inverse of M , solution of $Mc = e_0^n$
 - 2: $c_0 \leftarrow X^T c$
 - 3: $c_1 \leftarrow U_m c_0 - \sum_{i=0}^{\alpha-1} E_{0,i} F_{0\dots m-1,i}$
 - 4: **for** $1 \leq k \leq m-2$ **do**
 - 5: $c_{k+1} \leftarrow U_m c_k - c_{k-1} - \sum_{i=0}^{\alpha-1} E_{k,i} F_{0\dots m-1,i}$
 - 6: $S^{(m)}(x) = (c_0 || \dots || c_{m-1})$
-

THEOREM 5.3. Algorithm 3 is correct for $M = xI_n - T$ and T generic and uses

$$\tilde{O}\left(\frac{n^2}{m}\alpha^2 + mn\alpha\right)$$

operations in \mathbb{K} .

PROOF. Line 1 can be done in $\tilde{O}(\alpha^2 n)$ operations in the base ring, so $\tilde{O}\left(\frac{n^2}{m}\alpha^2\right)$ operations on \mathbb{K} [19, Corollary 5.3.3]. Each step of the for loop consists of a number of polynomial operations modulo $x^{2\lceil n/m \rceil + 1}$ linear in $m\alpha$ as U_m has only two non-zero entries on each row. Lines 2 to 5 can be done in $\tilde{O}(m^2\alpha)$ operations in the base ring, so $\tilde{O}(nm\alpha)$ operations on \mathbb{K} . \square

The minimal polynomial is then obtained the same way as in Section 4 which leads to Corollary 5.4.

COROLLARY 5.4. The characteristic polynomial of a generic $n \times n$ Toeplitz+Hankel-like matrix with displacement rank α can be computed in $\tilde{O}\left(n^{2-\frac{1}{\omega}}\alpha^{\frac{2(\omega-1)}{\omega}}\right)$ field operations when $\alpha = O\left(n^{\frac{\omega-2}{4-\omega}}\right)$, and $\tilde{O}\left(n^{\frac{3}{2}}\alpha^{\frac{3}{2}}\right)$ otherwise.

The complexity in n is the same as in the Toeplitz-like case but there is a stronger dependence in α as there is no known algorithm to compute the inverse of a Toeplitz+Hankel-like matrix in $O(n\alpha^{\omega-1})$, the best one depending on α^2 .

PROOF. The family of Hankel matrices presented in Section 6.2 now proves that for all generic Hankel-like matrix T , the matrix $\mathcal{H}^{(n)}$ is non-singular. The rest of the proof is similar to the Toeplitz-like case in Corollary 5.2.

Again the overall cost is that for computing the denominator and its determinant in $\tilde{O}(nm^{\omega-1})$ operations in \mathbb{K} plus the cost of computing the sequence H_k . We distinguish two cases:

If $\alpha = O\left(n^{\frac{\omega-2}{4-\omega}}\right)$: Setting $m = n^{\frac{1}{\omega}}\alpha^{\frac{2}{\omega}}$ so that $\alpha = O(m^{\omega-2})$ and the full cost is $\tilde{O}\left(n^{2-\frac{1}{\omega}}\alpha^{\frac{2(\omega-1)}{\omega}}\right)$.

If $\alpha = \Omega\left(n^{\frac{\omega-2}{4-\omega}}\right)$: Setting $m = n^{\frac{1}{2}}\alpha^{\frac{1}{2}}$ so that $\alpha = \Omega(m^{\omega-2})$ and the full cost is $\tilde{O}\left(n^{\frac{3}{2}}\alpha^{\frac{3}{2}}\right)$. \square

6 SPECIAL MATRICES FOR GENERICITY

The generic matrices T for which our algorithms output the characteristic polynomial are matrices such that $\mathcal{H}^{(n)} = \mathcal{H}_{1..n,1..n}$ is non-singular (Corollaries 5.2 and 5.4), where

$$\mathcal{H} = \left(V^T T^{i+j} W \right)_{0 \leq i, j \leq \lceil n/m \rceil - 1}$$

The first algorithm is Monte Carlo with matrices V and W sampled at random. In the second algorithm, however $V = W = X$ are fixed, and $\det \mathcal{H}^{(n)}$ is a polynomial in the coefficients of T . Toeplitz and Hankel matrices have $2n - 1$ independent coefficients. The coefficients of a Toeplitz-like or Hankel-like matrix of displacement rank α are themselves polynomials in the coefficients of its generators, so $\det \mathcal{H}^{(n)}$ is by composition a polynomial on the $2n\alpha$ coefficients of the $n \times \alpha$ generators of T .

In this section, we show that $\det \mathcal{H}^{(n)}$ is not uniformly zero on the space of Toeplitz (resp. Hankel) matrices by finding one Toeplitz (resp. Hankel) matrix for which $\mathcal{H}^{(n)}$ is non-singular. This shows the algorithm is correct for all matrices of each class except for those with coefficients in a certain variety of \mathbb{K}^{2n-1} . As the displacement rank of the matrices we show is 2 or less, they are Toeplitz-like (resp. Hankel-like) and can be represented with larger generators (padded with zeros). The algorithm is thus also correct for matrices with displacement rank $\alpha \geq 2$ whose generators' coefficients are not in a certain variety of $\mathbb{K}^{2n\alpha}$. Both matrices are also Toeplitz+Hankel and Toeplitz+Hankel-like so the same reasoning shows the algorithm is correct for all Toeplitz+Hankel matrices except for those with coefficients in a certain hypersurface of \mathbb{K}^{4n-2} and all Toeplitz+Hankel-like matrices with displacement rank $\alpha \geq 4$ except for those on a certain hypersurface of $\mathbb{K}^{2n\alpha}$.

6.1 A Toeplitz Point

Let

$$T = \begin{pmatrix} 0 & I_m \\ -I_{n-m} & 0 \end{pmatrix}$$

and $M(x) = xI_n - T$. Let $P(x) \in \mathbb{K}[x]^{n \times m}$ defined by:

$$\begin{aligned} P_{n-m+k, k} &= 1 & \text{for } 0 \leq k \leq m \\ P_{i, k} &= xP_{i+m, k} & \text{for } 0 \leq k \leq m, 0 \leq i \leq n - m - 1 \end{aligned}$$

With

$$D(x) = \begin{pmatrix} 0 & x^{\lfloor n/m \rfloor} I_{n \bmod m} \\ x^{\lfloor n/m \rfloor - 1} I_{-n \bmod m} & 0 \end{pmatrix}$$

we can write $P(x) = \begin{pmatrix} D(x)^T & R(x) & I_m \end{pmatrix}^T$. From there we have

$$M(x)P(x) = \begin{pmatrix} xD(x)^T - I_m & 0 \end{pmatrix}^T \text{ and thus}$$

$$X^T M^{-1}(x)X = X^T P(x) (xD(x) - I_m)^{-1}.$$

That is $X^T M^{-1}(x)X = D(x)Q^{-1}(x)$ with $Q(x) = xD(x) - I_m$. As $xI_m D(x) - I_m Q(x) = I_m$, the fraction DQ^{-1} is irreducible and

$$\det Q = \pm x^{\lfloor n/m \rfloor (n \bmod m) + (\lfloor n/m \rfloor - 1)(-n \bmod m)} - 1$$

from which we get $\deg \det Q = n$. By [23, Lemma 2.4], the matrix $\mathcal{H}^{(n)}$ is therefore non-singular.

6.2 A Hankel Point

Let $T_n = (I_n + Z_n^m)J_n$. For j such that $2j \leq \lceil n/m \rceil - 1$, rows jm to $(j+1)m - 1$ of $T_n^{2j}X$ are I_m and the following rows are 0. This can be seen by recursively applying the band matrix $T_n^2 = Z_n^m + I_n + Z_n^m Z_n^{mT} + Z_n^{mT}$ to X . By applying T_n to $T_n^{2j}X$ we get that the rows $n - (j+1)m$ to $n - jm - 1$ of $T_n^{2j+1}X$ are J_m , and the preceding rows are 0.

Let K_r be the first n columns of $(T^0 X | \dots | T^{\lceil n/m \rceil - 1} X)$. K_r is non-singular, as its columns can be permuted to get a matrix of the form

$$\begin{pmatrix} L_1^T & 0 \\ 0 & L_2 \end{pmatrix}$$

where L_1 and L_2 are lower triangular with ones on the diagonal. As T is symmetric, K_l defined as the first n rows of

$$(T^{0T} X | \dots | T^{(\lceil n/m \rceil - 1)T} X)^T$$

is also non-singular, as well as $\mathcal{H}^{(n)} = K_l K_r$.

REFERENCES

- [1] J. Alman and V. Vassilevska Williams. 2020. A Refined Laser Method and Faster Matrix Multiplication. arXiv:2010.05846 [cs.DS]
- [2] B. Beckermann and G. Labahn. 1994. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM J. Matrix Analysis and Applications* 15, 3 (1994), 804–823. <https://doi.org/10.1137/S0895479892230031>
- [3] A. Bostan, C.-P. Jeannerod, C. Moulleron, and É. Schost. 2017. On matrices with displacement structure: generalized operators and faster algorithms. *SIAM J. on Matrix Analysis and Applications* 38, 3 (2017), 733–775. <https://doi.org/10.1137/16M1062855>
- [4] W. Eberly, M. Giesbrecht, P. Giorgi, A. Storjohann, and G. Villard. 2007. Faster inversion and other black box matrix computation using efficient block projections. In *Proc. ISSAC*. ACM Press, 143–150. <https://doi.org/10.1145/1277548.1277569>
- [5] P. Giorgi, C. Jeannerod, and G. Villard. 2003. On the complexity of polynomial matrix computations. In *Proc. ISSAC* (Philadelphia, PA, USA). ACM Press, 135–142. <https://doi.org/10.1145/860854.860889>
- [6] Somit Gupta, Soumojit Sarkar, Arne Storjohann, and Johnny Valeriotte. 2012. Triangular x-basis decompositions and derandomization of linear algebra algorithms over $\mathbb{K}[x]$. *Journal of Symbolic Computation* 47, 4 (2012), 422–453. <https://doi.org/10.1016/j.jsc.2011.09.006>
- [7] G. Heinig, P. Jankowski, and K. Rost. 1988. Fast inversion algorithms of Toeplitz-plus-Hankel matrices. *Numer. Math.* 52, 6 (1988), 665–682.
- [8] Georg Heinig and Karla Rost. 1984. *Algebraic Methods for Toeplitz-like Matrices and Operator*. Springer, Birkhäuser Basel. <https://doi.org/10.1007/2F978-3-0348-6241-7>
- [9] G. Heinig and K. Rost. 2004. New fast algorithms for Toeplitz-plus-Hankel matrices. *SIAM J. Matrix Analysis and Applications* 25, 3 (2004), 842–857. <https://doi.org/10.1137/S0895479802410074>
- [10] C.-P. Jeannerod, V. Neiger, and G. Villard. 2020. Fast computation of approximant bases in canonical form. *J. Symb. Comput.* 98 (2020), 192–224. <https://doi.org/10.1016/j.jsc.2019.07.011>
- [11] T. Kailath. 1980. *Linear Systems*. Prentice-Hall.
- [12] T. Kailath, S.Y. Kung, and M. Morf. 1979. Displacement ranks of matrices and linear equations. *J. Mathematical Analysis and Applications* 68, 2 (1979), 395–407. [https://doi.org/10.1016/0022-247X\(79\)90124-0](https://doi.org/10.1016/0022-247X(79)90124-0)
- [13] E. Kaltofen. 1994. Asymptotically fast solution of Toeplitz-like singular linear systems. In *Proc. ISSAC* (Oxford, UK). ACM Press, 297–304. <https://doi.org/10.1145/190347.190431>
- [14] E. Kaltofen. 2000. Challenges of symbolic computation: my favorite open problems. *J. Symbolic Computation* 29, 6 (2000), 891–919. <https://doi.org/10.1006/jsc.2000.0370>
- [15] E. Kaltofen and B.D. Saunders. 1991. On Wiedemann's method of solving sparse linear systems. In *Proc. AAECC-9 (LNCS 539, Springer Verlag)*. 29–38.
- [16] E. Kaltofen and G. Villard. 2005. On the complexity of computing determinants. *Comput. Complex.* 13, 3 (2005), 91–130. <https://doi.org/10.1007/s00037-004-0185-3>
- [17] E. Kaltofen and G. Yuhasz. 2013. On the matrix Berlekamp-Massey algorithm. *ACM Trans. Algorithms* 9, 4 (2013), 33:1–33:24. <https://doi.org/10.1145/2500122>
- [18] F. Le Gall. 2014. Powers of Tensors and Fast Matrix Multiplication. In *Proc. ISSAC* (Kobe, Japan). ACM Press, 296–303. <https://doi.org/10.1145/2608628.2608664>

- [19] Victor Y. Pan. 2001. *Structured Matrices and Polynomials: Unified Superfast Algorithms*. Springer-Verlag, Berlin, Heidelberg.
- [20] A. Storjohann. 2003. High-order lifting and integrality certification. *J. Symb. Comput.* 36, 3-4 (2003), 613–648. [https://doi.org/10.1016/S0747-7171\(03\)00097-X](https://doi.org/10.1016/S0747-7171(03)00097-X)
- [21] M. Van Barel and A. Bultheel. 1992. A general module theoretic framework for vector M-Padé and matrix rational interpolation. 3 (1992), 451–462. <https://doi.org/10.1007/BF02141952>
- [22] G. Villard. 1997. *A study of Coppersmith's block Wiedemann algorithm using matrix polynomials*. RR 975 IM IMAG. <http://perso.ens-lyon.fr/gilles.villard/BIBLIOGRAPHIE/PDF/rr0497.pdf>
- [23] G. Villard. 2018. On computing the resultant of generic bivariate polynomials. In *Proc. ISSAC* (New York, NY, USA). ACM Press, 391–398. <https://doi.org/10.1145/3208976.3209020>