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Compartmentalisation of mathematical sectors: the case of continuous probability distributions and integrals

Thomas Hausberger · Charlotte Derouet · Reinhard Hochmuth · Gaetan Planchon

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Abstract This paper investigates the phenomenon of compartmentalisation of knowledge in the teaching and learning of continuous probability distributions and integral calculus at the secondary-tertiary transition in France. Using the Anthropological Theory of the Didactic (ATD), and in particular the key notion of praxeology, we investigate in which sense those two sectors may be described as compartmentalised in current textbooks. We then study, by means of a questionnaire, the educational effects of the compartmentalisation: do students' difficulties in completing "bridging tasks" (tasks that require to relate the two sectors) reflect the partial disconnections revealed by the praxeological analyses? The key notion of ostensive, combined with the role played by the technology in the sense of ATD, is used to interpret the data. Altogether, this study sheds light on the deficit of cognitive flexibility required to change mathematical sectors, which is understood as a result of deficient praxeologies developed within the institutions.

Keywords Continuous probability distributions · Integral calculus · Secondary-tertiary transition · Compartmentalisation of knowledge · Anthropological Theory of the Didactic

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1 Introduction

Continuous probability distributions (CPD) and integral calculus (IC) are two topics that are taught in France during the last year of upper high school (grade 12) in the scientific track (16-18 year olds). They constitute two sectors (in the sense of the Anthropological Theory of the Didactic, ATD) that belong to two different but closely related mathematical domains, probability theory and analysis respectively. Indeed, the continuous probability of an event and the definite integral with respect to a non-negative function are both defined as areas of suitable two-dimensional domains in the syllabus (in France; see section 3.2), and the formula

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

is the key for solving several standard tasks in CPD, where X represents a random variable and f its associated density function.

Much research is focused on the study of the teaching and learning of integral calculus at the end of high school and at the beginning of the university. Several researchers have worked on the links of the integral concept with the idea of accumulation (Thompson & Silverman, 2008; Kouropatov & Dreyfus, 2013, 2014), its Riemann sum interpretation (Wagner, 2018) or an introduction via the Fundamental Theorem of Calculus (Rosenthal, 1992; Thompson, 1994). In comparison, there is very little literature on CPD and, moreover, the available studies focus mainly on normal distributions (Batanero, Tauber & Meyer, 1999; Wilensky, 1997; Batanero, Tauber & Sánchez, 2004; Pfannkuch & Reading, 2006). Therefore, the teaching-learning phenomena generated by the interrelationship between CPD and IC are still to be investigated. A first stone was laid by Derouet and Parzys (2016) and Derouet (2019), who studied possible ways to introduce the density function at grade 12 such that students may construct this concept starting from considerations regarding histograms, and therefore might relate continuous probability to the integral.

In this paper, we consider aspects of interrelation between IC and CPD knowledge as a phenomenon that takes place within institutions (universities, schools, etc.) and therefore use ATD¹ as theoretical framework (see section 2). Our goal is to study the impact of choices made at the level of the teaching institution on the learning of mathematics: are students in France able to mobilise the IC sector to solve tasks in the CPD sector and vice versa at the transition from high school to higher education? In other words, may students' difficulties be interpreted as consequences of a form of institutional compartmentalisation of knowledge?

In cognitive psychology, the phenomenon of compartmentalisation of knowledge is generally understood as the composition of knowledge in separate and not intertwined parts. Mandl, Gruber and Renkl (1993), for example, have

¹ This theoretical framework puts into the fore the notion of *institution* and its consequences on teaching and learning.

differentiated between “three types of knowledge compartmentalisation: compartmentalisation of incorrect and correct concepts, compartmentalisation of several correct concepts, and compartmentalisation of symbol systems and real world entities” (p. 162). Our work relates to the first two types: we examine through a questionnaire on the one hand if correct concepts and practices from the two sectors CPD and IC are available and on the other hand if the students are able to connect these sectors.

In mathematics education research, the phenomenon of compartmentalisation has been considered in the context of geometric figures (Schoenfeld, 1988), the concept of addition of fractions (Vinner, Hershkowitz & Bruckheimer, 1981), the image and definition of functions (Vinner & Dreyfus, 1989) and the concept of quadratic functions (Eraslan, 2007). Using the ATD framework, phenomena of compartmentalisation have been investigated in Barbé, Bosch, Espinoza and Gascón (2005). In the domain of limits of functions, the authors could identify two completely disconnected mathematical organisations within the knowledge to be taught in Spanish high schools: a practical block of the algebra of limits without a technological-theoretical block and a knowledge block of the topology of limits, where the practical block was absent. This model was later extended by Winsløw (2015) to the case of the derivative functions and integrals and emphasised similar disconnections from an analysis of Danish textbooks. Recently, Wijayanti (2019) investigated compartmentalisation phenomena in Indonesian college textbooks regarding the two domains geometry and arithmetic. Finally, Kondratieva and Winsløw (2018) explored various possibilities in teaching to use theoretical concepts from analysis for a deeper understanding of somehow compartmentalised calculation objects such as angle, cosine and sine. The research is driven by Klein’s idea of a “Plan B” (a more holistic approach to mathematical domains) by opposition to Plan A which is described as the standard compartmentalised approach to mathematics into more or less self-contained entities.

As far as we know, there is no work to date offering a detailed ATD analysis of relationships between CPD and IC and, what is more, combining such an analysis of knowledge areas with a quantitative empirical analysis of data from student tests. The present paper expands earlier work presented at the INDRUM2018 conference (Derouet, Planchon, Hausberger & Hochmuth, 2018) by deepening both the theoretical framework, the praxeological reference model (in the sense of ATD, see section 2) and the data analysis. Although the data is limited to the French context, the literature review underlines that phenomena of compartmentalisation may be observed worldwide. Our methodology and tools for the analysis and interpretation of data related to knowledge compartmentalisation (in particular the ATD notion of ostensive which is new in the literature in such a use) may therefore be applied to a wide range of contexts. The relevance to university mathematics education research is not limited to the study of the secondary-tertiary transition as is evidenced by the works inscribed in the legacy of Klein who alerted to the educational effects of knowledge compartmentalisation at university level as early as 1908.

After the presentation of theoretical constructs from ATD, we describe the praxeological reference model (in the sense of ATD, see section 2) that we elaborated for the types of tasks in CPD and IC with a view to studying interrelations of the two sectors and in particular elucidate what is meant by compartmentalisation of knowledge in this case. We then present our empirical study: its methodology which builds on the elaboration of a questionnaire, the analysis of the data and interpretation of results using the theoretical framework. We finally draw some conclusions and perspectives opened up by this study.

2 Theoretical framework and research questions

In this section, we present the main theoretical constructs from ATD, namely the notions of *praxeology* and *praxeological reference model* (PRM) for a body of knowledge, and the notion of *ostensive* that we will relate to the didactic phenomenon of compartmentalisation. This theoretical background will provide us with the necessary tools and vocabulary to accurately state our research questions at the end of the section.

2.1 The modeling of knowledge according to ATD

ATD “postulates that any activity related to the production, diffusion or acquisition of knowledge should be interpreted as an ordinary human activity, and thus proposes a general model of human activity built on the key notion of praxeology” (Bosch & Gascón, 2014, p. 68). A praxeology II is represented by a quadruple $[T/\tau/\theta/\Theta]$: its praxis part (or know-how) consists of a type of tasks T together with a corresponding technique τ (useful to carry out the tasks $t \in T$ in the scope of τ). The logos part (or know-why) includes two levels of description and justification: the technology θ , i.e. a discourse on the technique, and the theory Θ , its ultimate foundation, which often unifies several technologies.

The elaboration of a reference epistemological model (Florensa, Bosch & Gascón, 2015) as sequences of praxeologies, for a given body of knowledge, is an important step in any research carried out in the ATD framework. It is the tool that will be used by the researcher to describe, analyse, put in question or design the specific contents that are at the core of a teaching and learning process. In order to build such a model, “mathematical praxeologies are described using data from the different institutions participating in the didactic transposition process, thus including historical, semiotic and sociological research, assuming the institutionalized and socially articulated nature of praxeologies” (loc. cit., p. 2637).

The elucidation of conditions that foster or conversely hinder the development of mathematical praxeologies within an institution is called, in ATD, the ecological study. The recognition of a large panel of influences at various levels

came to the theoretical construct of *scale of levels of didactic co-determinacy* (Bosch & Gascón, 2014, p. 73). Restrictions may indeed be related to the 1. subject (e.g. the standard normal distribution) 2. theme (normal distributions) 3. sector (continuous probability distributions) 4. domain (probability and statistics) 5. discipline (mathematics) 6. pedagogy (local teaching principles) 6. school (teaching institution) 7. society (state, ministry, region) 8. civilisation (western culture). This scale, below the level of the discipline, acknowledges the standard institutional division of the mathematical knowledge into a hierarchy of topics. It reflects both the division of mathematics (as a field of research) into increasingly specialised mathematical domains and sub-domains (as evidenced by the AMS classification) and the further division of the knowledge to be taught (the product of the didactic transposition of the scholarly knowledge) into teaching units. Furthermore, this hierarchy may be related to the components of praxeologies as follows: the theory usually gives a sector its unity, and the technology unifies a theme; topics are usually dedicated to working out a single type of tasks. In other words, the set of praxeologies that make up a theme may often correspond to what is called in ATD a *local mathematical organisation* (praxeologies unified under a common technology) and the set of all the praxeologies of a sector correspond to a *regional mathematical organisation* in the sense of ATD, which means that a common theory unifies all the praxeologies of the given sector.

In this article, we use this framework and speak of continuous probability distributions (CPD) and integral calculus (IC) as two sectors of the mathematical domains “probability and statistics” and “analysis”, respectively. This separation reflects the official syllabus at the last year of upper high school in France (the equivalent of grade 12) and the division into chapters in corresponding standard textbooks. But ATD does not take the knowledge to be taught for granted: this is where historical epistemology comes into play. In the context of compartmentalisation of knowledge, the epistemological study aims at providing a basis that may serve to problematise this division and its effects on the mathematical praxeologies developed within the teaching institution. The resulting praxeological reference model will be a sequence of mathematical praxeologies for the combined sectors CPD and IC that accounts for the observed existing or missing connectedness in a body of standard textbooks (see section 3).

2.2 Theoretical ideas on compartmentalisation based on the notion of ostensive

The ostensive/non-ostensive distinction introduced by Bosch and Chevallard (1999) is rooted in the modelling of mathematical activity as involving both signs and concepts. The adjective *ostensive* refers to any object “having a sensitive nature, a certain materiality, and which, in fact, acquires for the human subject a perceptive reality” (loc. cit. p. 10, our translation). In mathematics, signs are often signifying abstract entities: non-ostensive elements such as

ideas, intuitions and concepts, when they become socially shared, acquire the status of *non-ostensive* objects within an institution. This is the case for the mathematical objects “functions”, “primitive of a function”, “definite integral of a continuous function between two boundaries” or “probability of an event” within the school institution.

The distinction, once posed, immediately leads by its very definition to a dialectical game: non-ostensive objects emerge from the manipulation of ostensives, while this manipulation is guided and controlled by non-ostensives. This combined operation takes place at both the praxis and logos level of praxeologies, as technological discourse is being materialised through language and in particular mathematical signs.

Ostensive objects have, in institutional practices, “the power to evoke complexes of ostensive and non-ostensive objects with which they interrelate” (loc. cit., p. 25), what Bosch and Chevillard call “semiotic valence” of ostensive objects. All the praxeologies of the institution in which the ostensive participates as an instrument may be evoked. This is why semiotic valence and instrumental valence (what the ostensive effectively allows the subjects of the institution to accomplish) are linked. For example, dy/dx may or may not be manipulated as a quotient, depending on whether or not such manipulation techniques are available. The ostensive can therefore acquire instrumentality, if an appropriate technological work is undertaken to legitimise and control such techniques, or, on the contrary, lose it if the meaning of such a use is lost, i.e. if the technologies acquired in relation to this use become obsolete.

ATD thus posits that mathematical knowledge is conveyed through praxeologies that can be seen and accounted for through the ostensives that compose the tasks, techniques, technologies and theories. Each praxeology has a minimal system of ostensives in which it is set and some of which will serve as an activator when solving a task or a problem, through the semiotic valence of the ostensive. Accordingly, techniques that are applicable and known to the students may not be mobilised, not necessarily because of a loss of meaning, but because of a lack of activation. It is therefore a question of a loss of the operative valence of ostensives which is at stake, following a loss of their semiotic value. Several situations are conceivable: there may be an absence of one of the ostensives of the minimal system mentioned above. For example, the symbol P plays a fundamental role in the activation of praxeologies in probability theory. Or there may be competition from different ostensives, which refer to different sectors, therefore to different complexes of praxeologies. The learner then favours one of the sectors and fails to link the task to the expected praxeology that comes from the other sector. Such a case of complexification of the ostensive systems resulting in possible competition occurs, for example, with the ostensive $\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ which engages the integral concept as non-ostensive through the symbol \int_a^b and, perhaps less visibly or ostensibly, the non-ostensive “continuous probability distribution” through the density function $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ of the normal distribution.

In a similar way to the case of the differential highlighted by Bosch and Chevallard (1999), it can be hypothesised that technology (in the sense of ATD) plays a crucial role as a facilitator or inhibitor of the operative valence of ostensives. In the case described above, the issue is about being able to motivate the introduction of a missing ostensive (such as the symbol P), or to transfer techniques from one sector to another when the work has been placed by the learner in a given sector following her or his interpretation of ostensives. Indeed, technology guides and legitimizes the choice of techniques. It is reasonable to think that work at the level of the technological discourse, or even at the level of the theory which provides an epistemological foundation to the various praxeologies, constitutes a major lever for decompartmentalising praxeologies and encouraging their activation.

2.3 Research questions

We are now ready to set out the research questions that may be organized around two main issues:

1. *Praxeological characterisation of the CPD-IC compartmentalisation.*
CPD and IC are taught at the secondary-tertiary transition as two sectors of different mathematical domains, so they are compartmentalised in an obvious sense. What can we say on a deeper level? We make the hypothesis that the division of the curriculum in units as a result of the didactic transposition may lead to the teaching of mathematical organisations that are partially disconnected. This corresponds to what French sociologist Verret (1975, p. 140) called the *desynchronisation* of knowledge. Our first set of research questions will therefore be: how to characterize the compartmentalisation of CPD and IC in terms of praxeologies taught at grade 12 in France? With respect to the 4 components of main praxeologies in both sectors, what kinds of links are developed and what kinds of partial disconnections may be revealed?
2. *Effects of this compartmentalisation on the learning of mathematics.*
As an educational consequence of the partial disconnection of praxeologies, students may be incapable of transferring a technique from one sector to the other. We wish to document this phenomenon and provide explanatory hypotheses for such difficulties. Specifically, we make the hypothesis that ostensives play a major role in the activation of sectors and associated praxeologies. Our second set of research questions is therefore the following: are students able to use praxeologies from IC in the context of CPD and conversely (the converse being the more interesting case)? Are the data in line with the hypothesis that ostensives impact students' abilities to transfer techniques? Do students' difficulties reflect the partial disconnections revealed by the praxeological analysis?

3 A praxeological model for the interrelation of IC and CPD

In this section, we begin with an epistemological study on IC and CPD and their interrelation in the history of mathematics, and then present the results of our praxeological analyses of the chapters dedicated to IC and CPD in standard grade 12 textbooks in France. The combination of the two studies will allow us to draw our praxeological reference model (PRM), which will be discussed in order to answer the first set of research questions stated above.

3.1 Epistemological study

This study is based on the work of historians, mainly Jahnke (2003), Michel (1992) and Bourbaki (2007) for IC and Stigler (1986) for CPD. We also refer to high school curriculum reforms in the last decades to address the didactic transposition processes.

IC finds its roots in practical problems such as the measurement of agricultural land. These roots are related to what Freudenthal (1983) called “didactical phenomenology of mathematical structures” (although he himself did not treat the case of IC in that book). As early as 500 BC, Greek geometers were able to solve problems of quadratures for polygons: the area was determined by reduction to a square using geometric construction (Jahnke, 2003, p. 14). The axiom of continuity and the principle of exhaustion were used by Euclid to show that the disc area is proportional to the square of the diameter. Archimedes and then Cavalieri have proposed systematic methods for calculating flat areas by infinitesimal “slicing” and then infinite summation, which is now interpreted as an integral (Bourbaki, 2007, p. 3).

It was Cauchy who gave the first rigorous definition of the integral of a continuous function on a segment $[a; b]$, and thus founded the contemporary theory of integration (Dieudonné, 1992, p. 256). Bringing continuity on the fore, he provided a converging process for continuous functions by division of the interval, summing and going to the limit. This also solved the problem of existence of primitives for continuous functions. Indeed, Cauchy proved that the “area function” of a continuous function may serve as anti-derivative. This area function is the limit of the so-called Cauchy sums. This thus formalises the link between the area under the curve and the definite integral. The Cauchy integral of continuous functions easily extends to the case of piece-wise continuous functions. It is the integral that is taught in high school.

Riemann further investigated the type of functions for which the Cauchy process converged. He found a necessary and sufficient condition and stated what is nowadays called the Riemann criterion for integrability (Dieudonné, 1992, p. 259). This was a kind of culmination of the theory of integration, but at the same time it opened up a perspective of important development for the newborn measure theory. Indeed, the generalisation of the Riemann integral to multivariate functions was not straightforward, which prompted

Peano and Jordan to develop a new theory of measurement to address this problem (Michel, 1992).

The emergence of CPD may be situated in the 18th century with the theory of errors of measures in physics. In fact, the first explicit links between integrals and stochastics appeared in Laplace's works: the integral $\int_0^T e^{-t^2} dt$ was used to express the probability that the success in a Bernoulli test lies within a given interval centred on the observable frequency. Since a primitive was not known in terms of elementary functions, the values of the function were approximated using Daniel Bernoulli's table. But, although this has established a link between IC and CPD, we cannot consider that the notion of probability density function emerged from this work.

In 1757, Simpson introduced a link between the notions of area and probability, in relatively contemporary terms (Stigler, 1986, p. 96). He presented a geometrical model for the law of errors related to a measuring device. Laplace later formalised the relation between probability distribution and area under the curve. He highlighted conditions that nowadays define a density function. During the 19th century, histograms were drawn, which led to new curves of probability density functions. The Gaussian distribution was proposed by Gauss in 1809.

It was the development of measure theory which allowed the modern theory of probability to emerge. In fact, Borel pointed out in 1905 applications to probability theory the axiomatic definition of which was established later by Kolmogorov in 1933 (Jahnke, 2003, p. 282). It was Lebesgue, a student of Borel, who extended Borel's measure and drew connections between the Riemann integral and the newborn Lebesgue integral. As Bourbaki (2007, p. 287) pointed out, the progress of integration theory has allowed the development of probability theory: "formerly a pretext for riddles and paradoxes, probability has become a branch of integration theory since its axiomatisation by Kolmogorov, but an autonomous branch with its own methods and problems".

To summarise, the history of mathematics reveals close links between IC, CPD, and the measurement of magnitudes. The latter has served to problematise the development of IC and CPD and measure theory may be seen as a unifying axiomatic theory of both mathematical sectors. Let us now conclude by showing how these connections are reflected in the curriculum in France.

Since the "modern math" reform in the 1960s, the upper high school curriculum on IC has evolved in several steps. Since 2002, the definition of the integral of a positive continuous function has been the area under the curve. The current 2011 official syllabus² further specifies: "We rely on the intuitive notion of area encountered in lower high school and on the properties of additivity and invariance by translation and symmetry" (p. 7). Continuous random variables and associated probability density distributions are also defined in the 2011 syllabus from the notion of area. Thus, the notion of area has become central in the interrelation of IC and CPD.

² https://cache.media.education.gouv.fr/file/special_8_men/98/4/mathematiques_S_195984.pdf

3.2 Identification of praxeologies in standard textbooks

In France, students encounter both integrals and continuous probability distributions for the first time in grade 12. We begin by analysing the official syllabus (published in 2011) that follows a division into mathematical domains such as analysis (essentially calculus) and probability and statistics, and then provides a further division into two levels of headings. By a cross-analysis with textbooks, we relate this divisions to the ATD model of subjects/themes/sectors. Finally, we give a detailed account of the main praxeologies in IC and CPD encountered in the 12 textbooks that were in use in French classrooms at the time our experiment was conducted (2018). In particular, solved exercises allowed us to identify the specific techniques and related technological discourses.

The domain “analysis” contains several headings: sequences, limits of functions, continuity, complements in differential calculus, the sine and cosine functions, the exponential, the logarithm, and finally integration. In every textbook, integration is a full chapter. We consider it as a sector in the sense of ATD and call it IC, since we will see that types of tasks are mostly computational. Differential calculus is another sector, and both are related by the fundamental theorem of calculus (FTC) that is a major technological element of the IC logos.

The main subheadings of integration are the following: definition of the definite integral of a continuous and positive function (as the area under the curve), the FTC, primitive functions, extension to the case of general continuous functions, properties of the integral (linearity, positivity, additivity on intervals), mean value of a function (with application to extra-mathematical contexts). These subheadings suggest an organisation of the IC sector into 3 main themes. A first theme is dedicated to the definition of the definite integral in relation with areas. A second theme relates to anti-derivatives and their calculation. Both themes are connected by means of the FTC the proof of which uses the definition of the integral and properties of areas, thus technological elements that belong to theme 1. The third theme concerns the extension to continuous functions without restriction on the sign. The syllabus suggests that types of tasks should remain centred on the exact and approximate computation of integrals (or areas), yet powered by tools from the second theme. The subheading “properties of the integral” is related to the approximation of integrals: in ATD terms, these properties belong to the technology that justifies the praxis of approximation. Finally, the last subheading “mean value of a function” may lead to a last topic of theme 3 or to a new theme, depending on the emphasis given to applications outside mathematics.

Let us now describe the syllabus corresponding to the probability and statistics domain. It comprises 4 main headings: conditioning and independence, continuous probability distributions from examples, fluctuation intervals³, estimation. What we call the CPD sector thus corresponds to the second

³ The syllabus makes the distinction between “fluctuation” and confidence intervals. In the former, the frequency is to be estimated from the probability which is given; in the latter, the situation is reversed.

heading. We chose to call the sector CPD rather than “continuous probability theory” since the official syllabus insists on a bottom-up approach that focuses on concrete examples and their use to model phenomena rather than theoretical development. The subheadings are the following: continuous probability distribution on a bounded interval, the uniform distribution, its expectancy, exponential distributions, their expectancies (defined in terms of integrals), the standard normal distribution, the de Moivre-Laplace theorem (the Gauss distribution is introduced as a reasonable approximation of the binomial distribution $B(n, p)$ for large n), normal distributions in general. The Gauss distribution is used later to introduce students to inferential statistics (under the heading “fluctuation”): in ATD terms, it is – together with the de Moivre-Laplace theorem – a crucial element of the technology related to the praxeology the type of task of which is the determination of a confidence interval with confidence level 95%. In fact, the main types of tasks underlined in the syllabus (in terms of competencies) with respect to the uniform and exponential distributions is the (exact) calculation of a probability or an expectancy, which gives theme 1. Formulas are known, so the techniques can be directly borrowed from IC and the definition of continuous probability distributions in relation with areas (thus integrals) mainly serves as a logos. By contrast, in the case of the normal distributions, the types of tasks deal with the approximate computation of the probability of events and the syllabus mentions the use of software, thus a radical change in the techniques. The normal approximation of a binomial distribution is also a new type of task. These comments justify to call “normal distributions” a separate theme 2.

Let us now formulate more precisely the preceding discourse and account for the praxeologies as they are encountered in the textbooks. Let us also recall that our main goal is to study the interrelation of IC and CPD, which explains why we omit praxeologies that are not relevant. In coherence with the analysis of the syllabus, we denote T_I and T_P (related to IC and CPD respectively) the following main general types of tasks:

- T_I : compute an integral $\int_a^b f(x) dx$ for a positive continuous function f ;
- T_P : compute a probability $P(a \leq X \leq b)$ for a random variable X endowed with a density function f .

The two types of tasks are immediately related by the well-known formula:

$$P(a \leq X \leq b) = \int_a^b f(x) dx. \quad (\text{FFCPD})$$

We call it the fundamental formula of CPD (FFCPD). In view of the syllabus and the epistemological study, we also introduce a third main type of task, related to the measurement of magnitudes :

- T_A : compute an area.

The type of tasks T_I may be further split into two types of tasks, depending on the expected result: an exact value ($T_{I,exact}$) or an approximation ($T_{I,approx}$), and the same holds for the types of tasks T_P and T_A . In fact, there

is an interplay between the three types of tasks and their variants throughout the chapters on IC and CPD. We elaborate on this in the following section.

In the first theme of IC, $T_{A,exact}$ comes first in the fore, with elementary techniques that involve the formulae for elementary geometrical domains (e.g. a trapeze, a half disk). We call $\Pi_{A,elem}$ the corresponding praxeology. In order to compute an approximate value for the area under a curve, textbooks (e.g. “Math’x”, p. 245) emphasize the “rectangle method” that is thus taught as the standard technique τ_{rect} for the type of task $T_{A,approx}$. In view of the definition of the integral that is introduced in the context of such tasks, it is equivalent to compute either the definite integral or the area. We will thus call the praxeology Π_{rect} without reference to integration or areas. Whenever the limit of Riemann sums may be computed (e.g. in the case of the parabola), τ_{rect} may be enriched to produce an exact value. Otherwise, it is possible to implement the method in a software program. We will call Π_{rect}^{instr} this instrumented variant.

By contrast, $T_{I,exact}$ is the dominant type of task in the third theme of IC. The corresponding technique may be denoted $\tau_{I,prim}$ since it resorts to praxeologies dedicated to the computation of primitive functions. The standard technique at high school level is to use the “tabular of anti-derivatives”; the technology comprises the properties of the derivative and the theory is that of differential calculus. We call $\Pi_{I,prim}$ the praxeology corresponding to $T_{I,exact}$ and $\tau_{I,prim}$. Although its technology comprises properties of the definite integral such as additivity on the interval, the logos remains largely hidden under the algebraic computational aspects. This resonates with the findings of Winsløw (2015) in his analysis of Danish textbooks. As an application, the type of task $T_{A,exact}$ then reappears with more advanced examples such as domains delimited by two curves of given functions. This means that $T_{I,exact}$ is a sub-type of task used to solve these tasks, with technique $\tau_{I,prim}$. We shall denote $\Pi_{A,prim}$ the corresponding praxeology. Approximate values for such areas may also be obtained by means of a calculator (or software), more or less a blackbox, used to compute approximations of definite integrals. This instrumented praxeology will be denoted $\Pi_{I,approx}^{instr}$. We note that there is also an instrumented version $\Pi_{I,exact}^{instr}$ of $\Pi_{I,prim}$, that again relies on a blackbox (the algorithm used by the computer algebra system). Figure 1 provides an excerpt of a textbook and illustrates the hermeticity of such techniques that amount to clicking on specific buttons.

An interesting example of work on Π_{rect} and Π_{rect}^{instr} can be found in the textbook “Transmath” (figure 2). A specific meta-discourse on the function $f(t) = e^{-t^2/2}$ underlines both that this function admits anti-derivatives since it is continuous but that primitive functions F cannot be expressed in terms of standard functions. Therefore, the computation of areas under the curve for different segments $[0; x]$ is a way to draw a table of values for $F(x)$. This gives a new scope to the technique τ_{rect} . Nevertheless, authors of the textbook labelled this work “research exercise” and did not refer to this curve as the Gauss curve, pointing towards a forthcoming chapter on probability and statistics. This is

7 **Intégrales et primitives avec calculatrices ou logiciels**

Énoncé Calculer l'intégrale $J = \int_1^e \ln x \, dx$ avec une calculatrice ou un logiciel.

Solution

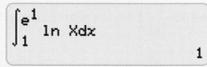
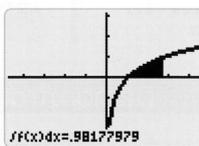
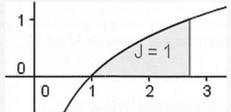
<p>Casio Graph 35 +</p>  <p>Par OPTN choisir CALC puis ∫dx.</p> <p>On peut aussi tracer la courbe puis appeler ensuite l'intégrale. L'aire est représentée sous la courbe comme pour la TI82 Stats ci-contre.</p>	<p>TI 82 Stats</p> <p>Tracer la courbe d'équation $y = \ln x$. Dans calculs (2nde trace) choisir l'intégrale (choix 7). Choisir au curseur, à peu près, la plus petite borne (le point de C d'abscisse 1) puis la plus grande (le point de C d'abscisse 2,7) et valider.</p>  <p>$\int f(x)dx = .88177979$</p>
<p>GeoGebra</p> <p>Entrer J=Intégrale[ln(x),1,e].</p> <p>On obtient J et le graphique suivant :</p> 	<p>Xcasfr</p> <p>On peut obtenir une primitive de $x \mapsto \ln x$ ou l'intégrale cherchée. Il s'agit ici d'une valeur exacte.</p> <pre> intégrer(ln(x)) x * ln(x) - x intégrer(ln(x),x,1,e) 1 </pre>

Figure 1 Instrumental techniques for T_I (textbook “Math’x”, p. 237)

Pratiquer le calcul approché
Mettre en œuvre un algorithme

→ **exercice 102**

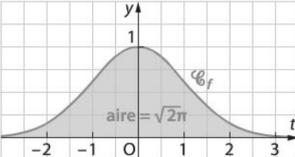
47 **Tabuler une primitive**

f est la fonction définie sur \mathbb{R} par : $f(t) = e^{-\frac{t^2}{2}}$.

\mathcal{C}_f désigne sa courbe représentative dans un repère orthogonal d'origine O .

On admet que le domaine sous la courbe, bien que non borné, a une aire finie égale à $\sqrt{2\pi}$ (u.a.).

f est une fonction continue sur \mathbb{R} , donc elle admet des primitives sur \mathbb{R} . Cependant, on ne peut pas exprimer une primitive de f à l'aide de fonctions usuelles.



OBJECTIFS Étudier la primitive F de f sur \mathbb{R} qui s'annule en 0 définie par : $x \mapsto F(x) = \int_0^x f(t) dt$.
Tabuler les valeurs de $F(a)$, où a est un nombre positif, avec une précision donnée.

Figure 2 An advanced usage of Π_{rect} : computing a table of values for an unknown primitive function (textbook “Transmath”, p. 212)

certainly an institutional sign of knowledge compartmentalisation. Although the official syllabus requires pointing out that no explicit formula is known for the anti-derivatives of e^{-x^2} , this is the only instance we could find of a textbook in which this instruction leads to an exercise.

Regarding CPD, types of tasks related to theme 1 include “show that a given function is a probability density function”. Density functions have been introduced as continuous functions the curve of which “fits” the histogram. An introductory activity on histograms thus serves as a heuristic to justify the definition of the probability of an event $X \in J$ for an interval J as the area under the curve of f , and also the condition that the total area under the

5 Calculer $P(a \leq X \leq b)$ où X suit la loi $\mathcal{N}(0; 1)$

Énoncé Sachant que X suit la loi $\mathcal{N}(0; 1)$, calculer $P(-1 \leq X \leq 2)$.

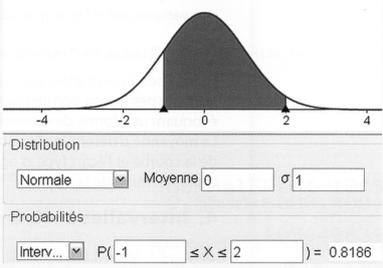
Solution
À l'aide d'une calculatrice ou d'un logiciel, on obtient $P(-1 \leq X \leq 2) \approx 0,82$.

Calculatrice CASIO Graph 35 +
Par choisir STAT, DIST, Norm, Normcd

Calculatrice TI 82Stats, 83
Par choisir normalFRéP

Tableur
Entrer

GeoGebra4
Par l'outil *Calculs Probabilités*



Distribution
Normale Moyenne 0 σ 1

Probabilités
Interv... $P(-1 \leq X \leq 2) = 0.8186$

Figure 3 Instrumental techniques for $T_{P,norm}$ (textbook “Math’x”, p. 407)

curve is 1. These definitions make up the theory for the main type of tasks T_P . Two cases need to be distinguished, depending on whether a primitive of the density function f is known ($T_{P,exact}$) or not. In theme 1, the generic technique $\tau_{P,prim}$ which consists in computing $\int_a^b f(x) dx$ by means of the praxeology $\Pi_{I,prim}$, is always applicable. The technology $\theta_{P,prim}$ is mainly composed of the FFCPD. As usual, we shall call $\Pi_{P,prim}$ the full praxeology. Moreover in theme 1, two particular cases are emphasised and lead to local techniques, as concrete formulas are available for $P(a \leq X \leq b)$ in the case of the uniform and exponential distributions. For instance, the technique $\tau_{P,exp}$ may be reduced to computing $e^{\lambda a} - e^{\lambda b}$ with technological argument $P(a \leq X \leq b) = e^{\lambda a} - e^{\lambda b}$ (praxeology denoted $\Pi_{P,exp}$). Altogether, the techniques are based on calculus, except in the case of a piece-wise affine density function f for which it is also possible to use $\Pi_{A,elem}$. This is the only case that connects to T_A . In fact, much time is dedicated in theme 1 to the interpretation of the uniform and exponential distributions as models related to probabilistic phenomena. Modelling praxeologies are not described in this article since they are out of our scope.

In theme 2 dedicated to the normal distributions, the type of tasks T_P takes the form $T_{P,norm}$, which is dealt with using a calculator or software. The corresponding praxeology should therefore be denoted $\Pi_{P,norm}^{instr}$. Students do not use the same button on calculators in IC and in CPD (Fig. 1 and 3), which may hinder the possibility for them to interpret the result of such a calculation as an approximate value of an area under the Gauss curve (thus the connection with Π_{rect}). To the exception maybe of the software Geogebra the environment of which combines a graphical representation of the curve and the relevant statistical parameters, we can make the hypothesis that the digital tools will mainly be used as a blackbox by students, and therefore the connections with IC are likely to be lost. In fact, the relationship with

3 Approximation normale d'une loi binomiale

Théorème 7

(admis)

Théorème de Moivre-Laplace

X est une variable aléatoire qui suit la loi $\mathcal{B}(n, p)$, et Z la variable aléatoire $\frac{X - E(X)}{\sigma(X)} = \frac{X - np}{\sqrt{np(1-p)}}$.

Animation

Pour tous nombres a et b tels que $a \leq b$, $\lim_{n \rightarrow \infty} P(a \leq Z \leq b) = \int_a^b \varphi(t) dt$.

Autrement dit, pour les grandes valeurs de n, la loi binomiale $\mathcal{B}(n, p)$ est très proche de la loi normale de même espérance np et de même variance $np(1-p)$.

➤ **Remarque.** Quand on a $n \geq 30$, $np \geq 5$, $n(1-p) \geq 5$, l'erreur sur les probabilités calculées est très faible. On ne fera l'approximation que lorsque ces trois conditions sont vérifiées.

➤ **Exemple.** $n = 36$, $p = 0,5$. Les trois conditions sont remplies. $E(X) = np = 18$; $V(X) = np(1-p) = 9$.

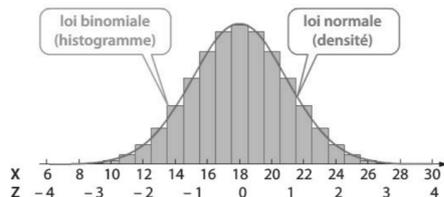


Figure 4 Praxeology of normal approximation (textbook “Transmath”, p. 411)

Π_{rect} is crucial throughout the theme at the logos level (through the notion of area). This is visible for instance in the praxeology Π_{norm_approx} of normal approximation of a binomial probability distribution (figure 4). But since the technique restricts to the verification that conditions for an application of $\Pi_{P,norm}^{instr}$ hold, students may not develop any knowledge on the interrelation of IC and CPD.

3.3 Our praxeological reference model and answers to the first set of research questions

Let us recall that our model is based on the study of standard textbooks used in grade 12 in France and is dedicated to the description of the teaching-learning of CPD and IC as it actually is (we do not plan any intervention at this stage).

Table 1 summarises the main local mathematical organisations (LMO) of the IC and CPD sectors which have been identified and described in the previous section. Each theme gives birth to a single LMO, which means that a shared technology unifies the different praxeologies included in the theme, except in the case of instrumented praxeologies the technologies of which are more or less a black box (underlying algorithms are not described; we thus call them “blind”, which results in a feeble control on the technique), to be set apart. In the case of CPD theme 2, the mathematical organisations presented in the table are centred around a main praxeology and therefore should be rigorously called punctual rather than local. In fact, our model is simplified and a more detailed account of variants of these main praxeologies may reveal that it is justified to call them local. In all cases, this is not our main point.

Table 1 Local mathematical organisations (LMO) of the IC and CPD sectors, with a focus on the logos blocks

Theme	Name of LMO	Praxeologies	Technology	Theory
IC theme 1	def. of the integral	$\Pi_{A,elem}$ Π_{rect} Π_{rect}^{instr}	def. of the definite integral and properties of areas	intuitive theory of areas
IC theme 3	exact calculation of integrals and areas	$\Pi_{I,prim}$ $\Pi_{A,prim}$	FTC, properties of the definite integral	differential calculus, def. of the definite and indefinite integrals
	“blind” instrumentation of IC	$\Pi_{I,approx}^{instr}$ $\Pi_{I,exact}^{instr}$		
CPD theme 1	exact calculation of probabilities	$\Pi_{P,prim}$ (uses $\Pi_{I,prim}$) $\Pi_{P,exp}$	FFCPD	integration theory, probability theory, def. of density function and continuous probability distributions in terms of areas
CPD theme 2	instrumented calculation of normal distributions	$\Pi_{P,norm}^{instr}$		probability theory, def. of normal distributions
	normal approximation	Π_{norm_approx}	de Moivre-Laplace theorem	

In this model, we note the following two main links between CPD and IC:

- at the level of the logos blocks, praxeologies in both sectors are anchored on the intuitive notion of area;
- at the level of the technique, $\Pi_{P,prim}$ uses $\Pi_{I,prim}$, so that, from an ecological point of view, CPD contributes to the thriving of such IC praxeologies.

Moreover, the measurement of magnitudes is not restricted to the logos: types of tasks in IC theme 1 are dedicated to the development of techniques to compute areas, leading to the praxeology Π_{rect} which is fundamental to conceptualise both the definite integral as the limit of sums of rectangular areas (Riemann sums) and the concept of area. The didactic transposition of CPD and IC operated in the syllabus thus follows the historical common roots of CPD and IC.

In fact, the links with the notion of area is much less developed in the CPD syllabus than in the case of IC: the main praxeology of CPD theme 1 is $\Pi_{P,prim}$. In the CPD sector, the notion of area is essentially part of the logos, whereas work on histograms (Derouet & Parzysz, 2016) could be an opportunity to deepen the relationships by introducing new types of tasks and techniques (for example, “Estimate a probability that models a situation from empirical data represented by a frequency histogram”). But such a “histogram praxeology” is missing in current textbooks, together with the opportunity to emphasise relationships with the praxeology Π_{rect} from IC and between both sectors.

Our model also shows signs of compartmentalisation. Contrary to $\Pi_{I,prim}$, Π_{rect} is not reinvested in CPD whereas the computation of a probability of the Gaussian distribution could be a great opportunity for that. This is due to the choice to emphasise instrumental techniques ($\Pi_{P,norm}^{instr}$) that we have described as blind due to a lack of technological discourse. The normal distribution is a prototypical example of a function a primitive of which cannot be expressed in terms of available elementary functions. This fact explains the choice of techniques to solve $T_{P,norm}$ and contributes also to the logos of $\Pi_{I,prim}$ (by complementing the statement that every continuous function admits a prim-

itive function). Textbooks more or less assume that this theoretical content belongs to the IC sector, and do not refer to a forthcoming study of the Gaussian distribution whenever it is presented. We argue that CPD would provide a relevant context to problematise the computation of $\int_a^b e^{-x^2/2} dx$. In a more spiralling approach, a praxeology dedicated to the calculation of values of anti-derivatives that cannot be computed using standard algebraic techniques from IC, based on Π_{rect} , could be developed fruitfully when the Gaussian distribution is brought forward in the context of probabilistic and statistical investigation of phenomena. Our model therefore reveals both a deficient logos for $\Pi_{P,norm}^{instr}$ and a second missed opportunity for de-compartmentalisation of IC and CPD.

This study therefore indicates how to enrich our model in view of the survey presented in section 4 by introducing the type of tasks $T_{I,norm}$: compute $\int_a^b e^{-\frac{(x-m)^2}{2\sigma^2}} dx$. As seen above, $T_{I,norm}$ can be solved in the IC sector through Π_{rect} but the most efficient technique consists in using $P(a \leq X \leq b) = \int_a^b f(x) dx$ from right to left, thus $\Pi_{P,norm}^{instr}$. How would students react in front of such a task that asks to bridge CPD and IC, yet in an unusual way? Indeed, the FFCPD is mainly used from left to right in relation to $\Pi_{P,prim}$. Such a task thus reverses the main ecological chain (IC is applied to CPD). Moreover, although the equality is symmetrical as an equivalence relation, it is not symmetrical as a sign which denotes a succession of operations in performing a computation. This reveals a further cognitive obstacle. Finally, this task is situated by its main ostensive \int_a^b in the IC sector. The deficient logos of $\Pi_{P,norm}^{instr}$ may hinder the possibility for students to connect the task with CPD. This questioning was a starting point for the elaboration of our questionnaire.

4 Empirical study

We conducted an empirical study in order to answer research question 2 on effects of the institutionalised compartmentalisation between the two sectors CPD and IC on students' learning. Our main approach is the interpretation of data obtained from students' work on a paper-pencil questionnaire, the items of which are built on the results of the praxeological analysis presented in section 3. The questionnaire covers tasks addressing techniques from praxeologies living within IC and CPD as well as tasks that require techniques (and technologies) from one sector to be transferred to the other sector. Since the textbook analysis shows almost no such migration of praxeological blocks (except the obvious connection between $\Pi_{P,prim}$ and $\Pi_{I,prim}$), we expect students to encounter difficulties in successfully completing the latter type of tasks. In the construction of items, we have paid particular attention to ostensives because, as explained in section 2, we expect that they play a significant role in students' mobilisation of techniques and praxeologies. Thus, we have elaborated a questionnaire both to check the availability of standard praxeologies

Question 1

Compute the exact value of the integral $\int_{-1}^4 e^{-2x} dx$.

Question 2

What are the properties that a function f must satisfy to be a probability density function associated to a random variable X with values in an interval I ?

Question 3

Give the domain of definition and the algebraic expression of the density function associated to:

- the exponential distribution of parameter λ ;
- the normal distribution of parameters 0 and 1.

Question 4

Let X be a random variable with exponential distribution of parameter 3.

- Compute the probability $P(1 \leq X \leq 5)$.
- Give a graphical interpretation of this probability. A picture may be drawn by hand. Reminder: the density function associated to an exponential distribution of parameter 3 is the function defined by $f(x) = 3e^{-3x}$, $x \in [0; +\infty[$.

Question 5

- Do you know a continuous function over an interval that does not admit primitives? If so, give an example. If not, justify.
- Do you know a continuous function over an interval that admits a primitive of which you do not explicitly know an expression? If so, give an example. If not, justify.

Question 6

Explain all the methods that you can use to determine an exact and/or approximate value of the following integral: $I = \int_{-0,5}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. You may restrict to give an idea of the method if its implementation is too complicated.

Question 7

Let A be the function defined on $[0, +\infty[$ by $A(\lambda) = \int_0^\lambda f(x) dx$ with $f(x) = xe^{-x}$. It can be proved that $\lim_{\lambda \rightarrow +\infty} \int_0^\lambda xe^{-x} dx = 1$. Based on this result, write down everything you can say about the function A and the function f .

Figure 5 Questionnaire submitted to students

of CPD and IC in the praxeological equipment of students and the capacity of students to complete bridging tasks.

In section 4.1, we present an a priori analysis of the items in view of the PRM worked out in section 3. Data collection and the sample are described in section 4.2. The results of the survey are analysed in section 4.3. Section 4.4 then summarises our answers to research question 2.

4.1 A priori analysis of the questionnaire

The questionnaire includes seven items (see Figure 5). The first group of items (from question 1 to 5) intend to “activate” the sectors CPD (questions 2, 3

and 4) and IC (questions 1 and 5). The second group of questions (6 and 7) represents bridging tasks: regarding ostensives they are located within the IC sector, but are aimed to mobilise the CPD sector. Thus questions 6 and 7 allow us to study the possible but questionable transfer of praxeologies from one sector to another in view of praxeologies the availability of which to students is checked by the first group of questions.

In the following, for each question we first clarify the relation to the PRM. Then, we describe the categories on which our preparation of the data from the survey and the subsequent analyses in section 4.2 are based.

Question 1

The routine task of type $T_{I,exact}$ aims at a straightforward calculation of a definite integral by FTC with respect to a standard function. We expect that students know a primitive and use the technique $\tau_{I,prim}$. The word “compute” rules out the technique $\tau_{I,exact}^{instr}$. We distinguish the following four categories of responses:

- FTC is mobilised with a valid primitive;
- the same with an invalid primitive;
- an incorrect technique is applied;
- no answer.

Question 2

This item asks for a characterisation of a density function on an interval I , which is a positive function with an area under the curve equal to 1. The latter property can be expressed in terms of “area under a curve” or by using the term “integral”. We distinguish five categories of responses:

- a complete characterisation of a density function is given;
- only the positivity of the function is mentioned;
- only the area or the integral equal to 1 is mentioned;
- irrelevant properties are mentioned;
- no answer.

Moreover, we distinguish whether students use ostensives related to the notion of area (the word “area”) or related to IC (the word “integral” or the sign \int). We do not take into account whether the students mention the continuity property of the density function, since rigour with respect to conditions for integrability of functions is not our focus.

Question 3

By asking for domains of definition and algebraic expressions, this item addresses elements of the logos block of $\Pi_{P,exp}$ and $\Pi_{P,norm}^{inst}$. It especially checks whether students have necessary prerequisites regarding semiotic valences of ostensives in bridging tasks 6 and 7. Availability of the algebraic expression of the exponential distribution will be an important indication for question 7 and of the normal distribution for question 6. We notice that question 4 provides the expression of the exponential distribution for $\lambda = 3$. We distinguish three categories for each sub-question:

- correct expression;
- false expression;
- no answer.

Question 4

Question 4a is a routine task of type T_P in the case of an exponential distribution. Two techniques can be used: $\tau_{P,prim}$, which applies $\tau_{I,prim}$, or $\tau_{P,exp}$. Thus, the item verifies whether students can mobilise IC praxeologies within the CPD sector, i.e., if they are able to mobilise the absent ostensive \int in the context of a routine task. We distinguish four categories of responses:

- $\tau_{P,prim}$ with a valid or invalid primitive mobilised;
- $\tau_{P,exp}$ with the correct or incorrect formula mobilised;
- an incorrect technique is used;
- no answer.

In question 4b, students are asked for a graphical interpretation of the probability $P(1 \leq X \leq 5)$ which is essential for linking the logos of II_P and of II_I . According to our PRM, this task should not be difficult for students. We distinguish two categories of responses:

- a correct (graphical or verbal) answer is given;
- an incorrect answer or no answer is given.

Question 5

This item aims to activate the IC sector, more precisely elements of the logos of $II_{I,prim}$ about primitives. We have also in mind the praxeology $II_{P,norm}$ which relates to the bridging question 6. The correct answer to question 5a refers to the theorem that every continuous function admits a primitive function. For question 5b, there are in principle many examples that could be mentioned by the students, like for example $\sin(1/x)$, which are in the realm of high school mathematics but tend not to be mentioned there in lessons and textbooks. But the function $x \mapsto \exp(-x^2)$ is mentioned as an example for functions which does not admit “explicit” primitives by standard functions in the curriculum. That is why we are looking by this question if the students propose this function or mention words like “density function of the normal distribution” in their answers. Question 5 is linked to the exercise in figure 2 which relates to the logos of $II_{I,prim}$ and the praxeology II_{rect} . If such a task has been dealt with in class, students could be able to answer the question with the normal density function. Unfortunately the formulation of question 5b is ambiguous: students could understand that they have to provide a continuous function which admits a primitive function for which they do personally not know an expression in terms of standard functions instead of that there is in general no such expression. It is therefore possible that they give examples such as $x \mapsto \ln(x)$.

The response categories for question 5a are:

- the theorem is referred to;
- a function is proposed that, according to them, do not admit primitives;
- no answer or false justifications;

and for question 5b:

- an example related to the normal distribution is proposed;
- other examples;
- no example.

Question 6

The open stated item represents an instance of the non routine type of task $T_{I,norm}$. It asks for any method to compute an exact or an approximate value for the Gaussian integral. Within the IC sector this item relates to $T_{I,prim}$ and/or $T_{I,approx}$. According to our PRM the techniques τ_{rect} , τ_{rect}^{instr} and $\tau_{I,approx}^{instr}$ could be available. We can also expect that a few students may use the technique $\tau_{I,prim}$ with a false primitive, or propose to use $\tau_{I,exact}^{instr}$. On the one hand this can be attributed to a deficient logoss regarding $\Pi_{I,prim}$. On the other hand it illustrates a deficient logoss regarding $\Pi_{P,norm}^{instr}$ as it is not recognised that the related technique could be applied here. If it is recognised that the integral is equal to $P(-0,5 \leq X \leq 1)$ (apply the FFPCD from right to left) with X a reduced centered normal random variable then also the technique $\tau_{P,norm}^{instr}$ could be used. The latter includes a sector change in which the student mobilises the ostensive P when the ostensive \int is present. We therefore distinguish three categories:

- students change the sector to CPD;
- students stay in the IC sector;
- no answer.

Let us now link answers to this question with answers to the items 3, 4 and 5. Item 3b allows to identify students who know the algebraic expression of the density function of the normal distribution. Within this group of students, we distinguish between students who change from the IC to the CPD sector and students who do not. In particular, we want to know whether knowing the normal density function is sufficient for connecting both sectors and to mobilise $\tau_{P,norm}^{instr}$ in an IC context. Moreover, we assume that if students can link the notions of area and probability, i.e. provide correct answers to question 4b, and can identify the normal density function as an example of a function with a primitive function which cannot be expressed in terms of standard functions, i.e. provide correct answers to question 5b, then they are more likely to mobilise the CPD sector.

Question 7

The ostensives of this item live apparently in the IC sector. We want to check whether students restrict their answer to the IC sector, for example with an interpretation related to areas, or if they recognise that A is both a primitive

for f and a probability ($P(0 \leq X \leq \lambda)$) associated with the random variable X of the density function f . In the latter case, students are able to change the sector. We distinguish three categories:

- students change the sector to CPD;
- students stay in the IC sector;
- no answer.

Moreover, answers to question 2 can be linked to answers to question 7: if students do not know the definition of a density function, then it is not very likely that they mobilise the CPD sector in question 7. On the other hand, a correct answer to question 2 may not be sufficient for mobilising the CPD sector.

4.2 Data collection and sample

The survey was launched in early September 2017 (the first week of classes) in two first year “higher education preparatory classes”. These classes which are part of the post-secondary education system aim at training selected students for enrolment in the *grandes écoles* (mainly engineering schools). The cohort under study is composed of students from a MPSI (mathematics, physics and engineering science) ($n_1 = 40$) and a PCSI (physics, chemistry and engineering science) ($n_2 = 42$) class of fairly prestigious institutions. Since the students are in selective classes, we assume that they are “good” students, i.e., if they have difficulties in answering the items correctly, this should also be the case for other students. In our data, we only considered students who have studied in French high schools during the past year because the constructed questionnaire takes into account the specific institutional context of French secondary education. The following analysis is based on a total of 82 completed questionnaires ($N = 82$). Students were given 40 minutes to complete the questionnaire.

4.3 Data analysis

In this section, we present the results of a descriptive data analysis of the survey and interpret them against the background of our a priori analysis of the items. We group the results according to the sector to which the items belong. We begin with the IC sector (questions 1 and 5), then go on to the CPD sector (questions 2, 3 and 4) and finally discuss the bridging items (questions 6 and 7).

IC sector (Questions 1 and 5)

The results regarding question 1 are presented in Table 2: 81 students use FTC and apply the technique $\tau_{I,prim}$. 84% provide a valid primitive and 77% obtain the correct result in the end. Only one student uses an incorrect technique. For the item in the IC sector, the technique $\tau_{I,prim}$ is well mastered by the students.

Table 2 Results on question 1

FTC		No FTC	
99%		1%	
valid primitive	invalid primitive	incorrect technique	no answer
84%	15%	1%	0%

Table 3 Results on question 5a

cite the theorem	propose a function (incorrect)	no answer or an incorrect justification
30%	23%	47%

Regarding question 5a (see Table 3) only 30% of the students cite the appropriate theorem (“every continuous function admits a primitive function”) to justify their answer. 23% of the students propose a function that, according to them, does not have a primitive, notably the absolute value function. The other students do not answer or give another false justification. The requested logos element of $\Pi_{I,exact}$ is therefore not well acquired by the students.

With respect to question 5b, overall 68% of the students propose examples, but a large part of these are irrelevant, for instance the functions \sqrt{x} or $|x|$. Only 18% of the students mention the density function of the normal distribution (either by verbal citation or by its algebraic expression). Among these 15 students, 9 explicitly refer to the normal distribution. The others give only the algebraic expression. 32% do not propose any example.

These results show that the logos of $\Pi_{I,prim}$ is incomplete for the majority of students although those elements are present in textbooks (see section 3). Moreover, since only 9 students explicitly refer to the normal distribution, we conclude that the majority of students do not mobilise knowledge from the CPD sector to enrich the IC sector, in this case the logos of $\Pi_{I,prim}$.

CPD sector (questions 2, 3 and 4)

Table 4 shows that only 35% of the students provide a complete list of properties characterising density functions. Among these students, only three (4% of the whole sample) refer to the area. The others express the related property by referring to the integral (most often with the ostensive \int). 31% mention only one of the properties. The rest of the students note irrelevant properties (18%) such as monotony or differentiability or do not give any answer (16%). Summarising, the definition of density functions is part of the logos of Π_P for only one third of the students approximately. At least two thirds of the students can come up with partial definitions.

On question 3 (see Table 5), at least 60% of the students can give the algebraic expression of the exponential distribution. However, in the interpretation of this rather high rate of success, one should take into account that the correct expression is mentioned for a specific case in question 4. For the nor-

Table 4 Results on question 2

complete definition	defi- nition	only function	positive	only area under the curve or integral equal to 1	irrelevant properties	no answer
35%		8%		23%	18%	16%
(4% area/ 31% integral)				(2% area/ 21% integral)		

Table 5 Results of question 3

	Correct expression	False expression	No answer
Exponential distribution	60%	22%	18%
Normal distribution	27%	62%	11%

Table 6 Results on question 4a

	$\tau_{P,prim}$	$\tau_{P,exp}$	incorrect technique	no answer
	71%	6%	7%	10%
correct answer	59%	4%	0%	0%

mal distribution, the rate of correct answers is significantly lower (only 27%). Interestingly, data on question 6 show that a few students who do not provide the formula for the normal distribution on question 3 are able to recognise the expression when it is presented. Given that this expression is mentioned explicitly in the syllabus and in textbooks, the low rate raises questions. We suppose that students' inability to recollect the formula is related to the fact that only the parameters of the distribution play a role in the application of the technique $\tau_{P,norm}^{instr}$ which is usually used. Thus, also in view of the answers to question 3, we note that the logos related to Π_P , i.e. the CPD sector, available to students seems rather fragile.

To the question 4a (see table 6) 63% of the students give a correct answer. 71% of the students mobilise the technique $\tau_{P,prim}$, with 83% finding the correct answer and the rest determining an incorrect primitive. 6% of students activate the technique $\tau_{P,exp}$. Among the students who mobilise $\tau_{P,prim}$, eight students apply an incorrect primitive, half of them also give an incorrect primitive in question 1. Four students try to mobilise $\tau_{P,exp}$ but use an incorrect formula. In total, the type of task T_P for exponential distribution is quite well mastered by the students. The technique $\tau_{P,prim}$ is mainly used. We can conclude that the FFCD is well known. Its application "from left to right" provides a solid basis for a move from the ostensive P to the ostensive f .

Regarding question 4b, 68% of the students are able to provide a graphical interpretation of the probability. They refer to the area under a curve either by a graphic or a verbal remark. This means that the link between the ostensive P and the ostensive *area* is also mastered by the majority of students.

Table 7 Results on question 6

CPD sector	IC sector only	No answer
20% (with the ostensive P : 13%)	71%	9%

Question 6

When working on this task, 71% of the students stay in the IC sector, 20% change the sector and 9% do not answer (see table 7). The majority of students, who remain in the IC sector, use FTC (46%) and 42% of them, i.e. 20% of the whole sample, mobilise $\tau_{I,prim}$ with an incorrect primitive. With a view to question 5, we can assume gaps in the logos of $\Pi_{I,prim}$ because these students are not able to identify the function $x \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ as a particular case of a function that does not possess an explicitly known expression for its primitive. From students changing sectors, 69% (13% of the whole sample) mention that the given integral is equal to the probability $P(-0,5 \leq X \leq 1)$, i.e. clearly apply FFCPD from right to left. Of this group, 64% (9% of all) refer to an application of the calculator, i.e. to the technique $\tau_{P,norm}^{instr}$.

What are the links with students' answers to the questions 3, 4 and 5? Among the students who know the algebraic expression of the normal density function (question 3b), however, 45% change the sector (mention the normal distribution), although only 27% of those students introduce the ostensive P (see table 8). Overall, this highlights that knowing the normal density function is not enough to connect both sectors in answering question 6. On the other hand, 10 out of the 15 students who change the sector know the algebraic expression of a normal density function. This means that this knowledge seems at least helpful for a sector change.

Regarding question 5b, 53% of the students who mention the normal density function (i.e. 8 students out of 15) activate the CPD sector in question 6. Moreover, 7 out of the 9 students who explicitly argue about the normal distribution change the sector. In addition, 7 students out of 15 change the sector without mentioning the normal density function in question 5b.

In view of the data, we managed to detect further links that were not envisaged in the a priori analysis. Regarding question 4b, all students (15) who do not provide the interpretation of probability as area under a curve remain in the IC sector. Likewise, all students (13) who do not know the properties of a density function in question 2 remain in the IC sector.

We interpret all those observations as evidence that a deficient logos in students' praxeologies hinders their ability to change sectors in question 6. The deficiency may concern the foundations of both CPD and IC on the notion of area, or the technological knowledge on the anti-derivative of normal distributions that orients the choice of the technique towards a sector change. In other words, the deficient logos is an explanatory factor for the compartmentalisation of students' knowledge.

Table 8 Links between results of question 6 and results of questions 3b and 5b

	Q3b : able to recollect the expression of the normal density function	Q5b : able to state that the anti-derivative of the normal distribution is unknown
	22/82 \simeq 27%	15/82 \simeq 18%
Change of sector in Q6 (with presence of the ostensive P)	10/22 (6)	8/15 (5)
Remark		4/15 propose to compute an exact value of the integral

Table 9 Results of question 7

CPD sector	IC sector only	No answer
48%	37%	16%

Question 7

We note that 48% of the whole sample (39 students) mobilise the CPD sector (see table 9). However, only 31 students (38% of all) actually mention that f is a density function. There are 13 students (16% of all) who interpret A as a probability and all of them but one have also identified f as a density function. The remaining 12 students who change the sector but neither interpret A nor f properly have in fact mistaken A as a probability density function (a confusion of the continuous probability with its density function).

Let us now discuss the links with the answers to question 2. We note that 34 out of the 39 students who change the sector in question 7 (87%) provide a (partially) correct answer to question 2. Reciprocally, only 5 out of the 38 students who do not answer or give an incorrect answer to question 2 change the sector in question 7. This means that a certain knowledge about density functions seems to be necessary to mobilise the CPD sector in view of question 7. On the other hand, knowledge of the definition of a density function is not a necessary precondition for mobilising the CPD sector.

4.4 Discussion of the results and answers to research question 2

Overall, the analyses of the data in section 4.3 largely coincide with the observations and conclusions drawn from the praxeological analysis in section 3. In the following, we summarise these results with a clear reference to the PRM, and then, against this background, we finally address the research question 2.

The results for questions 1 and 2 confirm for the IC sector that the students have solid knowledge of the technique $\tau_{I,prim}$, but that the logos of the related praxeology $\Pi_{I,prim}$ is underdeveloped. This assessment is further supported by the results for question 6: many students have not identified the normal

distribution as an example for a function that has a primitive, which cannot be expressed only by means of the standard functions. Furthermore, they have misapplied the technique to this non-standard case which relies outside the scope of the technique. This is a clear evidence of a deficiency of the technology in the sense of ATD.

Regarding the questions 2 to 4 from the CPD sector we can note the following. The data on 4a and 3a show that students master both techniques $\tau_{P,prim}$ and $\tau_{P,exp}$ and also the application of FFPCPD from left to right. Item 4b also shows that the concept of area is available in the students' knowledge. Considering that area is an important unifying element of the logos regarding T_P , T_I and T_A , mathematics teaching seems to have at least laid a foundation. With regard to questions 6 and 7, it can already be concluded that the logos blocks are weak in terms of the probability density function (results for question 2) and normal distributions, in particular regarding $\Pi_{P,norm}^{inst}$ (results for question 3). This points to rather weak semiotic valences with respect to those ostensives in the IC sector which were identified as important for bridging to the CPD sector in questions 6 and 7.

In working on question 6, only a few students have moved from the IC sector, where the ostensives of the item are mainly located, to the CPD sector. The application of FFPCPD from right to left seems not to be part of the students' repertoire. In view of the special case of normal distribution, this deficit can also be related to a weakly developed logos with regard to the technique $\tau_{P,norm}^{inst}$. In this respect, it probably also has an effect that in the context of normal distributions the possibility to mobilise $\Pi_{I,rect}$ in CPD is obviously not used in classrooms (see section 3.2). Moreover, there is an ultimately fragile logos with respect to $\Pi_{I,prim}$, which can be stated in view of the many incorrect solutions that have tried to apply the technique $\tau_{I,prim}$ out of its scope.

A comparison with the results for questions 3b and 5 confirms once again the importance of the logos regarding the normal distribution: knowing it is helpful, almost necessary, but not sufficient for a change of sectors. This can be stated analogously with regard to the logos element area (question 4b), which links the types of tasks T_P , T_I and T_A .

In question 7, a change from the IC to the CPD sector can be observed in almost half the students' solutions. Last but not least, we have found that a trained logos with regard to probability density functions is more likely to lead to a change of sectors, the lack of which severely hinders it.

Regarding research question 2, about effects of the institutionalised praxeologies on the compartmentalisation of learned knowledge, we may conclude as follows: students can mobilise IC sector practices in standard situations of the CPD sector. However, this application reveals the fragile logos of IC praxeologies. The reverse mobilisation of CPD praxis for IC tasks, however, is rarely successful. The data are also consistent with our theoretical analyses regarding the significance of ostensives. The disconnections shown in the praxeological analyses are clearly evident in the students' work on the tasks. In addition, fragile logos is generally observable in all task types and praxeologies.

5 General conclusion and ways forward

In this study, we have made the assumption that a lack at the level of the technology (in the sense of ATD) hinders students' ability to transfer techniques: in other words, a deficient logos may lead to a loss of the operative valence of ostensives, following the instability or loss of their semiotic value. As main results, we have found out that students can mobilise the IC sector praxis in standard situations of the CPD sector but that the converse, in the context of the bridging tasks where the ostensive P is absent, is rarely successful. Our analysis of the acquisition of crucial elements of the logos of CPD and IC supports the hypothesis that the deficiency of the logos plays a role regarding the missing links. These results coincide with the observation in textbooks that “blind” instrumented techniques (devoid of proper technology) are taught and connections between the notion of area and the concept of probability distribution are insufficiently developed.

In view of the theoretical elements that support the links, we are reasonably confident that extending the questionnaire to a wider population of students should make it possible to establish findings regarding the representativeness of the observations reported here. Another way of extending this research would consist in contrasting the results with a population of students that would be equipped, previously to taking the questionnaire, with the “histogram praxeology” (identified at the end of section 3.3). It is a major opportunity, yet underdeveloped in the current curriculum in France, to deepen the relationships between CPD, areas and integrals. An original teaching sequence introducing the integral concept by probability modelling tasks is designed and studied in (Derouet & Parzysz, 2016) and (Derouet, 2019). This can be seen as a first pioneering approach in this respect.

On a more general level, our study opens the road to similar works on other mathematical topics: students' difficulty in crossing mathematical sectors, which may be seen at first sight as a problem of cognitive flexibility, is understood as a result of deficient praxeologies developed within the institution. The institutional analyses are thus a first step in pointing to concrete proposals to improve the teaching with respect to a key aspect of the secondary-tertiary transition, if it is assumed that students' capacity to integrate compartmentalised knowledge in a coherent and operative whole is an important skill to be successful in post-secondary studies.

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⁴ Centre National de la Recherche Scientifique; <https://www.insmi.cnrs.fr/>

⁵ Didactique et Épistémologie des Mathématiques, liens avec l'Informatique et la Physique, dans le Supérieur; <https://demips.math.cnrs.fr/>

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