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From strategies to derivations and back.

An easy completeness proof for first-order intuitionistic dialogical logic

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Abstract

In this paper we give a new proof of the correspondence between the existence of a winning strategy for intuitionistic E-games and Intuitionistic validity for first-order logic. The proof is obtained by a direct mapping between formal E-strategies and derivations in a cut-free complete sequent calculus for first-order intuitionistic logic. Our approach builds on the one developed by Herbelin in his PhD dissertation and greatly simplifies the proof of correspondence given by Felscher in his classic paper.

1 Introduction

The art of persuasive debate, dialectics, and the science of valid inference, logic, have been intrinsically linked since their beginnings (Castelnérac & Marion, 2009, 2013; Marion & Rückert, 2016; Crubellier, Marion, McConaughy, & Rahman, 2019; Dutilh Novaes, 2005; Dutilh Novaes, 2020). At the dawn of the modern age the connection between the two disciplines seemed so clear that one of the first sentences pronounced by Doctor Faustus in Marlowe's work goes as follows

Is, to dispute well, logic's chiefest end? Affords this art no greater miracle?

Despite this ancient connection between the two disciplines, mathematical logic had to wait until the 60s of the last century to determine that the logical concept of validity could be expressed through the use of dialogical concepts and techniques. The German mathematician and philosopher Lorenzen (Lorenzen, 1960) proposed to analyze the meaning of the logical formulas and connectives through the concept of *debate*. Lorenzen argued that the meaning of a formula, or proposition, is to be determined by the way in which the assertion of the formula affects the outcome of a debate (win or loss) between two players. Lorenzen's debates can be succinctly presented as follows: two players participate in the debate: one player, the Proponent, who affirms that a certain formula holds, and another player, the Opponent, who contends the claim of the

Proponent. The debate starts by the Proponent affirming a certain formula. The Opponent takes his turn and attacks the claim made by the Proponent according to its logical form. The Proponent can, depending on the form of the attack made by the Opponent, either defend her previous claim or counterattack. The debate evolves following this pattern. The Proponent wins the debate if she has the last word, i.e., the defense against one of the attacks made by the Opponent is a formula that the Opponent cannot attack without violating the debate rules.

Lorenzen conceived dialogical logic as a foundation for the meaning of the intuitionistic logical connective and quantifiers. Subsequently, in his PhD dissertation, Lorenzen's student Lorenz introduced into dialogical logic concepts and techniques from game theory (Lorenz's PhD dissertation is reproduced in (Lorenz, 2021)). The study of the meaning of logical connectives in terms of debates, together with the game-theoretic focus introduced by Lorenz, led to the concept of winning strategies for debates (or dialogue-games). In particular, Proponent's winning strategies for certain particular classes of dialogue-games were connected to validity in intuitionistic logic:

Given a first-order formula A , A is IL valid if and only if there is a Proponent winning strategy for A .

Unfortunately almost 30 years of work were needed to get a first correct proof of this particular form of completeness theorem (Felscher, 1985). Felscher's approach to dialogical logic, an approach that will be followed in the rest of this paper, is proof-theoretically oriented: Felscher takes strategies as the primary concept of dialogical logic and shows that Proponent winning strategies can be algorithmically transformed into formal sequent calculus proofs. We briefly describe Felscher's proof of the completeness (or correspondence) theorem. Felscher's proof makes use of various intermediate notions that allow a winning strategy to be transformed into a proof in the sequent calculus LJ and vice versa. First, Felscher defines two types of dialogue games, called D-dialogues and E-dialogues. In the latter dialogues, the Opponent must react upon the last intervention of the Proponent, while in the former he is free to react against any preceding intervention of the Proponent. Second, Felscher gives an algorithm that converts D-dialogues into E-dialogues and, consequently, D-strategies into E-strategies. As a consequence, Felscher focuses on E-strategies. Winning strategies for E-dialogues are infinitely branching trees, in which each branch is an E-dialogue won by the Proponent (thus finite). The strategies are infinitely branching for the following reasons:

1. after any Proponent assertion of a universally quantified formula $\forall xA$, the Opponent can ask her to instantiate the formula $A[t/x]$ for any term t ;
2. after any Proponent attack against an existentially quantified formula $\exists xA$, the Opponent can defend against the attack by instantiating $A[t/x]$ for any term t .

On the contrary, proofs in the sequent calculus are finite trees. To prove the correspondence between strategies and proofs in the sequent calculus, Felscher considers a finitary canonical part of a strategy: its *skeleton*. Felscher obtains the skeleton of a strategy by considering only certain branches of the strategy, e.g., branches in which

the Opponent attacks a universally quantified formula $\forall xA$ by asking the Proponent to instantiate $A[w/x]$ with w the first variable (according to a fixed enumeration) that does not appear before the assertion of $\forall xA$. Although strategies skeletons are finite, they are not proof-like: in particular, they do not respect the *eigenvariable* condition of sequent calculus proofs¹. Felscher is thus forced to consider the *formal* E-skeleton of a strategy. Given a Strategy, Felscher obtains its formal E-skeleton by forcing the Proponent to immediately defend from an attack against an existentially quantified formula. To complete his proof, Felscher provides different algorithms: an algorithm that transforms a proof in the sequent calculus LJ into what Felscher calls IC-protableaux, and another algorithm that transforms an IC-protableaux into a formal E-skeleton. Since Felscher has already shown that any formal E-skeleton can be converted into an E-strategy, this concludes the proof. As one can appreciate merely from this description, Felscher’s proof is a big nut to crack. In fact, as the authors of (Alama, Knoks, & Uckelman, 2011) argue, “Felscher’s proof, though correct, is both complicated, with its introduction of the notion of protableaux, and difficult to understand.”

An improvement of Felscher’s work was produced by Herbelin in his PhD dissertation (Herbelin, 1995)². Herbelin presents a variant of the LJ sequent calculus, called LGQ, and proves that given a formula A , derivations of $\vdash A$ in the sequent calculus LGQ correspond to winning E-strategies for A and vice versa. Herbelin’s proofs is considerably simpler than Felscher’s but, unfortunately, it is limited to propositional logic. The objective of our paper is thus simple: we extend the work of Herbelin to first-order intuitionistic logic. We present a sequent calculus, SLJ, that is complete for first-order intuitionistic logic and we show that derivations in the sequent calculus SLJ corresponds to the formal skeleton of winning E-strategies and vice versa.

Overview

The rest of the paper is structured as follows: Sect. 2 introduces dialogical logic for intuitionistic logic: we define E-games (that we simply call games), formal E-strategies (that we simply call strategies) and prove some results about games and strategies. Sect. 3 introduces the sequent calculus SLJ: we prove some results about SLJ, in particular, that SLJ is sound and complete for intuitionistic first-order logic. In Sect. 4 we show how to transform a winning strategy for a formula A into a derivation of $\vdash A$ in the calculus SLJ. In Sect. 5 we show how to transform a derivation of $\vdash A$ in the calculus SLJ into a winning strategy for A . The paper will be as much self-contained as possible.

¹see section 2.8 for a discussion

²A natural correspondence between a variant of E-strategies for intermediate propositional logics and derivations in an hyper-sequent calculus is provided in (Fermüller, 2003). Building on the work of Fermüller (Alama et al., 2011) provides a natural correspondence between a variant of E-strategies and derivations in a complete sequent calculus for propositional classical logic. For a correspondence between strategies and tableaux proofs for first-order classical logic one can consult (Clerbout, 2014).

2 Dialogical Logic

2.1 First-order language, sequences

In this paper we will consider a standard first-order language \mathcal{L} over a signature $(\mathcal{R}, \mathcal{F})$. \mathcal{R} is a countable set of predicate variables (P, Q, R, S , etc. vary over predicate variables) and \mathcal{F} is a countable set of function symbols (f, g, h , etc. vary over function symbols). The two sets are disjoint and there is an arity function $\alpha : \mathcal{R} \cup \mathcal{F} \rightarrow \mathbb{N}$. Function symbols with arity 0 will be called constants and a, b, c, d , etc. will range over them. Predicate variables with arity 0 will be called propositional variables and X, Y, Z, W , etc. will range over them.

Let \mathcal{V} be a countable set of variable symbols (variables will be ranged over by x, y, z, w , etc.), terms are defined by the following grammar:

$$t := x \mid f(t_1, \dots, t_n)$$

where f is a function symbol with arity n . Let $\wedge, \vee, \Rightarrow, \forall, \exists$ be the symbols for the usual connective and quantifiers of first-order logic. And let \perp (falshood) be a predicate constant. Formulas are specified by the following grammar:

$$F := P(t_1, \dots, t_n) \mid \perp \mid F \wedge F \mid F \vee F \mid F \Rightarrow F \mid \forall x F \mid \exists x F$$

the predicate constant \perp and formulas of the form $P(t_1, \dots, t_n)$, where P is a predicate variable, will be called atomic formulas. The negation of a formula is defined as $\neg F \equiv F \Rightarrow \perp$. We will use capital roman letter from the beginning of the alphabet (A, B, C, D , etc.) to denote arbitrary formulas. The notion of free (resp. bound) variables of a term (resp. formula) will be the standard ones, as well as the notion of subformula, Gentzen-subformula and positive/negative subformula of a formula. If A is a formula and t a term, we denote by $A[t/x]$ the result of the capture avoiding substitution of the term t for each occurrence of the variable x in A . The depth $|A|$ of a formula A is the maximum length of a branch in its construction tree minus one.

In the present work, we will often talk about sequences. Informally, a sequence is a set of objects listed according to a certain specific order. Formally, a sequence is a mapping whose domain is a subset of the set \mathbb{N} of natural numbers. Every element of a sequence is thus a pair (n, x) where n is a natural number and x an element of some set. We will denote an element (n, x) as x_n and call n an index. We will consider that if A is a subset of natural numbers that index a set S and $n \in A$ then $m \in A$ for every $m < n$. Thus, the first element of a sequence will be indexed by 0 the second by 1 the third by 2 and so on. Sequences will be denoted by small case letters of the Greek alphabet σ, ρ, τ , etc. The length of a sequence is the number of elements in it. If ρ is a sequence, we write ρ_i to denote the element of ρ indexed by i . The parity of ρ_i is the parity of i , e.g., if $\sigma = \text{M O T H E R}$ then $\rho_0 = \text{M}$ and has parity 0, $\rho_3 = \text{H}$ and has parity 1. We will denote the empty sequence by ϵ . If $\sigma = x_0 x_1 \dots x_n$ is a sequence and $\rho = y_0 y_1 \dots y_m$ is another sequence, the expression $\sigma\rho$ will denote the sequence $x_0 x_1 \dots x_n y_{n+1} \dots y_{n+m}$. Given a sequence τ and a sequence ρ , τ is a prefix of ρ (written $\tau \sqsubseteq \rho$) iff there is a sequence σ such that $\rho = \tau\sigma$. If $\sigma \neq \epsilon$ then τ is a proper prefix of ρ . Given a sequence τ and a sequence ρ , τ is a suffix of ρ if there is a sequence σ such

that $\rho = \sigma\tau$. If $\tau \neq \epsilon$ then τ is a proper suffix of ρ . Let X be a set of sequences. The set X is said to be *prefix closed* whenever $\rho \in X$ implies $\sigma \in X$ for every $\sigma \sqsubseteq \rho$.

2.2 Argumentative dialogues: informal overview

Before entering into the formal matter of dialogical logic, let us give an informal example of an argumentative dialogue about the validity of a formula. Let A and B stand for two arbitrary atomic formulas.

0. **P**: I affirm that $A \wedge B \Rightarrow B$.
1. **O**: Let me assume, for the sake of the proof, that $A \wedge B$ holds, can you show that B holds?
2. **P**: You admitted that $A \wedge B$ holds, can you admit that B holds?
3. **O**: Indeed, I must admit that B holds.
4. **P**: Then I have nothing more to prove, you have admitted that B holds, if $A \wedge B$ holds.

We can see that the Proponent and the Opponent alternate in the dialogue. The dialogue is a sequence of interventions. Each intervention but the first consists in either an attack against a preceding intervention of the other player or a defense against an attack of the other player. For example **O** in intervention 1 attacks intervention 0 by asking **P** to show that B holds provided that $A \wedge B$ holds. **P**'s defense against 1 is the intervention 4. What counts as a question against an asserted formula A depends on its form. An answer to such a question is likewise dependent on the logical form of A . For example, in 2, **P** attacks the formula asserted in 1 by asking **O** to assert B . This is because if one admits that a conjunction holds, one must be ready to concede that both members of the conjunction hold. Summing up, an argumentative dialogue will be a sequence of alternated interventions made by the Proponent and the Opponent. Each intervention in the dialogue is an attack or a defense against a preceding intervention, the dialogue ends whenever the Opponent cannot produce a new intervention without falling in contradiction with what he already conceded. The content of the next subsection will be devoted to giving a formal content to this intuitive discussion. In subsection 2.3 we define what a question on a formula is and what counts as an answer to such a question. In subsection 2.4 we formally define what it means for an intervention in a dialogue to refer to another preceding intervention in the same dialogue (definitions 1 and 2). Finally, in subsection 2.5 we define (definition 3) the class of argumentative dialogues we are interested in (that we call games) and the conditions in which **P** wins in an argumentative dialogue.

2.3 Argumentation forms

The set of auxiliary symbols Aux is the smallest set containing the symbols $\wedge_1, \wedge_2, \vee, \exists$ and the expressions $\forall[t/x]$ for all terms in \mathcal{T} and variables x of \mathcal{L} .

Following the terminology of Felscher (Felscher, 2002), an argumentation form Arg is a function assigning to each non-atomic formula A a set of pairs consisting of one *question* and one *answer* with questions being either formulas or symbols in Aux and answers being formulas.³

$$\begin{aligned} Arg(A \Rightarrow B) &= \{(A, B)\} \\ Arg(A \wedge B) &= \{(\wedge_1, A), (\wedge_2, B)\} \\ Arg(A \vee B) &= \{(\vee, A), (\vee, B)\} \\ Arg(\forall xA) &= \{(\forall[t/x], A[t/x]) \mid t \in \mathcal{T}\} \\ Arg(\exists xA) &= \{(\exists, A[t/x]) \mid t \in \mathcal{T}\} \end{aligned}$$

Given a pair $(q, a) \in Arg(A)$, q is called a *question on A*. Given $(q, B) \in Arg(A)$, the formula B is called an *answer to the question q on A*. So, for example, if A is $B \wedge C$, both \wedge_1 and \wedge_2 are questions on A but only B is an answer to \wedge_1 and only C is an answer to \wedge_2 . If A is $B \vee C$, the symbol \vee is a question on A , and both B, C are answers to \vee . Consider the case where A is $B \Rightarrow C$. In this case B is a question on A and C is an answer to B .

2.4 Augmented sequence

A *defense move* is a pair $(!, A)$ where A is a formula. An *attack move* is a pair $(?, s)$ where s is either a formula or an auxiliary symbol. A *move* is either an attack move or a defense move. A move (\star, A) where A is a formula and $\star \in \{?, !\}$, is called *assertion move*. We will also say that the move asserts the formula A , or that A is the asserted formula of the move. Attack moves of the form $(?, \exists)$ are called *existential attacks*. Attack moves of the form $(?, \vee)$ are called *disjunctive attacks*. Let $\rho = m_0 m_1 \dots m_n \dots$ be a sequence of moves. An assertion move $\rho_j = (\star, A) \in \rho$ is called a *reprise* if and only if there is move $\rho_k \in \rho$ with $k < j$ such that $\rho_k = (\star', A)$ and ρ_j, ρ_k have different parities.

Definition 1. An augmented sequence is a non-empty sequence of moves ρ together with a function ϕ that is defined on each ρ_i with $i \geq 1$ and such that, for all $i \geq 1$, $\phi(\rho_i) = \rho_j$ for a $j < i$. The move $\phi(\rho_i)$ is called the *enabler* of ρ_i .

Definition 2. Let (ρ, ϕ) be an augmented sequence.

- An attack move $\rho_i = (?, s)$ is justified whenever $\phi(\rho_i)$ is of the form (\star, A) and s is a question on A ;
- a defense move $\rho_i = (!, B)$ is justified whenever $\phi(\rho_i)$ is of the form $(?, s)$, $\phi(\rho_i)$ is justified, $\phi(\phi(\rho_i)) = (\star, A)$ and B is an answer to the question s on A .

³The words “question” and “answer” are called “attack” and “defense” by Felscher in (Felscher, 2002); we deviate from this terminology because we will use the terms “attack” and “defense” exclusively for the moves in a game, avoiding possible confusion.

2.5 Games

Let (ρ, ϕ) be an augmented sequence, we say that a formula A appears in the augmented sequence if and only if there is a move $m \in \rho$ that asserts A . We say that a variable v appears in ρ whenever v occurs free in some asserted formula or there is a move $m = (?, \forall[v/x])$ in ρ . If ρ_i is an attack move and ρ_i is not the enabler of some defense move ρ_j then we will say that ρ_i is unanswered. Let us fix once and for all an enumeration $(v_i)_{i \in I}$ of the variables of \mathcal{L} .

Definition 3 (Game). A game \mathcal{G} for a formula A is an augmented sequence (ρ, ϕ) such that

1. $\rho_0 = (!, A)$ and for all $i > 0$ the move ρ_i is justified;
2. $\phi(\rho_i) = \rho_{i-1}$ if i is odd, $\phi(\rho_i) = \rho_j$ with j odd if i is even;
3. if $\rho_i = (\star, B)$ with B atomic formula and i even, then ρ_i is a reprise and $B \neq \perp$;
4. if ρ_i is an attack move of the form $(?, \forall[t/x])$ and i is odd then $t = v_k$ and v_k is the first variable in the enumeration $(v_i)_{i \in I}$ that does not appear in the prefix of ρ ending with ρ_{i-1} ;
5. if $\rho_i = (!, B[t/x])$ is a defense move, i is odd and ρ_{i-1} is of the form $(?, \exists)$ then $t = v_k$ and v_k is the first variable in the enumeration $(v_i)_{i \in I}$ that does not appear in the prefix of ρ ending with ρ_{i-1} ;
6. If ρ_k is a defense move and k is even then $\phi(\rho_k) = \rho_j$ is the unanswered attack move having the greatest odd index in the prefix of ρ ending with ρ_{k-1} .

In a game \mathcal{G} , moves ρ_i with i even are called **P**-moves. They are called **O**-moves otherwise. If $\mathcal{G}m$ is a game and m is a **P**-move we will write $\mathcal{G}m^{\mathbf{P}}$. We will write $\mathcal{G}m^{\mathbf{O}}$ otherwise. The length of a game $\mathcal{G} = (\rho, \phi)$ is the length of ρ .

Let $\mathcal{G} = (\rho, \phi)$ be a finite game and m be a move. The move m is legal for \mathcal{G} if and only if the augmented sequence $(\rho m, \psi)$ is a game, where $\psi|_{\rho} = \phi$ and $\psi(m) \in \rho$. $\psi|_{\rho}$ being the restriction of the function ψ to the sequence of moves ρ .

Definition 4. A game \mathcal{G} is won by **P** if and only if it is finite and either:

- the game is of the form $\mathcal{G}'m^{\mathbf{P}}$ and there is no move m' legal for \mathcal{G} ;
- the game is of the form $\mathcal{G}'m^{\mathbf{O}}$ and m asserts \perp .

In what follows we will often identify a game with the sequence of its moves by an abuse of notation.

Remark 1. The games of definition 3 are E-games according to the Felscher's nomenclature. In fact, condition 2 ensures that each **O**-move m is a reaction to the immediately preceding **P**-move n . Moreover, conditions 4 and 5 ensures that the strategies defined below will be skeletons according to Felscher's nomenclature. By including such conditions directly in the definition of game, we can skip many steps in the correspondence proof between sequent calculus derivations and winning strategies.

2.6 Some examples

We give some examples of games. A game will be represented as a table with two columns and as many rows as there are moves in the game. In the first column of the table we will write down the moves of the game. In the second column we will write the value of the function ϕ for the move in the first column. Let X, Y, Z be propositional variables, and R a binary predicate variable

$m_0 = (!, X \Rightarrow Y \Rightarrow ((Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z)))$	m_0
$m_1 = (?, X \Rightarrow Y)$	m_1
$m_2 = (!, (Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z))$	m_2
$m_3 = (?, Y \Rightarrow Z)$	m_3
$m_4 = (!, X \Rightarrow Z)$	m_4
$m_5 = (?, X)$	m_1
$m_6 = (?, X)$	m_6
$m_7 = (!, Y)$	m_3
$m_8 = (?, Y)$	m_8
$m_9 = (!, Z)$	m_5
$m_{10} = (!, Z)$	m_5

$m_0 = (!, X \vee Y \Rightarrow Y \vee X)$	m_0	$m_0 = (!, X \vee Y \Rightarrow Y \vee X)$	m_0
$m_1 = (?, X \vee Y)$	m_1	$m_1 = (?, X \vee Y)$	m_1
$m_2 = (?, \vee)$	m_2	$m_2 = (?, \vee)$	m_2
$m_3 = (!, X)$	m_1	$m_3 = (!, Y)$	m_1
$m_4 = (!, Y \vee X)$	m_4	$m_4 = (!, Y \vee X)$	m_4
$m_5 = (?, \vee)$	m_5	$m_5 = (?, \vee)$	m_4
$m_6 = (!, X)$	m_5	$m_6 = (!, Y)$	m_5

$m_0 = (!, X \Rightarrow \neg\neg X)$	m_0	$m_0 = (!, \neg\neg X \Rightarrow X)$	m_0
$m_1 = (?, X)$	m_1	$m_1 = (?, \neg\neg X)$	m_1
$m_2 = (!, \neg\neg X)$	m_2	$m_2 = (?, \neg X)$	m_2
$m_3 = (?, \neg X)$	m_3	$m_3 = (?, X)$	m_2
$m_4 = (?, X)$	m_4		
$m_5 = (?, \perp)$	m_4		

$m_0 = (!, (X \wedge \neg X) \Rightarrow Z)$	m_0	$m_0 = (!, X \vee Y \Rightarrow X)$	m_0
$m_1 = (?, X \wedge \neg X)$	m_1	$m_1 = (?, X \vee Y)$	m_1
$m_2 = (?, \wedge_1)$	m_2	$m_2 = (?, \vee)$	m_2
$m_3 = (!, X)$	m_1	$m_3 = (!, X)$	m_2
$m_4 = (?, \wedge_2)$	m_4	$m_4 = (!, X)$	m_1
$m_5 = (!, \neg X)$	m_5		
$m_6 = (?, X)$	m_6		
$m_7 = (!, \perp)$	m_6		

$m_0 = (!, \exists y \forall x R(x, y) \Rightarrow \forall x \exists y R(x, y))$	m_0
$m_1 = (?, \exists y \forall x R(x, y))$	m_1
$m_2 = (!, \forall x \exists y R(x, y))$	m_2
$m_3 = (?, \forall [w/x])$	m_3
$m_4 = (!, \exists y R(w, y))$	m_4
$m_5 = (?, \exists)$	m_5
$m_6 = (?, \exists)$	m_6
$m_7 = (!, \forall x R(x, z))$	m_7
$m_8 = (?, \forall [w/z])$	m_8
$m_9 = (!, R(w, z))$	m_9
$m_{10} = (!, R(w, z))$	m_{10}

Remark 2. All the games except the one for the formula $\neg\neg X \Rightarrow X$ are won by the Proponent: they either do not admit further Opponent's moves or they end with the Opponent asserting \perp . Remark that the last move of all games won by the Proponent ending in a Proponent's move are defense moves and assertions of an atomic formula. In all games formulas asserted by the Proponent are positive sub-formulas of the formula about which the game is played. Formulas asserted by the Opponent are negative sub-formulas of the formula about which the game is played. In each game atomic formulas asserted by the Proponent are both positives and negatives sub-formulas of the formula about which the game is played.

The Proponent cannot extend the game for the formula $\neg\neg X \Rightarrow X$ by the defense-move $m_4 = (!, X)$ with $\phi(m_4) = m_1$ because of condition 6 in the definition of game (definition 3). The two games for the formula $X \vee Y \Rightarrow X \vee Y$ have a common prefix, and they first differ on an Opponent's move. In one game the Opponent chooses to assert X in the defense move m_4 while in the other game the Opponent chooses to assert Y . In any case the Proponent wins.

The Proponent wins the game for the formula $X \vee Y \Rightarrow X$ even if this latter formula is not a tautology of first-order intuitionistic logic. Note that if the Opponent had chosen to assert the formula Y instead of the formula X on move 3 then the Proponent would have had no chance of winning. In the game for the formula $\exists y \forall x R(x, y) \Rightarrow \forall x \exists y R(x, y)$ the player does not defend immediately against the attack move m_5 . Instead, she delays her defense until the last move.

We systematize the observations on the games just made with some simple propositions.

Proposition 1. *Let A be an arbitrary formula and \mathcal{G} an arbitrary game for A . If (\star, B) is an assertion move in \mathcal{G} then B is a Gentzen-subformula of A .*

Proof. By induction on the length of \mathcal{G} . □

Proposition 2. *Let A be an arbitrary formula and $\mathcal{G} = \mathcal{G}' m^{\mathbf{P}}$ be a finite game for A . If \mathcal{G} is **won** by \mathbf{P} then m asserts an atomic Gentzen-subformula B of A .*

Proof. Suppose to reach a contradiction that the last move m of \mathcal{G} is not a defense move. Then it is an attack move of the form $(?, s)$. By definition of game, there is a

preceding **O**-move m_k that asserts some formula C , $\phi(m) = m_k$ and s is a question on C . Then the augmented sequence $\mathcal{G}\pi$, where $\pi = (!, D)$, $\phi(\pi) = m$ and D is an answer to the question s on the formula B , is a game. This contradicts the fact that \mathcal{G} is won by P . Thus m must be a defense move $(!, B)$. If B is not atomic we reach again a contradiction: in fact by adding a move $\pi = (?, s)$ to \mathcal{G} where s is a question on B we obtain a game. Thus we must conclude that B is an atomic formula. \square

Proposition 3. *Let A be an arbitrary formula and \mathcal{G} an arbitrary game for A . If (\star, B) is an assertion move in \mathcal{G} that is a **P**-move (resp. an **O**-move) then B is a positive (resp. negative) Gentzen-subformula of A .*

Proof. By induction on the length of \mathcal{G} . \square

Proposition 4. *Let A be an arbitrary formula and \mathcal{G} an arbitrary game for A . If (\star, B) is an assertion move made by **P** and B is an atomic formula, then B is both a negative and positive Gentzen-subformula of A .*

Proof. Direct consequence of Proposition 3 and of the condition 3 in the definition of game. \square

2.7 Strategies

As we have discussed in remark 2 the game for the formula $X \vee Y \Rightarrow X$ is won by the Proponent but by mere accident: if the Opponent had chosen to assert Y instead of X the Proponent would not have had a chance to win. This means that the Proponent *cannot* win a game on that formula *no matter how the Opponent choose* to act in the game. On the contrary the Proponent *can* win a game on the formula $X \vee Y \Rightarrow Y \vee X$ *no matter how* the Opponent chooses to act in the game. This means that there is a Proponent winning strategy for the formula $X \vee Y \Rightarrow Y \vee X$ and no winning strategy for the formula $X \vee Y \Rightarrow X$.

Intuitively speaking, a strategy for a game \mathcal{G} is a function. A function that specifies, at each moment of the game, which move a player must play according to the moves previously played (the history of the game). A strategy is *winning* when the player who is following the strategy wins whatever the history of the game is. As long as each move of the player following the strategy is determined by the strategy itself, it can be concluded that the game history varies only according to the moves of his Opponent. We informally describe how a strategy should operate and then formalize this notion. Imagine being engaged in a game \mathcal{G} , that the last move of \mathcal{G} was played according to the strategy, and that it is now your Opponent's turn to play. Your Opponent could extend the game in different ways: for example if you are playing chess, you are white and you just made your first move by moving a pawn to a certain position of the chessboard, black can in turn move a pawn or move a horse. If you are playing according to the strategy, the strategy should tell you how to react against either type of move. If black moves a pawn to $C6$ and you just moved your pawn to $C3$, then you move the horse to $H3$. If black moves a horse to $H6$ and you just moved your pawn to $C3$, then you move your pawn in $B4$. Therefore, a strategy can be viewed as a tree in which each node is a move in the game, the moves of my Opponent have at most one daughter, and my

moves have as many daughters as there are available moves for my Opponent. A tree can be seen as a prefix-closed set of sequence over an alphabet. Since our games are sequences over the alphabet of moves we can define strategies in the following manner:

Definition 5. A strategy \mathcal{S} for a formula A is a non empty prefix-closed set of games for A in which the following conditions are satisfied:

P-determinism: if \mathcal{G}_m^P and \mathcal{G}_n^P belongs to \mathcal{S} then $m = n$;

O-completeness: if $\mathcal{G}_m^P \in \mathcal{S}$ then $\mathcal{G}_m^P n^O \in \mathcal{S}$ for all moves n legal for \mathcal{G}_m^P ;

no delayed choices: if $\mathcal{G}_m^O \in \mathcal{S}$ and $m = (?, \exists)$ or $m = (?, \vee)$ then there is a move n such that $\mathcal{G}_m^O n^P \in \mathcal{S}$ and n is enabled by m .

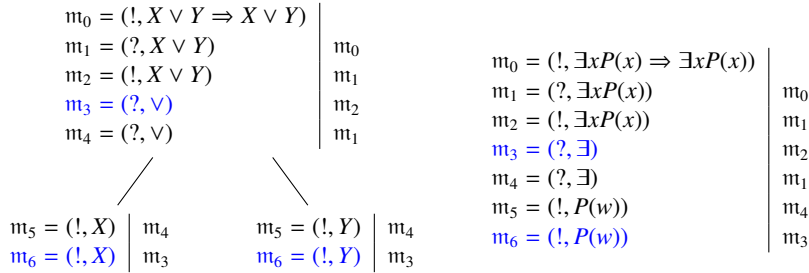
A strategy \mathcal{S} is winning if and only if every maximal, with respect to the prefix order, sequence of the strategy is a game won by **P**.

Proposition 5. Let \mathcal{S} be an arbitrary strategy and let \mathcal{G} be a game in \mathcal{S} that ends in a **P**-move. The family of moves $(m_{k_i})_{(i \in I)}$ such that $\mathcal{G}_{m_{k_i}} \in \mathcal{S}$ is finite.

Corollary 1. For any strategy \mathcal{S} , if \mathcal{S} is winning then \mathcal{S} is a finite tree.

2.8 On the “no delayed choices” condition

We would like to explain the intuition behind the *no delayed choices* condition in the definition 5 of strategy. As remarked by Felscher, without this condition the following counts as winning strategies for the formulas $X \vee Y \Rightarrow X \vee Y$ and $\exists x P(x) \Rightarrow \exists x P(x)$:



the “problem” with these two strategies is that they are not *proof-like*: they don’t correspond naturally to proofs in a standard intuitionistic sequent calculus like LJ or GKi⁴. In particular, the right-hand strategy would correspond to an incorrect proof in the sequent calculus, that is,

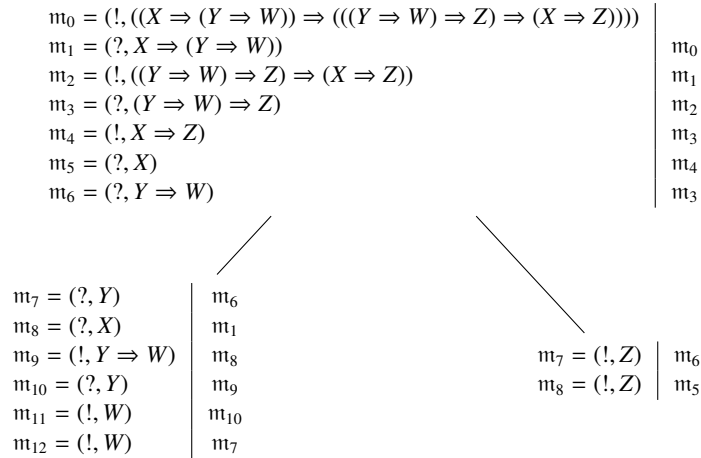
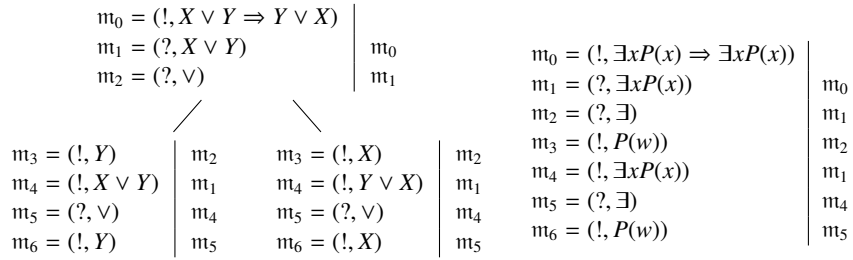
$$\frac{\frac{\frac{}{P(w) \vdash P(w)}{\text{Id}}}{\exists x P(x) \vdash P(w)} \exists^L}{\exists x P(x) \vdash \exists x P(x)} \exists^R$$

⁴The left-hand strategy is indeed proof-like if one considers an intuitionistic sequent-calculus in which multiple formulas can appear on the right of the \vdash symbol and a multiplicative version of the disjunction right-rule is adopted. This is the solution adopted by Herbelin in his PhD-dissertation. This kind of solution does not solve the eigenvariable problem. By this reason, we chose to deviate from Herbelin’s treatment.

we encounter this problem because **P** can indefinitely delay a defense move against an existential or a disjunctive attack. It is not easy to estimate the right dose of delay that should be conceded to **P**. In some cases, strategies with delayed defenses against existential or disjunctive attacks corresponds to proofs. In some other cases—as above—they do not. We settle things once and for all; **P** must immediately state a defense against an existential or disjunctive attack.

2.9 Some examples of strategies

Let X, Y, Z, W be propositional variables, P a unary predicate variable and R a binary predicate variable. A strategy will be represented as a tree of games.



$m_0 = (!, \exists y \forall x R(x, y) \Rightarrow \forall x \exists y R(x, y))$	m_0
$m_1 = (?, \exists y \forall x R(x, y))$	m_1
$m_2 = (!, \forall x \exists y R(x, y))$	m_1
$m_3 = (?, \forall [w/x])$	m_2
$m_4 = (?, \exists)$	m_1
$m_5 = (!, \forall x R(x, z))$	m_4
$m_6 = (?, \forall [w/x])$	m_5
$m_7 = (!, R(w, z))$	m_6
$m_8 = (!, \exists y R(w, y))$	m_3
$m_9 = (?, \exists)$	m_8
$m_{10} = (!, R(w, z))$	m_9

3 The sequent calculus SLJ

We now present the sequent calculus SLJ (strategic LJ). SLJ is a first-order version of the calculus LGQ studied by Herbelin in his Phd dissertation (Herbelin, 1995). LGQ is a Kleene style sequent calculus: the active formula of a left introduction rule is present in the premises of the rule. LGQ differs from a Kleene-style calculus like GKi (Troelstra & Schwichtenberg, 1996) because of a restriction on the use of the left introduction rule for the implication connective. SLJ is obtained from LGQ by adding the quantifier rules and imposing a restriction on the use of the right introduction rule for the disjunction and the existential quantifier connectives.

Definition 6. The sequent calculus SLJ is defined by the rules in Table 1. A sequent is an expression $\Gamma \vdash C$ where Γ is a finite (possibly empty) multiset of formulas and C is a formula. Greek upper-case letters Γ, Δ, \dots stand for multisets of formulas. In the Id-rule A is of the form $P(t_1, \dots, t_n)$ where P is a predicate variable with arity $n \geq 0$ and the t_i are terms. In the $\forall R$ and \exists^L -rules, the variable y does not occur in the conclusion sequent and it is called *eigenvariable* or proper parameter of the rule. In the \Rightarrow^L rule, the left-side premise of the rule is obtained by an Id-rule or a right introduction rule. In the \exists^R -rule as well as in the \forall^R -rule, the premise of the rule is obtained by an Id-rule or a right introduction rule. The bold formulas are called active formulas.

A derivation (also called proof) \mathcal{D} of a sequent $\Gamma \vdash C$ in SLJ is a finite tree of sequents constructed according to the rules of SLJ in which leaves are instances of Id-rules or \perp^L -rules, all sequents of the form $\Delta, \perp \vdash D$ are leaves and whose root, also called conclusion, is $\Gamma \vdash C$. The height $|\mathcal{D}|$ of a derivation \mathcal{D} is the number of nodes in its maximal branch minus 1. A sequent $\Gamma \vdash C$ is said to be *derivable* or *provable* in the sequent calculus SLJ whenever there exists a proof with conclusion $\Gamma \vdash C$.

We give two examples of pairs of derivations with the same root. In each pair, the left-hand proof-tree is a derivation in both SLJ and GKi while the other proof-tree is a derivation in GKi that it is not a derivation in SLJ.

$$\begin{array}{c}
 \frac{\frac{\frac{\forall x P(x), P(y) \vdash P(y)}{\forall x P(x), P(y) \vdash \exists x P(x)} \text{Id}}{\forall x P \vdash \exists x P(x)} \exists^R}{\forall x P \vdash \exists x P(x)} \forall^L
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\frac{\frac{\forall x P(x), P(y) \vdash P(y)}{\forall x P(x) \vdash P(y)} \text{Id}}{\forall x P \vdash \exists x P(x)} \forall^L}{\forall x P \vdash \exists x P(x)} \exists^R
 \end{array}$$

Table 1: The SLJ sequent calculus.

$$\begin{array}{c}
\frac{}{\Gamma, \perp \vdash C} \perp^L \qquad \frac{}{\Gamma, A \vdash A} \text{Id} \\
\\
\frac{\Gamma, A \Rightarrow B \vdash A, C \quad \Gamma, A \Rightarrow B, B \vdash C}{\Gamma, A \Rightarrow B \vdash C} \Rightarrow^L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B} \Rightarrow^R \\
\\
\frac{\Gamma, A, A \wedge B \vdash \Delta}{\Gamma, A \wedge B \vdash C} \wedge_1^L \qquad \frac{\Gamma, B, A \wedge B \vdash \Delta}{\Gamma, A \wedge B \vdash C} \wedge_2^L \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge^R \\
\\
\frac{\Gamma, A \vee B, A \vdash C \quad \Gamma, A \vee B, B \vdash C}{\Gamma, A \vee B \vdash C} \vee^L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee_1^R \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee_2^R \\
\\
\frac{\Gamma, A[y/x], \exists x A \vdash C}{\Gamma, \exists x A \vdash C} \exists^L \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \exists^R \\
\\
\frac{\Gamma, A[t/x], \forall x A \vdash C}{\Gamma, \forall x A \vdash C} \forall^L \qquad \frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A} \forall^R
\end{array}$$

$$\frac{}{\perp, X \vee Y \vdash P(y)} \perp^L \quad \frac{\frac{}{\perp, X \vee Y, Y \vdash P(y)} \perp^L \quad \frac{}{\perp, X \vee Y, X \vdash P(y)} \perp^L}{\perp, X \vee Y \vdash P(y)} \vee^L$$

3.1 Properties of SLJ

In this section we prove some properties of the sequent calculus SLJ. In particular, we prove that the sequent calculus SLJ is sound and complete for first-order intuitionistic logic. If the proof of a proposition is standard, we will omit it.

Proposition 6 (Inversion). *For any formula A and B , for any multiset of formulas Γ :*

1. *if there is a derivation \mathcal{D} of $\Gamma \vdash A \wedge B$ then there are derivations \mathcal{D}_1 of $\Gamma \vdash A$ and \mathcal{D}_2 of $\Gamma \vdash B$. Moreover $|\mathcal{D}_i| \leq |\mathcal{D}|$ for $i \in \{1, 2\}$;*
2. *if there is a derivation \mathcal{D} of $\Gamma \vdash A \Rightarrow B$ then there is a derivation \mathcal{D}_1 of $\Gamma, A \vdash B$. Moreover $|\mathcal{D}_1| \leq |\mathcal{D}|$;*
3. *if there is a derivation \mathcal{D} of $\Gamma \vdash \forall xA$ then there is a derivation \mathcal{D}_1 of $\Gamma \vdash A[y/x]$ where y is a variable that does not appear in Γ . Moreover $|\mathcal{D}_1| \leq |\mathcal{D}|$;*
4. *if there is a derivation \mathcal{D} of $\Gamma, A \vee B \vdash C$ then there are derivations \mathcal{D}_1 of $\Gamma, A \vee B, A \vdash C$ and \mathcal{D}_2 of $\Gamma, A \vee B, B \vdash C$. Moreover $|\mathcal{D}_i| \leq |\mathcal{D}|$ for $i \in \{1, 2\}$;*
5. *if there is a derivation \mathcal{D} of $\Gamma, \exists xA \vdash C$ then there is a derivation \mathcal{D}_1 of $\Gamma, \exists xA, A[y/x] \vdash C$ where y is a variable that neither appears in Γ nor in C . Moreover $|\mathcal{D}_1| \leq |\mathcal{D}|$.*

Proof. By induction on the height of $|\mathcal{D}|$ of \mathcal{D} . □

Corollary 2. *For any formula A for any multiset of formulas Γ , if the main connective of A is either \wedge , \Rightarrow or \forall and the sequent $\Gamma \vdash A$ is provable then there is a derivation \mathcal{D} of $\Gamma \vdash A$ in which A is active.*

Proposition 7. *Contraction and weakening are height preserving admissible in SLJ, i.e., for any formula A and C , for any multiset of formulas Γ :*

- *if there a derivation \mathcal{D} of $\Gamma, A, A \vdash C$ then there is a derivation \mathcal{D}_1 of $\Gamma, A \vdash C$ and $|\mathcal{D}_1| \leq |\mathcal{D}|$;*
- *if there is a derivation \mathcal{D} of $\Gamma \vdash C$ then there is a derivation \mathcal{D}_1 of $\Gamma, A \vdash C$ and $|\mathcal{D}_1| \leq |\mathcal{D}|$.*

Proof. By induction on the height of the derivation \mathcal{D} of $\Gamma, A, A \vdash C$ (resp $\Gamma \vdash C$). □

Proposition 8. *For any formula A and any multiset of formulas Γ there is a derivation \mathcal{D} of the sequent $\Gamma, A \vdash A$. Moreover, in the derivation \mathcal{D} , either the occurrence of A on the left of \vdash is active or the occurrence of A on the right of \vdash is active.*

Proof. Suppose that the proposition holds for all formulas B having height smaller than n , and let A be a formula of height n . We will detail the proof only for the case in which $A = C \Rightarrow D$. By induction hypothesis, there are proofs \mathfrak{D}_1 of $\Gamma, C \Rightarrow D, C \vdash C$ and \mathfrak{D}_2 of $\Gamma, C \Rightarrow D, D \vdash D$. We have a problem if in the derivation \mathfrak{D}_1 the active occurrence of C is the one on the left of the turnstile, i.e. \mathfrak{D}_1 ends in a left introduction rule. If the main connective of C is \forall, \wedge or \Rightarrow then by corollary 2 we can conclude that there is a derivation \mathfrak{D}'_1 of $\Gamma, C \Rightarrow D, C \vdash C$ in which the occurrence of C on the right of the turnstile is active. Thus we can apply \Rightarrow^L on \mathfrak{D}'_1 and \mathfrak{D}_2 to obtain the wanted result. If the main connective of C is \exists or \vee then C has, respectively, the form $\exists x C_1$ or $C_1 \vee C_2$. Let us consider the second case. By induction hypothesis there are derivations of $\Gamma', C_1 \vee C_2, C_1 \vdash C_1$ and $\Gamma', C_1 \vee C_2, C_2 \vdash C_2$ where $\Gamma' = \Gamma, C_1 \vee C_2 \Rightarrow D$. First we construct the two following derivations

$$\begin{array}{c} \vdots \\ \frac{\Gamma', C_1 \vee C_2, C_1 \vdash C_1}{\Gamma', C_1 \vee C_2, C_1 \vdash C_1 \vee C_2} \vee_1^R \quad \frac{\vdots \mathfrak{D}'_2}{\Gamma, C_1 \vee C_2 \Rightarrow D, D, C_1 \vdash D} \\ \Rightarrow^L \frac{\frac{\Gamma', C_1 \vee C_2, C_1 \vdash C_1 \vee C_2}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vdash D}}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vdash D} \end{array} \left. \vphantom{\frac{\Gamma', C_1 \vee C_2, C_1 \vdash C_1 \vee C_2}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vdash D}} \right\} \mathfrak{D}_A$$

$$\begin{array}{c} \vdots \\ \frac{\Gamma', C_1 \vee C_2, C_2 \vdash C_2}{\Gamma', C_1 \vee C_2, C_1 \vdash C_1 \vee C_2} \vee_2^R \quad \frac{\vdots \mathfrak{D}''_2}{\Gamma, C_1 \vee C_2 \Rightarrow D, D, C_2 \vdash D} \\ \Rightarrow^L \frac{\frac{\Gamma', C_1 \vee C_2, C_1 \vdash C_1 \vee C_2}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_2 \vdash D}}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_2 \vdash D} \end{array} \left. \vphantom{\frac{\Gamma', C_1 \vee C_2, C_1 \vdash C_1 \vee C_2}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_2 \vdash D}} \right\} \mathfrak{D}_B$$

where \mathfrak{D}'_2 and \mathfrak{D}''_2 are obtained from \mathfrak{D}_2 by weakening admissibility. We can now construct a derivation of $\Gamma, C_1 \vee C_2 \Rightarrow D \vdash C_1 \vee C_2 \Rightarrow D$

$$\frac{\frac{\frac{\vdots \mathfrak{D}_A}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vdash D} \quad \frac{\vdots \mathfrak{D}_B}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_2 \vdash D}}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vee C_2 \vdash D} \vee^L}{\Gamma, C_1 \vee C_2 \Rightarrow D \vdash C_1 \vee C_2 \Rightarrow D} \Rightarrow^R$$

□

Proposition 9. For any formula A, B for any multiset of formulas, Γ , the sequents

1. $\Gamma, A, A \Rightarrow B \vdash B$
2. $\Gamma, A \vdash A \vee B$
3. $\Gamma, B \vdash A \vee B$
4. $\Gamma, A[y/x] \vdash \exists x A$

are derivable in SLJ. In (4) y is a variable that does not appear in $\Gamma, \exists x A$.

Proof. It is an immediate consequence of propositions 6,7 and 8

□

Proposition 10. For any formula A and F , for any multiset of formulas Γ , the cut rule

$$\frac{\Gamma \vdash A \quad \Gamma, A \vdash F}{\Gamma \vdash F} \text{ cut}$$

is admissible in SLJ that is, if $\Gamma \vdash A$ and $\Gamma, A \vdash F$ are provable then $\Gamma \vdash F$ is provable.

Proof. By nested inductions on the depth of A , (the cut-formula) the height of the derivation \mathfrak{D}_1 of $\Gamma \vdash A$ and the height of the derivation \mathfrak{D}_2 of $\Gamma, A \vdash F$. More precisely, we appeal to the induction hypothesis either with a strictly smaller cut-formula, or with an identical cut-formula and two derivations, one of which is strictly smaller while the other stays the same; the cut admissibility proof follows the usual path of case analysis on the active formula of \mathfrak{D}_1 and \mathfrak{D}_2 .

We detail two cases. Suppose that the cut-formula A is $B \Rightarrow C$. We have a derivation \mathfrak{D}_1 of $\Gamma, B \Rightarrow C \vdash F$ and a derivation \mathfrak{D}_2 of $\Gamma \vdash B \Rightarrow C$. Moreover, suppose that in both derivations the cut-formula $B \Rightarrow C$ is active. This means that \mathfrak{D}_1 and \mathfrak{D}_2 have the form

$$\frac{\begin{array}{c} \vdots \mathfrak{D}_{1,1} \\ \Gamma, B \Rightarrow C \vdash B \end{array} \quad \begin{array}{c} \vdots \mathfrak{D}_{1,2} \\ C, B \Rightarrow C, \Gamma \vdash F \end{array}}{\Gamma, B \Rightarrow C \vdash F} \Rightarrow^L \quad \frac{\begin{array}{c} \vdots \mathfrak{D}_{2,1} \\ \Gamma, B \vdash C \end{array}}{\Gamma \vdash B \Rightarrow C,} \Rightarrow^R$$

we obtain a derivation of $\Gamma \vdash F$ as follows: we first construct a derivation \mathfrak{D}_A of $\Gamma \vdash C$, and a derivation \mathfrak{D}_B of $C, \Gamma \vdash F$, using the admissibility of the cut rule either on derivations that are smaller than \mathfrak{D}_2 or \mathfrak{D}_1 or on formulas that are smaller than $B \Rightarrow C$. In what follows \mathfrak{D}'_2 is obtained from \mathfrak{D}_2 by height preserving admissibility of the weakening rule. For the sake of clarity, we underline the cut-formula of each cut-rule instance.

$$\left. \frac{\begin{array}{c} \vdots \mathfrak{D}_{1,1} \\ \Gamma, B \Rightarrow \underline{C} \vdash B, \end{array} \quad \begin{array}{c} \vdots \mathfrak{D}_2 \\ \Gamma \vdash \underline{B} \Rightarrow C, \end{array}}{\text{cut} \frac{\Gamma \vdash \underline{B}}{\Gamma \vdash C}} \text{ cut} \quad \begin{array}{c} \vdots \mathfrak{D}_{2,1} \\ \Gamma, \underline{B} \vdash C \end{array} \right\} \mathfrak{D}_A$$

$$\left. \begin{array}{c} \vdots \mathfrak{D}_{1,2} \\ C, \Gamma, \underline{B} \Rightarrow C \vdash F \end{array} \quad \begin{array}{c} \vdots \mathfrak{D}'_2 \\ C, \Gamma \vdash \underline{B} \Rightarrow C \end{array} \right\} \mathfrak{D}_B$$

We then put together \mathfrak{D}_A and \mathfrak{D}_B by using an instance of the cut rule with cut-formula C . C is a formula strictly smaller than $B \Rightarrow C$, thus this application of the cut-rule is allowed by the induction hypothesis.

$$\frac{\begin{array}{c} \vdots \mathfrak{D}_A \\ \Gamma \vdash \underline{C} \end{array} \quad \begin{array}{c} \vdots \mathfrak{D}_B \\ \Gamma, \underline{C} \vdash F \end{array}}{\Gamma \vdash F} \text{ cut}$$

Suppose that $B \Rightarrow C$ is not active in one of the two derivations \mathfrak{D}_1 and \mathfrak{D}_2 . Suppose, for example, that it is not active in \mathfrak{D}_2 e.g., $\Gamma = \Gamma', D \vee E$ and \mathfrak{D}_2 has the form:

$$\frac{\begin{array}{c} \vdots \mathfrak{D}_{2,1} \\ \Gamma', D \vee E, D \vdash B \Rightarrow C \end{array} \quad \begin{array}{c} \vdots \mathfrak{D}_{2,2} \\ \Gamma', D \vee E, E \vdash B \Rightarrow C \end{array}}{\Gamma', D \vee E \vdash B \Rightarrow C} \vee L$$

We want to obtain a derivation of the sequent $\Gamma', D \vee E \vdash F$. A proof of it can be constructed as follows. We first construct two derivations \mathfrak{D}_A and \mathfrak{D}_B of the sequents $\Gamma', D \vdash F$ and $\Gamma', E \vdash F$. In what follows \mathfrak{D}'_1 and \mathfrak{D}'_2 are obtained from \mathfrak{D}_1 by height-preserving admissibility of weakening.

$$\text{cut} \frac{\begin{array}{c} \vdots \mathfrak{D}'_1 \\ \Gamma', D, D \vee E, B \Rightarrow C \vdash F \end{array} \quad \begin{array}{c} \vdots \mathfrak{D}_{2,1} \\ \Gamma', D, D \vee E \vdash B \Rightarrow C \end{array}}{\Gamma', D, D \vee E \vdash F} \left. \vphantom{\frac{\vdots \mathfrak{D}'_1}{\Gamma', D, D \vee E, B \Rightarrow C \vdash F}} \right\} \mathfrak{D}_A$$

$$\text{cut} \frac{\begin{array}{c} \vdots \mathfrak{D}'_2 \\ \Gamma', E, D \vee E, B \Rightarrow C \vdash F \end{array} \quad \begin{array}{c} \vdots \mathfrak{D}_{2,2} \\ \Gamma', E, D \vee E \vdash B \Rightarrow C \end{array}}{\Gamma', E, D \vee E \vdash F} \left. \vphantom{\frac{\vdots \mathfrak{D}'_2}{\Gamma', E, D \vee E, B \Rightarrow C \vdash F}} \right\} \mathfrak{D}_B$$

We put together the two derivations σ_1 and σ_2 using a $\vee L$ rule and we obtain the wanted derivation of the sequent $\Gamma', D \vee E \vdash F$

$$\frac{\begin{array}{c} \vdots \mathfrak{D}_A \\ \Gamma', D, D \vee E \vdash F \end{array} \quad \begin{array}{c} \vdots \mathfrak{D}_B \\ \Gamma', E, D \vee E \vdash F \end{array}}{\Gamma', D \vee E \vdash F} \vee L$$

□

We now prove that SLJ is sound and complete for intuitionistic logic. In order to prove this fact we show that a sequent $\Gamma \vdash C$ is provable in SLJ if and only if it is provable in the sequent calculus system GKi (Troelstra & Schwichtenberg, 1996). GKi is sound and complete for intuitionistic logic, and it is obtained from SLJ by dropping the restriction on the left introduction rule for the implication, right introduction rule for disjunction and right introduction rule for existential quantification of SLJ.

Proposition 11. *For any formula C , for any multiset of formulas Γ , the sequent $\Gamma \vdash C$ is provable in SLJ if and only if it is provable in GKi.*

Proof. Each derivation in SLJ is a derivation in GKi, thus one side of the proof is for free.

For the other side, suppose that for each derivation \mathfrak{D}_i in GKi with height n and conclusion $\Gamma' \vdash C'$ there is a derivation \mathfrak{D}_s in SLJ having the same conclusion. Let \mathfrak{D} be a derivation of $\Gamma \vdash C$ in GKi having height $n + 1$ and let R be the last rule application

of \mathcal{D} . If R is not \Rightarrow^L , \exists^R nor \forall^R we have just to apply the induction hypothesis on the premises of R .

If R is a \Rightarrow^L -rule, then the conclusion of \mathcal{D} is $\Gamma, A \Rightarrow B \vdash C$ and, by induction hypothesis, we have a SLJ derivation \mathcal{D}_1 with conclusion $\Gamma, A \Rightarrow B \vdash A$ and another SLJ derivation \mathcal{D}_2 with conclusion $B, \Gamma, A \Rightarrow B \vdash C$. We can construct a SLJ derivation of the sequent $\Gamma, A \Rightarrow B \vdash C$ as follows

$$\frac{\frac{\frac{\vdots \mathcal{D}_1}{\Gamma, A \Rightarrow B \vdash A} \quad \frac{\frac{\vdots \mathcal{D}_A}{A, \Gamma, A \Rightarrow B \vdash B} \quad \frac{\vdots \mathcal{D}'_2}{B, \Gamma, A, A \Rightarrow B \vdash C}}{A, \Gamma, A \Rightarrow B \vdash C} \text{cut}}{\Gamma, A \Rightarrow B \vdash C} \text{cut}}$$

Where the derivation \mathcal{D}_A exists by proposition 9, and \mathcal{D}'_2 is obtained from \mathcal{D}_2 by admissibility of weakening.

If R is \exists^R or \forall^R then, the conclusion of \mathcal{D} is of the form $\Gamma \vdash \exists xA$ or, respectively, $\Gamma \vdash A_1 \vee A_2$. By induction hypothesis, we have a SLJ derivation \mathcal{D}_1 of its premise $\Gamma \vdash C$, where C is either $A[t/x]$ for some term t , or A_i ($i \in \{1, 2\}$). We treat only the \exists case.

$$\frac{\frac{\vdots \mathcal{D}_1}{\Gamma \vdash A[t/x]} \quad \frac{\vdots \mathcal{D}_A}{\Gamma, A[t/x] \vdash \exists xA}}{\Gamma \vdash \exists xA} \text{cut}$$

Here $\mathcal{D}_A = \mathcal{D}'_A[t/y]$, and \mathcal{D}'_A is a proof of $\Gamma, A[y/x] \vdash \exists xA$ in which y does not appear in $\Gamma, \exists xA$; the derivation \mathcal{D}'_A exists in virtue of proposition 9. \square

In virtue of the above proposition, we can conclude that the following theorem holds.

Theorem 1. *The sequent calculus SLJ is sound and complete for first-order intuitionistic logic.*

4 From strategies to derivations

In this section, we will prove that given a winning strategy \mathcal{S} for a formula A , \mathcal{S} can be transformed into a SLJ proof of $\vdash A$. In fact, we have designed the calculus SLJ in such a way that derivations in SLJ have the ‘shape’ of winning strategies. In the two tables below (tables 2 and 3) we show the correspondence between SLJ-rules and **P**-moves. For each **P**-move, we also show the **O** attack or defense moves that are justified by the **P**-move⁵. We can see that a **P** defense move asserting A corresponds to a SLJ right introduction rule in which A is active. The no delayed choices condition forces **P** defending immediately against a disjunctive or existential attack. This restricted way of playing corresponds to the restrictions on the use of the \forall^R and \exists^R -rules of SLJ. In fact, the sequent calculus SLJ obeys a *focusing principle* (Andreoli, 1992): whenever we apply (bottom-up) an \exists^R or a \forall^R rule over a sequent $\Gamma \vdash \exists xB$ (resp., $\Gamma \vdash A \vee B$) we are obliged to apply right-rules until an implication, a conjunction, or a universally quantified formula occupies the position of $\exists xB$ (resp. $A \vee B$).

⁵We consider that eigenvariables of the \forall^R and \exists^L -rules are chosen according to the enumeration $(v_i)_{i \in I}$.

Rule	P-move	justified O-move
$\frac{}{\Gamma, A \vdash A} \text{Id}$	$(!, A)$	None
$\frac{}{\Gamma, \perp \vdash C} \perp^L$	None	(\star, \perp)
$\frac{\Gamma, A \Rightarrow B \vdash A \quad \Gamma, A \Rightarrow B, B \vdash C}{\Gamma, A \Rightarrow B \vdash C} \Rightarrow^L$	$(?, A)$	$(!, B)$ or $(?, q)$
$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow^R$	$(!, A \Rightarrow B)$	$(?, A)$
$\frac{\Gamma, A, A \wedge B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge_1^L$	$(?, \wedge_1)$	$(!, A)$
$\frac{\Gamma, B, A \wedge B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge_2^L$	$(?, \wedge_2)$	$(!, B)$
$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge^R$	$(!, A \wedge B)$	$(?, \wedge_1)$ or $(?, \wedge_2)$
$\frac{\Gamma, A, A \vee B \vdash C \quad \Gamma, B, A \vee B \vdash C}{\Gamma, A \vee B \vdash C} \vee^L$	$(?, \vee)$	$(!, A)$ or $(!, B)$
$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee_1^R$	$(!, A \vee B)$	$(?, \vee)$
$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee_2^R$	$(!, A \vee B)$	$(?, \vee)$

Table 2: Correspondence between SLJ-rules and P-moves, connectives.

$\frac{\Gamma, A[y/x], \exists xA \vdash C}{\Gamma, \exists xA \vdash C} \exists^L$	$(?, \exists)$	$(!, A[y/x])$
$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists xA} \exists^R$	$(!, \exists xA)$	$(?, \exists)$
$\frac{\Gamma, A[t/x], \forall xA \vdash C}{\Gamma, \forall xA \vdash C} \forall^L$	$(?, \forall[t/x])$	$(!, A[t/x])$
$\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall xA} \forall^R$	$(!, \forall xA)$	$(?, \forall[y/x])$

Table 3: Correspondence between SLJ-rules and **P**-moves, quantifiers.

Let S be a strategy for a formula F and let \mathcal{G} be a game in S . We define the **O**-sequence $\mathcal{G}|_O$ of \mathcal{G} to be the subsequence of \mathcal{G} obtained by forgetting all its **P**-moves, i.e, if $\mathcal{G} = m_0 m_1 \cdots m_{2n}$ the **O**-sequence of \mathcal{G} is $m_1 \cdots m_{2n-1}$. We define the **O**-tree $S|_O$ of a strategy S to be the prefix-closed set of sequences

$$S|_O = \{\mathcal{G}|_O \mid \mathcal{G} \in S\}$$

Let S be a strategy. We define a function Φ from $S|_O$ to a tree of sequent τ . The function Φ associates a sequent $\Gamma_{\mathcal{G}|_O} \vdash C_{\mathcal{G}|_O}$ to each $\mathcal{G}|_O$ in $S|_O$. We recall that ϵ denotes the empty sequence.

If $\mathcal{G}|_O = \epsilon$ then $\Gamma_{\mathcal{G}|_O} = \emptyset$ and $C_{\mathcal{G}|_O} = F$;

if $\mathcal{G}|_O = \mathcal{G}'|_O(!, A)$ then $\Gamma_{\mathcal{G}|_O} = \Gamma_{\mathcal{G}'|_O}, A$ and $C_{\mathcal{G}|_O} = C_{\mathcal{G}'|_O}$;

if $\mathcal{G}|_O = \mathcal{G}'|_O(?, A)$ then $\Gamma_{\mathcal{G}|_O} = \Gamma_{\mathcal{G}'|_O}, A$ and $C_{\mathcal{G}'|_O}$ have the form $A \Rightarrow B$ we put $C_{\mathcal{G}|_O} = B$;

if $\mathcal{G}|_O = \mathcal{G}'|_O(?, \wedge_1)$ then $\Gamma_{\mathcal{G}|_O} = \Gamma_{\mathcal{G}'|_O}$ and $C_{\mathcal{G}'|_O}$ have the form $A \wedge B$ we put $C_{\mathcal{G}|_O} = A$;

if $\mathcal{G}|_O = \mathcal{G}'|_O(?, \wedge_2)$ then $\Gamma_{\mathcal{G}|_O} = \Gamma_{\mathcal{G}'|_O}$ and $C_{\mathcal{G}'|_O}$ have the form $A \wedge B$ we put $C_{\mathcal{G}|_O} = B$;

if $\mathcal{G}|_O = \mathcal{G}'|_O(?, \forall[w/x])$ then $\Gamma_{\mathcal{G}|_O} = \Gamma_{\mathcal{G}'|_O}$ and $C_{\mathcal{G}'|_O}$ have the form $\forall xA$ we put $C_{\mathcal{G}|_O} = A[w/x]$;

if $\mathcal{G}|_O = \mathcal{G}'|_O(?, \vee)$ then $\Gamma_{\mathcal{G}|_O} = \Gamma_{\mathcal{G}'|_O}$ and $C_{\mathcal{G}'|_O}$ have the form $A_1 \vee A_2$ we put $C_{\mathcal{G}|_O} = A_i$ where A_i is the formula asserted by the move $m \in \mathcal{G}$, such that $\phi(m) = (?, \vee)$;

if $\mathcal{G}|_{\mathbf{O}} = \mathcal{G}'|_{\mathbf{O}}(?, \exists)$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \Gamma_{\mathcal{G}'|_{\mathbf{O}}}$ and $C_{\mathcal{G}'|_{\mathbf{O}}}$ have the form $\exists xA$ we put $C_{\mathcal{G}|_{\mathbf{O}}} = A[t/x]$ where $A[t/x]$ is the formula asserted by the move $m \in \mathcal{G}$, $\phi(m) = (?, \exists)$.

We now prove that given a winning strategy \mathcal{S} , $\Phi(\mathcal{S}|_{\mathbf{O}})$ is *almost* a derivation in SLJ; all leaves of $\Phi(\mathcal{S}|_{\mathbf{O}})$ are instances of Id-rules or \perp^L rules (proposition 13), moreover the tree of sequents $\Phi(\mathcal{S}|_{\mathbf{O}})$ respect the variable conditions of the \forall^R and \exists^L rules of SLJ (proposition 14).

Proposition 12. *Let \mathcal{S} be an arbitrary winning strategy and \mathcal{G} be an arbitrary game in \mathcal{S} . If \mathcal{G} ends in a \mathbf{P} defense move that asserts a formula A , then the sequent $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ associated to the \mathbf{O} -restriction $\mathcal{G}|_{\mathbf{O}}$ of \mathcal{G} by the function Φ is of the form $\Gamma \vdash A$.*

Proof. By induction on the length of \mathcal{G} . □

Proposition 13. *Let \mathcal{S} be an arbitrary winning strategy for a formula A and \mathcal{G} an arbitrary maximal branch in \mathcal{S} :*

1. *If $\mathcal{G} = \mathcal{G}'m^{\mathbf{O}}$, then m asserts \perp and the sequent $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ associated to the \mathbf{O} -restriction $\mathcal{G}|_{\mathbf{O}}$ of \mathcal{G} by the function Φ is of the form $\Gamma, \perp \vdash B$ with B Gentzen-subformula of A ;*
2. *if $\mathcal{G} = \mathcal{G}'m^{\mathbf{P}}$ then $m = (!, C)$ with C atomic Gentzen-subformula of A and the sequent $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ associated to the \mathbf{O} -restriction $\mathcal{G}|_{\mathbf{O}}$ of \mathcal{G} by the function Φ is of the form $\Gamma', C \vdash C$.*

Proof. (1) is a direct consequence of definition 4 and of the definition of the function Φ . (2) derives from condition 3 in definition 3 and from proposition 12. □

Proposition 14. *let \mathcal{S} be an arbitrary winning strategy and \mathcal{G} be a game in \mathcal{S} . Suppose that \mathcal{G} ends in an \mathbf{O} -move that is either:*

1. *an attack against a universal quantifier $(?, \forall[w/x])$;*
2. *or a defense against an existential attack $(!, A[w/x])$.*

Then the variable w does not appear in the sequent associated by the function Φ to the \mathbf{O} -restriction $\mathcal{G}'|_{\mathbf{O}}$ of the proper prefix \mathcal{G}' of \mathcal{G} .

Proof. Both (1) and (2) are granted by the conditions 4 and 5 in definition 3 and by the no delayed choices condition in definition 5 of strategy. □

We are now ready to prove the main result of this section. We have just shown that we can associate a tree of sequents with each winning strategy. In addition, we have shown that the above-mentioned sequent tree is almost a proof in SLJ: all its leaves are instances of Id-rules or \perp^L -rules, and it respects the variable restriction on the \forall^R and \exists^L -rules.

Theorem 2. Let S be an arbitrary winning strategy and $S|_{\mathbf{O}}$ by its \mathbf{O} -tree. To each sequence of \mathbf{O} -moves $\mathcal{G}|_{\mathbf{O}}$ in $S|_{\mathbf{O}}$ we can associate a derivation $\mathcal{D}_{\mathcal{G}|_{\mathbf{O}}}$ of $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$, where $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ is the sequent associated by the function Φ to $\mathcal{G}|_{\mathbf{O}}$.

Proof. Let $\mathcal{G}|_{\mathbf{O}}$ be an arbitrary element of $S|_{\mathbf{O}}$. Suppose that the induction hypothesis holds for each suffix $\mathcal{G}'|_{\mathbf{O}}$ of $\mathcal{G}|_{\mathbf{O}}$ in $S|_{\mathbf{O}}$. We consider the last \mathbf{P} -move m_{2n} of the game $\mathcal{G} \in S$ such that the \mathbf{O} -restriction of \mathcal{G} is $\mathcal{G}|_{\mathbf{O}}$.

We only prove some of the cases that are not straightforward:

1. if m_{2n} is a defense move $(!, \exists xA)$, then there are many cases, depending on the form of A . We treat only two cases:

if A is atomic then $\mathcal{G}|_{\mathbf{O}}(?, \exists)$ is maximal in $S|_{\mathbf{O}}$; we associate with $\mathcal{G}|_{\mathbf{O}}$ the following derivation in which A is active.

$$\frac{\overline{\Gamma, A \vdash \mathbf{A}} \text{ Id}}{\Gamma, A \vdash \exists xA} \exists^R$$

If $A = B \vee C$ then $\mathcal{G}|_{\mathbf{O}}' = \mathcal{G}|_{\mathbf{O}}(?, \exists)(?, \vee) \in S|_{\mathbf{O}}$ because of condition 5 in definition 5. This means in particular that the formula $(B \vee C)[t/x]$ is active in the derivation that we associate with $\mathcal{G}|_{\mathbf{O}}$

$$\frac{\overline{\Gamma \vdash (\mathbf{A} \vee \mathbf{B})[t/x]} \text{ Id}}{\Gamma \vdash \exists x(\mathbf{A} \vee \mathbf{B})} \exists^R$$

2. If m_{2n} is an attack $(?, A)$ on the assertion $A \Rightarrow C$, then there are many cases depending on the form of A . We only treat two cases:

If A is atomic, then the immediate suffix of $\mathcal{G}|_{\mathbf{O}}$ is $\mathcal{G}|_{\mathbf{O}}(!, C)$ for which the proposition hold by hypothesis. We associate it with the following derivation.

$$\frac{\overline{\Gamma, A \Rightarrow C, A \vdash \mathbf{A}} \text{ Id} \quad \overline{\Gamma, A \Rightarrow C, C \vdash F} \text{ Id}}{\Gamma, A \Rightarrow C \vdash F} \Rightarrow^L$$

If $A = (A_1 \Rightarrow A_2)$, then $\mathcal{G}|_{\mathbf{O}}$ has two immediate suffixes: $\mathcal{G}|_{\mathbf{O}}(?, A_1)$ and $\mathcal{G}|_{\mathbf{O}}(!, C)$, for which the proposition holds by hypothesis. We associate the following derivation to $\mathcal{G}|_{\mathbf{O}}$

$$\frac{\overline{\Gamma, (A_1 \Rightarrow A_2) \Rightarrow C, A_1 \vdash A_2} \text{ Id} \quad \overline{\Gamma, (A_1 \Rightarrow A_2) \Rightarrow C, C \vdash F} \text{ Id}}{\Gamma, (A_1 \Rightarrow A_2) \Rightarrow C \vdash A_1 \Rightarrow A_2} \Rightarrow^R \quad \frac{\overline{\Gamma, (A_1 \Rightarrow A_2) \Rightarrow C, C \vdash F} \text{ Id}}{\Gamma, (A_1 \Rightarrow A_2) \Rightarrow C \vdash F} \Rightarrow^L$$

□

5 From derivations to strategies

Turning a proof \mathcal{D} of the sequent $\vdash F$ into a winning strategy \mathcal{S} for F is easier. To do so we describe a procedure, that we call $p2s$ (from a proof in SLJ to a strategy). The procedure $p2s$ explores the proofs \mathcal{D} starting from the root and proceeding by level order traversal. The order of traversal of daughters is irrelevant. The procedure associates to \mathcal{D} a prefix closed set of games for the formula F .

Theorem 3. *Let F be an arbitrary formula and \mathcal{D} be an arbitrary derivation of $\vdash F$ in SLJ. There is a function $p2s$ such that $p2s(\mathcal{D})$ is a winning strategy \mathcal{S} for F .*

Proof. Let x be an arbitrary node in the proof \mathcal{D} of the formula F having depth n , and let $\Gamma \vdash C$ be the sequent that decorates x . Suppose that

1. the branch $r = x_0, \dots, x_n = x$ of the derivation from the root r of \mathcal{D} to x is already associated with a prefix closed set S_x of games for the formula F . Each \mathcal{G} in S_x in which the last move of \mathbf{P} is the assertion of a complex formula or an attack move ends in an \mathbf{O} -move.
2. For each formula B in Γ there is an \mathbf{O} -assertion move (\star, B) in some game G in S_x .
3. The prefix closed set S_x is a strategy for F .

The prefix closed set of games S_{a_1} associated with a_1 where a_1 is any daughter of x is defined as follows:

1. if a_1 is obtained by an identity rule $\Gamma, A \vdash A$ then $S_{a_1} = S_x \cup \{\mathcal{G}(!, A)\}$ where A is the active formula of the identity rule and \mathcal{G} is a maximal game in S_x such that $(!, A)$ is legal for \mathcal{G} .
2. If a_1 is labelled with a sequent obtained from a right introduction rule with active formula A :

if A is not a conjunction nor a universally quantified formula then $S_{a_1} = S_x \cup \{\mathcal{G}(!, A)(?, s)\}$ where \mathcal{G} is a maximal game in S_x such that $(!, A)$ is legal for \mathcal{G} and $(?, s)$ is an attack move such that s is the unique question on A ;

if A is $\forall xA'$ then S_{a_1} is $S_x \cup \{\mathcal{G}(!, \forall xA), (?, \forall[w/x])\}$ where \mathcal{G} is a maximal game in S_x such that $(!, A)$ is legal for \mathcal{G} and the variable w in $(?, \forall[w/x])$ is the variable that appears in the premise of a_1 but not in a_1 ;

if A is $B \wedge C$ then $S_{a_1} = S_x \cup \{\mathcal{G}(!, B \wedge C)(?, \wedge_1)\} \cup \{\mathcal{G}(!, B \wedge C)(?, \wedge_2)\}$ where \mathcal{G} is a maximal game in S_x such that the \mathbf{P} -move $(!, B \wedge C)$ is legal for \mathcal{G} .

3. If a_1 is labelled with a sequent obtained from a left introduction rule with active formula A :

if A is $B_1 \wedge B_2$ then $S_{a_1} = S_x \cup \{\mathcal{G}(?, \wedge_i)(!, B_i)\}$ where B_i is the direct subformula of $B_1 \wedge B_2$ that appears in the premise of a_1 but not in a_1 and \mathcal{G} is a maximal game in S_x such that the **P**-move $(?, \wedge_i)$ is legal for \mathcal{G} ;

if A is $\forall xB$ then $S_{a_1} = S_x \cup \{\mathcal{G}(?, \forall[t/x])(!, B[t/x])\}$ where $B[t/x]$ is the formula occurrence that appears in the premise of a_1 but not in a_1 and \mathcal{G} is a maximal game in S_x such that the **P**-move $(?, \forall[t/x])$ is legal for \mathcal{G} ;

if A is $B \vee C$ then $S_{a_1} = S_x \cup \{\mathcal{G}(?, \vee)(!, B)\} \cup \{\mathcal{G}(?, \vee)(!, C)\}$ where \mathcal{G} is a maximal game in S_x such that the **P**-move $(?, \vee)$ is legal for \mathcal{G} ;

if A is $\exists xB$, $S_{a_1} = S_x \cup \{\mathcal{G}(?, \exists)(!, B[w/x])\}$ where \mathcal{G} is a maximal game in S_x such that the **P**-move $(?, \exists)$ is legal for \mathcal{G} and $B[w/x]$ is a formula that appears in the premise of a_1 but not in a_1 ;

if A is $B \Rightarrow C$, $S_{a_1} = S_x \cup \{\mathcal{G}(?, B), (? , q_1)\} \cup \dots \cup \{\mathcal{G}(?, B), (? , q_n)\} \cup \{\mathcal{G}(?, B), (!, C)\}$. Where \mathcal{G} is a maximal game in S_x such that the **P**-move $(?, B)$ is legal for \mathcal{G} where each q_i is a question on B .

Remark that the procedure jumps from a node v of the proof-tree obtained by a \Rightarrow^L rule to the daughter of the daughter of v . It is easy to check that conditions 1,2 and 3 are respected after the application of the procedure. □

6 Conclusion and Future Works

We have proved that there is a natural correspondence between formal E-strategies and derivations of a complete sequent calculus for first-order intuitionistic logic. We hope that the simplicity of our approach will help other researchers to better appreciate the dialogical logic approach.

As we have previously mentioned, the sequent calculus SLJ obeys a *focusing principle* (Andreoli, 1992): whenever we apply (bottom-up) an \exists^L -rule or a \forall^R -rule over a sequent $\Gamma \vdash C$ we are obliged to apply right-rules until an implication, a conjunction, or a universally quantified formula is at the place of C . It is quite surprising that (skeletons of) formal E-strategies naturally correspond to this type of calculus. The fact that Felscher himself did not notice such correspondence is explained by noticing that focusing was not known at the time he worked on dialogical logic.

Concerning future works: given a winning strategy for A and a winning strategy for $A \Rightarrow B$, we can conclude that there is a winning strategy for B . Simply because the cut-rule is admissible in SLJ and we can thus translate the winning strategies for $A, A \Rightarrow B$ into two derivations and use the cut rule to obtain a derivation of B . However, it would be much more interesting to define an analogous of the cut-rule directly on strategies. We think that this could be obtained by relaxing the definition of game in order to let the Proponent assert, at any point of the game, an arbitrary formula C . After Proponent's assertion of C , the Opponent can continue the game by either attacking C or by asserting C in turn. The cut-elimination theorem for strategies would be obtained by proving that the set of formulas admitting winning strategies containing this kind of games is equal to the set of formulas admitting regular winning strategies.

Dialogical logic heavily influenced the game semantics approach (Hyland, 1997; Abramsky, Jagadeesan, & Malacaria, 2000). Game semantics has been used in order to give denotational models of many programming languages. In particular, there is a natural correspondence between winning strategies in the Hyland-Ong setting of game semantics and the simply-typed λ -calculus. Despite the fact that dialogical logic and game semantics have a lot in common, no one, at least to our knowledge, has tried to investigate the question in detail. We think that shedding light on this subject could be fruitful for the development of both game semantics and dialogical logic.

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