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Kalikow decomposition for counting processes with stochastic intensity *

Tien Cuong PHI

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Abstract

We propose a new Kalikow decomposition and the corresponding Perfect Simulation algorithm for continuous time multivariate counting processes, on potentially infinite networks. We prove the existence of such a decomposition in various cases. This decomposition is not unique and we discuss the choice of the decomposition in terms of algorithmic efficiency. We apply these methods on several examples: linear Hawkes process, age dependent Hawkes process, exponential Hawkes process.

Keywords: Kalikow decomposition, counting process, Hawkes process, Perfect Simulation algorithm

MSC 2010 subject classification: 60G55, 60K35

1 Introduction

Multivariate point (or counting) processes on networks have been used to model a large variety of situations : social networks [14], financial prices [2], genomics [22], etc. One of the most complex network models comes from neuroscience where the number of nodes can be as large as billions [17, 16, 23]. Several counting process models have been used to model such large networks: Hawkes processes [12, 11], Galves-Löcherbach models [10] etc. If the simulation of such large and potentially infinite networks is of fundamental importance in computational neuroscience [16, 17], also the existence of such processes in stationary regime and within a potentially infinite network draws a lot of interest (see [10, 19, 13] in discrete or continuous time).

Kalikow decompositions [15] have been introduced and mainly used in discrete time. Such a decomposition provides a decomposition of the transition probabilities into a mixture of more elementary transitions. The whole idea is that even if the process is complex (infinite memory, infinite network), the elementary transitions look only at what happens in a finite neighborhood in time and space. Once the decomposition is proved for a given process, this can be

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used to write a Perfect Simulation algorithm [10, 13, 19]. Here the word "perfect" refers to the fact that it is possible in finite time to simulate what happens on one of the nodes of the potentially infinite network in a stationary regime. The existence of such an algorithm guarantees in particular the existence of the process in stationary regime. Most of the papers referring to Kalikow decomposition and Perfect Simulation are theoretical and aim at proving the existence of such processes on infinite networks in stationary regime [13, 10].

In the present paper, we propose to go from discrete to continuous time. Therefore, we will decompose conditional intensities rather than transition probabilities. This leads to serious difficulties that usually prevent a more practical application of the Perfect Simulation algorithm. Indeed, up to our knowledge, the only work dealing with continuous time counting processes, is the one by Hodara and Löcherbach [13]. Their decomposition is constructed under the assumption that there is a dominating Poisson process on each of the nodes, from which the points of the processes under interest can be thinned by rejection sampling (see also [18] for another use of thinning in simulation of counting processes). To prove the existence of a Kalikow decomposition and go back to a more classic discrete time setting, the authors need to freeze the dominating Poisson process, leading to a mixture, in the Kalikow decomposition, that depends on the realization of the dominating Poisson process. Such a mixture is not accessible in practice, and this prevents the use of their Perfect Simulation algorithm for more concrete purposes than mere existence.

More recently, in a previous computational article [17], we have used another type of Kalikow decomposition, which does not depend on the dominating Poisson process. This leads to a Perfect Simulation algorithm, which can be used as a concrete way for Computational Neuroscience to simulate neuronal networks as an open physical system, where we do not need to simulate the whole network to simulate what happens in a small part [17].

In the present work, we want to go further, by proposing an even more general Kalikow decomposition, which does not assume the existence of a dominating Poisson process at all. We also prove (and this is not done in [17]) that such decomposition exists on various interesting examples, even if these decompositions are not unique. Finally we propose a corresponding Perfect Simulation algorithm and we discuss its efficiency with respect to the decomposition that is used.

The paper is organized as follows. In Section 2, we introduce the basic notation and give the precise definition of a Kalikow decomposition. In Section 3, we present two methods to obtain a Kalikow decomposition for a counting process having stochastic intensity. The first method is very general and based on adequate choices of neighborhoods and weights. As an application, we study a classical example, the linear Hawkes process [12, 11], and a very recent and promising process, the age dependent Hawkes process [20]. Though very simple cases of Hawkes processes are treated, it is worth to note that our method can be generalized very easily. The second method exists only for Hawkes processes and is based on Taylor expansion. Finally, in Section 4, we present a modified Perfect Simulation algorithm based on the Kalikow decomposition written in

Section 3, and we discuss the efficiency of the algorithm with respect to the Kalikow decomposition.

2 Notation and Kalikow decomposition

2.1 Notation and Definition

We start this section by recalling the definition of counting processes and stochastic intensity. We refer the reader to [4] and [9] for more complete statements.

Let \mathbf{I} be a countable index set. A counting process $Z^i, i \in \mathbf{I}$, can be described by its sequence of jump times in \mathbb{R} , $(T_n^i)_{n \in \mathbb{Z}}$ [4]. Consider $(\mathcal{F}_t)_{t \in \mathbb{R}}$ the past filtration of the process $Z = (Z^i)_{i \in \mathbf{I}}$:

$$\mathcal{F}_t = \sigma(Z_s^i, i \in \mathbf{I}, s \leq t).$$

Since a point process is fully characterized by its arrival times [4], we can denote by \mathcal{X} the canonical path space of Z :

$$\mathcal{X} = \{(\{t_n^i\}_{n \in \mathbb{Z}})_{i \in \mathbf{I}} \text{ such that } \forall n, i, \quad t_n^i < t_{n+1}^i \text{ and } t_0^i \leq 0 < t_1^i\},$$

where $\{t_n^i\}_{n \in \mathbb{Z}}$ denotes a possible realization of $(T_n^i)_{n \in \mathbb{Z}}$. Denote \mathcal{X}_t the canonical path space of Z before time t :

$$\mathcal{X}_t = \mathcal{X} \cap (-\infty, t)^I.$$

A past configuration x_t is an element of \mathcal{X}_t which is a realization of arrival times of Z before t .

Under suitable assumptions, the evolution of the point process Z^i with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}}$ is fully characterized by its stochastic intensity which depends on the past configuration, see Proposition 7.2.IV of [9]. Hence, in this paper, for any $x_t \in \mathcal{X}_t$, given that the past before time t is x_t , we denote by $\phi_{i,t}(x_t)$ the corresponding stochastic intensity of the process Z^i at time t for any $i \in \mathbf{I}$. More precisely, for any $x_t \in \mathcal{X}_t$, we have

$$\mathbb{P}(Z^i \text{ has jump in } [t, t + dt) \mid \text{past before time } t = x_t) = \phi_{i,t}(x_t)dt.$$

Together with the path space \mathcal{X}_t , we denote \mathbf{V}_t a countable collection of finite space-time neighborhoods v_t , in which each neighborhood v_t is a Borel subset of $I \times (-\infty, t)$. More precisely, we call v_t a finite neighborhood if there exists a finite subset $J \subset I$ and a finite interval $[a, b]$ such that:

$$v_t \subset J \times [a, b].$$

For convenience, we denote \mathcal{X} the canonical path space of Z before time 0 instead of \mathcal{X}_0 . In addition, a past configuration before 0 is denoted by x and the intensity at time 0 is $\phi_i(x)$ instead of $\phi_{i,0}(x_0)$. Moreover, if $x = (\{t_n^i\}_{n \in \mathbb{Z}_-})_{i \in \mathbf{I}} \in \mathcal{X}$, then for any $i \in \mathbf{I}$, we denote the point measure associated to index i by

$$dx_s^i = \sum_{m \in \mathbb{Z}^-} \delta_{t_m^i}(ds)$$

in which $\delta_t(\cdot)$ is a Dirac measure at t . Throughout this article, without further mentioning, the integral \int_a^b stands for $\int_{[a,b]}$ with $a, b \in \mathbb{R}$.

Let $x_t = (\{t_n^i\}_{n \in \mathbb{Z}})_{i \in \mathbf{I}} \in \mathcal{X}_t$ and denote $x_t^{\leftarrow t} := (\{t_n^i - t\}_n)_i \in \mathcal{X}$ the shifted configuration. In this paper, if we do not mention otherwise, we always consider time homogeneous point processes, which in our setting can be defined as follows.

Definition 1. For a given $i \in \mathbf{I}$, a counting process Z^i with stochastic intensity $(\phi_{i,t}(x_t))_{t \in \mathbb{R}}$ is said to be time homogeneous if

$$\phi_{i,t}(x_t) = \phi_i(x_t^{\leftarrow t})$$

for all $t \in \mathbb{R}$ and $x_t \in \mathcal{X}_t$.

Before giving the definition of a Kalikow decomposition, we introduce the definition of a cylindrical function as follows.

Definition 2. For any neighborhood $v_t \in \mathbf{V}_t$ and $x_t, y_t \in \mathcal{X}_t$, we say $x_t \stackrel{v_t}{=} y_t$ whenever $x_t = y_t$ in v_t . This means that, for all $i \in \mathbf{I}, n \in \mathbb{Z}$, such that $t_n^i \in x_t$ and $(i, t_n^i) \in v_t$, we have $t_n^i \in y_t$ and vice-versa. A real valued function f is called cylindrical in v_t if $f(x_t) = f(y_t)$ for any $x_t \stackrel{v_t}{=} y_t$, and we usually stress the dependence in v_t by denoting $f^{v_t}(x_t)$.

In what follows, we give the definition of the Kalikow decomposition for a counting process Z^i with stochastic intensity $\phi_{i,t}(x_t)$.

Definition 3. We say a time homogeneous process Z^i for some $i \in \mathbf{I}$ admits the Kalikow decomposition with respect to (w.r.t) a neighborhood family $(\mathbf{V}_t)_{t \in \mathbb{R}}$ and a sequence of subspaces $(\mathcal{Y}_t)_{t \in \mathbb{R}}$ of \mathcal{X}_∞ , if for all t , the intensity $\phi_{i,t}(x_t)$ admits a convex decomposition for any past configuration $x_t \in \mathcal{X}_t \cap \mathcal{Y}_t$, that is, for any $v_t \in \mathbf{V}_t$ there exists a cylindrical function $\phi_{i,t}^{v_t}(\cdot)$ on v_t taking values in \mathbb{R}^+ and a probability density function $\lambda_{i,t}(\cdot)$ such that

$$\forall x_t \in \mathcal{X}_t \cap \mathcal{Y}_t, \quad \phi_{i,t}(x_t) = \lambda_{i,t}(\emptyset) \phi_{i,t}^\emptyset + \sum_{v_t \in \mathbf{V}_t, v_t \neq \emptyset} \lambda_{i,t}(v_t) \phi_{i,t}^{v_t}(x_t) \quad (2.1)$$

with $\lambda_{i,t}(\emptyset) + \sum_{v_t \in \mathbf{V}_t, v_t \neq \emptyset} \lambda_{i,t}(v_t) = 1$.

We say process $Z = (Z^i)_{i \in \mathbf{I}}$ satisfies a Kalikow decomposition w.r.t $(\mathbf{V}_t)_{t \in \mathbb{R}}$ and $(\mathcal{Y}_t)_{t \in \mathbb{R}}$ if for all $i \in \mathbf{I}$ each process Z^i satisfies a Kalikow decomposition w.r.t $(\mathbf{V}_t)_{t \in \mathbb{R}}$ and $(\mathcal{Y}_t)_{t \in \mathbb{R}}$. For simplicity, all the conventions of $(\mathcal{X}_t)_{t \in \mathbb{R}}$ will be used for $(\mathcal{Y}_t)_{t \in \mathbb{R}}$, such as \mathcal{Y}_0 being replaced by \mathcal{Y} , etc.

Remark 1. Note that the function $\lambda_{i,t}(\cdot)$ in Definition 3 is a deterministic function, that is why this decomposition is unconditional, whereas in [13], $\lambda_{i,t}(\cdot)$ was depending on the dominating Poisson processes (see Introduction). Secondly, we do not restrict ourself to a bounded intensity, which is a notable improvement compared to [17].

In the following subsection, we show that we can translate the decomposition at time 0 to any time t when dealing with time homogeneous processes.

2.2 From the decomposition at time 0 to the decomposition at any time t .

In the context of time homogeneous process, for any $t \in \mathbb{R}$, we define

$$\mathcal{Y}_t = \mathcal{Y}^{\rightarrow t} = \{x + t | x \in \mathcal{Y}\}.$$

For all $i \in \mathbf{I}$, assume that Equation (2.1) is satisfied at time $t = 0$ for any $x \in \mathcal{X} \cap \mathcal{Y}$, we then prove that, for time homogeneous processes, this equation is achieved at any time t , for any $x_t \in \mathcal{X}_t \cap \mathcal{Y}_t$, for a particular choice of \mathbf{V}_t . Hence, to show that a counting process $Z^i, i \in \mathbf{I}$ satisfies a Kalikow decomposition, it is then sufficient to write the Kalikow decomposition at time 0 only.

Consider a neighborhood family at time 0, \mathbf{V}_0 . Take a neighborhood $v \in \mathbf{V}_0$ and for $t \geq 0$, denote $v^{\rightarrow t} = \{(i, u + t) : (i, u) \in v\}$, i.e, shift to the right the time component of the neighborhood v by t and define

$$\mathbf{V}_t = \mathbf{V}^{\rightarrow t} := \{v^{\rightarrow t} : v \in \mathbf{V}_0\}. \quad (2.2)$$

Since Equation (2.1) is satisfied at time 0 for any $x \in \mathcal{X} \cap \mathcal{Y}$ and since $x_t^{\leftarrow t} \in \mathcal{X} \cap \mathcal{Y}$, this implies that

$$\phi_i(x_t^{\leftarrow t}) = \lambda_{i,0}(\emptyset)\phi_{i,0}^\emptyset + \sum_{v \in \mathbf{V}_0, v \neq \emptyset} \lambda_{i,0}(v)\phi_{i,0}^v(x_t^{\leftarrow t}).$$

Define the cylindrical function $\varphi_{i,t}^{v^{\rightarrow t}}$, that is cylindrical on $v^{\rightarrow t}$ such that $\varphi_{i,t}^{v^{\rightarrow t}}(x_t) = \phi_{i,0}^v(x_t^{\leftarrow t})$ and $\varphi_{i,t}^\emptyset := \phi_{i,0}^\emptyset$. Moreover, for any nonempty neighborhood v in \mathbf{V}_0 , we consider $\lambda_{i,t}(v^{\rightarrow t}) = \lambda_{i,0}(v)$. Since Z^i is a time homogeneous process, by Definition 1, we have that

$$\phi_{i,t}(x_t) = \lambda_{i,t}(\emptyset)\varphi_{i,t}^\emptyset + \sum_{v^{\rightarrow t} \in \mathbf{V}_t, v^{\rightarrow t} \neq \emptyset} \lambda_{i,t}(v^{\rightarrow t})\varphi_{i,t}^{v^{\rightarrow t}}(x_t) \quad (2.3)$$

with

$$\lambda_{i,t}(\emptyset) + \sum_{v^{\rightarrow t} \in \mathbf{V}_t, v^{\rightarrow t} \neq \emptyset} \lambda_{i,t}(v^{\rightarrow t}) = 1.$$

This shows that Equation (2.1) is satisfied at any time t . Thus, by definition, Z^i admits a Kalikow decomposition w.r.t the neighborhood family $(\mathbf{V}_t)_{t \in \mathbb{R}}$ where $\mathbf{V}_t = \mathbf{V}^{\rightarrow t}$. In addition, $(\mathbf{V}_t)_{t \in \mathbb{R}}$ is completely determined once the neighborhood family at time 0, \mathbf{V}_0 , is determined. From now on, it is sufficient to mention \mathbf{V}_0 whenever we speak about the Kalikow decomposition.

2.3 About the subspace \mathcal{Y}

The role of the subspace \mathcal{Y} is to make the Kalikow decomposition achievable. There are many possible choices for such a subspace, depending on the model under consideration. In this paper, we focus for instance on the choice $\mathcal{Y} = \mathcal{X}^{>\delta}$, the subspace of \mathcal{X} where the distance between any two consecutive possible jumps is greater than δ . More precisely,

$$\mathcal{X}^{>\delta} = \{x = (\{t_n^i\}_{n \in \mathbb{Z}^-})_{i \in \mathbf{I}} \text{ such that } \forall n, i \quad t_{n+1}^i - t_n^i > \delta \text{ and } t_0^i \leq 0\}. \quad (2.4)$$

Another possible choice of a subspace \mathcal{Y} that we consider guarantees that the intensity is finite, by introducing

$$\mathcal{Y} = \{x \in \mathcal{X}, \forall i \quad \phi_i(x) < \infty\}.$$

In practice, either the process is known, namely, the formula of the intensity is given, and we dispose of theoretical results that guarantee for instance that $x \in \mathcal{Y}$ almost surely, or we handle the decomposition by creating a certain threshold. Before the process reaches this threshold, we are able to achieve the Kalikow decomposition by following one of the methods described in the following section. However, after having reached this threshold we can not say anything any more. This depends of course on the initial condition, and if we start for instance with no point on $(-\infty, 0)$, the intensity is in many cases likely to be finite at least at time 0. To proceed further with a Perfect Simulation algorithm is more tricky since we need to check that the intensity remains finite during all steps of the algorithm. This will be the main object of a future work.

3 Main results

3.1 The first method

In this section, we present step by step a general method to prove the existence of a Kalikow decomposition for the counting process $Z^i, i \in \mathbf{I}$. We start by discussing the relevant family of neighborhoods.

For any subset $J \subset \mathbf{I}$ we say a process Z^i is locally dependent on a subprocess $Z_J := (Z^j)_{j \in J}$ if the intensity $\phi_i(x)$ is a cylindrical function on $J \times (-\infty, 0)$. Roughly speaking, a process Z^i is locally dependent on a subprocess Z_J if the information of process Z^j with $j \in J$ is compulsory to compute the intensity of process Z^i . For short, we say i is locally dependent on J . A more formal definition of local dependence can be found at [8]. Denote

$$\mathcal{V}_{\rightarrow i} := \{j \in \mathbf{I} \text{ such that } i \text{ is locally dependent on } j\} \quad (3.1)$$

and $\mathcal{S}^i := \mathcal{V}_{\rightarrow i} \times (-\infty, 0)$.

We denote by (\mathbf{P}, \leq) a countable, ordered set. To simplify notation, in what follows, we consider \mathbf{P} to be \mathbb{N} , but depending on the concrete examples, different sets \mathbf{P} will be considered, for example $\mathbb{N} \times \mathbb{N}$, etc.

Define a family of finite space-time neighborhoods at time 0 associated to the dynamic of Z^i :

$$\mathbf{V}^i = \{(v_k^i)_{k \in \mathbf{P}} \subset \mathbf{I} \times (-\infty, 0) \text{ such that } \cup_k v_k^i = \mathcal{S}^i\} \quad (3.2)$$

with the convention that $v_0^i = \emptyset$.

In order to prove that Z^i admits a Kalikow decomposition w.r.t the neighborhood family \mathbf{V}^i and a convenient subspace \mathcal{Y} , we use the following condition. Once this condition is fulfilled, we will show in Proposition 1 below that we can derive the corresponding Kalikow decomposition.

Assumption 1. *There exists a non-negative sequence of functions $(\Delta_k^i(x))_{k \in \mathbf{P}}$ which are cylindrical on v_k^i such that*

$$\sum_{k=0}^n \Delta_k^i(x) \rightarrow \phi_i(x)$$

as $n \rightarrow \infty$, for every $x \in \mathcal{X} \cap \mathcal{Y}$.

Now, we present a method to obtain a Kalikow decomposition under Assumption 1.

Step 1: For any $k \geq 0$, take $\Delta_k^i(x)$ from Assumption 1, it then guarantees that $\phi_i(x)$ can be written as follows:

$$\phi_i(x) = \Delta_0^i + \sum_{k \geq 1} \Delta_k^i(x)$$

where $\Delta_0^i := \Delta_0^i(x)$ is a constant, that is, it does not depend on x since $v_0^i = \emptyset$. Note that, for $k \geq 1$, by definition, $\Delta_k^i(x)$ is cylindrical on v_k^i and non negative.

Step 2: Define a deterministic nonnegative sequence $(\eta_k^i)_{k \in \mathbf{P}}$ such that

$$\sum_{k \geq 0} \eta_k^i = 1.$$

Obviously, $\phi_i(x)$ can be written as:

$$\phi_i(x) = \eta_0^i \frac{\Delta_0^i}{\eta_0^i} + \sum_{k \geq 1} \eta_k^i \frac{\Delta_k^i(x)}{\eta_k^i}$$

with the convention that $0/0 = 1$. We can set η_k^i to 0 whenever $\Delta_k^i(x)$ equals 0 for all x and even discard the corresponding neighborhood from \mathbf{V}^i .

Proposition 1. *Consider a point process $Z^i, i \in \mathbf{I}$, with intensity at time 0, $\phi_i(x)$, for any $x \in \mathcal{X}$. For the neighborhood family \mathbf{V}^i defined in (3.2), if $\phi_i(x)$ satisfies Assumption 1, then Z^i has the following Kalikow decomposition with*

respect to \mathbf{V}^i and the subspace \mathcal{Y} :

$$\begin{cases} \lambda_i(\emptyset) = \eta_0^i \\ \phi_i^\emptyset = \frac{\Delta_0^i}{\eta_0^i} \\ \lambda_i(v_k^i) = \eta_k^i \\ \phi_i^{v_k^i}(x) = \frac{\Delta_k^i(x)}{\eta_k^i} \end{cases}$$

for any choice of non negative weights $(\eta_k^i)_{k \in \mathbf{P}}$ such that

$$\sum_{k \geq 0} \eta_k^i = 1.$$

Remark 2. From Step 2, it is clear that such a Kalikow decomposition is not unique. However, to perform the Perfect Simulation algorithm (see Section 4), $(\eta_k^i)_{k \in \mathbf{P}}$ can not be chosen arbitrarily. These weights need to satisfy several conditions, for example: the stopping condition, see Proposition 5 below, which enables the algorithm to end. On the other hand, for the simulation purposes, these weights should be chosen carefully to avoid the explosion of $\phi_i^{v_k^i}(x)$. In addition, they also influence the mean number of total simulated points of this algorithm, and therefore the efficiency of the algorithm (see Section 4). To illustrate this point more clearly, one example is considered in Section 4.

3.2 Examples of the first method

3.2.1 Linear Hawkes process.

In the following, we consider a linear Hawkes process [12, 11], where the intensity at time 0, $\phi_i(x)$, is given as follows. For any $x \in \mathcal{X}$,

$$\phi_i(x) = \mu_i + \sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dx_s^j,$$

where $h_{ji}(\cdot)$ measures the local dependence of process Z^i on Z^j and μ_i refers to the spontaneous rate of process Z^i .

We consider the following assumption for the method to work.

Assumption 2. For all $i, j \in \mathbf{I}$, $h_{ji}(\cdot)$ is a non negative function.

Denote ϵ an arbitrary, fixed and positive number. Then we have

$$\phi_i(x) = \mu_i + \sum_{j \in \mathbf{I}} \sum_{n \in \mathbf{N}^*} \int_{-n\epsilon}^{-n\epsilon + \epsilon} h_{ji}(-s) dx_s^j.$$

Furthermore, in this example, we have

$$\mathcal{V}_{\rightarrow i} = \{j \in \mathbf{I} \text{ such that } h_{ji} \neq 0\}.$$

Since \mathbf{I} is at most countable, we may enumerate the indices in \mathbf{I} by j_l for $l \in \mathbb{N}^*$. In this example we put $\mathbf{P} = \mathbb{N}^* \times \mathbb{N}^*$, and we consider the neighborhood family \mathbf{V}^{atom} as follows: for $l, n \geq 1$

$$v_{(l,n)}^i := \{j_l\} \times [-n\epsilon, -n\epsilon + \epsilon).$$

We define, for all $l, n \geq 1$:

$$\Delta_{(l,n)}^i(x) := \int_{-n\epsilon}^{-n\epsilon+\epsilon} h_{j_l i}(-s) dx_s^{j_l} \geq 0,$$

and $\Delta_{(0,0)}^i := \Delta_{(0,0)}^i(x) := \mu_i$. Applying the Monotone Convergence theorem, we obtain

$$\Delta_{(0,0)}^i + \sum_{(l,n)=(1,1)}^{(L,N)} \Delta_{(l,n)}^i(x) \rightarrow \phi_i(x),$$

when $(L, N) \rightarrow (\infty, \infty)$.

Notice that the intensity $\phi_i(x)$ is finite if and only if the series converges. However, what we do here is more general, since we allow the intensity to be infinite.

Thus, Assumption 1 is fulfilled. As a consequence, relying on Proposition 1, we conclude that Z^i has a Kalikow decomposition with respect to \mathbf{V}^{atom} with

$$\begin{cases} \lambda_i(\emptyset) = \eta_{(0,0)} \\ \phi_i^\emptyset = \frac{\mu_i}{\eta_{(0,0)}} \\ \lambda_i(v_{(l,n)}^i) = \eta_{(l,n)} \\ \phi_i^{v_{(l,n)}^i}(x) = \frac{\int_{-n\epsilon}^{-n\epsilon+\epsilon} h_{j_l i}(-s) dx_s^{j_l}}{\eta_{(l,n)}}, \end{cases}$$

for any choice of non negative weights $(\eta_{(l,n)}^i)_{(l,n) \in \mathbb{N}^* \times \mathbb{N}^*}$ such that

$$\eta_{(0,0)}^i + \sum_{l,n \geq 1} \eta_{(l,n)}^i = 1.$$

Remark 3. *Note that under mild conditions (see for instance [7]) it is well-known that the process exists in a stationary regime. On the other hand, for our purpose it is sufficient to consider the subspace $\mathcal{Y} = \{x \in \mathcal{X} \mid \forall i \quad \phi_i(x) < \infty\}$ such that for all $x \in \mathcal{X} \cap \mathcal{Y}$, $\phi_i(x)$ and $\phi_i^v(x)$ are finite.*

3.2.2 Age dependent Hawkes process with hard refractory period.

In this section, we are interested in writing a Kalikow decomposition for Age dependent Hawkes processes with hard refractory period. Up to our knowledge, this process was first introduced in [5] and no Kalikow decomposition has been

proved, even in a conditional framework, for this type of process. In our setting, the stochastic intensity of an Age dependent Hawkes process with hard refractory of length $\delta > 0$ can be written as follows. For any $i \in \mathbf{I}$ and $x \in \mathcal{X}$,

$$\phi_i(x) = \psi_i \left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dx_s^j \right) \mathbb{1}_{a_0^i(x) > \delta} \quad (3.3)$$

with the convention that $a_0^i(x) = -L_0^i(x)$ and

$$L_0^i(x) = \sup\{t_k^i \in x \text{ such that } t_k^i < 0\} = t_{-1}^i$$

is the last jump before 0 of process Z^i , if it has at least one jump. Otherwise, we put $L_0^i(x) = -\infty$. Again, we have

$$\mathcal{V}_{\cdot \rightarrow i} = \{j \in \mathbf{I} \text{ such that } h_{ji} \neq 0\} \cup \{i\}.$$

By definition of stochastic intensity (3.3), we observe that the distance of any two consecutive jumps have to be larger than δ . This observation leads us to consider the subspace $\mathcal{Y} = \mathcal{X}^{>\delta}$ that is introduced in (2.4). To prove a Kalikow decomposition, we consider the following assumptions.

Assumption 3. (i) For all $i, j \in \mathbf{I}$, $h_{ji}(\cdot)$ is a non negative, non increasing L_1 function. Moreover, for every i

$$\sum_{j \in \mathbf{I}} \|h_{ji}\|_{L_1} < \infty.$$

(ii) For every i , $\psi_i(\cdot)$ is an increasing, non negative continuous function.

On the other hand, we build a neighborhood family \mathbf{V}^{nested} by introducing a non decreasing sequence $(V_i(k))_{k \geq 0}$ of finite subsets of \mathbf{I} such that

$$V_i(0) = \emptyset, V_i(1) = \{i\}, V_i(k-1) \subset V_i(k) \text{ and } \cup_k V_i(k) = \mathcal{V}_{\cdot \rightarrow i} \cup \{i\}. \quad (3.4)$$

Consider $\mathbf{P} = \mathbf{N}$ and set $v_k^i = V_i(k) \times [-k\delta, 0)$, then we obtain by construction an increasing, nested neighborhood sequence $(v_k^i)_{k \in \mathbf{N}}$. By applying the methodology of Section 3.1, we prove the following Proposition.

Proposition 2. Consider $\mathcal{X}^{>\delta}$ defined as in (2.4). For any past configuration $x \in \mathcal{X}$, suppose that the intensity is of the form

$$\phi_i(x) = \psi_i \left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dx_s^j \right) \mathbb{1}_{a_0^i(x) > \delta}.$$

If Assumption 3 is fulfilled, then the corresponding Kalikow decomposition with respect to \mathbf{V}^{nested} and $\mathcal{X}^{>\delta}$ is for $k \geq 1$

$$\begin{cases} \lambda_i(v_k^i) = \eta_k^i \\ \phi_i^{v_k^i}(x) = \frac{\Delta_k^i(x)}{\eta_k^i} \end{cases}$$

with

(i) any positive sequence $(\eta_k^i)_{k \geq 1}$ such that $\sum_{k \geq 1} \eta_k^i = 1$,

(ii)

$$\begin{aligned} \Delta_k^i(x) &= \psi_i \left(\sum_{j \in V_i(k)_{-k\delta}} \int_{-k\delta}^0 h_{ji}(-s) dx_s^j \right) \mathbf{1}_{a_0^i(x) > \delta} \\ &\quad - \psi_i \left(\sum_{j \in V_i(k-1)_{-k\delta+\delta}} \int_{-k\delta+\delta}^0 h_{ji}(-s) dx_s^j \right) \mathbf{1}_{a_0^i(x) > \delta} \end{aligned}$$

for $k \geq 2$ and $\Delta_i^1(x) = \psi_i(0) \mathbf{1}_{a_0^i(x) > \delta}$.

Remark 4. Note that $\lambda_i(\emptyset)$ does not appear in the above Kalikow decomposition. Amazingly, this does not cause any problems for the Perfect Simulation algorithm (see Section 4). We stress that this is one of the main differences with [13], the other one being that the decomposition does not depend on the dominating Poisson processes.

Proof. For any $x \in \mathcal{X}^{>\delta}$, $i \in \mathbf{I}$ and $k \geq 2$, consider

$$\begin{aligned} \Delta_k^i(x) &= \psi_i \left(\sum_{j \in V_i(k)} \int_{-k\delta}^0 h_{ji}(-s) dx_s^j \right) \mathbf{1}_{a_0^i(x) > \delta} \\ &\quad - \psi_i \left(\sum_{j \in V_i(k-1)} \int_{-k\delta+\delta}^0 h_{ji}(-s) dx_s^j \right) \mathbf{1}_{a_0^i(x) > \delta}. \end{aligned} \quad (3.5)$$

By Assumption 3, $(\Delta_k^i(x))_{k \in \mathbf{N}}$ is well-defined, non negative and cylindrical on v_k^i . Moreover, we set $\Delta_0^i = 0$ as well $\eta_0^i = 0$ and $\Delta_i^1(x) = \psi_i(0) \mathbf{1}_{a_0^i(x) > \delta}$. Let

$$r_i^{[n]}(x) := \sum_{k=1}^n \Delta_k^i(x) = \psi_i \left(\sum_{j \in V_i(n)} \int_{-n\delta}^0 h_{ji}(-s) dx_s^j \right) \mathbf{1}_{a_0^i(x) > \delta}$$

and let us show that $r_i^{[n]}(x) \rightarrow \phi_i(x)$ when $n \rightarrow \infty$. Consider the inner-term of the parenthesis,

$$\sum_{j \in V_i(n)} \int_{-n\delta}^0 h_{ji}(-s) dx_s^j = \int_{I \times \mathbb{R}^-} h_{ji}(-s) \mathbf{1}_{(j,s) \in v_n^i} dx_s^j d\kappa_j,$$

where we denote $d\kappa$ the counting measure on the discrete set \mathbf{I} .

We have that $(h_{ji}(-s) \mathbf{1}_{(j,s) \in v_n^i})_{n \in \mathbf{Z}}$ is non negative and non decreasing sequence in n . In addition, it converges to $h_{ji}(-s) \mathbf{1}_{(j,s) \in \mathcal{V}_{\rightarrow i} \times (-\infty, 0)}$ as $n \rightarrow \infty$. Moreover, since $\psi_i(\cdot)$ is a continuous and increasing function, the Monotone convergence theorem for Lebesgue Stieltjes measures implies that $r_i^{[n]}(x) \rightarrow \phi_i(x)$ as $n \rightarrow \infty$. As a consequence, Assumption 1 is fulfilled, and by Proposition 1, the conclusion follows. \square

Remark 5. Denote $D_i^k(x) = \{z \in \mathcal{X}^{>\delta} : z \stackrel{v_k^i}{=} x\}$. Clearly,

$$\begin{aligned} \psi_i \left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dz_s^j \right) \mathbf{1}_{a_0^i(x) > \delta} &= \psi_i \left(\sum_{j \in V_i(k)} \int_{-k\delta}^0 h_{ji}(-s) dz_s^j + \right. \\ &\quad \left. + \sum_{j \in V_i(k)} \int_{-\infty}^{-k\delta} h_{ji}(-s) dz_s^j + \sum_{j \notin V_i(k)} \int_{-\infty}^0 h_{ji}(-s) dz_s^j \right) \mathbf{1}_{a_0^i(x) > \delta}. \end{aligned}$$

By Assumption 3, we conclude that,

$$\inf_{z \in D_i^k(x)} \psi_i \left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dz_s^j \right) \mathbf{1}_{a_0^i(x) > \delta} = \psi_i \left(\sum_{j \in V_i(k)} \int_{-k\delta}^0 h_{ji}(-s) dx_s^j \right) \mathbf{1}_{a_0^i(x) > \delta}.$$

The equality is achieved when we consider the configuration z having points only inside the neighborhood v_k^i .

Hence, for $k \geq 2$, we observe that

$$\begin{aligned} \Delta_k^i(x) &= \inf_{z \in D_i^k(x)} \psi_i \left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dz_s^j \right) \mathbf{1}_{a_0^i(x) > \delta} \\ &\quad - \inf_{z \in D_i^{k-1}(x)} \psi_i \left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dz_s^j \right) \mathbf{1}_{a_0^i(x) > \delta}. \end{aligned}$$

The above prescription corresponds to the classical method of obtaining a Kalikow decomposition in discrete time, discussed in [13, 10].

3.3 Another method for nonlinear Hawkes processes

In this section, we present a second method to prove the existence of a Kalikow decomposition for nonlinear Hawkes processes [3]. In the setting of this section, we suppose that the intensity of the nonlinear Hawkes process Z^i , $i \in \mathbf{I}$, is given by

$$\phi_i(x) = \psi_i \left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dx_s^j \right)$$

for any $x \in \mathcal{X}$.

We work under the following assumptions.

Assumption 4. (i) For any $i, j \in \mathbf{I}$, the function $h_{ji}(\cdot)$ is non negative and belongs to L_1 . Moreover, for every i , we have

$$\sum_{j \in \mathbf{I}} \|h_{ji}\|_{L_1} < \infty.$$

(ii) For every $i \in \mathbf{I}$, $\psi_i(\cdot)$ is an analytic function on \mathbb{R} with radius of convergence about 0 which is given by K , for some positive K . Moreover, its derivative of order n , $\psi_i^{(n)}(0)$, is non negative for all $n \geq 1$, and $\psi_i(0)$ is non negative as well.

In this section, to develop our series using Taylor expansion, we choose

$$\mathcal{Y}^K = \left\{ x \mid \sup_i \left(\sum_j \int_{-\infty}^0 h_{ji}(-s) dx_s^j \right) < K \right\}.$$

We now describe our method. We fix ϵ an arbitrary positive parameter, and we put $D = \mathbf{I} \times \mathbf{N}$. An index in D is $\alpha = (j, n)$ where $j \in \mathbf{I}$ and $n \in \mathbf{N}$. We put $|\alpha| := |j| + n$. For any $x \in \mathcal{X} \cap \mathcal{Y}^K$, we introduce

$$a_\alpha(x) := \int_{-(n+1)\epsilon}^{-n\epsilon} h_{ji}(-s) dx_s^j.$$

Note that, whenever $x \in \mathcal{X} \cap \mathcal{Y}^K$, we have $a_\alpha(x)$ converges to 0 when $|\alpha| \rightarrow \infty$. Furthermore, we have

$$\phi_i(x) = \psi_i \left(\sum_{j \in \mathbf{I}} \sum_{n \in \mathbf{N}} \int_{-(n+1)\epsilon}^{-n\epsilon} h_{ji}(-s) dx_s^j \right) = \psi_i \left(\sum_{j \in \mathbf{I}} \sum_{n \in \mathbf{N}} a_{jn}(x) \right) = \psi_i \left(\sum_{\alpha \in D} a_\alpha(x) \right).$$

We suppose moreover that

Assumption 5. For any $i \in \mathbf{I}$, there exist a deterministic sequence $(\eta_{\alpha_k}^i)_{\alpha_k \in D} \in (0, 1)$ such that

$$\sum_{n \geq 1} \sum_{\alpha_1, \dots, \alpha_n \in D} \eta_{\alpha_1}^i \dots \eta_{\alpha_n}^i < 1.$$

Let us consider $\mathbf{P} = \cup_{n=1}^{\infty} D^n$ where $D^n := D \times D \times \dots \times D$ (n times). We construct the neighborhood family \mathbf{V}^{Taylor} by defining for $\alpha_{1:k} = (\alpha_1, \dots, \alpha_k)$,

$$v_{\alpha_{1:k}}^i = v_{(\alpha_1, \dots, \alpha_k)}^i := \cup_{l=1}^k w_{\alpha_l}, \quad (3.6)$$

where $w_{\alpha_l} := \{j\} \times [-(n+1)\epsilon, n\epsilon]$ if $\alpha_l = (j, n)$, and $v_0^i := \emptyset$.

Proposition 3. Consider a non linear Hawkes process $Z^i, i \in \mathbf{I}$, with intensity $\phi_i(x)$ given by

$$\phi_i(x) = \psi_i \left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dx_s^j \right).$$

Under Assumption 4 and 5, Z^i has a Kalikow decomposition with respect to

the neighborhood family \mathbf{V}^{Taylor} defined by (3.6) and \mathcal{Y}^K as follows:

$$\begin{cases} \lambda_i(\emptyset) &= 1 - \sum_{k \geq 1} \sum_{\alpha_1, \dots, \alpha_k \in D} \eta_{\alpha_1}^i \cdots \eta_{\alpha_k}^i \\ \phi_i^\emptyset &= \frac{\psi_i(0)}{\lambda_i(\emptyset)} \\ \lambda_i(v_{\alpha_1:k}^i) &= \eta_{\alpha_1}^i \cdots \eta_{\alpha_k}^i \\ \phi_i^{v_{\alpha_1:k}^i}(x) &= \frac{\psi_i^{(k)}(0)}{k!} \frac{a_{\alpha_1}(x)}{\eta_{\alpha_1}^i} \cdots \frac{a_{\alpha_k}(x)}{\eta_{\alpha_k}^i}, \end{cases}$$

for any weights $(\eta_{\alpha_k}^i)_{\alpha_k \in D}$ satisfying Assumption 5 and for $\alpha = (j, n)$, a_α defined by

$$a_\alpha(x) := \int_{-(n+1)\epsilon}^{-n\epsilon} h_{ji}(-s) dx_s^j.$$

Proof. From Assumption 4, (ii), and noticing that for any $x \in \mathcal{X} \cap \mathcal{Y}^K$, we have $0 \leq \sum_{j \in \mathbf{I}} \sum_{n \in \mathbf{N}} a_{jn}(x) = \sum_{\alpha \in D} a_\alpha(x) < K$, we have a Taylor expansion of $\phi_i(x)$:

$$\begin{aligned} \phi_i(x) &= \psi_i \left(\sum_{j \in \mathbf{I}} \sum_{n \in \mathbf{N}} a_{jn}(x) \right) = \psi_i \left(\sum_{\alpha \in D} a_\alpha(x) \right) \\ &= \psi_i(0) + \sum_{k \geq 1} \frac{\psi_i^{(k)}(0)}{k!} \left(\sum_{\alpha = (j,n) \in D} a_\alpha(x) \right)^k \\ &= \psi_i(0) + \sum_{k \geq 1} \sum_{\alpha_1 \dots \alpha_k \in D} \frac{\psi_i^{(k)}(0)}{k!} a_{\alpha_1}(x) \cdots a_{\alpha_k}(x). \end{aligned}$$

Thus, to obtain the Kalikow decomposition, take $(\eta_{\alpha_k}^i)_{\alpha_k \in D}$ from Assumption 5, then the conclusion follows. \square

Remark 6. The parameter $(\eta_{\alpha_k}^i)_{\alpha_k \in D}$ plays the same role here as $(\eta_k^i)_{k \in \mathbf{N}}$ in the previous section. Again, the choice of this parameter is discussed in Section 4.

In the following, we give several examples in which we can apply the second method.

3.4 Examples of second method

In this section, we always consider a deterministic sequence $(\eta_{\alpha_k}^i)_{\alpha_k \in D}$ that satisfies Assumption 5.

Example 1. Exponential Hawkes process with $\psi_i(\cdot) := \exp(\cdot)$. In this case we have $K = \infty$, $\psi_i(0) = 1$ and $\psi_i^{(k)}(0) = 1$ for $k \geq 1$. Therefore,

$$\begin{aligned}\phi_i(x) &= \exp\left(\sum_{j \in \mathbf{I}} \int_{-\infty}^0 h_{ji}(-s) dx_s^j\right) \\ &= 1 + \sum_{k \geq 1} \sum_{\alpha_1, \dots, \alpha_k \in D} \frac{1}{k!} a_{\alpha_1}(x) \dots a_{\alpha_k}(x).\end{aligned}$$

Then, the Kalikow decomposition of Z^i with respect to the neighborhood family \mathbf{V}^{Taylor} in (3.6) and \mathcal{Y}^∞ is given by

$$\begin{cases} \lambda_i(\emptyset) &= 1 - \sum_{k \geq 1} \sum_{\alpha_1, \dots, \alpha_k \in D} \eta_{\alpha_1}^i \dots \eta_{\alpha_k}^i \\ \phi_i(\emptyset) &= \frac{1}{\lambda_i(\emptyset)} \\ \lambda_i(v_{\alpha_1:k}^i) &= \eta_{\alpha_1}^i \dots \eta_{\alpha_k}^i \\ \phi_i^{v_{\alpha_1:k}^i}(x) &= \frac{1}{k!} \times \frac{a_{\alpha_1}(x)}{\eta_{\alpha_1}^i} \dots \frac{a_{\alpha_k}(x)}{\eta_{\alpha_k}^i}. \end{cases}$$

Example 2. We consider the non-linear Hawkes process with $\psi_i(u) = ch(u)$ where $ch(u) = \frac{e^u + e^{-u}}{2}$. Then $K = \infty$, $\psi_i(0) = 1$, and for $k \geq 1$, we have $\psi_i^{(2k)}(0) = 0$ and $\psi_i^{(2k-1)}(0) = \frac{1}{2}$.

Therefore, Z^i has the following Kalikow decomposition with respect to the neighborhood family \mathbf{V}^{Taylor} in (3.6) and \mathcal{Y}^∞ is given by

$$\begin{cases} \lambda_i(\emptyset) &= 1 - \sum_{k \geq 1} \sum_{\alpha_1, \dots, \alpha_{2k-1}} \eta_{\alpha_1}^i \dots \eta_{\alpha_{2k-1}}^i \\ \phi_i(\emptyset) &= \frac{1}{\lambda_i(\emptyset)} \\ \lambda_i(v_{\alpha_1:2k-1}^i) &= \eta_{\alpha_1}^i \dots \eta_{\alpha_{2k-1}}^i \\ \phi_i^{v_{\alpha_1:2k-1}^i}(x) &= \frac{1}{2(2k-1)!} \times \frac{a_{\alpha_1}(x)}{\eta_{\alpha_1}^i} \dots \frac{a_{\alpha_{2k-1}}(x)}{\eta_{\alpha_{2k-1}}^i}. \end{cases}$$

Remark 7. It is well-known that the Exponential Hawkes process may explode in finite time with probability non null [6] (that is, produce an infinite number of jumps within a finite time interval). To perform the Perfect Simulation, in this case, we shall consider the process having empty past before time 0, in other words, the process starting from 0. However, the difficulty lies on the fact that we need to check that the process stays in \mathcal{Y}^∞ . To avoid this difficulty, we shall not consider the Exponential Hawkes process or other potentially explosive processes in the following section. The discussion of potentially explosive process will be the main object of upcoming work, as we mentioned earlier.

4 Modified Perfect Simulation algorithm

In the following, we present a Perfect Simulation algorithm that simulates the process Z^i on an interval $[0, tmax]$, for some fixed $tmax > 0$ in the stationary

regime. Our algorithm is a modification of the method described in [17]. The procedure consists of backward and forward steps. In the backward steps, thanks to the Kalikow decomposition, we create a set of ancestors, which is a list of all the points that might influence the point under consideration. On the other hand, in the forward steps, where we go forward in time, by using the thinning method [18] we give the decision (keep or reject) to each visited point based on its neighborhood until the state of all considered points is decided. Further discussion can be found in [10, 13, 17].

For this algorithm to work, we introduce a subspace \mathcal{Y} as follows:

$$\mathcal{Y} = \{x \in \mathcal{X} \text{ such that } \forall v, i : \phi_i^v(x) \leq \Gamma_i\},$$

where Γ_i is a positive constant. We assume that the process generated by the algorithm stays in \mathcal{Y} almost surely. For example, in the case of age dependent Hawkes processes in Section 3.2.2, we show in Proposition 7 below that $\Delta_k^i(x)$ are bounded. Thus, with an adequate choice of weights $(\lambda_i(\cdot))_{i \in \mathbf{I}}$ and $(\Gamma_i)_{i \in \mathbf{I}}$, the algorithm remains in \mathcal{Y} almost surely. Moreover, for any $x \in \mathcal{Y}$, we have

$$\phi_i(x) = \lambda_i(\emptyset)\phi_i^\emptyset + \sum_{\emptyset \neq v \in \mathbf{V}^i} \lambda_i(v)\phi_i^v(x) \leq \Gamma_i,$$

which means that the intensity is bounded.

Furthermore, for any t and $x_t \in \mathcal{Y}_t$, by the time homogeneity assumption,

$$\phi_{i,t}(x_t) = \phi_i(x_t^{\leftarrow t}) \leq \Gamma_i.$$

4.1 Backward procedure

Initial step. Fix i , set the initial time to be 0.

Step 1. Move to the first possible jump T of Z^i after 0 by taking

$$T \leftarrow 0 + \text{Exp}(\Gamma_i).$$

Step 2. Recall that \mathbf{V}^i is the neighborhood family associated to index i . Independently of anything else, pick a random neighborhood $V_{i,T}$ of (i, T) according to the distribution $(\lambda_i(v))_{v \in \mathbf{V}^i}$, which corresponds to the Kalikow decomposition of intensity $\phi_{i,T}(x_T)$ in (2.1) where $x_T \in \mathcal{Y}_T$.

More precisely, this means that we attach to (i, T) a random variable $V_{i,T}$ with values in $(\mathbf{V}^i)^{\rightarrow T}$. For any $v \in \mathbf{V}^i$,

$$\mathbb{P}(V_{i,T} = v^{\rightarrow T}) = \lambda_i(v).$$

Assume that $V_{i,T} = v^{\rightarrow T}$ and define the projection to the second coordinate of v by

$$\pi_j(v) := \{t \in \mathbb{R} \mid (j, t) \in v\},$$

for any $j \in \mathbf{I}$. Notice that if for some j , $(j, t) \notin v$ for all t , then $\pi_j(v) = \emptyset$.

Simulate Poisson processes in $v \rightarrow T$, that is for each $j \in \mathbf{I}$, we simulate a Poisson process Π_j with intensity Γ_j on $\pi_j(v \rightarrow T)$. Put these points in $\mathcal{C}_{i,T}^1$, call it the first set of ancestors of (i, T) :

$$\mathcal{C}_{i,T}^1 = \bigcup_{j \in \mathbf{I}} \{(j, t) | t \in \Pi_j(\pi_j(v \rightarrow T))\}.$$

Step 3. Recursively, we define the n^{th} set of ancestors of (i, T) ,

$$\mathcal{C}_{i,T}^n = \bigcup_{(j,s) \in \mathcal{C}_{i,T}^{n-1}} \mathcal{C}_{j,s}^1 \setminus (\mathcal{C}_{i,T}^1 \cup \dots \cup \mathcal{C}_{i,T}^{n-1}),$$

and for each $(j, t') \in \mathcal{C}_{i,T}^n$, we perform Step 1 and Step 2.

The backward scheme stops when $\mathcal{C}_{i,T}^n = \emptyset$. We denote

$$N_{i,T} = \inf\{n : \mathcal{C}_{i,T}^n = \emptyset\}.$$

The genealogy of (i, T) is given by

$$\mathcal{C}_{i,T} = \bigcup_{k=1}^{N_{i,T}} \mathcal{C}_{i,T}^k.$$

Remark 8. Let us emphasize that $\lambda_i(\emptyset)$ does not need to be strictly positive in the present Kalikow decomposition. This means that there is a chance that at every step of the Backward steps, we need to simulate a Poisson process in a non empty neighborhood. However, if there is no point simulated in these intervals, then we do nothing in the next step. Hence, the Backwards steps end. This is one of the main advantage of this Kalikow decomposition with respect to [13, 10].

Remark 9. Notice that, comparing to the original Backward procedure [13, 10], here, we start by simulating a point in the "future" by adding an exponential random variable to the initial time. Unfortunately, due to this additional step, the algorithm always requires that the intensity is bounded in \mathcal{Y} . This is one of the main drawback of this new algorithm.

4.2 Forward procedure

We now attach to each point in $\mathcal{C}_{i,T}$ a random variable χ whose value is either 0 or 1, where 1 means that this point is accepted.

For any point (i, t) in the Backward steps, we know its neighborhood which contains all the points that might influence the state of (i, t) , but we do not know yet the state χ of the points in the neighborhood.

We start with the point $(j, s) \in \mathcal{C}_{i,T}$ which is the smallest in time, so that its associated neighborhood is either empty ($v = \emptyset$) or non empty but without any point of the Poisson process in it.

Step 1. Assign $\chi_{j,s}$ by Bernoulli variable with parameter $\frac{\phi_{j,s}^{V_{j,s}}(x_s)}{\Gamma_j}$ where $V_{j,s}$ is the neighborhood of (j, s) . If $\chi_{j,s} = 1$, this means we accept this point.

Step 2. Move to the next point of $\mathcal{C}_{i,T}$ in increasing time order. Repeat Step 1:2 until $\chi_{i,T}$ is determined.

Update step. Update the starting time of the initial step by T . Repeat the Backward and Forward procedures until the starting time is greater than $tmax$.

Remark 10. *When implementing the algorithm on a computer, it is worth noticing that, whenever we simulate in an interval that intersects with previously simulated intervals, we will not simulate in the intersecting parts. In particular, no additional points is simulated in $\{i\} \times (0, T)$ where (i, T) is the first simulated point.*

4.3 Do we construct the right intensity?

For any $i \in \mathbf{I}$, we denote the arrival times of Π_i in Step 2 of the Backward steps by $(\tau_n^i)_{n \in \mathbb{Z}}$, with τ_1^i being the first positive time. Indeed even if we have simulated it on the randomly generated neighborhood, since the intensity is always the same, we can assume that all these points come only from one single Poisson process.

As in Step 1 of the Forward steps, we attach to each point of Π_i a stochastic mark χ given by,

$$\chi_n^i = \begin{cases} 1 & \text{if } \tau_n^i \text{ is accepted,} \\ 0 & \text{otherwise.} \end{cases}$$

In addition, for any $i \in \mathbf{I}$, define $\mathcal{Z}^i = (\tau_n^i, \chi_n^i)_{n \in \mathbb{Z}}$ an E -marked point process with $E = \{0; 1\}$. In particular, following the notation in Chapter VIII of [4], for any $i \in \mathbf{I}$, let

$$d\mathcal{Z}_t^i(\text{mark}) = \sum_{n \in \mathbb{Z}} \mathbb{1}_{\chi_n^i = \text{mark}} \delta_{\tau_n^i}(dt) \quad \text{for } \text{mark} \in E,$$

$$\mathcal{F}_t^{\mathcal{Z}} = \bigvee_{i \in \mathbf{I}} \sigma(\mathcal{Z}_s^i(0), \mathcal{Z}_s^i(1); s < t) \quad \text{and} \quad \mathcal{F}_{t-}^{\mathcal{Z}(1)} = \bigvee_{i \in \mathbf{I}} \sigma(\mathcal{Z}_s^i(1); s < t).$$

Moreover note that $(\mathcal{Z}_t^i(1))_{t \in \mathbb{R}}$ is the counting process associated to the accepted points of the algorithm. With these notations, we can prove the following result. This proof is already done in [17] but we add it here for sake of completeness.

Proposition 4. *If we suppose that the process $(\mathcal{Z}_t^i(1))_{t \in \mathbb{R}}$ stays in $(\mathcal{Y}_t)_{t \in \mathbb{R}}$ almost surely then it admits $\phi_{i,t}(x_t)$ as $\mathcal{F}_{t-}^{\mathcal{Z}(1)}$ -predictable intensity.*

Proof. Take C_t a non negative predictable process with respect to $\mathcal{F}_t^{\mathcal{Z}^i(1)}$, that is $\mathcal{F}_{t-}^{\mathcal{Z}(1)}$ measurable and therefore $\mathcal{F}_{t-}^{\mathcal{Z}}$ measurable. We have, for any $i \in \mathbf{I}$,

$$\mathbb{E} \left(\int_0^\infty C_t d\mathcal{Z}_t^i(1) \right) = \sum_{n=1}^\infty \mathbb{E} (C_{\tau_n^i} \mathbb{1}_{\chi_n^i=1}).$$

Note that by Theorem T35 at Appendix A1 of [4], any point T should be understood as a stopping time, and that by Theorem T30 at Appendix A2 of [4],

$$\mathcal{F}_{T-}^{\mathcal{Z}} = \bigvee_{j \in \mathbf{I}} \sigma\{\tau_m^j, \chi_m^j \text{ such that } \tau_m^j < T, T\}.$$

Therefore,

$$\mathbb{E} \left(\int_0^\infty C_t d\mathcal{Z}_t^i(1) \right) = \sum_{n=1}^\infty \mathbb{E} \left(C_{\tau_n^i} \mathbb{E}(\mathbf{1}_{\chi_n^i=1} | \mathcal{F}_{\tau_n^i-}^{\mathcal{Z}}, V_n^i) \right) = \sum_{n=1}^\infty \mathbb{E} \left(C_{\tau_n^i} \frac{\phi_{i, \tau_n^i}^{V_n^i}(x_{\tau_n^i})}{\Gamma_i} \right),$$

where we denote V_n^i for the neighborhood of τ_n^i .

Let us now integrate with respect to the choice V_n^i , which is independent of anything else.

$$\begin{aligned} \mathbb{E} \left(\int_0^\infty C_t d\mathcal{Z}_t^i(1) \right) &= \sum_{n=1}^\infty \mathbb{E} \left(C_{\tau_n^i} \frac{\lambda_i(\emptyset) \phi_{i, \tau_n^i}^\emptyset + \sum_{v \in \mathbf{V} \rightarrow \tau_n^i, v \neq \emptyset} \lambda_i(v) \times \phi_{i, \tau_n^i}^v(x_{\tau_n^i})}{\Gamma_i} \right) \\ &= \mathbb{E} \left(\int_0^\infty C_t \frac{\phi_{i,t}(x_t)}{\Gamma_i} d\Pi_i(t) \right). \end{aligned}$$

Since Π_i is a Poisson process with respect to $\mathcal{F}_t^{\mathcal{Z}}$ with intensity Γ_i , and $C_t \frac{\phi_{i,t}(x_t)}{\Gamma_i}$ is $\mathcal{F}_{t-}^{\mathcal{Z}}$ measurable, we finally have that

$$\mathbb{E} \left(\int_0^\infty C_t d\mathcal{Z}_t^i(1) \right) = \mathbb{E} \left(\int_0^\infty C_t \phi_{i,t}(x_t) dt \right),$$

which ends the proof. \square

4.4 Why does the Backward steps end?

We construct a tree with root (i, T) . For each point $(j_{T'}, T')$ in the tree, the points which are simulated in $V_{j_{T'}, T'}$ (Step 2 of the Backward steps) define the children of $(j_{T'}, T')$ in the tree. This forms the tree $\tilde{\mathcal{T}}$.

Let us now build a tree $\tilde{\mathcal{C}}$ with root (i, T) (that includes the previous tree) by mimicking the procedure in the Backward steps, except that we simulate – independently on anything else – on the whole neighborhood even if it has a part that intersects with previous neighborhoods (if they exist) (Step 3 of the Backward steps). By doing so, we make the number of children larger but at each node, they are independent of anything else.

If the tree $\tilde{\mathcal{C}}$ goes extinct, then so does the tree $\tilde{\mathcal{T}}$, and the backward part of the algorithm terminates.

To formulate a sufficient criterion that implies the almost sure extinction of $\tilde{\mathcal{C}}$, let us denote the product measure P on $\mathbf{I} \times \mathbb{R}$, defined on the product sets that generate the Borel subsets of $\mathbf{I} \times \mathbb{R}$, as follows.

$$P(J \times A) := \sum_{j \in J} \Gamma_j \mu(A)$$

for any $J \subset \mathbf{I}$, where A is a Borel subset of \mathbb{R} and where μ is the Lebesgue measure. The following proposition is already proved in [17], but without precisely defining P .

Proposition 5. *If*

$$\sup_{i \in \mathbf{I}} \sum_{k \geq 1} P(v_k^i) \lambda_i(v_k^i) < 1 \quad (4.1)$$

then the Backward steps in the Perfect Simulation algorithm terminate almost surely in finite time.

Remark 11. *For any neighborhood v , we have*

$$\sum_{j \in v} \mathbb{E}(\Pi_j(\pi_j(v))) = P(v).$$

This implies that $\sum_{k \geq 1} P(v_k^i) \lambda_i(v_k^i)$ is the mean number of children issued from one point of type i . Then, the condition (4.1) says that the number children in each step should be less than one in average, for the tree to go extinct almost surely. That is a very classical result in Branching process [1].

4.5 The complexity of the algorithm in terms of η

In this section, we study the effect of $(\lambda_i(v_k^i))_{i,k}$ on the number of points simulated by our algorithm. Until the end of this section, we suppose that

Assumption 6. *The index set \mathbf{I} is finite, namely*

$$|\mathbf{I}| = N < \infty.$$

Let us mention several notations that will be useful in the sequel. We denote e_i the i -th unit vector of \mathbb{R}^N , $\mathbf{1}$ is the vector $(1, 1, \dots, 1)^T$ and μ always stands for the Lebesgue measure. Finally, by a positive vector, we mean that all its components are positive.

Coming back to our problem, since the tree $\tilde{\mathcal{C}}$ dominates $\tilde{\mathcal{T}}$, to give an upper bound of the number of points of $\tilde{\mathcal{T}}$, it is sufficient to study the number of points of $\tilde{\mathcal{C}}$. Recall that the root of the tree $\tilde{\mathcal{C}}$ is of type $i \in \mathbf{I}$. For $n \geq 1$, write $K_i(n)$ for the vector containing the numbers of ancestors in the n^{th} set of ancestors of a single point of index i . We consider

$$K_i(n) = \begin{pmatrix} \text{number of ancestors in the } n^{\text{th}} \text{ set of ancestors of type } 1 \\ \text{number of ancestors in the } n^{\text{th}} \text{ set of ancestors of type } 2 \\ \dots \\ \text{number of ancestors in the } n^{\text{th}} \text{ set of ancestors of type } N \end{pmatrix},$$

with the convention that $K_i(0) = e_i$ for every i . For every $j \in \mathbf{I}$, denote $X_i^j := K_i^j(1)$ the number of ancestors of type j in the first set of ancestors with initial point of type i . Moreover, as we discussed earlier, X_i^j is the cardinal of the points that a Poisson process of intensity Γ_j puts on $\pi_j(V_i)$, where V_i is the random neighborhood from a point of type i . In other words, if we denote \mathcal{P} the Poisson distribution, conditioning on $V_i = v$, we have

$$X_i^j \sim \mathcal{P}(\Gamma_j \mu(\pi_j(v))).$$

In addition, we denote $X_i = (X_i^1, X_i^2, \dots, X_i^N)^T$ and for any $\theta \in \mathbb{R}^N$, we consider the log-Laplace transform of X_i :

$$\phi_i(\theta) := \log \mathbb{E}_i \left(e^{\theta^T X_i} \right)$$

where we denote \mathbb{P}_i the law of a random tree \tilde{C} , whose root is of type i and \mathbb{E}_i the corresponding expectation.

For $n \geq 2$, given the population in the first set of ancestors X_i , we have the following relationship between the $(n-1)^{th}$ set of ancestors and the n^{th} set of ancestors in \tilde{C} ,

$$K_i(n) = \sum_{j \in \mathbf{I}} \sum_{p=1}^{X_i^j} K_j^{(p)}(n-1),$$

where $K_j^{(p)}(n-1)$ is the vector of the number of ancestors in the $(n-1)^{th}$ set of ancestors issued from the p -th point of type j in the first generation. In addition, for $p = 1, \dots, X_i^j$, we have that the $K_j^{(p)}(n-1)$'s are independent copies of $K_j(n-1)$. Note that this equation is trivial for $n = 1$.

Moreover, we consider

$$W_i(n) = \sum_{k=0}^n K_i(k)$$

the total number of ancestors in the first n set of ancestors.

The log Laplace transform associated to the random vector $W_i(n)$ is given by:

$$\Phi_i^{(n)}(\theta) := \log \mathbb{E}_i \left(e^{\theta^T W_i(n)} \right).$$

Finally we put $\Phi^{(n)}(\theta) = (\Phi_1^{(n)}(\theta), \dots, \Phi_N^{(n)}(\theta))^T$.

Denote the derivative of $\phi(\cdot)$ at 0 by the matrix $M = D\phi(0)$. Now, let us take a closer look to the function $\phi_i(\theta)$,

$$\phi_i(\theta) = \log \mathbb{E}_i \left(e^{\theta^T X_i} \right) = \log \mathbb{E}_i \left(\prod_{j=1}^N e^{\theta_j X_i^j} \right).$$

Therefore the gradient matrix M satisfies $M_{ji} = \mathbb{E}_i(X_i^j)$ for $i, j \in \mathbf{I}$. Recall that, conditioning on $V_i = v$, $X_i^j \sim \mathcal{P}(\Gamma_j \mu(\pi_j(v)))$, therefore we have

$$M_{ji} = \sum_k \Gamma_j \mu(\pi_j(v_k^i)) \lambda_i(v_k^i). \quad (4.2)$$

Introduce $W_i(\infty) := \lim_{n \rightarrow \infty} W_i(n)$, if it exists, and let $W_i = \mathbf{1}^T W_i(\infty)$ be the total number of points of \tilde{C} . The log-Laplace transform of $W_i(\infty)$ is given by

$$\Phi_i(\theta) := \log \mathbb{E}_i \left(e^{\theta^T W_i(\infty)} \right),$$

and we write again $\Phi(\theta) = (\Phi_1(\theta), \Phi_2(\theta), \dots, \Phi_N(\theta))^T$.

With this approach, we can prove the following exponential inequality. This result is inspired by Lemma 1 of [21]. Define $\|\cdot\|_1$ a 1-norm on \mathbb{R}^N and the ball with respect to this norm on \mathbb{R}^N by $B(\cdot)$.

Proposition 6. *Grant Assumption 6 and assume that the weights $(\lambda_i(\cdot))_{i \in \mathbf{I}}$ satisfy the following condition:*

$$\sup_{i \in \mathbf{I}} \sum_{k \geq 1} P(v_k^i) \lambda_i(v_k^i) < 1.$$

Suppose moreover that there exists a positive number r that depends on the matrix M such that, for all positive vectors θ belonging to $B(0, r)$, we have

$$\sup_i \phi_i(\theta) = \sup_i \sum_j \log \left(\sum_k \lambda_i(v_k^i) \exp[(e^{\theta_j} - 1) \Gamma_j \mu(\pi_j(v_k^i))] \right) < \infty.$$

Then for all $\theta \in B(0, r)$, we conclude that $\Phi_i(\theta) < \infty$ and moreover

$$\Phi(\theta) = \theta + \phi(\Phi(\theta)).$$

In particular, for $\vartheta \in \mathbb{R}_+$ such that $\theta = \vartheta \mathbf{1} \in B(0, r)$, there exists a constant c_0 that depends on M and i such that

$$\mathbb{P}(W_i > \mathbb{E}(W_i) + x) \leq c_0 e^{-\vartheta x} \quad (4.3)$$

for all $x > 0$.

Remark 12. *From Chapter V of [1], we have*

$$\mathbb{E}(K_i(n)) = M^n \mathbb{E}(K_i(0)) = M^n e_i.$$

Therefore, the total number of points simulated of the tree \tilde{C} with initial point of type i is

$$\mathbb{E}(W_i(n)) = \mathbb{E} \left(\sum_{k=0}^n K_i(k) \right) = \left(\sum_{k=0}^n M^k \right) e_i.$$

It leads us to conclude that,

$$\mathbb{E}(W_i) = \mathbf{1}^T \left(\sum_{k=0}^{\infty} M^k \right) e_i,$$

where the matrix M is defined in (4.2).

To conclude, if we think the complexity of the algorithm in terms of the number of simulated points, W_i is a good bound, and by (4.3) we see that we can grasp the complexity of the algorithm by studying (4.3).

Proof. First, we prove that $\phi_i(\theta)$ is well defined for all $\theta \in B(0, r)$. Indeed, since $(X_i^j)_{j=1, \dots, N}$ are independent, we have

$$\begin{aligned} \phi_i(\theta) &= \log \mathbb{E}_i \left(\prod_j \exp(\theta_j X_i^j) \right) \\ &= \sum_j \log \mathbb{E}_i \left(\exp(\theta_j X_i^j) \right). \end{aligned}$$

Moreover, by conditioning on the first random neighborhood of type i , V_i , we have

$$X_i^j | V_i = v_k^i \sim \mathcal{P}(\Gamma_j \mu(\pi_j(v_k^i))).$$

Therefore, we conclude that

$$\mathbb{E}_i \left(\exp(\theta_j X_i^j) | V_i = v_k^i \right) = \exp[(e^{\theta_j} - 1) \Gamma_j \mu(\pi_j(v_k^i))].$$

Finally, we obtain that

$$\phi_i(\theta) = \sum_j \log \left(\sum_k \lambda_i(v_k^i) \exp[(e^{\theta_j} - 1) \Gamma_j \mu(\pi_j(v_k^i))] \right) < \infty.$$

In the following, we prove that $\Phi_i^{(n)}(\theta)$ satisfies the following recursion

$$\Phi_i^{(n)}(\theta) = \theta^T K_i(0) + \phi_i(\Phi^{(n-1)}(\theta)).$$

Indeed, by definition of $W_i(n)$ we have

$$\mathbb{E}_i \left(e^{\theta^T W_i(n)} \right) = e^{\theta^T K_i(0)} \mathbb{E}_i \left(e^{\theta^T \sum_{k=1}^n K_i(k)} \right).$$

In addition, from the definition of $K_i(k)$ we obtain,

$$\begin{aligned} \mathbb{E}_i \left(e^{\theta^T \sum_{k=1}^n K_i(k)} \right) &= \mathbb{E}_i \left(e^{\theta^T \sum_{k=1}^n \sum_{j=1}^N \sum_{p=1}^{X_i^j} K_j^{(p)}(k-1)} \right) \\ &= \mathbb{E}_i \left(\prod_{j=1}^N e^{\theta^T \sum_{k=1}^n \sum_{p=1}^{X_i^j} K_j^{(p)}(k-1)} \right) \\ &= \mathbb{E}_i \left[\mathbb{E} \left(\prod_{j=1}^N e^{\theta^T \sum_{k=1}^n \sum_{p=1}^{X_i^j} K_j^{(p)}(k-1)} | X_i \right) \right]. \end{aligned}$$

Since conditioning on X_i , $K_j^{(p)}(k-1)$'s are independent, we have that

$$\begin{aligned}
\mathbb{E}_i \left(e^{\theta^T \sum_{k=1}^n K_i(k)} \right) &= \mathbb{E}_i \left[\prod_{j=1}^N \mathbb{E} \left(e^{\theta^T \sum_{k=1}^n \sum_{p=1}^{X_i^j} K_j^{(p)}(k-1)} \mid X_i^j \right) \right] \\
&= \mathbb{E}_i \left[\prod_{j=1}^N \prod_{p=1}^{X_i^j} \mathbb{E} \left(e^{\theta^T \sum_{k=1}^n K_j^{(p)}(k-1)} \mid X_i^j \right) \right] \\
&= \mathbb{E}_i \left[\prod_{j=1}^N \prod_{p=1}^{X_i^j} \mathbb{E} \left(e^{\theta^T \sum_{k=1}^n K_j^{(p)}(k-1)} \right) \right] \\
&= \mathbb{E}_i \left[\prod_{j=1}^N \left(\mathbb{E} \left(e^{\theta^T W_j(n-1)} \right) \right)^{X_i^j} \right] \\
&= \mathbb{E}_i \left[\prod_{j=1}^N e^{\Phi_j^{(n-1)}(\theta) X_i^j} \right] = \mathbb{E}_i \left[e^{\Phi^{(n-1)}(\theta)^T X_i} \right] = e^{\phi_i(\Phi^{(n-1)}(\theta))}.
\end{aligned}$$

Finally, we conclude that

$$\Phi_i^{(n)}(\theta) = \theta^T K_i(0) + \phi_i(\Phi^{(n-1)}(\theta)).$$

This equation holds for every i , therefore we have

$$\Phi^{(n)}(\theta) = \theta + \phi(\Phi^{(n-1)}(\theta)). \quad (4.4)$$

Let us consider the column sums of the matrix M :

$$\sum_{j \in \mathbf{I}} M_{ji} = \sum_{j \in \mathbf{I}} \mathbb{E}_i(X_i^j) = \sum_{k \geq 1} P(v_k^i) \lambda_i(v_k^i).$$

Hence, $\|M\|_1 = \sup_{\|x\|_1 \leq 1} \{ \|Mx\|_1 \} = \sup_i \sum_{j \in \mathbf{I}} |M_{ji}| < 1$ where $\|\cdot\|_1$ is the induced norm for matrix on $\mathbb{R}^{N \times N}$. Therefore, $\|D\phi(0)\|_1 \leq C < 1$. Moreover the norm is continuous and $D\phi(s)$ is likewise, there is a $r > 0$ such that, for $\|s\|_1 \leq r$,

$$\|D\phi(s)\|_1 \leq C < 1.$$

Hence, $\phi(s)$ is Lipschitz continuous in the ball $B(0, r)$ and moreover $\phi(0) = 0$, which implies that

$$\|\phi(s)\|_1 \leq C \|s\|_1$$

for $\|s\|_1 \leq r$.

Moreover, take θ such that $\frac{\|\theta\|_1}{1-C} \leq r$, hence $\|\theta\|_1 \leq r$. By induction we can show that

$$\|\Phi^{(n)}(\theta)\|_1 \leq \|\theta\|_1(1 + C + \dots + C^n) \leq r < \infty.$$

In addition, $(W_i(n))_{n \geq 1}$ is a positive, increasing vector sequence. Here, for any $u, v \in \mathbb{R}^N$, we say $u \geq v$ if $(u - v)$ is a positive vector.

For any θ such that $\frac{\|\theta\|_1}{1 - C} \leq r$, by using the theorem of monotone convergence, we have

$$\Phi^{(n)}(\theta) \rightarrow \Phi(\theta)$$

when $n \rightarrow \infty$. Moreover, $\|\Phi(\theta)\|_1 < \infty$, and passing to the limit $n \rightarrow \infty$ in equation (4.4), we conclude that

$$\Phi(\theta) = \theta + \phi(\Phi(\theta)).$$

In particular, choosing $\theta = \vartheta \mathbf{1} \in B(0, r)$ with $\vartheta \in \mathbb{R}$, we have

$$\mathbb{E}_i(e^{\vartheta W_i}) < \infty,$$

where $W_i = \mathbf{1}^T W_i(\infty)$ is the total number of point of \tilde{C} .

The last point is concluded by using Markov's inequality, that ends the proof. \square

4.6 Efficiency of the algorithm and discussion of the choice of the weights η 's on a particular example

Finally, to illustrate the effect of η to the complexity of Perfect Simulation algorithm, we consider the age dependent Hawkes process with hard refractory period of length δ . We consider the setting of Proposition 2 and assume moreover that $\psi_i(\cdot)$ is an L -Lipschitz function, where L is the Lipschitz constant. In this case we obtain an explicit upper bound for $\Delta_k^i(x)$, for any $x \in \mathcal{X}^{>\delta}$.

Proposition 7. *For any $k \geq 1$, we have*

$$\Delta_k^i(x) \leq L \times \left[\sum_{j \in V_i(k) \setminus V_i(k-1)} \sum_{0 \leq m < k} h_{ji}(m\delta) + \sum_{j \in V_i(k-1)} h_{ji}((k-1)\delta) \right].$$

To prove this proposition we use the following lemma, which is a particular case of Lemma 2.4 in [20].

Lemma 1. *For any $0 \leq k < l, j \in \mathbf{I}, z \in \mathcal{X}^{>\delta}, t \in \mathbb{R}$ we have*

$$\int_{t-l\delta}^{t-k\delta} h_{ji}(t-s) dz_s^j \leq \sum_{k \leq m < l} h_{ji}(m\delta).$$

Proof of Lemma 1. For any j , fixed $\epsilon > 0$, we have

$$\begin{aligned} \int_{[t-l\delta, t-k\delta-\epsilon]} h_{ji}(t-s) dz_s^j &= \sum_{k \leq m < l} \int_{[t-(m+1)\delta-\epsilon, t-m\delta-\epsilon]} h_{ji}(t-s) dz_s^j \\ &\leq \sum_{k \leq m < l} h_{ji}(m\delta + \epsilon) \end{aligned}$$

by Assumption 3 and the fact that there is at most one jump in the interval of length δ . Therefore,

$$\begin{aligned}
\int_{[t-l\delta, t-k\delta)} h_{ji}(t-s) dz_s^j &= \lim_{\epsilon \downarrow 0} \int_{[t-l\delta, t-k\delta-\epsilon]} h_{ji}(t-s) dz_s^j \\
&\leq \lim_{\epsilon \downarrow 0} \sum_{k \leq m < l} h_{ji}(m\delta + \epsilon) \\
&\leq \sum_{k \leq m < l} \lim_{\epsilon \downarrow 0} h_{ji}(m\delta + \epsilon) \\
&\leq \sum_{k \leq m < l} h_{ji}(m\delta),
\end{aligned}$$

by using the theorem of monotone convergence and the fact that $h_{ji}(\cdot)$ is a decreasing function according to Assumption 3. This completes the proof of Lemma 1. \square

Proof of Proposition 7. By (ii) of Assumption 3, we have

$$\Delta_i^k(x) \leq L \times \left[\sum_{j \in V_i(k) \setminus V_i(k-1)} \int_{-k\delta}^0 h_{ji}(-s) dx_s^j + \sum_{j \in V_i(k-1)} \int_{-k\delta}^{-k\delta+\delta} h_{ji}(-s) dx_s^j \right].$$

Applying Lemma 1 we conclude that

$$\Delta_i^k(x) \leq L \times \left[\sum_{j \in V_i(k) \setminus V_i(k-1)} \sum_{0 \leq m < k} h_{ji}(m\delta) + \sum_{j \in V_i(k-1)} h_{ji}((k-1)\delta) \right]$$

which ends the proof. \square

Remark 13. In addition, from (i) of Assumption 3, we have

$$\sum_{j \in V_i(k) \setminus V_i(k-1)} \sum_{0 \leq m < k} h_{ji}(m\delta) + \sum_{j \in V_i(k-1)} h_{ji}((k-1)\delta) \rightarrow 0$$

as $k \rightarrow \infty$. Therefore $\Delta_k^i(x) \rightarrow 0$ when $k \rightarrow \infty$.

In what follows, to simplify the computation, we consider that $\mathbf{I} = \mathbb{Z}$, and for all i , we set

1. $h_{ji}(t) = \beta_{ji} \exp(-\alpha t)$ where β_{ji}, α are positive constants for all j, i . In addition, we take $\alpha = \frac{1}{\delta}$ and $\beta_{ji} = \frac{1}{2|j-i|^\gamma}$ for $j \neq i$ with a positive number γ and $\beta_{ii} = 1$.
2. $V_i(0) = \emptyset, V_i(1) = \{i\}, \dots, V_i(k) = \{i-k+1, \dots, i, i+1, \dots, i+k-1\} \forall k \geq 2$.

3. $\eta_k^i = \eta_k = c_\eta \frac{1}{k^p}, \forall k \geq 1$, where p is a positive constant and c_η is a normalization constant.

From Proposition 7, we can choose

$$\begin{aligned} \Gamma_i &= \sup \left\{ \frac{L}{c_\eta}, \sup_{k \geq 2} \left[\frac{Lk^p}{c_\eta} \left(\frac{1 - e^{-k}}{1 - e^{-1}} \sum_{j \in \{i-k+1, i+k-1\}} \beta_{ji} + e^{-k+1} \sum_{j \in V_i(k-1)} \beta_{ji} \right) \right] \right\} \\ &= \sup \left\{ \frac{L}{c_\eta}, \sup_{k \geq 2} \left[\frac{Lk^p}{c_\eta} \left(\frac{1 - e^{-k}}{(k-1)^\gamma (1 - e^{-1})} + e^{-k+1} \left(1 + \sum_{m=1}^{k-2} \frac{1}{m^\gamma} \right) \right) \right] \right\} := \Gamma. \end{aligned}$$

Since we define η_k^i are independent of i , so we can consider that all points have the same type. Instead of comparing the Backward Steps to a multitype branching process, we may therefore put $K(k) := K_i(k) := \sum_j K_i^j(k)$ and $W(n) := W_i(n) := \sum_j W_i^j(n)$ such that $K(k)$ is the total number of ancestors in the k^{th} set of ancestors and $W(n)$ is the total number of ancestors in the first n set of ancestors. Hence, all the vector $K(k) := K_i(k)$ and $W(n) = W_i(n)$ in Proposition 6 now are numbers.

Moreover, $v_k^i = V_i(k) \times [-k\delta, 0)$ and we set

$$\begin{aligned} \zeta &= \sum_{k' \geq 1} P(v_{k'}^i) \lambda(v_{k'}^i) = \sum_{k' \geq 1} \eta_{k'} k' \delta \sum_{j \in V_i(k')} \Gamma_j \\ &= \sum_{k' \geq 1} c_\eta \frac{1}{k'^p} k' \delta (2k' - 1) \Gamma := f(p). \end{aligned}$$

By a classical well-known result in Branching processes [1], we have for $k \geq 1$

$$\mathbb{E}(K(k)) = \zeta^k.$$

Therefore, the total expected number of simulated points $W := W(\infty)$ is given by

$$\mathbb{E}(W) = \frac{1}{1 - \zeta}.$$

Now we are looking for values of p such that:

- (a) $(\eta_k)_k$ defines a probability,
- (b) the algorithm stays in \mathcal{Y} ,
- (c) the Kalikow decomposition exists (Proposition 2),
- (d) the branching process goes extinct in finite time almost surely.

Once all these condition are fulfilled, we choose p such that it minimizes $f(p)$, so that the total number of simulated points is minimal.

$$(a) : \sum_k \eta_k = 1 \Rightarrow p > 1$$

$$(b) : \Gamma < \infty \Rightarrow p < \gamma$$

$$(c) : \sum \int h_{ji} < \infty \Rightarrow \gamma > 1$$

$$(d) : \zeta < 1 \Rightarrow p - 2 > 1$$

Finally, we conclude that $3 < p < \gamma$. Moreover, we want to choose p such that $f(p)$ to be smallest possible, which means that we need to take p largest possible. Hence, we take p close to γ .

5 Conclusion

Prior research has investigated the Perfect Simulation based on a conditional Kalikow decomposition in continuous time point processes [13]. However, it is impossible to implement this study in practice. In the present study, we continue our work in [17], to extend the class of point processes which satisfies a new Kalikow decomposition. This decomposition plays a vital role to build a tractable algorithm in practice. In addition, we also improve the results in [13] and [17] on the existence of the Kalikow decomposition. Most notably, this is the first study to our knowledge to establish a Kalikow decomposition for a stochastic intensity in general context. Our results provide a general method to write a Kalikow decomposition for large class of point processes, for example: linear Hawkes processes, exponential Hawkes processes and including very complex Hawkes processes: age dependent Hawkes processes. However, some limitations are worth noting. Although, we succeed to write the Kalikow decomposition for a variety of Hawkes processes, we still have to restrict ourselves to a bounded intensity to implement the Perfect Simulation. This is due to the simulation of the first jump in the Backward Steps. Future work should focus on removing totally the upper bound of the intensity and extending the Perfect Simulation to the unbounded intensities.

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