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From strategies to derivations and back.

An easy completeness proof for first order intuitionistic dialogical logic

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Abstract

In this paper we give a new proof of the correspondence between the existence of a winning strategies for intuitionistic E-games and Intuitionistic validity for first order logic. The proof is obtained by a direct mapping between formal E-strategy and derivations in a cut-free complete sequent calculus for first order intuitionistic logic. Our approach builds on the one developed by Herbelin in his PhD dissertation and greatly simplifies the proof of correspondence given by Felscher in his classic paper

1 Introduction

The art of persuasive debate, dialectics, and the science of valid inference, logic, have been intrinsically linked since their beginnings (Castelnérac & Marion, 2009, 2013; Marion & Rückert, 2016; Novaes, 2005). At the dawn of the modern age the connection between the two disciplines seemed so clear that one of the first sentences pronounced by Doctor Faustus in Marlowe's work goes as follows

Is, to dispute well, logic's chiefest end? Affords this art no greater miracle?

Despite this ancient connection between the two disciplines, mathematical logic had to wait until the 50s of the last century to determine that the logical concept of validity could be expressed through the use of dialogical concepts and techniques. The German mathematician and philosopher Lorenzen (Lorenzen, 1958) proposed to analyze the concept of validity of a formula A through the concept of winning strategy in a particular type of two-player game. This type of game is nothing more than an argumentative dialogue between a player, called Proponent, who affirms the validity of a certain formula A and another player, called Opponent, who contends its validity. The argumentative dialogue starts by the Proponent affirming a certain formula. The Opponent takes his turn

and attacks the claim made by the Proponent according to its logical form. The Proponent can, depending on the form of the attack made by the Opponent, either defend his previous claim or counter attack. The debate evolves following this pattern. The Proponent wins the debate if he has the last word, i.e., the defence against one of the attacks made by the Opponent is a proposition that the Opponent cannot attack without violating the debate rules.

Dialogical logic was initially conceived by Lorenzen as a foundation for intuitionistic logic (IL). Lorenzen's idea was the following: it is possible to define a natural class of dialogue games in which given a formula A , the Proponent can always win a game on A , *no matter how the opponent choose to act in the debate*, if and only if A is IL-valid. This intuition was formalized by saying that, given a certain class of dialogue games, and a formula A

A is IL valid if and only if, there is a winning strategy for the proponent for the formula A in the class of games under consideration.

Unfortunately almost 40 years of work were needed to get a first correct proof of the completeness theorem (Felscher, 1985). Felscher's proof makes use of various intermediate notions that allow a winning strategy to be transformed into a proof in the sequent calculus LJ and vice versa. First, Felscher defines two types of dialogue games, called D-dialogues and E-dialogues. Second he gives an algorithm that converts D-strategies into formal E-strategies (strategies that respect the eigen-variable condition). Third, algorithms are given which transform derivations of the sequent calculus LJ into what Felscher calls IC-protableaux. Felscher concludes his proof by providing an algorithm to transform an IC-protableaux into an E-strategy. As one can see merely from this description Felscher's proof is a big nut to crack.

An improvement of Felscher's work was produced by Herbelin in his PhD dissertation (Herbelin, 1995)¹. Herbelin presents a variant of the LJ sequent calculus, called LJQ, and proves that, given a formula A , derivations for A in the sequent calculus LJQ corresponds to winning E-strategies for A and viceversa. Herbelin's proofs is considerably simpler then Felscher's but, unfortunately, it is limited to propositional logic. The objective of our paper is thus simple: we extend the work of Herbelin to first order intuitionistic logic. We present a sequent calculus, LJs, that is complete for first order intuitionistic logic and we show that derivations in the sequent calculus LJs corresponds to *formal* winning E-strategies and viceversa.

Overview

The rest of the paper is structured as follows: Sect. 2 introduces dialogical logic for intuitionistic logic: we define E-games — that we simply call games —

¹A natural correspondence between a variant of E-strategies for intermediate propositional logics and derivations in an hyper-sequent calculus is provided in (Fermüller, 2003). Building on the work of Fermüller (Alama, Knoks, & Uckelman, 2011) provides a natural correspondence between a variant of E-strategies and derivations in a complete sequent calculus for propositional classical logic. For a correspondence between strategies and tableaux proofs for first order classical logic one can consult (Clerbout, 2014)

formal E-strategies — that we simply call strategies — and prove some results about games and strategies. Sect. 3 introduces the sequent calculus LJs: we prove some results about LJs, in particular, that LJs is sound and complete for intuitionistic first order logic. In Sect. 4 we show how to transform a winning strategy for a formula A into a derivation of A in the calculus LJs; In Sect. 5 we show how to transform a derivation of A in the calculus LJs into a winning strategy for A . The paper will be as much self contained as possible.

2 Dialogical Logic

2.1 First order language

In this paper we will consider a standard first order language \mathcal{L} over a signature $(\mathcal{R}, \mathcal{F})$. \mathcal{R} is a countable set of predicate variables (P, Q, R, S , etc. varies over predicate variables) and \mathcal{F} is a countable set of function symbols (f, g, h , etc. varies over function symbols). The two sets are disjoint and there is an arity function $a : \mathcal{R} \cup \mathcal{F} \rightarrow \mathbb{N}$. Function symbols with arity 0 will be called constants and ranged over by a, b, c, d , etc. Predicate variables with arity 0 will be called propositional variables and ranged over by X, Y, Z, W , etc.

Let \mathcal{V} be a countable set of variable symbols (variables will be ranged over by x, y, z, w , etc.) Terms are defined by the following grammar

$$t ::= x \mid f(t_1, \dots, t_n)$$

where f is a function symbol with arity n . Let $\wedge, \vee, \Rightarrow, \forall, \exists$ be the symbols for the usual connective and quantifiers of first order logic. And let \perp (falsehood) be a predicate constant

Formulas are specified by the following grammar

$$F ::= P(t_1, \dots, t_n) \mid \perp \mid F \wedge F \mid F \vee F \mid F \Rightarrow F \mid \forall x F \mid \exists x F$$

The predicate constant \perp and formulas of the form $P(t_1, \dots, t_n)$, where P is a predicate variable, will be called atomic formulas. The negation of a formula is defined as $\neg F \equiv F \Rightarrow \perp$. We will use capital roman letter from the beginning of the alphabet (A, B, C, D , etc.) to denote arbitrary formulas. The notion of free (resp. bound) variables of a term (resp. formula) will be the usual ones, as well as the notion of subformula, Gentzen-subformula and positive/negative subformula of a formula. The depth $|A|$ of a formula A is the maximum length of a branch in its construction tree.

In the present work we will often talk about sequences. A sequence, formally, is a mapping whose domain is a subset of the set \mathbb{N} of natural numbers. Informally a sequence is a set of objects listed according to a certain specific order. The length of a sequence is the number of elements in it. Given two sequences t, t' t is a prefix (resp. suffix) of t' if and only if, there is a sequence r such that $t' = tr$ (resp. $t'r = t$). If r is not the empty sequence ϵ then t is said to be a proper prefix (resp. proper suffix) of t' . Given a set of sequences S , a sequence $t \in S$ is said to be maximal in S whenever there is no proper suffix of t in S .

2.2 Argumentative dialogues: informal overview

Before entering into the formal matter of dialogical logic let us give an informal example of an argumentative dialogue about the validity of a formula. Let A and B stand for two arbitrary formulas.

0. **P**: I affirm that $A \wedge B \Rightarrow B$
1. **O**: Let me assume, for the sake of the proof, that $A \wedge B$ holds, can you show that B holds?
2. **P**: You admitted that $A \wedge B$ holds, can you admit that B holds?
3. **O**: Indeed, I must admit that B holds.
4. **P**: Then I have nothing more to prove, you have admitted that B holds, if $A \wedge B$ holds.

We can see that the Proponent and the Opponent alternates in the dialogue. The dialogue is a sequence of interventions. Each intervention but the first consist in either an attack against a preceding intervention of the other player or a defence against an attack of the other player. For example **O** in intervention 1 attacks intervention 0 by asking **P** to show that B holds provided that $A \wedge B$ holds. **P**'s defence against 1 is the intervention 4. What counts as a question against an asserted formula A , and what counts as an answer to such a question depends upon the logical form of A . For example in 2 **P** attacks the formula asserted in 1 by asking **O** to assert B . This is because if one admit that a conjunction holds he must be ready to concede that both members of the conjunction holds. Resuming: An argumentative dialogue will be a sequence of alternated interventions made by the Proponent and the Opponent. Each intervention in the dialogue is an attack or a defence against a preceding intervention, the dialogue ends whenever the Opponent cannot produce a new intervention without falling in contradiction with what he already conceded. The content of the next subsection will be devoted to give a formal content to this intuitive discussion. In subsection 2.3 we define what a question on a formula is and what counts as an answer to such a question. In subsection 2.4 we formally define what it means for an intervention in a dialogue to refer to another preceding intervention in the same dialogue (definitions 1 and 2). Finally in subsection 2.5 we define (definition 3) the class of argumentative dialogues we are interested in (the we call games) and when does **P** wins in an argumentative dialogue.

2.3 Argumentation forms

The set of auxiliary symbols Aux is the smallest set containing the symbols $\wedge_1, \wedge_2, \vee, \exists$ and the expressions $\forall[t/x]$ for all terms in \mathcal{T} and variables x in \mathcal{L} .

Following the terminology of Felscher (Felscher, 2002), an argumentation form Arg is a function assigning to each non atomic formula A a set of pairs

consisting of one *question* and one *answer* with questions being either formulas or symbols in Aux and answers being formulas²

$$\begin{aligned} Arg(A \Rightarrow B) &= \{(A, B)\} \\ Arg(A \wedge B) &= \{(\wedge_1, A), (\wedge_2, B)\} \\ Arg(A \vee B) &= \{(\vee, A), (\vee, B)\} \\ Arg(\forall x A) &= \{(\forall[t/x], A[t/x]) \mid t \in \mathcal{T}\} \\ Arg(\exists x A) &= \{(\exists, A[t/x]) \mid t \in \mathcal{T}\} \end{aligned}$$

Given a formula A , a question q that belongs to a couple $(q, a) \in Arg(A)$ is called a *question on A* . Given a formula A and a question q on A , a formula B is called an *answer to the question q on the formula A* whenever the couple (q, B) is an element of $Arg(A)$. So, for example, if A is $B \wedge C$, both \wedge_1 and \wedge_2 are question on A but only B is an answer to \wedge_1 and only C is an answer to \wedge_2 . If $A = B \vee C$, the symbol \vee is a question on A , and both B, C are answers to \vee . Consider the case where A is $B \Rightarrow C$. In this case B is a question on A and C is an answer to B .

2.4 Augmented sequence

A *defence move* is a couple $(!, A)$ where A is a formula. An *attack move* is a couple $(?, s)$ where s is either a formula or an auxiliary symbol. A *move* is either an attack move or a defence move. A move (\star, A) where A is a formula and $\star \in \{?, !\}$, is called *assertion move*. We will also say that the move asserts the formula A or that A is the asserted formula of the move. Attack moves of the form $(?, \exists)$ are called *existential attacks*. Attack moves of the form $(?, \vee)$ are called *disjunctive attacks*. Let $\rho = m_0, m_1, \dots, m_n \dots$ be a sequence of moves. We denote by ρ_i the i th move of the sequence. The parity of ρ_i is the parity of i . An assertion move $\rho_j = (\star, A)$ is called a *reprise* if and only if there is move $\rho_k \in \rho$ with $k < j$ such that $\rho_k = (\star', A)$ and ρ_j, ρ_k have different parities

Definition 1. An *augment sequence* is a non empty sequence of move ρ together with a function ϕ that is defined on each ρ_i with $i > 1$ and such that, for all i , $\phi(\rho_i) = \rho_j$ for a $j < i$. The move $\phi(\rho_i)$ is called the *enabler* of ρ_i .

Definition 2. Let (ρ, ϕ) be an augmented sequence.

- An attack move $\rho_i = (?, s)$ is justified whenever $\phi(\rho_i)$ is of the form (\star, A) and s is a question on A .
- A defence move $\rho_i = (!, B)$ is justified whenever $\phi(\rho_i)$ is of the form $(?, s)$, $\phi(\rho_i)$ is justified, $\phi(\phi(\rho_i)) = (\star, A)$ and B is an answer to the question s on A

²The words “question” and “answer” are called “attack” and “defence” by Felscher in (Felscher, 2002); we deviate from this terminology because we will use the terms “attack” and “defence” exclusively for the moves in a game, avoiding possible confusion.

2.5 Games

Let (ρ, ϕ) be an augmented sequence, we say that a formula A appears in the augmented sequence if and only if there is a move $m \in \rho$ that asserts A . We say that a variable v appears in ρ whenever v occurs free in some asserted formula or there is a move $m = (?, \forall[v/x])$ in ρ . If ρ_i is an attack move and ρ_i is not the enabler of some defence move ρ_j then we will say that ρ_i is unanswered.

Fix an enumeration $(v_i)_{i \in I}$ of the variables of \mathcal{L}

Definition 3 (Game). A game \mathcal{G} for a formula A is an augmented sequence (ρ, ϕ) such that

1. $\rho_0 = (!, A)$ and for all $i > 0$ the move ρ_i is justified.
2. $\phi(\rho_i) = \rho_{i-1}$ if i is odd, $\phi(\rho_i) = \rho_j$ with j odd if i is even.
3. if $\rho_i = (\star, B)$ with B atomic formula and i even then ρ_i is a reprise and $B \neq \perp$
4. if ρ_i is an attack move of the form $(?, \forall[v_k/x])$ and i is odd then v_k is the first variable in the enumeration $(v_i)_{i \in I}$ that does not appear in the prefix of ρ ending with ρ_{i-1}
5. if $\rho_i = (!, B[v_k/x])$ is a defence move, i is odd and ρ_{i-1} is of the form $(?, \exists)$ then v_k is the first variable in the enumeration $(v_i)_{i \in I}$ that does not appear in the prefix of ρ ending with ρ_{i-1}
6. If ρ_k is a defence move and k is even then $\sigma(\rho_k) = \rho_j$ is the unanswered attack move having greatest odd index in the prefix of ρ ending with ρ_{k-1}

In a game \mathcal{G} moves ρ_i with i even are called **P**-moves. They are called **O**-moves otherwise. If $\mathcal{G}m$ is a game and m is **P**-move we will write $\mathcal{G}m^{\mathbf{P}}$. We will write $\mathcal{G}m^{\mathbf{O}}$ otherwise.

Let $\mathcal{G} = (\rho, \phi)$ be a finite game and m be a move. The move m is legal for \mathcal{G} if and only if the augmented sequence $(\rho m, \sigma)$ is a game, where $\sigma|_{\rho} = \phi$ and $\sigma(m) \in \rho$

Definition 4. A game \mathcal{G} is won by **P** if and only if it is finite and either

- the game is of the form $\mathcal{G}'m^{\mathbf{P}}$ and there is no move m' legal for \mathcal{G}
- the game is of the form $\mathcal{G}'m^{\mathbf{O}}$ and m asserts \perp

In what follows we will often identify a game with the sequence of its moves by an abuse of notation.

2.6 Some examples

We give some examples of games. A game will be represented as a table with two columns and as many rows as there are moves in the game. In the first column of the table we will write down the moves of the game. In the second column we will write the value of the function ϕ for the move in the first column. Let X, Y, Z be propositional variables, and R a binary predicate variable

$$\begin{array}{l}
 m_0 = (!, X \Rightarrow Y \Rightarrow ((Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z))) \\
 m_1 = (?, X \Rightarrow Y) \\
 m_2 = (!, (Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z)) \\
 m_3 = (?, Y \Rightarrow Z) \\
 m_4 = (!, X \Rightarrow Z) \\
 m_5 = (?, X) \\
 m_6 = (?, X) \\
 m_7 = (!, Y) \\
 m_8 = (?, Y) \\
 m_9 = (!, Z) \\
 m_{10} = (!, Z)
 \end{array} \left| \begin{array}{l}
 m_0 \\
 m_1 \\
 m_1 \\
 m_2 \\
 m_3 \\
 m_4 \\
 m_1 \\
 m_6 \\
 m_3 \\
 m_8 \\
 m_5
 \end{array} \right.$$

$$\begin{array}{l}
 m_0 = (!, X \vee Y \Rightarrow Y \vee X) \\
 m_1 = (?, X \vee Y) \\
 m_2 = (?, \vee) \\
 m_3 = (!, X) \\
 m_4 = (!, Y \vee X) \\
 m_5 = (?, \vee) \\
 m_6 = (!, X)
 \end{array} \left| \begin{array}{l}
 m_0 \\
 m_1 \\
 m_2 \\
 m_1 \\
 m_4 \\
 m_5
 \end{array} \right. \quad
 \begin{array}{l}
 m_0 = (!, X \vee Y \Rightarrow Y \vee X) \\
 m_1 = (?, X \vee Y) \\
 m_2 = (?, \vee) \\
 m_3 = (!, Y) \\
 m_4 = (!, Y \vee X) \\
 m_5 = (?, \vee) \\
 m_6 = (!, Y)
 \end{array} \left| \begin{array}{l}
 m_0 \\
 m_1 \\
 m_2 \\
 m_1 \\
 m_4 \\
 m_5
 \end{array} \right.$$

$$\begin{array}{l}
 m_0 = (!, X \Rightarrow \neg\neg X) \\
 m_1 = (?, X) \\
 m_2 = (!, \neg\neg X) \\
 m_3 = (!, \neg X) \\
 m_4 = (!, X) \\
 m_5 = (?, \perp)
 \end{array} \left| \begin{array}{l}
 m_0 \\
 m_1 \\
 m_2 \\
 m_3 \\
 m_4
 \end{array} \right. \quad
 \begin{array}{l}
 m_0 = (!, \neg\neg X \Rightarrow X) \\
 m_1 = (?, \neg\neg X) \\
 m_2 = (?, \neg X) \\
 m_3 = (?, X)
 \end{array} \left| \begin{array}{l}
 m_0 \\
 m_1 \\
 m_2
 \end{array} \right.$$

$$\begin{array}{l}
 m_0 = (!, (X \wedge \neg X) \Rightarrow Z) \\
 m_1 = (?, X \wedge \neg X) \\
 m_2 = (?, \wedge_1) \\
 m_3 = (!, X) \\
 m_4 = (?, \wedge_2) \\
 m_5 = (!, \neg X) \\
 m_6 = (?, X) \\
 m_7 = (!, \perp)
 \end{array} \left| \begin{array}{l}
 m_0 \\
 m_1 \\
 m_2 \\
 m_1 \\
 m_4 \\
 m_5 \\
 m_6
 \end{array} \right. \quad
 \begin{array}{l}
 m_0 = (!, X \vee Y \Rightarrow X) \\
 m_1 = (?, X \vee Y) \\
 m_2 = (?, \vee) \\
 m_3 = (!, X) \\
 m_4 = (!, X)
 \end{array} \left| \begin{array}{l}
 m_0 \\
 m_1 \\
 m_2 \\
 m_1
 \end{array} \right.$$

$m_0 = (!, \exists y \forall x R(x, y) \Rightarrow \forall x \exists y R(x, y))$	m_0
$m_1 = (?, \exists y \forall x R(x, y))$	m_1
$m_2 = (!, \forall x \exists y R(x, y))$	m_2
$m_3 = (?, \forall [w/x])$	m_2
$m_4 = (!, \exists y R(w, y))$	m_3
$m_5 = (?, \exists)$	m_4
$m_6 = (?, \exists)$	m_1
$m_7 = (!, \forall x R(x, z))$	m_6
$m_8 = (?, \forall [w/z])$	m_7
$m_9 = (!, R(w, z))$	m_8
$m_{10} = (!, R(w, z))$	m_5

Remark 1. All the games except the one for the formula $\neg\neg X \Rightarrow X$ are won by the Proponent: they either do not admit further Opponent's moves or they end with an Opponent's assertion of \perp . Remark that the last move of all games won by the Proponent ending in a Proponent's move are defence moves and assertions of an atomic formula. In all games formulas asserted by the Proponent are positive sub-formulas of the formula about which the game is played. Formulas asserted by the Opponent are negative sub-formulas of the formula about which the game is played. In each game atomic formulas asserted by the Proponent are both positives and negatives sub-formulas of the formula about which the game is played.

The proponent cannot extend the game for the formula $\neg\neg X \Rightarrow X$ by the defence-move $m_4 = (!, X)$ with $\phi(m_4) = m_1$ because of condition 6 in the definition of game (definition 3). The two games for the formula $X \vee Y \Rightarrow X \vee Y$ have a common prefix and they first differ on an Opponent's move. In one game the Opponent chooses to assert X in the defence move m_4 while in the other game the Opponent chooses to assert Y . In any case the Proponent wins.

The Proponent wins the game for the formula $X \vee Y \Rightarrow X$ even if this latter formula is not a tautology of first order intuitionistic logic. Note that if the Opponent had chosen to assert the formula Y instead of the formula X on move 3 then the Proponent would have had no chance of winning. In the game for the formula $\exists y \forall x R(x, y) \Rightarrow \forall x \exists y R(x, y)$ the player does not defend immediately against the attack move m_5 . Instead he delays his defence until the last move.

We systematize the observations on the games just made with some simple propositions

Proposition 1. *Let A be an arbitrary formula and \mathcal{G} and arbitrary game for A . If (\star, B) is an assertion move in \mathcal{G} then B is a Gentzen subformula of A*

Proof. By induction on the length of \mathcal{G} □

Proposition 2. *Let A be an arbitrary formula and $\mathcal{G} = \mathcal{G}'m^{\mathbf{P}}$ be a finite game for A . if \mathcal{G} is **won** by \mathbf{P} then m asserts an atomic Gentzen subformula B of A .*

Proof. Suppose, to reach a contradiction that the last move m of \mathcal{G} is not a defence move. Then it is an attack move of the form $(?, s)$. By definition of game there is a preceding **O**-move m_k that asserts some formula C , $\phi(m_n) = m_k$ and s is a question on C . Then the augmented sequence $\mathcal{G}m$ where $m = (!, D)$, $\phi(m) = m_n$ and D is an answer to the question s on the formula B , is a game. This contradicts the fact that \mathcal{G} is won by P . Thus m_n must be a defence move $(!, B)$. If B is not atomic we reach again a contradiction: in fact by adding a move $m = (?, s)$ to \mathcal{G} where s is a question on B we obtain a game. Thus we must conclude that B is an atomic formula \square

Proposition 3. *Let A be an arbitrary formula and \mathcal{G} an arbitrary game for A . If (\star, B) is an assertion move in \mathcal{G} that is a **P**-move (resp. an **O**-move) then B is a positive (resp. negative) gentzen-subformula of A .*

Proof. By induction on the length of \mathcal{G} \square

Proposition 4. *Let A be an arbitrary formula and \mathcal{G} an arbitrary game for A . If (\star, B) is an assertion move made by **P** and B is an atomic formula then B is both a negative and positive gentzen subformula of A .*

Proof. Direct consequence of Proposition 3 and of the condition 3 in the definition of game. \square

2.7 Strategies

As we have discussed in remark 1 the game for the formula $X \vee Y \Rightarrow X$ is won by the Proponent but by mere accident: If the Opponent had chosen to assert Y instead of X the Proponent would not have had a chance to win. This means that the Proponent *cannot* win a game on that formula *no matter how the Opponent choose* to act in the game. On the contrary the Proponent *can* win a game on the formula $X \vee Y \Rightarrow Y \vee X$ *no matter how* the Opponent choose to act in the Game. This means that there is a Proponent winning strategy for the formula $X \vee Y \Rightarrow Y \vee X$ and no winning strategy for the formula $X \vee Y \Rightarrow X$

Intuitively speaking a strategy for a game \mathcal{G} is a function. A function that specifies, at each moment of the game, which move a player must play according to the moves previously played (the history of the game). A strategy is *winning* when the player that follows the strategy wins whatever the history of the game is. As long as each move of the player that follows the strategy is determined by the strategy itself, it can be concluded that the game history varies only according to the moves of his opponent. We informally describe how a strategy should operate and then formalize this notion. Imagine being engaged in a game \mathcal{G} , that the last move of \mathcal{G} was played according to the strategy, and that it is now your opponent's turn to play. Your opponent could extend the game in different ways: for example if you are playing chess, you are white and you just made your first move by moving a pawn to a certain position of the chessboard, black can in turn move a pawn or move a horse. If you are playing according to the strategy, the strategy should tell you how to react against either type of

move. If black moves a pawn to $C6$ and you just moved your pawn to $C3$ then move the horse to $H3$. If black moves a horse to $H6$ and you just moved your pawn to $C3$ then move your pawn in $B4$. Therefore, a strategy can be viewed as tree in which each node is a move in the game, the moves of my opponent have at most one daughter, and my moves have as many daughters as there are available moves for my opponent. A tree can be seen as a prefix-closed set of sequence over an alphabet. Since our games are sequences over the alphabet of moves we can define strategies in the following manner:

Definition 5. A strategy \mathcal{S} for a formula A is a non empty prefix-closed set of games for A such that

1. if $\mathcal{G}m^{\mathbf{P}}$ and $\mathcal{G}n^{\mathbf{P}}$ belongs to \mathcal{S} then $m = n$
2. if $\mathcal{G} = \mathcal{G}'m^{\mathbf{P}} \in \mathcal{S}$ then $\mathcal{G}n^{\mathbf{O}} \in \mathcal{S}$ for all moves n legal for \mathcal{G}
3. if $\mathcal{G} = \mathcal{G}'m^{\mathbf{O}} \in \mathcal{S}$ and $m = (?, \exists)$ or $m = (?, \vee)$ then $\mathcal{G}n^{\mathbf{P}} \in \mathcal{S}$ and n is enabled by m

A strategy \mathcal{S} is winning if and only if every maximal sequence of the strategy is a game won by \mathbf{P} .

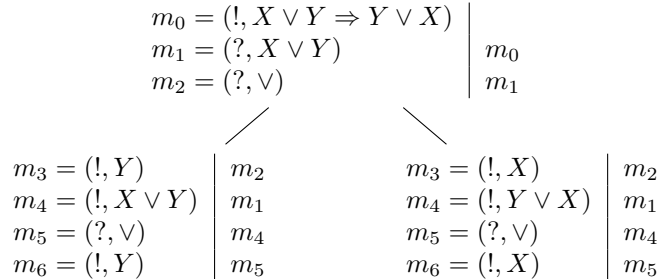
Condition 3 in the definition above precludes the Proponent to delay a defence against an existential attack. As a consequence a game like the one for the formula $\exists y \forall x R(x, y) \Rightarrow \forall x \exists y R(x, y)$ presented in subsection 2.6 cannot belong to a strategy.

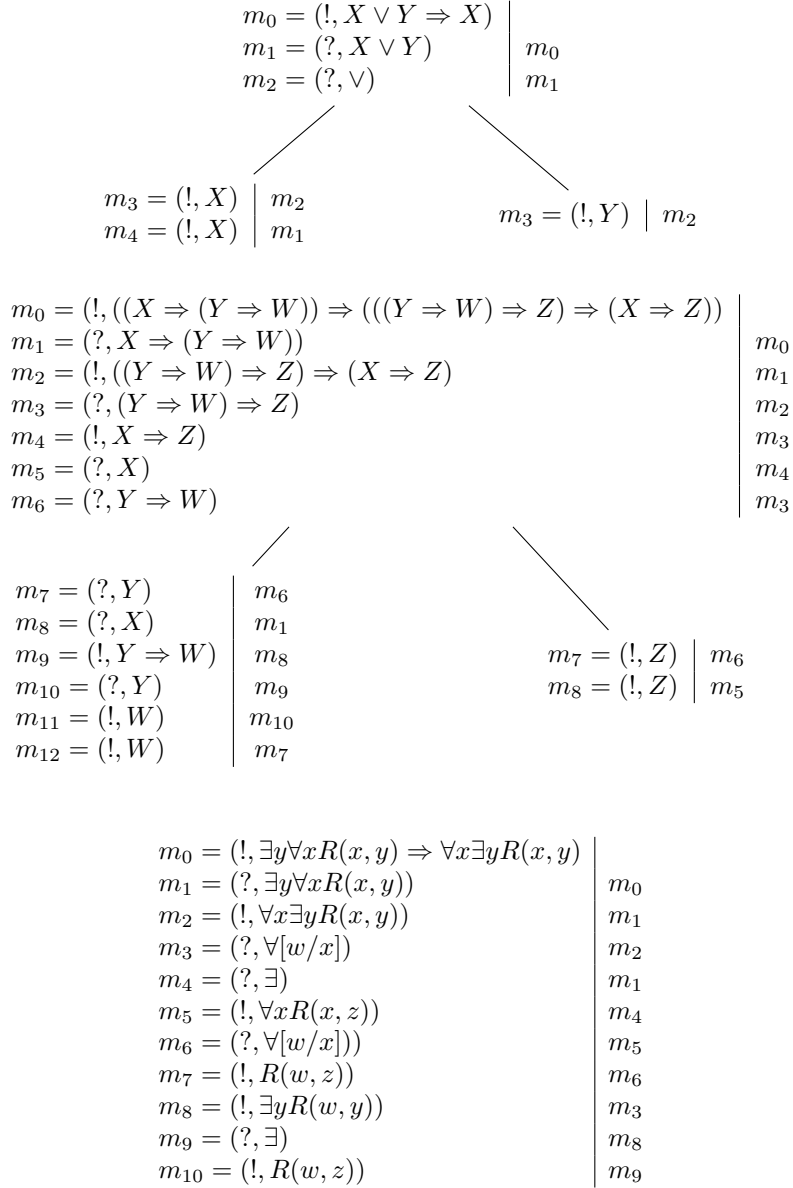
Proposition 5. Let \mathcal{S} be an arbitrary strategy and let \mathcal{G} be a game in \mathcal{S} that ends in \mathbf{P} -move. The family of moves $(m_{k_i})_{(i \in I)}$ such that $\mathcal{G}m_{k_i} \in \mathcal{S}$, is a finite family

Corollary 1. For all strategy \mathcal{S} , if \mathcal{S} is winning then \mathcal{S} is a finite tree.

2.8 Some examples of strategies

Let X, Y, Z, W be propositional variables and R be a binary predicate variable. A strategy will be represented as a tree of games.





3 The sequent calculus LJs

We now present the sequent calculus LJs. (Strategic LJ). LJs is a first order version of the calculus LJQ studied by Herbelin in his Phd dissertation (Herbelin, 1995). LJQ is a Kleene style sequent calculus: the active formula of a left introduction rule is present in the premises of the rule. LJQ differs from a Kleene-style calculus like GK_i (Troelstra & Schwichtenberg, 1996) because of a

Table 1: The LJs sequent calculus.

$$\begin{array}{c}
\frac{}{\Gamma, A \vdash \mathbf{A}} Id \qquad \frac{}{\perp, \Gamma \vdash C} \perp L \\
\\
\frac{\Gamma, A \vdash B}{\Gamma \vdash \mathbf{A} \Rightarrow \mathbf{B}} \Rightarrow R \qquad \frac{\Gamma, A \Rightarrow B \vdash A \quad \Gamma, A \Rightarrow B, B \vdash C}{\Gamma, \mathbf{A} \Rightarrow \mathbf{B} \vdash C} \Rightarrow L \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash B,}{\Gamma \vdash \mathbf{A} \wedge \mathbf{B}} \wedge R \qquad \frac{\Gamma, A_i, A_1 \wedge A_2 \vdash C}{\Gamma, \mathbf{A}_1 \wedge \mathbf{A}_2 \vdash C} \wedge L_i \\
\\
\frac{\Gamma \vdash A_i}{\Gamma \vdash \mathbf{A}_1 \vee \mathbf{A}_2} \vee R_i \qquad \frac{\Gamma, A \vee B, A \vdash C \quad \Gamma, A \vee B, B \vdash C}{\Gamma, \mathbf{A} \vee \mathbf{B} \vdash C} \vee L \\
\\
\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists \mathbf{x} \mathbf{A}} \exists R \qquad \frac{\Gamma, A[y/x], \exists x A \vdash C}{\Gamma, \exists \mathbf{x} \mathbf{A} \vdash C} \exists L \\
\\
\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall \mathbf{x} \mathbf{A}} \forall R \qquad \frac{\Gamma A[t/x], \forall x A \vdash C}{\Gamma, \forall \mathbf{x} \mathbf{A} \vdash C} \forall L
\end{array}$$

restriction on the use of the left introduction rule for the implication connective. LJs is obtained from LJQ by adding the quantifier rules and imposing a restriction on the use of the right introduction rule for the disjunction and the existential quantifier connectives.

Definition 6. The sequent calculus LJs is defined by the rules in Table 1. A sequent is an expression $\Gamma \vdash C$ where Γ is a finite (possibly empty) multiset of formulas and C is a formula. Greek upper-case letters Γ, Δ, \dots stand for multisets of formulas. In the Id -rule A is of the form $P(t_1, \dots, t_n)$ where P is a predicate variable with arity $n \geq 0$ and the t_i are terms. In the $\forall R$ and $\exists L$ rules the variable y does not occur in the conclusion sequent. In the $\Rightarrow L$ rule the left-side premise of the rule is obtained by an Id -rule or a right introduction rule. In the $\exists R$ -rule as well as in the $\vee R$ -rule the premise of the rule is obtained by an Id -rule or a right introduction rule. The bold formulas are called active formulas.

A derivation (or a proof) π of a sequent $\Gamma \vdash C$ in LJs is a tree of sequents

constructed according to the rules of LJs in which leaves are instances of *Id*-rules or $\perp L$ -rules, all sequents of the form $\Delta, \perp \vdash D$ are leaves and whose root, also called conclusion, is $\Gamma \vdash C$. The height of a derivation π is the number of nodes in its maximal branch minus 1.

A sequent $\Gamma \vdash C$ is said to be *derivable* or *provable* in the sequent calculus LJs whenever there exists a proof with conclusion $\Gamma \vdash C$.

We now state some propositions that are true about the sequent calculus LJs. Whereas the proof of one of the proposition is "standard" we will omit the detail of the proof.

Proposition 6 (Inversion). *For all formula A, B for all multiset of formulas Γ*

1. *If there is a derivation π of $\Gamma \vdash A \wedge B$ then there are derivation π_1 of $\Gamma \vdash A$ and π_2 of $\Gamma \vdash B$. Moreover the derivations π, π_1, π_2 have the same height*
2. *If there is a derivation π of $\Gamma \vdash A \Rightarrow B$ then there is a derivation π_1 of $\Gamma, A \vdash B$. Moreover π, π_1 have the same height*
3. *If there is a derivation π of $\Gamma \vdash \forall x A$ then there is a derivation π_1 of $\Gamma \vdash A[y/x]$ where y is a variable that does not appear in Γ . Moreover π, π_1 have the same height*
4. *If there is a derivation π of $\Gamma, A \vee B \vdash C$ then there are derivation π_1 of $\Gamma, A \vee B, A \vdash C$ and π_2 of $\Gamma, A \vee B, B \vdash C$. Moreover the derivations π, π_1, π_2 have the same height*
5. *If there is a derivation π of $\Gamma, \exists x A \vdash C$ then there is a derivation π_1 of $\Gamma, \exists x A, A[y/x] \vdash C$ where y is a variable that does not appear in Γ nor in C . Moreover π, π_1 have the same height*

Proof. By induction on the height of π □

Corollary 2. *For all formulas A for all multiset of formula Γ and Δ if the main connective of A is either \wedge, \Rightarrow or \forall and the sequent $\Gamma \vdash A, \Delta$ is provable then there is a derivation π of $\Gamma \vdash A, \Delta$ in which A is active.*

Proposition 7. *Contraction and weakening are height preserving admissible in LJs, i.e., for all formula A, C for all multiset of formulas Γ*

- *if there a derivation π of $\Gamma, A, A \vdash C$ and π has height n then there is a derivation ρ of $\Gamma, A \vdash C$ and the height of ρ is n*
- *if there is a derivation π of $\Gamma \vdash C$ and π has height n then there is a derivation ρ of $\Gamma, A \vdash C$ and the height of ρ is n*

Proof. By induction on the height of the derivation π of $\Gamma, A, A \vdash C$ (resp $\Gamma \vdash C$) □

Proposition 8. *For all formula A and all multiset of formulas Γ and Δ there is a derivation π of the sequent $\Gamma, A \vdash A, \Delta$. Moreover in the derivation π either the occurrence of A on the left of \vdash is active or the occurrence of A on the right of \vdash is active.*

Proof. Suppose that the proposition holds for all formulas B having height smaller than n and let A be a formula of height n . We will detail the proof only for the case in which $A := C \Rightarrow D$. By induction hypothesis there are proofs of ρ_1 of $\Gamma, C \Rightarrow D, C \vdash C$ and ρ_2 of $\Gamma, C \Rightarrow D, D \vdash D$. We have a problem if in the derivation ρ_1 the active occurrence of C is the one on the left of the turnstile, i.e. ρ_1 ends in a left introduction rule. If the main connective of C is \forall, \wedge or \Rightarrow then by corollary 2 we can conclude that there is a derivation ρ'_1 of $\Gamma, C \Rightarrow D, C \vdash C$ in which the occurrence of C on the right of the turnstile is active. Thus we can apply $\Rightarrow L$ on ρ'_1 and ρ_2 to obtain the wanted result. If the main connective of C is \exists or \vee then C has, respectively, the form $\exists x C_1$ or $C_1 \vee C_2$. Let us consider the second case. By induction hypothesis there are derivations of $\Gamma', C_1 \vee C_2, C_1 \vdash C_1$ and $\Gamma', C_1 \vee C_2, C_2 \vdash C_2$ where $\Gamma' = \Gamma, C_1 \vee C_2 \Rightarrow D$. First of all we construct the two following derivations.

$$\pi_A \left\{ \frac{\begin{array}{c} \vdots \\ \Gamma', C_1 \vee C_2, C_1 \vdash C_1 \end{array} \quad \begin{array}{c} \vdots \rho'_2 \\ \Gamma, C_1 \vee C_2 \Rightarrow D, D, C_1 \vdash D \end{array}}{\Gamma, C_1 \vee C_2, C_1 \vdash C_1 \vee C_2} \quad \Gamma, C_1 \vee C_2 \Rightarrow D, D, C_1 \vdash D}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vdash D}$$

$$\pi_B \left\{ \frac{\begin{array}{c} \vdots \\ \Gamma', C_1 \vee C_2, C_2 \vdash C_2 \end{array} \quad \begin{array}{c} \vdots \rho''_2 \\ \Gamma, C_1 \vee C_2 \Rightarrow D, D, C_2 \vdash D \end{array}}{\Gamma', C_1 \vee C_2, C_1 \vdash C_1 \vee C_2} \quad \Gamma, C_1 \vee C_2 \Rightarrow D, D, C_2 \vdash D}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_2 \vdash D}$$

Where ρ'_2 and ρ''_2 are obtained from ρ by weakening admissibility. We can now construct a derivation of $\Gamma, C_1 \vee C_2 \Rightarrow D \vdash C_1 \vee C_2 \Rightarrow D$

$$\frac{\begin{array}{c} \vdots \pi_A \\ \Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vdash D \end{array} \quad \begin{array}{c} \vdots \pi_B \\ \Gamma, C_1 \vee C_2 \Rightarrow D, C_2 \vdash D \end{array}}{\Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vee C_2 \vdash D} \quad \Gamma, C_1 \vee C_2 \Rightarrow D, C_1 \vee C_2 \vdash D}{\Gamma, C_1 \vee C_2 \Rightarrow D \vdash C_1 \vee C_2 \Rightarrow D}$$

□

Proposition 9. *For all formula A, B for all multiset of formulas, Γ , the sequents*

1. $\Gamma, A, A \Rightarrow B \vdash B$
2. $\Gamma, A \vdash A \vee B$

3. $\Gamma, B, \vdash A \vee B$

4. $\Gamma, A[y/x] \vdash \exists x A$

Are derivable in LJs. In (4) y is a variable that does not appears in $\Gamma, \exists x A$

Proof. It is an immediate consequence of propositions 8 and 6. □

Proposition 10. For all formulas A, C for all multiset of formulas Γ the cut rule

$$\frac{\Gamma \vdash A \quad \Gamma, A \vdash C}{\Gamma \vdash C} \text{ cut}$$

is admissible in LJs i.e.,

- If $\Gamma \vdash A$ is provable and $\Gamma, A \vdash C$ is provable then $\Gamma \vdash C$ is provable

Proof. By nested inductions on the depth of A , (the cut formula) the height of the derivation π of $\Gamma \vdash A$ and the height of the derivation ρ of $\Gamma, A \vdash C$. More precisely, we appeal to the induction hypothesis either with a strictly smaller cut formula, or with an identical cut formula and two derivations, one of which is strictly smaller while the other stays the same; the cut admissibility proof follows the usual path of case analysis on the active formula of π and ρ .

We detail two cases. Suppose that the cut-formula A is $B \Rightarrow C$. We have a derivation π of $\Gamma, B \Rightarrow C \vdash F$ and a derivation ρ of $\Gamma \vdash B \Rightarrow C$. Moreover suppose that in both derivation the cut formula $B \Rightarrow C$ is active. This means that π and ρ have the form

$$\frac{\begin{array}{c} \vdots \pi_1 \\ \Gamma, B \Rightarrow C \vdash B \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ C, B \Rightarrow C, \Gamma \vdash F \end{array}}{\Gamma, B \Rightarrow C \vdash F} \quad \frac{\begin{array}{c} \vdots \rho_1 \\ \Gamma, B \vdash C, \Delta \end{array}}{\Gamma \vdash B \Rightarrow C,}$$

We obtain a derivation of $\Gamma \vdash F$ as follows: we first construct a derivation σ_1 of $\Gamma \vdash C$, and a derivation σ_2 of $C, \Gamma \vdash F$, using the admissibility of the cut rule either on derivations that are smaller then ρ or π or on formulas that are smaller then $B \Rightarrow C$. In what follows ρ' is obtained from ρ by height preserving admissibility of the weakening rule. For the sake of clarity we underline the cut-formula of each cut-rule instance.

$$\left. \begin{array}{c} \begin{array}{c} \vdots \pi_1 \\ \Gamma, \underline{B \Rightarrow C} \vdash B, \end{array} \quad \begin{array}{c} \vdots \rho \\ \Gamma \vdash \underline{B \Rightarrow C}, \end{array} \\ \hline \Gamma \vdash \underline{B} \quad \Gamma, \underline{B} \vdash C \end{array} \right\} \sigma_1$$

$$\left. \frac{\begin{array}{c} \vdots \pi_2 \\ C, \Gamma, \underline{B} \Rightarrow C \vdash F \end{array} \quad \begin{array}{c} \vdots \rho' \\ C, \Gamma \vdash \underline{B} \Rightarrow C \end{array}}{C, \Gamma \vdash F} \right\} \sigma_2$$

We then put together σ_1 and σ_2 by using an instance of the cut rule with cut-formula C . Since C is a formula strictly smaller than $B \Rightarrow C$ this application of the cut-rule is allowed by the induction hypothesis.

$$\frac{\begin{array}{c} \vdots \sigma_1 \\ \Gamma \vdash \underline{C} \end{array} \quad \begin{array}{c} \vdots \sigma_2 \\ \Gamma, \underline{C} \vdash F \end{array}}{\Gamma \vdash F}$$

Now suppose that $B \Rightarrow C$ is not active in one of the two derivations π and ρ . Suppose it is not active in ρ e.g., $\Gamma = \Gamma', D \vee E$ and ρ has the form

$$\frac{\begin{array}{c} \vdots \rho_1 \\ \Gamma', D \vee E, D \vdash B \Rightarrow C \end{array} \quad \begin{array}{c} \vdots \rho_2 \\ \Gamma', D \vee E, E \vdash B \Rightarrow C \end{array}}{\Gamma', D \vee E \vdash B \Rightarrow C}$$

We want to obtain a derivation of the sequent $\Gamma', D \vee E \vdash F$. A derivation of this sequent can be constructed as follows. We first construct two derivations σ_1 and σ_2 of the sequents $\Gamma', D \vdash F$ and $\Gamma', E \vdash F$. In what follows π' and π'' are obtained from π by height-preserving admissibility of weakening

$$\left. \frac{\begin{array}{c} \vdots \pi' \\ \Gamma', D, D \vee E, \underline{B} \Rightarrow C \vdash F \end{array} \quad \begin{array}{c} \vdots \rho'_1 \\ \Gamma', D, D \vee E \vdash \underline{B} \Rightarrow C \end{array}}{\Gamma', D, D \vee E \vdash F} \right\} \sigma_1$$

$$\left. \frac{\begin{array}{c} \vdots \pi'' \\ \Gamma', E, D \vee E, \underline{B} \Rightarrow C \vdash F \end{array} \quad \begin{array}{c} \vdots \rho'_2 \\ \Gamma', E, D \vee E \vdash \underline{B} \Rightarrow C \end{array}}{\Gamma', E, D \vee E \vdash F} \right\} \sigma_2$$

We put together the two derivations σ_1 and σ_2 using a $\vee L$ rule and we obtain the wanted derivation of the sequent $\Gamma', D \vee E \vdash F$

$$\frac{\begin{array}{c} \vdots \sigma_1 \\ \Gamma', D, D \vee E \vdash F \end{array} \quad \begin{array}{c} \vdots \sigma_2 \\ \Gamma', E, D \vee E \vdash F \end{array}}{\Gamma', D \vee E \vdash F} \vee L$$

□

We now prove that LJs is sound and complete for intuitionistic logic. In order to prove this fact we show that a sequent $\Gamma \vdash C$ is provable in LJs if and only if, it is provable in the sequent calculus system GK_i (Troelstra & Schwichtenberg, 1996). GK_i is sound and complete for intuitionistic logic, and it is obtained from LJs by dropping the restriction on the left introduction rule for the implication, right introduction rule for disjunction and right introduction rule for existential quantification of LJs.

Proposition 11. *For all formula C , for all multiset of formulas Γ , the sequent $\Gamma \vdash C$ is provable in LJs if and only if it is provable in GK_i*

Proof. Each derivation in LJs is a derivation in GK_i, thus one side of the proof is for free.

For the other side, suppose that for each derivation ρ in GK_i with height n and conclusion $\Gamma' \vdash C'$ there is a derivation ρ' in LJs having the same conclusion. Let π be a derivation of $\Gamma \vdash C$ in GK_i having height $n + 1$ and let R be the last rule application of π . If R is not $\Rightarrow L$, $\exists R$ nor $\forall R$ we have just to apply the induction hypothesis on the premises of R .

If R is $\Rightarrow L$ then the conclusion of π is $\Gamma, A \Rightarrow B \vdash C$ and, by induction hypothesis we have an LJs derivation ρ_1 with conclusion $\Gamma, A \Rightarrow B \vdash A$ and another LJs derivation ρ_2 with conclusion $B, \Gamma, A \Rightarrow B \vdash C$. We can construct a LJs derivation of the sequent $\Gamma, A \Rightarrow B \vdash C$ as follows

$$\frac{\frac{\frac{\vdots \rho_1}{\Gamma, A \Rightarrow B \vdash A} \quad \text{cut} \quad \frac{\frac{\frac{\vdots \pi}{A, \Gamma, A \Rightarrow B \vdash B} \quad \frac{\vdots \rho'_2}{B, \Gamma, A, A \Rightarrow B \vdash C}}{A, \Gamma, A \Rightarrow B \vdash C}}{\Gamma, A \Rightarrow B \vdash C} \text{cut}}{\Gamma, A \Rightarrow B \vdash C} \text{cut}}$$

Where the derivation π exists by proposition 9, and ρ'_2 is obtained from ρ_2 by admissibility of weakening

If R is $\exists R$ or $\forall R$ then, the conclusion of π is of the form $\Gamma \vdash \exists x A$ or, respectively, $\Gamma \vdash A_1 \vee A_2$. By induction hypothesis, we have a LJs derivation ρ of its premise $\Gamma \vdash C$. Where C is either $A[t/x]$ for some term t , or A_i ($i \in \{1, 2\}$). We treat only the \exists case.

$$\frac{\frac{\frac{\vdots \rho}{\Gamma \vdash A[t/x]} \quad \text{cut} \quad \frac{\vdots \pi}{\Gamma, A[t/x] \vdash \exists x A}}{\Gamma \vdash \exists x A} \text{cut}}$$

Where $\pi = \pi'[t/y]$, and π' is a proof of $\Gamma, A[y/x] \vdash \exists x A$ in which y does not appear in $\Gamma, \exists x A$; the derivation π' exists in virtue of proposition 9 \square

4 From strategies to derivations

Let \mathcal{S} be a strategy for a formula F and let \mathcal{G} be a game in \mathcal{S} . We define the **O**-sequence $\mathcal{G}|_{\mathbf{O}}$ of \mathcal{G} to be the subsequence of \mathcal{G} obtained by forgetting all its **P**-moves, i.e, if $\mathcal{G} = m_0, m_1, \dots, m_n$ the **O**-sequence of \mathcal{G} is m_1, \dots, m_{n-1} . We define the **O**-tree $\mathcal{S}|_{\mathbf{O}}$ of a strategy \mathcal{S} to be the prefix closed set of sequence

$$\mathcal{S}|_{\mathbf{O}} = \{\mathcal{G}|_{\mathbf{O}} \mid \mathcal{G} \in \mathcal{S}\}$$

Let \mathcal{S} be a strategy. We define a function Φ from $\mathcal{S}|_{\mathbf{O}}$ to a tree of sequent τ . The function Φ associate a sequent $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ to each $\mathcal{G}|_{\mathbf{O}}$ in $\mathcal{S}|_{\mathbf{O}}$. Let us denote the empty sequence by ϵ

- if $\mathcal{G}|_{\mathbf{O}} = \epsilon$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \emptyset$ and $C_{\mathcal{G}|_{\mathbf{O}}} = F$
- if $\mathcal{G}|_{\mathbf{O}} = \mathcal{G}'|_{\mathbf{O}}(!, A)$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \Gamma_{\mathcal{G}'|_{\mathbf{O}}}, A$ and $C_{\mathcal{G}|_{\mathbf{O}}} = C_{\mathcal{G}'|_{\mathbf{O}}}$
- if $\mathcal{G}|_{\mathbf{O}} = \mathcal{G}'|_{\mathbf{O}}(?, A)$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \Gamma_{\mathcal{G}'|_{\mathbf{O}}}, A$ and $C_{\mathcal{G}'|_{\mathbf{O}}}$ have the form $A \Rightarrow B$ we put $C_{\mathcal{G}|_{\mathbf{O}}} = B$
- if $\mathcal{G}|_{\mathbf{O}} = \mathcal{G}'|_{\mathbf{O}}(?, \wedge_1)$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \Gamma_{\mathcal{G}'|_{\mathbf{O}}}$ and $C_{\mathcal{G}'|_{\mathbf{O}}}$ have the form $A \wedge B$ we put $C_{\mathcal{G}|_{\mathbf{O}}} = A$
- if $\mathcal{G}|_{\mathbf{O}} = \mathcal{G}'|_{\mathbf{O}}(?, \wedge_2)$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \Gamma_{\mathcal{G}'|_{\mathbf{O}}}$ and $C_{\mathcal{G}'|_{\mathbf{O}}}$ have the form $A \wedge B$ we put $C_{\mathcal{G}|_{\mathbf{O}}} = B$
- if $\mathcal{G}|_{\mathbf{O}} = \mathcal{G}'|_{\mathbf{O}}(?, \forall[w/x])$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \Gamma_{\mathcal{G}'|_{\mathbf{O}}}$ and $C_{\mathcal{G}'|_{\mathbf{O}}}$ have the form $\forall x A$ we put $C_{\mathcal{G}|_{\mathbf{O}}} = A[w/x]$
- if $\mathcal{G}|_{\mathbf{O}} = \mathcal{G}'|_{\mathbf{O}}(?, \vee)$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \Gamma_{\mathcal{G}'|_{\mathbf{O}}}$ and $C_{\mathcal{G}'|_{\mathbf{O}}}$ have the form $A_1 \vee A_2$ we put $C_{\mathcal{G}|_{\mathbf{O}}} = A_i$ where A_i is the formula asserted by the move $m \in \mathcal{G}$, \mathcal{G} in \mathcal{S} such that $\phi(m) = (?, \vee)$.
- if $\mathcal{G}|_{\mathbf{O}} = \mathcal{G}'|_{\mathbf{O}}(?, \exists)$ then $\Gamma_{\mathcal{G}|_{\mathbf{O}}} = \Gamma_{\mathcal{G}'|_{\mathbf{O}}}$ and $C_{\mathcal{G}'|_{\mathbf{O}}}$ have the form $\exists x A$ we put $C_{\mathcal{G}|_{\mathbf{O}}} = A[t/x]$ where $A[t/x]$ is the formula asserted by the move $m \in \mathcal{G}$, \mathcal{G} in \mathcal{S} such that $\phi(m) = (?, \exists)$.

We now prove that given a winning strategy \mathcal{S} , $\Phi(\mathcal{S}|_{\mathbf{O}})$ is *almost* a derivation in LJs; all leaves of $\Phi(\mathcal{S}|_{\mathbf{O}})$ are instances of *Id* rules or $\perp L$ rules (proposition 13) and that $\Phi(\mathcal{S}|_{\mathbf{O}})$ respect the variable condition of the $\forall R$ and $\exists L$ rules of LJs (proposition 14)

Proposition 12. *Let \mathcal{S} be an arbitrary winning strategy and \mathcal{G} be an arbitrary game in \mathcal{S} . If \mathcal{G} ends in a **P** defence move that asserts a formula A then the sequent $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ associated to the **O**-restriction $\mathcal{G}|_{\mathbf{O}}$ of \mathcal{G} by the function Φ is of the form $\Gamma \vdash A$*

Proof. By induction on the length of \mathcal{G} □

Proposition 13. *Let \mathcal{S} , be an arbitrary winning strategy for a formula A and \mathcal{G} an arbitrary maximal branch in \mathcal{S} .*

1. If $\mathcal{G} = \mathcal{G}'m^{\mathbf{O}}$ then m asserts \perp and the sequent $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ associated to the \mathbf{O} -restriction $\mathcal{G}|_{\mathbf{O}}$ of \mathcal{G} by the function Φ is of the form $\Gamma, \perp \vdash B$ with B gentzen-subformula of A
2. if $\mathcal{G} = \mathcal{G}'m^{\mathbf{P}}$ then $m = (!, C)$ with C atomic gentzen subformula of A and the sequent $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ associated to the \mathbf{O} -restriction $\mathcal{G}|_{\mathbf{O}}$ of \mathcal{G} by the function Φ is of the form $\Gamma', C \vdash C$

Proof. (1) is a direct consequence of definition 4 and of the definition of the function Φ . (2) derives from condition 3 in definition 3 and proposition 12 □

Proposition 14. *let \mathcal{S} be an arbitrary winning strategy and \mathcal{G} be a game in \mathcal{S} . Suppose that \mathcal{G} ends in \mathbf{O} -move that is either*

1. an attack against a universal quantifier $(?, \forall[w/x])$
2. or a defence against an existential attack $(!, A[w/x])$.

Then the variable w does not appear in the sequent associated by the function Φ to the \mathbf{O} -restriction $\mathcal{G}'|_{\mathbf{O}}$ of the proper prefix \mathcal{G}' of \mathcal{G}

Proof. Both (1) and (2) are granted by the conditions 4 and 5 in definition 3 and by condition 3 in definition 5 □

We are now ready to prove the main result of this section. We have just shown that we can associate a tree of sequents with each winning strategy. In addition, we have shown that the above mentioned sequent tree is "almost" a proof in LJs: all its leaves are instance of *id* rules or \perp rules of LJs and it respects the variable restriction on the $\forall R$ and $\exists L$ rules of LJs.

Theorem 1. *Let \mathcal{S} an arbitrary winning strategy and let $\mathcal{S}|_{\mathbf{O}}$ by its \mathbf{O} -tree. To each sequence of \mathbf{O} -moves $\mathcal{G}|_{\mathbf{O}}$ in $\mathcal{S}|_{\mathbf{O}}$ we can associate a derivation $\pi_{\mathcal{G}|_{\mathbf{O}}}$ of $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$, where $\Gamma_{\mathcal{G}|_{\mathbf{O}}} \vdash C_{\mathcal{G}|_{\mathbf{O}}}$ is the sequent associated by the function Φ to $\mathcal{G}|_{\mathbf{O}}$*

Proof. Let $\mathcal{G}|_{\mathbf{O}}$ be an arbitrary element of $\mathcal{S}|_{\mathbf{O}}$. Suppose that the induction hypothesis holds for each suffix $\mathcal{G}'|_{\mathbf{O}}$ of $\mathcal{G}|_{\mathbf{O}}$ in $\mathcal{S}|_{\mathbf{O}}$. We consider the last move \mathbf{P} -move m_{2n} of the game $\mathcal{G} \in \mathcal{S}$ such that the \mathbf{O} -restriction of \mathcal{G} is $\mathcal{G}|_{\mathbf{O}}$.

We only prove some of the cases that are not straightforward

1. if m_{2n} is a defence move $(!, \exists xA)$ then there are many cases, depending on the form of A . We treat only two cases
 - if A is atomic then $\mathcal{G}|_{\mathbf{O}}(?, \exists)$ is maximal in $\mathcal{S}|_{\mathbf{O}}$; we associate with $\mathcal{G}|_{\mathbf{O}}$ the following derivation in which A is active.

$$\frac{\overline{\Gamma, A \vdash A} \text{ Id}}{\Gamma, A \vdash \exists xA}$$

- if $A = B \vee C$ then $\mathcal{G}|_{\mathbf{O}'} = \mathcal{G}|_{\mathbf{O}}(? , \exists)(? , \vee) \in \mathcal{S}|_{\mathbf{O}}$ because of condition 3 in definition 5. This means in particular that the formula $(B \vee C)[t/x]$ is active in the derivation that we associate with $\mathcal{G}|_{\mathbf{O}}$

$$\frac{\begin{array}{c} \vdots \pi_{\mathcal{G}|_{\mathbf{O}'}} \\ \Gamma \vdash (A \vee B)[t/x] \end{array}}{\Gamma \vdash \exists x(A \vee B)}$$

2. if m_{2_n} is an attack $(?, A)$ on the assertion $A \Rightarrow C$, then there are many cases depending on the form of A . We again only treat two cases

- if A is atomic then the immediate suffix of $\mathcal{G}|_{\mathbf{O}}$ is $\mathcal{G}|_{\mathbf{O}}(!, C)$ for which the proposition hold by hypothesis. We associate it with the following derivation.

$$\frac{\frac{\Gamma, A \Rightarrow C, A \vdash A}{\Gamma, A \Rightarrow C \vdash F} \text{Id} \quad \begin{array}{c} \vdots \pi_{\mathcal{G}|_{\mathbf{O}}(!, C)} \\ \Gamma, A \Rightarrow C, C \vdash F \end{array}}{\Gamma, A \Rightarrow C \vdash F} \Rightarrow L$$

- if $A = (A_1 \Rightarrow A_2)$ then $\mathcal{G}|_{\mathbf{O}}$ has two immediate suffixes: $\mathcal{G}|_{\mathbf{O}}(? , A_1)$ and $\mathcal{G}|_{\mathbf{O}}(!, C)$, for which the proposition holds by hypothesis. We associate the following derivation to $\mathcal{G}|_{\mathbf{O}}$.

$$\frac{\frac{\begin{array}{c} \vdots \pi_{\mathcal{G}|_{\mathbf{O}}(? , A_1)} \\ \Gamma, (A_1 \Rightarrow A_2) \Rightarrow C, A_1 \vdash A_2 \end{array}}{\Gamma, (A_1 \Rightarrow A_2) \Rightarrow C \vdash A_1 \Rightarrow A_2} \Rightarrow R \quad \begin{array}{c} \vdots \pi_{\mathcal{G}|_{\mathbf{O}}(!, C)} \\ \Gamma, (A_1 \Rightarrow A_2) \Rightarrow C, C \vdash F \end{array}}{\Gamma, (A_1 \Rightarrow A_2) \Rightarrow C \vdash F} \Rightarrow L$$

□

5 From derivations to strategies

Turning a derivation π of a formula F into a winning strategy \mathcal{S} for F is easier. To do so we describe a procedure, that we call *p2s* (from a Proof in LJs to a strategy). The procedure *p2s* explore the proofs π starting from the root and proceeding by level order traversal. The order of traversal of daughters is irrelevant. The procedure associate to π a prefix closed set of games for the formula F .

Theorem 2. *Let F be an arbitrary formula and π be an arbitrary derivation of F in LJs. There is a function *p2s* such that *p2s*(π) is a winning strategy \mathcal{S} for F*

Proof. Let x be an arbitrary node the proof π of the formula F having depth n , and let $\Gamma \vdash C$ be the sequent that decorates x . Suppose that

1. the branch $r = x_o, \dots x_n = x$ of the derivation from the root r of π to x is already associated with a prefix closed set S_x of games for the formula F . Each \mathcal{G} in S_x in which the last move of \mathbf{P} is the assertion of a complex formula or an attack move ends in a \mathbf{O} -move
2. For each formula B in Γ there is an \mathbf{O} -assertion move (\star, B) in some game G in S_x
3. The prefix closed set S_x is a strategy for F

The prefix closed set of games S_{a_1} associated with a_1 where a_1 is any daughter of x is defined as follows:

1. if a_1 is obtained by an identity rule $\Gamma, A \vdash A$ then $S_{a_1} = S_x \cup \{\mathcal{G}(!, A)\}$ where A is the active formula of the identity rule and \mathcal{G} is a maximal game in S_x such that $(!, A)$ is legal for \mathcal{G} .
2. If a_1 is labelled with a sequent obtained from a right introduction rule with active formula A .
 - If A is not a conjunction nor a universal formula then $S_{a_1} = S_x \cup \{\mathcal{G}(!, A)(?, s)\}$ where \mathcal{G} is a maximal game in S_x such that $(!, A)$ is legal for \mathcal{G} and $(?, s)$ is an attack move such that s is the unique question on A
 - if A is $\forall xA'$ then S_{a_1} is $S_x \cup \{\mathcal{G}(!, \forall xA), (?, \forall[w/x])\}$ where \mathcal{G} is a maximal game in S_x such that $(!, A)$ is legal for \mathcal{G} and the variable w in $(?, \forall[w/x])$ is the variable that appears in the premise of a_1 but not in a_1 .
 - if A is $B \wedge C$ then $S_{a_1} = S_x \cup \{\mathcal{G}(!, B \wedge C)(?, \wedge_1)\} \cup \{\mathcal{G}(!, B \wedge C)(?, \wedge_2)\}$ where \mathcal{G} is a maximal game in S_x such that the \mathbf{P} -move $(!, B \wedge C)$ is legal for \mathcal{G}
3. If a_1 is labelled with a sequent obtained from a left introduction rule with active formula A .
 - if A is $B_1 \wedge B_2$ then $S_{a_1} = S_x \cup \{\mathcal{G}(?, \wedge_i)(!, B_i)\}$ where B_i is the direct sub-formula of $B_1 \wedge B_2$ that appears in the premise of a_1 but not in a_1 and \mathcal{G} is a maximal game in S_x such that the \mathbf{P} -move $(?, \wedge_i)$ is legal for \mathcal{G} .
 - if A is $\forall xB$ then $S_{a_1} = S_x \cup \{\mathcal{G}(?, \forall[t/x])(!, B[t/x])\}$ where $B[t/x]$ is the formula occurrence that appears in the premise of a_1 but not in a_1 and \mathcal{G} is a maximal game in S_x such that the \mathbf{P} -move $(?, \forall[t/x])$ is legal for \mathcal{G} .
 - If A is $B \vee C$ then $S_{a_1} = S_x \cup \{\mathcal{G}(?, \vee)(!, B)\} \cup \{\mathcal{G}(?, \vee)(!, C)\}$ where \mathcal{G} is a maximal game in S_x such that the \mathbf{P} -move $(?, \vee)$ is legal for \mathcal{G} .

- if A is $\exists xB$, $S_{a_1} = S_x \cup \{\mathcal{G}(?, \exists)(!, B[w/x])\}$ where \mathcal{G} is a maximal game in S_x such that the **P**-move $(?, \exists)$ is legal for \mathcal{G} and $B[w/x]$ is a formula that appears in the premise of a_1 but not in a_1 .
- if A is $B \Rightarrow C$, $S_{a_1} = S_x \cup \{\mathcal{G}, (? , B), (? , q_1)\} \cup \dots \cup \{\mathcal{G}, (? , B), (? , q_n)\} \cup \{\mathcal{G}, (? , B), (! , C)\}$. Where \mathcal{G} is a maximal game in S_x such that the **P**-move $(?, B)$ is legal for \mathcal{G} each q_i is a question on B .

Remark that the procedure jumps from a node v of the proof-tree obtained by a $\Rightarrow L$ rule to the daughter of the daughter of v . It is easy to check that conditions 1,2 and 3 are respected after the application of the procedure. \square

6 Conclusion and Future Works

We proved that there is a natural correspondence between formal E-strategies and derivations of a complete sequent calculus for first order intuitionistic logic. We hope that the simplicity of our approach will help other researchers to appreciate more the dialogical logic approach.

The attentive reader has surely remarked that the sequent calculus LJs obeys a *focusing principle* (Andreoli, 1992): whenever we apply (bottom-up) an $\exists R$ or an $\forall R$ rule over a sequent $\Gamma \vdash C$ we are obliged to apply right-rules until an implication a conjunction or an universally quantified formula is at place of C . It is quite surprising that formal E-strategies naturally corresponds to this type of calculus. The fact that Felscher himself did not notice such correspondence is explained by remarking that focusing was not known at the time he worked on dialogical logic.

Concerning the future works: given a winning strategy for A and a winning strategy for $A \Rightarrow B$ we can conclude that there is a winning strategy for B . Simply because the cut-rule is admissible in LJs and we can thus translate the winning strategies for $A, A \Rightarrow B$ into two derivations and use the cut rule to obtain a derivation of B . However it would be much more interesting to define an analogous of the cut-rule directly on strategies. We think that this could be obtained by relaxing the definition of game in order to let the proponent assert, at any point of the game, an arbitrary formula C . After the Proponent assertion of C , the Opponent can continue the game by either attacking C or by asserting C in turn. The cut-elimination theorem for strategies would be obtained by proving that the set of formulas admitting winning strategies that contains this kind of games is equal to class of formulas admitting 'regular' winning strategies.

Dialogic logic heavily influenced the game semantics approach (Hyland, 1997; Abramsky, Jagadeesan, & Malacaria, 2000). Game semantics has been used in order to give denotational models of many programming languages. In particular there is a natural correspondence between winning strategies in the Hyland-Ong setting of game semantics and the simply-typed λ -calculus. Despite the fact that dialogical logic and game semantics share a lot of point in common no

one, at least to our knowledge, has tried to investigate the question in detail. We think that shedding light on this subject could be fruitful for the development of both game semantics and dialogical logic.

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