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# Estimation of the potential in a 1D wave equation via exponential observers

Constantinos Kitsos, Mathieu Bajodek, and Lucie Baudouin

**Abstract**—The problem of estimation of the unknown potential in a 1-dimensional wave equation via state observers is considered in this work. The potential is supposed to depend on the space variable only and be polynomial. The main observation information is the value of the solution of the wave equation in a subinterval of the domain, including also some of its higher-order spatial derivatives. The method we propose to estimate the potential includes turning it into a new state as in finite-dimensional parameter estimation approaches. However, in this infinite dimensions setting, this requires an indirect approach that is introduced, including an infinite-dimensional state transformation. Sufficient conditions allow the design of an internal semilinear observer for the resulting cascade system, corresponding to the observed subinterval, which estimates the potential everywhere in an exponentially fast manner.

**Keywords:** 1-d wave equation, nonlinear inverse problem, reconstruction of the potential in the wave equation, nonlinear observer design, infinite-dimensional observers

## I. INTRODUCTION

Coefficient inverse problems for partial differential equations (PDEs) [10] have gained significant interest during the last decades. More particularly, the inverse problem of reconstructing the unknown potential in a wave equation, has been addressed in several works. It is a nonlinear inverse problem and some of its applications come from the need to recover properties of the medium described by the potential and include acoustic waves, ocean and seismic prospection, medical imaging, and geophysics [8], [12], [16], [24].

The main approach to deal with the solvability of an inverse problem is a uniqueness [19] and a stability property [26] to ensure the identifiability of the coefficient of interest. Carleman estimates more recently lead to the algorithmic reconstruction of the potential of the wave equation [4], [5] and are based on particular boundary measurements. Observation and observer design results for the wave equation have already been obtained for instance in [3], [20], [23]. In [13] and references therein the problem of reconstruction of initial data using observers is considered.

The present paper deals with a different problem than the aforementioned classical reconstruction approaches, namely the online estimation of the potential of the 1-dimensional wave equation via state observers. This work proposes an approach, which is based on a classical trick for constant parameter estimation in finite dimensions, where the state of the system is extended with the unknown parameter, in order to write it as a nonlinear cascade system in a

“canonical” form, for which observer methodology can be easily applied [7]. In the present case, where the role of the parameter is played by the space-varying potential, a similar approach turns out not to be so direct due to the properties of differential operator of the extended coupled hyperbolic system. This indirect approach eventually leads to the asymptotic estimation in time of the potential. In [15], an in-domain constant parameter is estimated for a hyperbolic system via adaptive observer, however, similarly as in the finite dimensions [25], the estimation is not necessarily exponential. Contrary to this approach, in this work the estimation of the potential is exponential.

The main contribution here is a novel observer-based approach for the reconstruction of the potential of the wave equation. A proposed indirect methodology includes the introduction of an infinite-dimensional transformation that maps the wave equation into a semilinear system of three coupled PDEs, whose differential operator is decomposed into a part with the same elements on the diagonal and a part acting on the observation of the solution (considered as known). To this end, we assume that the potential is an even polynomial in space and that observations include the solution and its higher order spatial derivatives in a subinterval of the domain, coming from appropriately strong regularity assumptions. An internal observer is designed in the observed subinterval that provides the estimation of the potential arbitrarily fast. The main idea of the current approach relies on a recently introduced methodology in [17], [18] to deal with the observer design of under-observed systems of coupled PDEs. It eventually results in an extension of the classical finite-dimensional parameter estimation approach to the infinite-dimensional problem of the potential estimation for the wave equation.

The paper is organized as follows. In Section II, we introduce the estimation problem and we present the observer design approach that leads to solvability of the main observer problem. In Section III, we present the proof of our main theorem, along with an illustrative simulation. Finally, in Section IV, we include some conclusions and perspectives.

*Notation:* We adopt the notation  $\mathbb{R}_+ := [0, +\infty)$ . For a given  $w$  in  $\mathbb{R}^n$ ,  $|w|$  denotes its usual Euclidean norm. For a given constant matrix  $A$  in  $\mathbb{R}^{n \times n}$ ,  $A^\top$  denotes its transpose,  $|A| := \sup\{|Aw|, |w| = 1\}$  is its induced norm and  $\text{Sym}(A) = \frac{A+A^\top}{2}$  stands for its symmetric part. By  $\underline{\text{eig}}(A)$  we denote the minimum eigenvalue of a symmetric matrix  $A$ . By  $I_n$  we denote the identity matrix of dimension  $n$ . By  $\text{diag}\{a_i\}_{i=1,\dots,k}$  we denote the diagonal matrix with elements  $a_1, \dots, a_k$  on its diagonal and by  $\text{antidiag}_k(A_i)_{i=1,\dots,k}$  the block-antidiagonal matrix consist-

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ing of  $k$  matrices  $A_i$  placed on antidiagonal block-row  $i$ . For given  $u : \mathbb{R}_+ \times [0, L] \rightarrow \mathbb{R}^n$  and time  $t \geq 0$  we use the notation  $u(t)(x) := u(t, x)$ , for all  $x$  in  $[0, L]$  to refer to the profile at certain time and with  $\partial_t^i u$  (resp.  $\partial_x^i u$ ) we refer to its partial derivative with respect to  $t$  (resp.  $x$ ) of order  $i$ . For a  $q$ -times continuously differentiable mapping  $u : [0, L] \rightarrow \mathbb{R}$  (in  $C^q$ ), we adopt the notation  $\|u\|_{C^q[0, L]} := \sum_{i=0}^q \max\{|\partial_x^i u(x)|, x \in [0, L]\}$  for the  $q$ -norm.  $L^2(0, L; \mathbb{R}^n)$  denotes the space of equivalence classes of measurable functions  $u : [0, L] \rightarrow \mathbb{R}^n$ , for which  $\|u\|_{L^2(0, L; \mathbb{R}^n)} := \left(\int_0^L |u(x)|^2 dx\right)^{1/2} < +\infty$ . By  $H^q(0, L; \mathbb{R}^n)$  we denote the Sobolev space of functions in  $L^2(0, L; \mathbb{R}^n)$  with all their weak derivatives up to order  $q$  in  $L^2(0, L; \mathbb{R}^n)$ , equipped with the norm  $\|u\|_{H^q(0, L; \mathbb{R}^n)} := \left(\int_0^L \sum_{i=0}^q |\partial_x^i u(x)|^2 dx\right)^{1/2}$ . By  $\mathbb{1}_{[a, b]}$  we denote the characteristic function of the set  $[a, b]$ . By  $B(\mathcal{X})$  we denote the space of bounded linear operators from  $\mathcal{X}$  to  $\mathcal{X}$ .

## II. PROBLEM STATEMENT AND REQUIREMENTS

Let us consider the following wave equation

$$\begin{cases} \partial_t^2 z(t, x) = \partial_x^2 z(t, x) - q(x)z(t, x), & (t, x) \in \mathbb{R}_+ \times (0, L) \\ z(t, 0) = h_1(t), \quad z(t, L) = h_2(t), & t \in \mathbb{R}_+, \\ z(0, x) = z^0(x), \quad \partial_t z(0, x) = z^1(x), & x \in (0, L), \end{cases} \quad (1)$$

assuming that potential  $q$  and initial datum  $(z^0, z^1)$  are unknown. The problem we wish to solve here is summarized as follows.

**Problem:** *Under which assumptions and observations, can we design a state observer system, which estimates the unknown wave potential function  $q$ , exponentially in time  $t$ ?*

To the best of our knowledge, this problem has not been given solutions yet. In this paper, we avoid a standard approach as in parameter identification problems for finite dimensions, see for instance [27]. In [15], an in-domain constant uncertainty is estimated for a class of hyperbolic systems via an adaptive observer with a measurement of the state in the entire domain. Such approaches might lead to asymptotic convergence of the observer scheme to the unknown parameters, but not necessarily exponentially, as we plan here. Additionally, the problem of estimation of a spatially varying parameter, as the potential  $q = q(x)$  here, is even more complicated. To solve this problem and force the observer to asymptotic convergence, we make some assumptions, including the polynomial nature of  $q$  and some strong regularity of the solution. Dropping these assumptions is still an open problem.

Let us first assume that the potential is an unknown polynomial in the space variable, i.e.,

$$q(x) = \sum_{i=0}^N q_i x^i, \quad \forall x \in [0, L], \quad (2)$$

with a priori known degree  $N \in \{0, 2, 4, \dots\}$  and unknown coefficients  $q_i \in \mathbb{R}$ . Note that the case of polynomial-type

potential is not uncommon in physical realizations, especially in quantum mechanics, see for instance [9], where the use of a decaic potential is justified in the Schrödinger equation.

Consider now a given  $\delta \in (0, L]$ . We assume that initial condition localized on  $[0, \delta]$  has some additional regularity than the usual one assumed for the wave equation. This will induce existence of solutions in  $[0, \delta]$  of sufficiently strong regularity. More precisely, let us assume that  $\mathbb{1}_{[0, \delta]} z^0 \in H^{3N+3}(0, \delta)$ ,  $\mathbb{1}_{[0, \delta]} z^1 \in H^{3N+2}(0, \delta)$ , while in the rest of the domain we assume more commonly  $z^0 \in H^1(0, L)$ ,  $z^1 \in L^2(0, L)$ . We also suppose that  $h_1$  and  $z(\cdot, \delta)$  are of class  $C^{3N+3}(\mathbb{R}_+)$  and on the  $\{x = L\}$  boundary we have  $h_2 \in C^1(\mathbb{R}_+)$ . Assuming also some compatibility conditions of order  $3N+3$  for initial condition restricted in  $[0, \delta]$ , namely for  $\mathbb{1}_{[0, \delta]} z^0, \mathbb{1}_{[0, \delta]} z^1$ , we may invoke well-known existence/uniqueness arguments (see for instance [21], following also a transposition method, since we have inhomogeneous boundary conditions), along with the extra regularity and, thus, we can easily deduce the existence of solutions  $\mathbb{1}_{[0, \delta]} z \in C^{3N+1}(\mathbb{R}_+; H^{3N+3}(0, \delta)) \cap C^{3N+2}(\mathbb{R}_+; H^{3N+2}(0, \delta)) \cap C^{3N+3}(\mathbb{R}_+; H^{3N+1}(0, \delta))$ ,  $z \in C^0(\mathbb{R}_+, H^1(0, L)) \cap C^1(\mathbb{R}_+, L^2(0, L))$ .

**Assumption 1 (Observations):** *As a main observation, we consider the solution of the wave equation in  $[0, \delta]$*

$$\mathbb{1}_{[0, \delta]} z(t, \cdot), \quad t \in \mathbb{R}_+, \quad (3a)$$

for some  $\delta \in (0, L]$ . Given the extra regularity of the system, we consider first the following derivatives (in the weak sense) of higher order as measurements on the boundaries. That is

$$\partial_x^i \partial_t z(t, l), \quad i \in \{0, \dots, 2N+1\}, \quad l = 0, \delta, \quad (3b)$$

$$\partial_x^i z(t, l), \quad i \in \{0, \dots, 3N+2\}, \quad l = 0, \delta, \quad (3c)$$

Second, we need also the internal spatial derivatives (in the weak sense) of order up to  $N+1$  of the solution localized on  $[0, \delta]$ , meaning

$$\mathbb{1}_{[0, \delta]} \partial_x^i z(t, \cdot), \quad i \in \{1, \dots, N+1\}. \quad (3d)$$

Notice that in Assumption 1 we avoid observations of noncausal nature, namely, time derivatives of the solution, except for the first time derivative (3b) on the boundaries, which is natural as it is a part of the state  $(z, \partial_t z)$  of the wave. Time derivatives are not in general available as measurements in observer designs. However, the spatial derivatives of the solution in a part of the domain, which are assumed known, can be measured causally and one can use them as observations in conjunction with the extra regularity. Note also that to solve (offline) the inverse problems for the potential of the wave equation as in [4], we only need the first spatial derivative of the solution on the boundaries as observation. However, this online estimation problem via observers appears to be more demanding.

We now make the following assumption.

**Assumption 2:** *We have*

$$\inf_{(t, x) \in \mathbb{R}_+ \times [0, \delta]} z(t, x) > 0, \quad (4)$$

and the initial condition  $z^0$  is compatible with (4). Moreover, there exists  $c > 0$ , such that

$$\|z(t, \cdot)\|_{H^{N+1}(0, \delta)} \leq c, \quad \forall t \geq 0. \quad (5)$$

In the sequel, the lower bound coming from (4), as well as the upper bound  $c$  in (5) are considered a priori known.

The nature of the above assumption is revealed in classical works on the solvability of inverse problems for the potential of the wave equation, where some slightly weaker conditions are assumed [4], [5], i.e. that the infimum of the initial condition only is strictly positive. It turns out that the stronger Assumption 2 is a sufficient condition for the observer design that we will present hereafter.

A convenient approach in the parameter identification, in order to design an observer, is to write (1) as a cascade system. We are indeed trying to recover the trick followed in finite dimension, where the system's state is extended by  $\partial_t q = 0$ , see [7], which leads to the design of an exponential observer for the extended system. However, this is not standard for the infinite dimension as it is shown next.

In the first step, let us rewrite (1) as a first-order hyperbolic system as follows:

$$\begin{cases} \partial_t u_1(t, x) = \partial_x u_1(t, x) + u_2(t, x), \\ \partial_t u_2(t, x) = -\partial_x u_2(t, x) - u_3(x)u_1(t, x), \\ \partial_t u_3(x) = 0, \end{cases} \quad (6)$$

where

$$u_1 = -z, \quad u_2 = \partial_x z - \partial_t z, \quad u_3 = q. \quad (7)$$

Notice that the above system has a nonlinear cascade form. Using the abstract realization of (6), the diagonal part of system's operator is equal to  $\text{diag}\{\partial_x, -\partial_x, -\partial_x^{N+1}\}$ , where we added a fictitious element  $-\partial_x^{N+1}$ , since we automatically have  $\partial_x^{N+1}q = 0$ , due to the polynomial nature of  $q$ . Observe that this operator has three distinct elements on its diagonal. Our goal is to design an observer for the system written in this above cascade form, but it turns out that we need a second step for this, as we explain next.

Internal observability of under-observed linear cascade systems has been studied for instance in [2], [22]. In these works, it is shown that the presence of distinct elements on the diagonal of the main differential operator adds difficulty. This observability problem reveals the difficulty that may subsequently appear in observer designs for such systems, taking also into account the nonlinearity of (6). Besides this, solutions to the problem of observer design of under-observed cascade nonlinear infinite-dimensional systems has been given in [17]. Here, we get inspired by this work and we employ an analogous strategy, noting that (6) is under-observed, since we only measure the first state in  $[0, \delta]$ . The goal is to write system (6) in a form where its differential operator is decomposed into a part with the same elements on the diagonal plus an operator acting only on the first state, which is the observation  $z$ . By this modification in the dynamics, as it was shown in [17], [18] for coupled systems of PDEs, the problem of observer design can be solved.

Following the previous reasoning, in order to write a system in an appropriate form for observer design, let us apply a lower triangular infinite-dimensional state transformation  $\mathcal{T} \in B(\mathcal{X})$ , invertible, with inverse  $\mathcal{T}^{-1} \in B(\mathcal{X})$ , where  $\mathcal{X} = H^{3N+3}(0, \delta) \times H^{2N+2}(0, \delta) \times H^{2N+2}(0, \delta)$ . Transformation  $\mathcal{T} : \mathbb{1}_{[0, \delta]}u \rightarrow v$  (with  $u = (u_1 \ u_2 \ u_3)^\top$ ,  $v = (v_1 \ v_2 \ v_3)^\top$ ) satisfies

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 \\ \partial_x - \partial_x^{N+1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (8)$$

Notice that for  $N = 0$ , we have  $\mathcal{T} = I_3$ , implying that we keep the original variables unchanged. This transformation is inspired by previous works [17], [18], applied to systems of coupled PDEs, in order to be written in a suitable form for observer design purposes. System writes in the new state variable  $v$  as a cascade system of PDEs of order  $2N + 2$ , locally in  $[0, \delta]$ , as follows:

$$\begin{aligned} \partial_t v(t, x) = & -\partial_x^{N+1}v(t, x) + A(-Cv(t, x))v(t, x) \\ & + \mathcal{K}Cv(t, x), \quad (t, x) \in \mathbb{R}_+ \times (0, \delta) \end{aligned} \quad (9)$$

where

$$\begin{aligned} A(Cv) := & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & Cv \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{K} := \begin{pmatrix} 2\partial_x^{N+1} \\ \partial_x^2 - \partial_x^{2N+2} \\ 0 \end{pmatrix}, \\ C := & (1 \ 0 \ 0). \end{aligned} \quad (10)$$

As we mentioned earlier, differential operator associated to system (9) consists of a part  $-\partial_x^{N+1}$  (same elements on the diagonal), which is desired for the observer convergence proof, plus an undesired operator  $\mathcal{K}$  acting on the first state  $v_1 = -z$  only. This second part will be copied in the observer dynamics, but acting on the measurement  $\mathbb{1}_{[0, \delta]}z$ , in order to get rid of it in the observer error dynamics. This decomposition of the differential operator into the previous two parts and the subsequent elimination of  $\mathcal{K}Cv$  reveals the reason why we assumed in Assumption 1 knowledge of  $z$  and of its spatial derivatives in  $[0, \delta]$ . It is worth also noting that  $q_i = \frac{1}{i!}\partial_x^i v_3(0)$ , which is our target in the reconstruction of the potential  $q : x \mapsto \sum_{i=0}^N q_i x^i$ .

We also associate to system (9) the following boundary conditions for  $t \geq 0$ , which result from boundary conditions of (1), in conjunction with transformations (7)-(8), viz.

$$\partial_x^j v_1(t, l) = -\partial_x^j z(t, l), \quad \forall j \in \{0, \dots, 2N + 1\}, l = 0, \delta; \quad (11a)$$

$$\begin{aligned} \partial_x^j v_2(t, l) = & \partial_x^{N+1+j} z(t, l) - \partial_t \partial_x^j z(t, l), \\ & \forall j \in \{0, \dots, 2N + 1\}, l = 0, \delta; \end{aligned} \quad (11b)$$

$$\begin{aligned} \partial_x^j v_3(0) = & \sum_{k=j}^N \sigma_{jk} \partial_x^k v_3(\delta), \quad \forall j \in \{0, \dots, N\}, \\ \partial_x^j v_3(l) = & 0, \quad \forall j \in \{N + 1, \dots, 2N + 1\}, l = 0, \delta, \end{aligned} \quad (11c)$$

$$\text{where } \sigma_{jk} := \frac{(-\delta)^{k-j}}{(k-j)!}, \quad \forall j \leq k \leq N, \quad (11d)$$

while (11c) is in view of the polynomial properties of  $v_3 = q$ , invoking its Taylor representation. Notice, that

boundary conditions (11a), (11b) come naturally by the transformation (8), in conjunction with the strong system regularity. These boundary conditions involve mappings that have been considered as measurements and are, thus, known. Furthermore, they are selected to be more numerous than the order  $N + 1$  of the PDE system (9), as we require solutions of (9) and of the corresponding observer coming afterwards, belonging to  $H^{2N+2}(0, \delta; \mathbb{R}^3)$ .

Now, for system (9), written in such an appropriate form, we are in a position to propose the following observer:

$$\begin{aligned} \partial_t \hat{v}(t, x) &= -\partial_x^{N+1} \hat{v}(t, x) + A(z(t, x)) \hat{v}(t, x) - \mathcal{K}z(t)(x) \\ &\quad - \Theta P^{-1} C^\top (z(t, x) + C \hat{v}(t, x)) \text{ in } \mathbb{R}_+ \times (0, \delta). \end{aligned} \quad (12)$$

In the above dynamics of the observer, we introduced a gain matrix  $\Theta := \text{diag} \{ \theta, \theta^2, \theta^3 \}$  (as in the ‘‘high-gain observers’’ in finite dimension), with  $\theta > 0$  a tuning parameter, and  $P$  positive definite symmetric, satisfying a Lyapunov inequality for  $(t, x) \in \mathbb{R}_+ \times [0, \delta]$  of the form

$$\text{Sym}(PA(z(t, x))) - C^\top C \leq -\frac{\eta}{2} I_3, \quad (13)$$

for some constant  $\eta > 0$ . Such an inequality is always feasible for  $A(z)$  and  $C$  given by (10) (structured as an observability canonical form of a finite-dimensional system) and under Assumption 2. The reader can refer to [14] for the feasibility of inequalities of such a type in the finite-dimensional case.

We now associate to the observer state  $\hat{v}$  its spatial derivatives  $\partial_x^i \hat{v}$  up to order  $i = N + 1$ . Overall, the observer state  $\hat{v}^{[N+1]} := (\hat{v}, \partial_x \hat{v}, \dots, \partial_x^{N+1} \hat{v})^\top$  satisfies the following dynamics, coming from differentiation of (12) and incorporating also the case  $i = 0$  corresponding to (12):

$$\begin{aligned} \partial_t \partial_x^i \hat{v} &= -\partial_x^{N+1+i} \hat{v} + \sum_{k=0}^i \binom{i}{k} A(\partial_x^{i-k} z) \partial_x^k \hat{v} - \mathcal{K} \partial_x^i z \\ &\quad - \Theta P^{-1} C^\top (\partial_x^i z + C \partial_x^i \hat{v}), \quad i \in \{0, \dots, N+1\}, \end{aligned} \quad (14)$$

with boundary conditions ( $l$  being 0 or  $\delta$ )

$$\begin{aligned} \partial_x^j \hat{v}_1(t, 0) &= -\partial_x^j z(t, 0) + \sum_{k=j}^N \sigma_{jk} \partial_x^k (\hat{v}_1(t, \delta) + z(t, \delta)), \\ &\quad \forall j \in \{2i, \dots, N\}, \\ \partial_x^j \hat{v}_1(t, l) &= -\partial_x^j z(t, l), \quad \forall j \in \{N+1, \dots, N+i\}; \\ \partial_x^j \hat{v}_2(t, 0) &= \partial_x^{N+1+j} z(t, 0) - \partial_t \partial_x^j z(t, 0) \\ &\quad + \sum_{k=j}^N \sigma_{jk} \partial_x^k [\hat{v}_2(t, \delta) - (\partial_x^{N+1} - \partial_t) z(t, \delta)], \quad \forall j \in \{2i, \dots, N\}, \\ \partial_x^j \hat{v}_2(t, l) &= \partial_x^{N+1+j} z(t, l) - \partial_t \partial_x^j z(t, l), \quad \forall j \in \{N+1, \dots, N+i\}; \\ \partial_x^j \hat{v}_3(t, 0) &= \sum_{k=j}^N \sigma_{jk} \partial_x^k \hat{v}_3(t, \delta), \quad \forall j \in \{2i, \dots, N\}, \\ \partial_x^j \hat{v}_3(t, l) &= 0, \quad \forall j \in \{N+1, \dots, N+i\}. \end{aligned} \quad (15)$$

**Remark 1:** Solutions to (13) require a priori knowledge of upper and lower bounds of the solutions to the wave equation coming from Assumption 2. This assumption will

be relaxed in our future works, in conjunction with results known from finite dimension, as for instance in [6]. Additionally, we note that solvability of (13) requires from (4) that the solution is strictly positive and lower bounded in  $[0, \delta]$ . In case we had a strictly negative solution instead, we would make a change of coordinates  $(v_1, v_2, v_3) \mapsto (-v_1, -v_2, v_3)$ , in order to follow a similar observer design.

We are now in a position to state our main result.

**Theorem 1:** Consider wave equation (1) with an unknown polynomial potential  $q(\cdot)$  of known even degree  $N$ , as in (2). Assume that observations in Assumption 1 are available and that Assumption 2 holds. Let  $P \succ 0$  satisfying (13) for some  $\eta > 0$ . Denote also  $v^0 = \begin{pmatrix} -z^0 \\ \partial_x^{N+1} z^0 - z^1 \\ q \end{pmatrix}$ .

Then, the following result holds about the unique solutions  $\hat{v}$  to observer (14)-(15), assuming also that initial condition  $\hat{v}^0(\cdot) := \hat{v}(0, \cdot)$  belongs to  $H^{N+1}(0, \delta; \mathbb{R}^3)$  and satisfies compatibility conditions of order  $N + 1$ : For  $\theta > 0$ , observer (14) provides an estimation for the potential  $q$  via  $\hat{v}_3$ , as there exist  $\ell, \kappa > 0$ , such that

$$|q_i - \frac{1}{i!} \partial_x^i \hat{v}_3(t, 0)| \leq \ell \theta^2 e^{-\theta \kappa t} \|v^0 - \hat{v}^0\|_{H^{N+1}(0, \delta; \mathbb{R}^3)}, \quad (16)$$

for  $i$  in  $\{0, \dots, N\}$ . In other words,  $\hat{q}(t, x) = \sum_{i=0}^N \frac{\partial_x^i \hat{v}_3(t, 0)}{i!} x^i$  approaches the unknown polynomial potential  $q$  on  $[0, L]$  as time grows to infinity.

### III. PROOF OF THEOREM 1

In this Section, we first prove Theorem 1 and then we provide an illustrative simulation.

*Proof:* Consider observer (14)-(15) localized on  $(0, \delta)$ . First, we can easily deduce global existence-uniqueness of solutions to (14)-(15) for  $z$  being uniformly bounded by Assumption 2. For this, we might invoke classical arguments from the theory of PDEs (Lumers-Philipp's theorem for instance). More particularly, for initial condition  $\hat{v}^0 \in H^{N+1}(0, \delta; \mathbb{R}^3)$ , satisfying compatibility conditions of order  $N + 1$ , there exist unique classical solutions  $\hat{v}$  to (14)-(15), belonging to  $C^{2N}(\mathbb{R}_+; H^{2N+2}(0, \delta; \mathbb{R}^3)) \cap C^{2N+1}(\mathbb{R}_+; H^{2N+1}(0, \delta; \mathbb{R}^3)) \cap C^{2N+2}(\mathbb{R}_+; H^{2N}(0, \delta; \mathbb{R}^3))$ .

Let us define the scaled observer error by  $\varepsilon := \Theta^{-1}(\hat{v} - v)$ . Then,  $E^{[N+1]} := (\varepsilon \quad \partial_x \varepsilon \quad \dots \quad \partial_x^{N+1} \varepsilon)^\top$  satisfies the following equations in  $\mathbb{R}_+ \times (0, \delta)$ :

$$\begin{aligned} \partial_x^i \partial_t \varepsilon &= -\partial_x^{N+1+i} \varepsilon + \theta \sum_{k=0}^{i-1} \binom{i}{k} A(\partial_x^{i-k} z) \partial_x^k \varepsilon \\ &\quad + \theta (A(z) - P^{-1} C^\top C) \partial_x^i \varepsilon, \quad i \in \{0, \dots, N+1\}, \end{aligned} \quad (17a)$$

$$\partial_x^j \varepsilon(t, 0) = \sum_{k=j}^N \sigma_{jk} \partial_x^k \varepsilon(t, \delta), \quad \forall j \in \{2i, \dots, N\}, \quad (17b)$$

$$\partial_x^j \varepsilon(t, l) = 0, \quad \forall j \in \{N+1, \dots, N+i\}, \quad l = 0, \delta. \quad (17c)$$

Let us now introduce a Lyapunov functional  $\mathcal{V} : H^{N+1}(0, \delta) \rightarrow \mathbb{R}_+$  by

$$\mathcal{V}[\varepsilon] = \sum_{i=0}^{N+1} \rho_i \int_0^\delta \partial_x^i \varepsilon^\top(x) P \partial_x^i \varepsilon(x) dx, \quad (18)$$

with  $\rho_0 = 1$  and  $\rho_i > 0$ , for  $i$  in  $\{1, \dots, N+1\}$ , to be chosen appropriately, and matrix  $P \succ 0$  chosen to satisfy (13). Taking the time derivative of  $V(t) := \mathcal{V}[\varepsilon(t)]$ ,  $t \geq 0$  along the solutions of (17), we obtain

$$\begin{aligned} \dot{V} = & - \sum_{i=0}^{N+1} \rho_i \int_0^\delta 2\partial_x^{N+1+i} \varepsilon^\top(x) P \partial_x^i \varepsilon(x) dx \\ & - \theta \int_0^\delta \left( E^{[N+1]}(x) \right)^\top \mathcal{M}[z](x) E^{[N+1]}(x) dx, \end{aligned} \quad (19)$$

where  $\mathcal{M}[z] := (\mathcal{M}_{ij})_{(N+2) \times (N+2)}[z]$ , with

$$\begin{aligned} \mathcal{M}_{ii}[z] = & -2\rho_{i-1} (\text{Sym}(PA(z)) - C^\top C), \forall i \in \{1, \dots, N+2\}, \\ \mathcal{M}_{ij}[z] = & \mathcal{M}_{ji}^\top[z] = -\rho_{i-1} \binom{i-1}{j-1} PA(\partial_x^{i-j} z), \\ & \forall i \in \{j+1, \dots, N+2\}, j \in \{1, \dots, N+2\}. \end{aligned}$$

Note that  $\mathcal{M}$  contains  $H^{N+1}(0, \delta)$  functions, satisfying Assumption 2. Now, by using the Schur complement, (13), and the uniform boundedness of  $z$  in  $H^{N+1}(0, \delta)$ , we can always choose  $\rho_i \in (0, 1]$ , for  $i$  in  $\{1, \dots, N+1\}$ , such that  $\mathcal{M}$  becomes positive for all  $z$  in its domain. More precisely, there exist  $\rho_i \in (0, 1]$ , such that

$$\gamma := \inf_{\|z\|_{H^{N+1}(0, \delta)} \leq c} \frac{w^\top \mathcal{M}[z] w}{|w|^2} > 0, \quad \forall w \in \mathbb{R}^{3N+6} \setminus \{0\}.$$

Next, by applying repeated integrations by parts in (19) and by the fact that  $N$  is even, one obtains

$$\dot{V} \leq \sum_{i=0}^{N+1} \rho_i [\Pi_i(x)]_0^\delta - \theta \frac{\gamma}{|P|} V, \quad (20)$$

where

$$\begin{aligned} \Pi_i(x) := & \left( \partial_x^i E^{[N]}(x) \right)^\top \mathcal{P} \left( \partial_x^i E^{[N]}(x) \right), \\ \mathcal{P} := & \text{antidiag}_{N+1} \left( (-1)^j P \right)_{j=1, \dots, N+1}. \end{aligned}$$

Moreover, the boundary conditions (17b)-(17c) lead to

$$[\Pi_i(x)]_0^\delta = \left( \partial_x^i E^{[N]}(\delta) \right)^\top (\mathcal{P} - \Sigma^\top \mathcal{P} \Sigma) \left( \partial_x^i E^{[N]}(\delta) \right),$$

for  $i$  in  $\{0, \dots, N+1\}$  and where  $\Sigma$  is the block matrix  $\Sigma := (\sigma_{jk} I_3)_{j,k=0, \dots, N}$ , while  $\sigma_{jk}$  are given by (11d) for  $j \leq k \leq N$  and  $\sigma_{jk} \equiv 0$  for  $k < j \leq N$ . After appropriate calculations and substituting boundary condition (17c), it turns out that  $[\Pi_i(x)]_0^\delta = 0$ , for all  $i$  in  $\{0, \dots, N+1\}$ . Inequality (20) can then be rewritten as

$$\dot{V} \leq -2\theta\kappa V; \quad \kappa := \frac{\gamma}{2|P|}. \quad (21)$$

By use of transformations (7)-(8), and (21), we obtain

$$\begin{aligned} & \|z(t, \cdot) + \hat{v}_1(t, \cdot)\|_{H^{N+1}(0, \delta)} \\ & + \|\partial_x^{N+1} z(t, \cdot) - \partial_t z(t, \cdot) - \hat{v}_2(t, \cdot)\|_{H^{N+1}(0, \delta)} \\ & + \|q(\cdot) - \hat{v}_3(t, \cdot)\|_{H^{N+1}(0, \delta)} \\ & \leq \gamma_1 \theta^2 e^{-\theta\kappa t} \|v^0(\cdot) - \hat{v}^0(\cdot)\|_{H^{N+1}(0, \delta; \mathbb{R}^3)}, \end{aligned} \quad (22)$$

where  $\gamma_1 := \frac{1}{\min\{\rho_i, i=0, \dots, N+1\}} \sqrt{3 \frac{|P|}{\underline{\text{eig}}(P)}}$ .

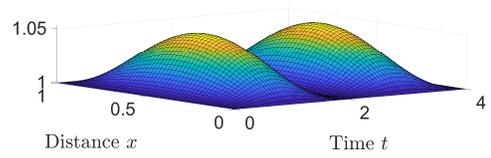


Fig. 1: Simulation of the solution to (1).

This stability inequality indicates that  $-\hat{v}_1$  provides a time estimation of  $\mathbb{1}_{[0, \delta]} z$  and  $\hat{v}_3$  an estimation of  $q$  in  $H^{N+1}$  spatial norm. We observe that this inequality has lead to a phenomenon of loss of derivatives. Indeed, in order to estimate the solution of the wave equation, we require stronger regularity ( $3N+3$ ) than the order of the convergent derivatives of the observer error ( $N+1$ ). Such phenomena appear when studying the controllability of underactuated coupled systems [2]. Finally, by invoking the compact embedding  $H^1(0, \delta) \hookrightarrow C^0[0, \delta]$  (the Rellich-Kondrachov theorem [1]), we get from (22)

$$\|q - \hat{v}_3\|_{C^N[0, \delta]} \leq c_0 \gamma_1 \sqrt{N+1} \theta^2 e^{-\theta\kappa t} \|v^0 - \hat{v}^0\|_{H^{N+1}(0, \delta; \mathbb{R}^3)}$$

with  $c_0 > 0$  an embedding constant. The latter shows that the coefficients of  $q(\cdot)$ , given by  $q_i = \frac{q^{(i)}(0)}{i!}$ , are approximated by  $\frac{1}{i!} \partial_x^i \hat{v}_3(t, 0)$  for all  $i$  in  $\{0, \dots, N\}$  in an exponentially fast manner. By use of (7), (8), we deduce (16), with  $\ell = \frac{1}{i!} c_0 \gamma_1 \sqrt{N+1}$ . ■

### Simulation

We illustrate here the solvability of the inverse problem of reconstruction of the potential via the proposed observer.

Consider a signal  $z$  which satisfies the wave equation (1) with a potential  $q(x) = x(x-1)$ , for  $x \in [0, 1]$ . To satisfy compatibility conditions and Assumption 2, the boundary and initial conditions are fixed to  $h_1(t) = h_2(t) = 1$  and  $z^0(x) = 1$ ,  $z^1(x) = 0$ . The solution  $z$  is plotted in Fig. 1 on  $[0, 4] \times [0, 1]$ . From measurements of the wave  $z$  and its three first spatial derivatives on  $\mathbb{R}_+ \times [0, \delta]$ , the objective is to estimate coefficients  $q_0 = 0$ ,  $q_1 = -1$  and  $q_2 = 1$ .

The observer (14)-(15) with  $\theta = 1$  and  $\delta = 0.1$  leads to exponential stabilization of the error  $\varepsilon = \hat{v} - v$  governed by (17) on  $\mathbb{R}_+ \times [0, 0.1]$ . To find numerically a solution for the error system, we exhibit an explicit numerical scheme which is consistent and stable under the Courant–Friedrichs–Lewy condition with  $\frac{\Delta t}{\Delta x^3}$  chosen sufficiently small [11]. Taking  $P = 0.1 \begin{pmatrix} 5 & -3 & -2 \\ -3 & 6 & -4 \\ -2 & -4 & 17 \end{pmatrix}$  satisfying condition (13) with  $\eta = 0.2$ , Fig. 2 shows that the  $H^3(0, \delta)$  norm of  $\varepsilon_3$  is exponentially convergent. Then, Fig. 3 emphasizes that the error on each coefficients of the potential  $\hat{q}_i = \frac{1}{i!} \partial_x^i \varepsilon_3(t, 0)$  are converging towards zero.

In future works, the numerical scheme will be improved to deal with some implementation issues, mostly for higher orders  $N$ .

## IV. CONCLUSION

The problem of estimation of an unknown polynomial potential of the 1-d wave equation was solved via an internal

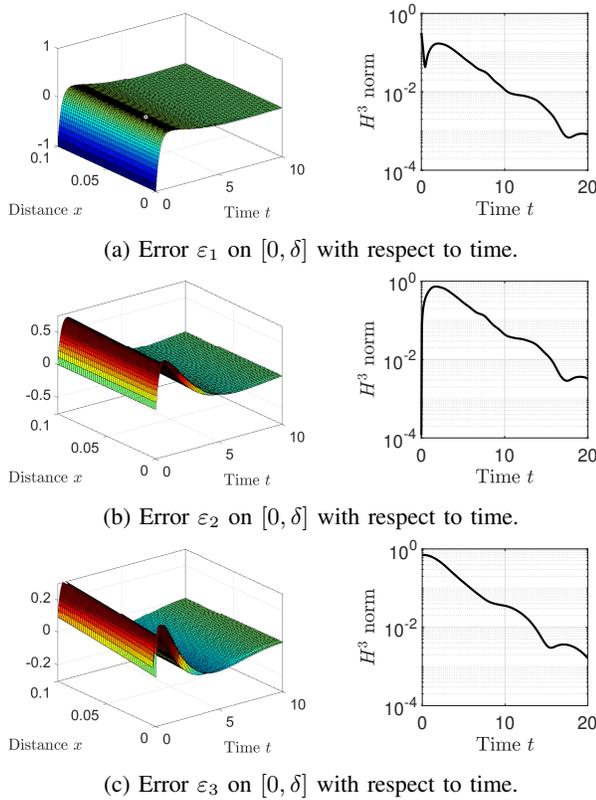


Fig. 2: Simulation of error system (17).

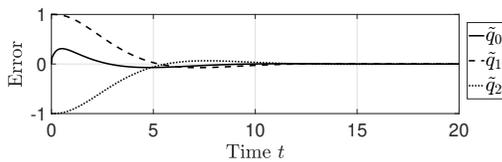


Fig. 3: Errors of the coefficients of the potential.

exponential observer in an observed subinterval of the domain. For this, the wave equation was written as a semilinear cascade system of PDEs of higher order. The observation included the solution of the wave and its spatial derivatives on a subinterval of the domain. In future works, we will consider the simultaneous estimation of the potential and solution to the wave equation via a composite observer. We will also investigate the extension of this approach to the 2-d wave equation and some links of such an observer approach with the internal observability of cascade systems.

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