



A pocket-sized introduction to dynamics

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A pocket-sized introduction to dynamics

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1

Cartesian motion

The Greek word $\deltaυναμις$ means physical force or power. Dynamics consists of

- kinematics: the study of motion of bodies ($\kappaινημα = \text{motion}$);
- kinetics: the study of action of forces and resulting motion ($\kappaινητικος = \text{movable}$).

Definitions

In kinematics, we define the position, velocity, and acceleration of bodies.

Position x is the location of a body. In dynamics problems, location is usually a function of time. For 1-dimensional situations, we use x with reference to some origin, for 3-dimensional situations, we use Cartesian (x, y, z) or polar (r, ϑ, φ) coordinates.

Velocity $\vec{v} = \dot{x}$ is the rate of change of position x with time. Here, the dot $\dot{}$ stands for $\frac{d}{dt}$.

Acceleration $\vec{a} = \dot{v} = \ddot{x}$ is the rate of change of speed with time. Velocity and acceleration have magnitude and direction, *i.e.*, they are vectors.

Jerk $\dot{a} = \ddot{v} = \dddot{x}$ is the rate of change of acceleration with time.

Jounce $\ddot{a} = \ddot{v} = \ddot{x}$ is the rate of change of jerk with time. So, the jounce equals the fourth time derivative of the position.

Speed is the magnitude of the velocity.

Deceleration $-a = -\dot{v} = -\ddot{x}$ is the opposite of acceleration.

Given any acceleration function,

$$v(t) = \int a(t) dt. \quad (1.1)$$

For constant acceleration, i.e., $\dot{a} = 0$,

$$v(t) = \int a dt = v(0) + at \quad (1.2)$$

and

$$x(t) = \int v(t) dt = x(0) + v(0)t + \frac{1}{2}at^2. \quad (1.3)$$

Substituting $at = v(t) - v(0)$ from (1.2) into (1.3) yields

$$x(t) = \frac{v(t) + v(0)}{2} t. \quad (1.4)$$

Note that

$$a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}. \quad (1.5)$$

Thus,

$$\int a dx = \int v dv \quad (1.6)$$

or

$$ax = \frac{v^2}{2} + c. \quad (1.7)$$

If $v = v(0)$ when $x = 0$, $c = -\frac{v(0)^2}{2}$. Thus,

$$v(t)^2 = v(0)^2 + 2ax. \quad (1.8)$$

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Projectiles

Consider a projectile launched with a velocity v at an angle θ relative to the normal of ground level. The vertical component of the velocity at launch is $v \sin \theta$, whereas the vertical component of the velocity at maximum height is zero. The projectile is subjected to a constant acceleration $a = -g$, where g is the gravity acceleration. In the absence of resistive forces, there is no horizontal acceleration and since the gravity acceleration is perpendicular to the horizontal component of the velocity, the horizontal component stays constant at $v \cos \theta$.

Since the travel time from launch to maximum height is half of the total travel time t , we consider only this motion to compute the total travelled distance. Then, assuming no air resistance,

$$v \sin \theta - g \left(\frac{1}{2} t \right) = 0 \quad (2.1)$$

or

$$t = \frac{2v \sin \theta}{g}. \quad (2.2)$$

Hence, the maximum range of the projectile is

$$x = \frac{2v^2 \sin \theta \cos \theta}{g} = \frac{v^2 \sin 2\theta}{g}. \quad (2.3)$$

Forces and projectiles on inclined planes

Consider an object of mass m on an inclined plane, tilted at an angle θ . If the object is stationary, the net force on the object must be zero. The weight component along the plane $F_{\parallel} = mg \sin \theta$ is balanced by the friction. The weight component perpendicular to the plane $F_{\perp} = mg \cos \theta$.

3

Conservation of energy

The potential energy of a ball of mass m on top of a ramp with height h is

$$E_p = m g h, \quad (3.1)$$

whereas its kinetic energy is zero. At the bottom of the ramp, the kinetic energy of the ball is

$$E_k = \frac{1}{2} m v^2, \quad (3.2)$$

whereas its potential energy is zero. Hence, if all energy is transferred lossless into motion, the velocity at the bottom of the ramp must be

$$v = \sqrt{2 g h}. \quad (3.3)$$

The work done starting from rest equals the integral of the force F to which the ball has been subjected over the path it has travelled:

$$\begin{aligned} \int F dx &= \int m \frac{dv}{dt} dx = \int m \frac{dv}{dx} \frac{dx}{dt} dx \\ &= \int m v \frac{dv}{dx} dx = \int m v dv = \frac{1}{2} m v^2 = E_k. \end{aligned} \quad (3.4)$$

4

Mass on a spring

Consider an object of a mass m attached by a massless elastic spring to a motionless vertical wall and resting on top of a horizontal frictionless surface. Let us define this equilibrium position as $x = 0$. The mass is pulled back over a distance Δx by a force F . For a stationary mass,

$$F = s x, \quad (4.1)$$

where s is the stiffness of the spring. The elastic potential energy gained equals the work done by stretching the elastic object:

$$E_P = \int_0^{\Delta x} F dx = \int_0^{\Delta x} s x dx = \left[\frac{1}{2} s x^2 \right]_0^{\Delta x} = \frac{1}{2} s \Delta x^2. \quad (4.2)$$

After release, we can define the excursion $x(t)$ around the equilibrium by

$$m \ddot{x} = -s x \quad (4.3)$$

or

$$a = -\frac{s}{m} x. \quad (4.4)$$

Using (1.5), we get

$$v \frac{dv}{dx} = -\frac{s}{m} x. \quad (4.5)$$

This can be rearranged to

$$\int v \, dv = \int -\frac{s}{m} x \, dx \quad (4.6)$$

or

$$\frac{v^2}{2} = -\frac{s}{m} \frac{x^2}{2} + c. \quad (4.7)$$

Since $v(0) = 0$ and $x(0) = \Delta x$, $c = \frac{s}{m} (\Delta x)^2$, yielding

$$v = \frac{dx}{dt} = \sqrt{\frac{s}{m} (\Delta x)^2 - \frac{s}{m} x^2}, \quad (4.8)$$

from which it follows that

$$\sqrt{\frac{m}{s}} \int \frac{1}{\sqrt{(\Delta x)^2 - x^2}} dx = \int dt. \quad (4.9)$$

Using $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$, this becomes

$$\sqrt{\frac{m}{s}} \arcsin \frac{x}{\Delta x} = t + c \quad (4.10)$$

or

$$\arcsin \frac{x}{\Delta x} = \sqrt{\frac{s}{m}} (t + c). \quad (4.11)$$

Hence, a solution for (4.3) is given by

$$x = \Delta x \sin \left(\sqrt{\frac{s}{m}} (t + c) \right). \quad (4.12)$$

This can be further simplified to

$$x = \Delta x \sin (\omega_0 t + \phi), \quad (4.13)$$

where

$$\omega_0 = \sqrt{\frac{s}{m}} \quad (4.14)$$

is the natural (or resonance) frequency of the system in radians per time unit, and

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}, \quad (4.15)$$

where f_0 is the resonance frequency in cycles per time unit and T_0 is the resonance period. Since the excursion is maximal at $x = 0$, the phase $\phi = \frac{1}{2}\pi$. Thus, (4.13) becomes

$$x = \Delta x \cos (\omega_0 t). \quad (4.16)$$

5

Circular motion

For circular motion, we can express position, velocity, and acceleration in a similar way as for Cartesian motion.

Definitions

Angle θ is the difference in angular position with reference to an original at radius r , usually expressed in degrees ($^\circ$) or radians (rad). 360° corresponds to 2π rad.

Angular velocity $\omega = \dot{\theta}$ is the rate of change of angle θ with time.

Angular acceleration $\alpha = \dot{\omega} = \ddot{\theta}$ is the rate of change of angular velocity with time.

At a constant angular velocity, the distance travelled around a circle is $r\theta = r\omega t$. For constant acceleration, *i.e.*, $\dot{\alpha} = 0$, analogous to (1.2),

$$\omega(t) = \int \alpha \, dt = \omega(0) + \alpha t. \quad (5.1)$$

Analogous to (1.3),

$$\theta(t) = \int \omega(t) \, dt = \theta(0) + \omega(0)t + \frac{1}{2}\alpha t^2. \quad (5.2)$$

Circular motion using vectors

Consider a point on a circle of radius r around the origin. The vector describing this point is given by

$$\vec{r} = r \cos \theta \hat{x} + r \sin \theta \hat{y}, \quad (5.3)$$

where θ is the angle between \vec{r} and the x-axis. For constant ω , we can define $\theta = \omega t$. Thus, (5.3) becomes

$$\vec{r} = r \cos \omega t \hat{x} + r \sin \omega t \hat{y}. \quad (5.4)$$

The velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = -\omega r \sin \omega t \hat{x} + \omega r \cos \omega t \hat{y}. \quad (5.5)$$

So,

$$v = |\vec{v}| = \omega r \quad (5.6)$$

or

$$\omega = \frac{v}{r}. \quad (5.7)$$

The inner product of the radius and the velocity is

$$\langle \vec{r}, \vec{v} \rangle = \vec{r} \cdot \frac{d\vec{r}}{dt} = -\omega r^2 \cos \omega t \sin \omega t + \omega r^2 \sin \omega t \cos \omega t = 0. \quad (5.8)$$

Hence, \vec{v} is at right angles to \vec{r} , i.e., tangential to the circle. The acceleration

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -\omega^2 r \cos \omega t \hat{x} - \omega^2 r \sin \omega t \hat{y} = -\omega^2 \vec{r}. \quad (5.9)$$

The acceleration is constant in magnitude and centripetal, i.e., directed towards the centre. The magnitude of the central force is

$$F_c = m\omega^2 r = \frac{mv^2}{r}. \quad (5.10)$$

6

Law of gravitation and geostationary satellites

The gravitational force is

$$F_g = \frac{M_\oplus m G}{r^2}, \quad (6.1)$$

where M_\oplus is the mass of the earth, m is the mass of an object subjected to the earth's gravity, $G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ is the gravitational constant, and r is the centre-to-centre distance of the earth and the object.

A geostationary satellite is one that orbits once a day, *i.e.*, its period is $T = 24 \times 60 \times 60 \text{ s}$. The gravitational force on the satellite must equal the radial force required to keep it in circular orbit:

$$m \omega^2 r = \frac{M_\oplus m G}{r^2}. \quad (6.2)$$

Hence, given $M_\oplus = 5.977 \times 10^{24} \text{ kg}$,

$$r = \sqrt[3]{\frac{M_\oplus G}{\omega^2}} = \sqrt[3]{\frac{M_\oplus G T^2}{4\pi^2}} = 4.22 \times 10^7 \text{ m}. \quad (6.3)$$

Since the radius of the earth at the equator is 6378 km, the height of a geostationary satellite must approximate 35,900 km.

7

Vehicles on bends and bumps

On a circular bend, the acceleration towards the centre is provided by the friction. Vertically there is a reaction force equal to the weight. The horizontal friction force is given by μmg , where μ is the coefficient of friction. To avoid skidding, the horizontal friction force has to be greater than the centripetal force in the opposite direction, *i.e.*,

$$\mu mg \geq \frac{mv^2}{r}. \quad (7.1)$$

The resultant force on the tyres has a direction

$$\theta = \arctan \frac{\mu mg}{mg} = \arctan \mu. \quad (7.2)$$

By banking the road or track at this angle, the resultant force appears perpendicular to the road or track.

Consider a vehicle moving at a constant horizontal speed v that encounters a bump in the track with a profile given by

$$y(x) = \frac{h}{2} \left(1 - \cos \frac{2\pi x}{L} \right), \quad 0 \leq x \leq L. \quad (7.3)$$

After substitution $x = vt$ this becomes

$$y(t) = \frac{h}{2} \left(1 - \cos \frac{2\pi vt}{L} \right), \quad 0 \leq t \leq \frac{L}{v}. \quad (7.4)$$

Hence, the vertical component of the velocity

$$\dot{y}(t) = \frac{h}{2} \frac{2\pi v}{L} \sin \frac{2\pi vt}{L} = \frac{h\pi v}{L} \sin \frac{2\pi vt}{L} \quad (7.5)$$

and the vertical acceleration owing to the path

$$\ddot{y}(t) = \frac{2h\pi^2 v^2}{L^2} \cos \frac{2\pi vt}{L}. \quad (7.6)$$

The vertical force balance is

$$m\ddot{y} = -mg + F_r, \quad (7.7)$$

where F_r is the reaction force. So, the total vertical force exerted on the vehicle is given by

$$F_r = mg + m \frac{2h\pi^2 v^2}{L^2} \cos \frac{2\pi vt}{L} = mg \left(1 + \frac{2h\pi^2 v^2}{gL^2} \cos \frac{2\pi vt}{L} \right). \quad (7.8)$$

8

Motion on a vertical circle

Consider a vehicle moving downhill and accelerating by gravity without any braking or rolling friction. If the hill is described by a semicircle of radius r , then at a certain angle, the vehicle leaves the circular path and becomes a projectile. Assume the path is frictionless. The change in potential energy between the top and in intermediate point on the path at θ rad is given by

$$E_p = mgr(1 - \cos \theta). \quad (8.1)$$

Conservation of energy gives

$$\frac{1}{2}mv_0^2 + mgr(1 - \cos \theta) = \frac{1}{2}mv^2, \quad (8.2)$$

where v_0 is the initial velocity. The force balance along the normal is given by

$$F_g - F_n = F_c, \quad (8.3)$$

where F_n is the resultant normal force. Hence,

$$F_n = mg \cos \theta - \frac{mv^2}{r} = mg \cos \theta - \frac{m}{r} (v_0^2 + 2gr(1 - \cos \theta)). \quad (8.4)$$

At the critical angle where the vehicle leaves the circular path, $F_n = 0$. Therefore, (8.4) simplifies to

$$v_0^2 + 2gr = 3gr \cos \theta \quad (8.5)$$

and the critical angle is

$$\theta = \arccos \left(\frac{v_0^2}{3gr} + \frac{2}{3} \right). \quad (8.6)$$

9

Tangential acceleration

When the angular velocity varies, there is a tangential acceleration

$$a_t = \frac{dv}{dt} = r\dot{\omega} = r\alpha. \quad (9.1)$$

Since the acceleration towards the centre is

$$a_c = \frac{v^2}{r} = r\omega^2, \quad (9.2)$$

the resultant acceleration is

$$a = \sqrt{a_t^2 + a_c^2} = r\sqrt{\dot{\omega}^2 + \omega^4} \quad (9.3)$$

at an angle

$$\theta = \arctan \frac{a_c}{a_t} = \arctan \frac{\omega^2}{\alpha}. \quad (9.4)$$

10

Gear wheels and belt drives

Consider two gear wheels with numbers of teeth N_1 and N_2 , and angular rotation velocities ω_1 and ω_2 , respectively. Since the same number of teeth must pass per unit of time,

$$\omega_1 N_1 = \omega_2 N_2 \quad (10.1)$$

or

$$\omega_2 = \frac{N_1}{N_2} \omega_1. \quad (10.2)$$

Here, $\frac{N_1}{N_2}$ is referred to as the gear ratio. Similarly, for a belt drive with wheels of radii r_1 and r_2 , respectively,

$$\omega_2 = \frac{r_1}{r_2} \omega_1. \quad (10.3)$$

11

Rolling

Consider a wheel of radius r rolling on a horizontal surface in positive direction (clockwise) at constant speed without slipping. Let's define A as the point that touches the surface, B as the point $\frac{1}{2}\pi$ rad further on the wheel in clockwise direction, C as the top of the wheel, and O as the centre of the wheel, as schematically represented in Fig. 11.1. The velocities of these points are defined as v_A , v_B , v_C , and v_O , respectively. The wheel must move at a velocity v_O . The velocity of points A, B, and C relative to O is $\omega\vec{r}$ and its direction perpendicular to \vec{r} . Thus, for each point, the velocity vector equals $\omega\vec{r} + \vec{v}_O$, e.g., $v_A = -\omega r + v_O$. Since the wheel does not slip, $v_A = 0$. Thus,

$$v_O = \omega r. \quad (11.1)$$

In B, $\omega\vec{r}$ and \vec{v}_O are perpendicular. Therefore,

$$v_B = \sqrt{(\omega r)^2 + (\omega r)^2} = \sqrt{2}\omega r. \quad (11.2)$$

Obviously, at C,

$$v_C = 2\omega r. \quad (11.3)$$

If the wheel is accelerating in positive direction,

$$v_O = \omega(t) r \quad (11.4)$$

and

$$a_O = \frac{dv_O}{dt} = \dot{\omega}r = \alpha r. \quad (11.5)$$

The acceleration of points A, B, C relative to O is $\alpha \vec{r} + \vec{a}_c = \alpha \vec{r} + \omega^2 \vec{r}$. At A, the horizontal acceleration is $-\alpha r + \alpha r = 0$. So only the centripetal acceleration remains.

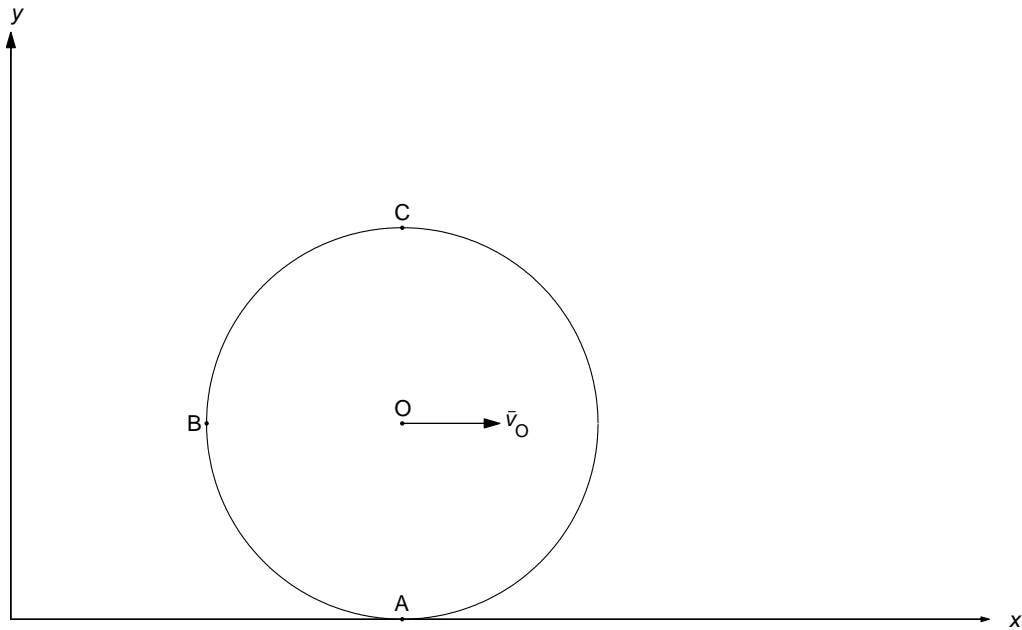


Figure 11.1: A wheel rolling on a horizontal surface in positive direction (clockwise) at constant speed without slipping.

12

Moment of a force

The moment of a force about a point is the product of the force and the perpendicular distance from the point of reference to the line of action of the force.

Centre of mass or centroid

The centre of mass G of n elemental masses m_i at fixed positions $(\vec{r}_i)_G$ relative to G is such, that the sum of the products of mass and distance, *i.e.*, the moments of the masses about G ,

$$\sum_{i=1}^n m_i (\vec{r}_i)_G = 0. \quad (12.1)$$

For an arbitrarily shaped body,

$$\int \vec{r} \, dm = 0. \quad (12.2)$$

Consider some external coordinate origin at \vec{r}_G from G . Then, the positions of the element masses relative to the origin is

$$\vec{r}_i = \vec{r}_G + (\vec{r}_i)_G. \quad (12.3)$$

Applying Newton's second law to all of the elemental masses yields a resultant force

$$\vec{F} = \sum_{i=1}^n m_i \vec{a}_i = \sum_{i=1}^n m_i \ddot{\vec{r}}_i = \sum_{i=1}^n m_i \ddot{\vec{r}}_G + \sum_{i=1}^n m_i \left(\ddot{\vec{r}}_i \right)_G = \ddot{\vec{r}}_G \sum_{i=1}^n m_i. \quad (12.4)$$

This means that the translational motion of a rigid body can be modelled as the motion of the total mass at the centroid. The force acting on a body relates to the moment by

$$\vec{F} \times \vec{r} = \vec{M}. \quad (12.5)$$

13

Moment of inertia

The rotational motion of a system is related to the moment of forces as follows:

$$M = I \alpha, \quad (13.1)$$

where I is the moment of inertia about a certain origin, M is the moment of forces about the origin, and α is the angular acceleration relative to the origin. For a system containing n elementary masses m_i ,

$$I = \sum_{i=1}^n m_i r_i^2. \quad (13.2)$$

Since $I = (mr)r$, the moment of inertia is also referred to as the secondary moment of mass. For an arbitrarily shaped body,

$$I = \int r^2 dm. \quad (13.3)$$

Cylindrical shaft

Consider a cylindrical shaft length L , radius R , and density ρ rotating about its length axis. The mass of the shaft is

$$m = \pi R^2 L \rho. \quad (13.4)$$

The mass of an elementary tube of radius r and thickness dr

$$dm = 2\pi r L dr \rho. \quad (13.5)$$

The moment of inertia for the elementary tube is

$$dI = r^2 dm = 2\pi r^3 L dr \rho. \quad (13.6)$$

Integrating over all elemental tubes from 0 to R gives

$$I = \int_0^R 2\pi r^3 L \rho dr = 2\pi L \rho \left[\frac{r^4}{4} \right]_0^R = \frac{1}{2} \pi R^4 L \rho = \frac{1}{2} m R^2. \quad (13.7)$$

Moments of inertia for some simple shapes

The moment of inertia of a body rotating about the i -axis is given by I_i .

1. Solid cylinder of mass m , radius R , and length in z -direction L
 $I_x = I_y = \frac{1}{12} m(3R^2 + L^2);$
 $I_z = \frac{1}{2} m R^2.$
2. Thin wheel of mass m and radius R rotating about the z -axis
 $I_x = I_y = \frac{1}{2} m R^2;$
 $I_z = \frac{1}{2} m R^2.$
3. Hollow cylinder of mass m , external radius R , internal radius r , and length in z -direction L
 $I_z = \frac{1}{2} m(3R^2 + r^2).$
4. Solid sphere of mass m and radius R
 $I_x = I_y = I_z = \frac{2}{5} m R^2.$
5. Rectangular bar of mass m , width in x -direction b , height in y -direction h , and length in z -direction L
 $I_x = \frac{1}{12} m(h^2 + L^2);$
 $I_y = \frac{1}{12} m(b^2 + L^2);$
 $I_z = \frac{1}{12} m(b^2 + h^2).$

14

Parallel axis theorem

Consider an axes of rotation parallel to the axes of rotation through the centroid G. Consider the plane perpendicular to both axes and through G. We define C as the point where the new axis of rotation cuts the plane, which has a distance r_G to G. The distance from C to an elemental mass m_i in the plane is r_i , whereas the distance from G to m_i is $(r_i)_G$. We define \vec{x}_i and \vec{y}_i such, that

$$\begin{aligned}\vec{x}_i + \vec{y}_i &= (\vec{r}_i)_G, \\ \overrightarrow{(r_G + x_i)} + \vec{y}_i &= \vec{r}_i, \\ \vec{x}_i \cdot \vec{y}_i &= 0.\end{aligned}\tag{14.1}$$

Furthermore, we define the angle θ_i between $(\vec{r}_i)_G$ and the line $\overrightarrow{(r_G + x_i)}$ through C and G. Here,

$$\begin{aligned}\frac{y_i}{x_i} &= \tan \theta_i, \\ x_i &= (r_i)_G \cos \theta_i, \\ y_i &= (r_i)_G \sin \theta_i.\end{aligned}\tag{14.2}$$

According to the Pythagorean theorem,

$$\begin{aligned}
 r_i^2 &= (r_G + x_i)^2 + y_i^2 \\
 &= \left(r_G^2 + 2r_G (r_i)_G \cos \theta_i + (r_i)_G^2 \cos^2 \theta_i \right) + (r_i)_G^2 \sin^2 \theta_i \\
 &= r_G^2 + (r_i)_G^2 + 2r_G (r_i)_G \cos \theta_i.
 \end{aligned} \tag{14.3}$$

Now we can reformulate the inertia moment for rotation about C with respect to the inertia moment about G for n elemental masses:

$$\begin{aligned}
 I_C &= \sum_{i=0}^n r_i^2 m_i = \sum_{i=0}^n \left((r_i)_G^2 + r_G^2 + 2r_G (r_i)_G \cos \theta_i \right) m_i \\
 &= \sum_{i=0}^n (r_i)_G^2 m_i + \sum_{i=0}^n r_G^2 m_i + 2r_G \sum_{i=0}^n x_i m_i \\
 &= I_G + \sum_{i=0}^n r_G^2 m_i = I_G + r_G^2 \sum_{i=0}^n m_i.
 \end{aligned} \tag{14.4}$$

Thus, for an arbitrarily shaped body of mass m ,

$$I_C = I_G + r_G^2 \int dm = I_G + m r_G^2. \tag{14.5}$$

Rotation energy

Combining (3.4) and (5.10), the work done during rotation is

$$\int F dr = \int m \omega^2 r dr = \frac{1}{2} m \omega^2 r^2 = \frac{1}{2} I \omega^2. \tag{14.6}$$

Thus, the total kinetic energy of a body translating at a velocity v and rotating at an angular velocity ω is

$$E_K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2. \tag{14.7}$$

15

Momentum and impulse

Momentum is the product of mass and velocity:

$$\vec{G} = m \vec{v}. \quad (15.1)$$

Because of the conservation of (linear) momentum,

$$\sum_i \vec{G}_i = \sum_i m_i \vec{v}_i = \text{constant}. \quad (15.2)$$

Impulse is defined by

$$\vec{J} = \int \vec{F} dt = \int_{\vec{v}_1}^{\vec{v}_2} m \frac{d\vec{v}}{dt} dt = m [\vec{v}]_{\vec{v}_1}^{\vec{v}_2} = m\vec{v}_2 - m\vec{v}_1. \quad (15.3)$$

Hence, impulse is the change in momentum, whereas force is the rate of change of momentum:

$$\vec{F} = \frac{d\vec{J}}{dt} = \frac{d}{dt} (m\vec{v}_2 - m\vec{v}_1). \quad (15.4)$$

Momentum and impulse are useful concepts in problems that involve time-varying forces, e.g., impacts, as long as the force variation is known. Alternatively, the impulse can be used for finding average forces in the velocities are known.

Impact

Impact problems involving separation tend to be situations in which force variations are difficult to specify. Consider an object of mass m_1 travelling at a velocity u_1 and an object of mass m_2 travelling over the same path at a velocity u_2 . Assume that $u_2 > u_1$. After impact, the respective velocities are v_1 and v_2 . However, $v_2 < v_1$. Conservation of (linear) momentum gives

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2. \quad (15.5)$$

For given masses and initial velocities, there are two unknowns. Another relationship is the coefficient of restitution. It gives the ratio of the separation velocity to the approach velocity

$$C_R = \frac{v_1 - v_2}{u_2 - u_1}. \quad (15.6)$$

Table 15.1 shows typical coefficients of restitution for some materials.

Table 15.1: Coefficients of restitution.

Materials	C_R
Glass on glass	0.95
Steel on steel	0.7
Wood on wood	0.5
Lead on lead	0.15

16

Free vibrations

Consider an undamped horizontal mass m on a spring with stiffness s (4.3):

$$m \ddot{x} + s x = 0 \quad (16.1)$$

or

$$\ddot{x} + \omega_0^2 x = 0, \quad (16.2)$$

where $\omega_0 = \sqrt{\frac{s}{m}}$ is the resonance frequency of the system (4.14). The general solution of this ordinary differential equation is

$$x = A \cos \omega_0 t + B \sin \omega_0 t. \quad (16.3)$$

If $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$,

$$\begin{aligned} x &= x_0 \cos \omega_0 t + \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t \\ &= C \sin(\omega_0 t + \phi) \\ &= C (\sin \omega_0 t \cos \phi + \cos \omega_0 t \sin \phi) \\ &= C \sin \phi \cos \omega_0 t + C \cos \phi \sin \omega_0 t. \end{aligned} \quad (16.4)$$

Since $C \sin \phi = x_0$ and $C \cos \phi = \frac{\dot{x}_0}{\omega_0}$,

$$x = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}} \sin \left(\omega_0 t + \arctan \frac{x_0 \omega_0}{\dot{x}_0} \right). \quad (16.5)$$

Damped free vibrations

Consider a mass-spring-dashpot system, for which the damping force is proportional to the velocity. Then (16.1) becomes

$$m\ddot{x} + \beta\dot{x} + s x = 0, \quad (16.6)$$

where β is the mechanical resistance. Using (4.14) and the damping coefficient

$$\zeta = \frac{\beta}{2m\omega_0}, \quad (16.7)$$

we can express the system in terms of its damping and resonance frequency:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0. \quad (16.8)$$

Assume $x = Ae^{\lambda t}$. Then,

$$\lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2 = 0. \quad (16.9)$$

Thus,

$$\lambda = \frac{-2\zeta\omega_0 \pm \sqrt{4\zeta^2\omega_0^2 - 4\omega_0^2}}{2} = \omega_0 \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right) \quad (16.10)$$

and

$$x = A_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_0 t} + A_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_0 t}. \quad (16.11)$$

Now, different forms of motion result from different ζ values and this is the reason for the assumed form of ζ . The following three conditions of damping exist:

1. Overdamped condition $\zeta > 1$

Since $\sqrt{\zeta^2 - 1}$ is real, (16.9) has two distinct roots.

2. Critically damped condition $\zeta = 1$

There is only one root: $\lambda = -\omega_0$. Therefore,

$$x = Ae^{-\omega_0 t}. \quad (16.12)$$

3. Underdamped condition $\zeta < 1$

Since $\sqrt{\zeta^2 - 1}$ is imaginary, the solution is complex:

$$\begin{aligned} x &= A_1 e^{(-\zeta + j\sqrt{1-\zeta^2})\omega_0 t} + A_2 e^{(-\zeta - j\sqrt{1-\zeta^2})\omega_0 t} \\ &= \left[A_1 e^{j\omega_0 t\sqrt{1-\zeta^2}} + A_2 e^{-j\omega_0 t\sqrt{1-\zeta^2}} \right] e^{-\zeta\omega_0 t}. \end{aligned} \quad (16.13)$$

Making use of $e^{j\theta} = \cos \theta + j \sin \theta$, we can reformulate this to

$$\begin{aligned}
 x &= [A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}] e^{-\zeta\omega_0 t} \\
 &= [A_1 (\cos \omega_d t + j \sin \omega_d t) + A_2 (\cos \omega_d t - j \sin \omega_d t)] e^{-\zeta\omega_0 t} \\
 &= [(A_1 + A_2) \cos \omega_d t + j(A_1 - A_2) \sin \omega_d t] e^{-\zeta\omega_0 t} \\
 &= [B_1 \cos \omega_d t + B_2 \sin \omega_d t] e^{-\zeta\omega_0 t} \\
 &= C e^{-\zeta\omega_0 t} \sin(\omega_d t + \phi),
 \end{aligned} \tag{16.14}$$

where, ω_d is the damped natural frequency

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2}. \tag{16.15}$$

So the excursion amplitude decays by $e^{-\zeta\omega_0 t}$. The damped period is

$$T_d = \frac{2\pi}{\omega_d}. \tag{16.16}$$

The damping coefficient can be deduced by measuring the excursion amplitudes $x_1 = x(t_1)$ and $x_2 = x(t_2)$ at two times whose difference $t_2 - t_1 = T_d$:

$$\delta \equiv \log_e \frac{x_1}{x_2} = \zeta\omega_0 T_d = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}}. \tag{16.17}$$

Thus,

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}. \tag{16.18}$$

17

Forced vibrations

Consider a mass-spring-dashpot system, whose mass is subjected to a periodic force F with amplitude F_0

$$F = F_0 \sin \omega t. \quad (17.1)$$

The excursion of the mass can be described by adding F into (16.6):

$$-s x - \beta \dot{x} + F_0 \sin \omega t = m \ddot{x}. \quad (17.2)$$

Using (4.14) and (16.7), we can express the mass-forced system in terms of its damping and resonance frequency:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \frac{F_0 \sin \omega t}{m}. \quad (17.3)$$

Now consider a mass-spring-dashpot system, whose end is subjected to a periodic excursion x_b with amplitude b

$$x_b = b \sin \omega t. \quad (17.4)$$

The excursion of the mass can be described by

$$-s(x - x_b) - \beta \dot{x} = m \ddot{x} \quad (17.5)$$

or

$$\ddot{x} + \frac{\beta}{m} \dot{x} + \frac{s}{m} x = \frac{s b \sin \omega t}{m}. \quad (17.6)$$

A base-forced system can be expressed in terms of its damping and resonance frequency by

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = \frac{s b \sin \omega t}{m}. \quad (17.7)$$

Undamped forced vibrations

Consider a mass-forced mass-spring-dashpot system. If the system is undamped, (17.3) reduces to

$$\ddot{x} + \omega_0^2x = \frac{F_0 \sin \omega t}{m}. \quad (17.8)$$

The solution consists of the complementary function and the particular integral. The complementary function is the solution of

$$\ddot{x} + \omega_0^2x = 0, \quad (17.9)$$

which has been shown in (16.4) and (16.5):

$$\begin{aligned} x &= x_0 \cos \omega_0 t + \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t \\ &= \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}} \sin \left(\omega_0 t + \arctan \frac{x_0 \omega_0}{\dot{x}_0} \right). \end{aligned} \quad (17.10)$$

This is called the transient solution. The particular integral represents the steady-state solution. It can be determined assuming the same form as the driving function

$$x = X \sin \omega t. \quad (17.11)$$

Hence, substituting in (17.8) gives

$$-\omega^2 X \sin \omega t + \omega_0^2 X \sin \omega t = \frac{F_0 \sin \omega t}{m}. \quad (17.12)$$

Dividing through by $\omega_0^2 \sin \omega t$ and rearranging results into

$$\left(1 - \left(\frac{\omega}{\omega_0} \right)^2 \right) X = \frac{F_0}{m \omega_0^2} = \frac{F_0}{s} \quad (17.13)$$

or

$$X = \frac{\frac{F_0}{s}}{1 - \left(\frac{\omega}{\omega_0} \right)^2}. \quad (17.14)$$

So the steady-state solution of (17.8) is

$$x = \frac{\frac{F_0}{s}}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \sin \omega t. \quad (17.15)$$

The static deflection of a mass under a static load F_0 is denoted by

$$\delta_s = \frac{F_0}{s}. \quad (17.16)$$

Now, the severity of the vibration can be expressed in terms of the amplitude ratio or magnification factor

$$M = \frac{X}{\delta_s} = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2}. \quad (17.17)$$

Damped forced vibrations

Consider a damped mass-forced mass-spring-dashpot system. The excursion around equilibrium is given by (17.3)

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = \frac{F_0 \sin \omega t}{m}. \quad (17.18)$$

The complimentary function is a solution of

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = 0, \quad (17.19)$$

which has been shown in (16.14):

$$x = Ce^{-\zeta\omega_0 t} \sin(\omega_d t + \phi). \quad (17.20)$$

The transient part of the solution damps out. The particular integral can be determined assuming the form

$$x = A \cos \omega t + B \sin \omega t = X \sin(\omega t - \phi). \quad (17.21)$$

Substituting in (17.18) gives

$$\begin{aligned} -\omega^2 X \sin(\omega t - \phi) + 2\zeta\omega_0 \omega X \cos(\omega t - \phi) \\ + \omega_0^2 X \sin(\omega t - \phi) = \frac{F_0 \sin \omega t}{m}. \end{aligned} \quad (17.22)$$

Dividing through by ω_0^2 and rearranging results into

$$\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) X \sin(\omega t - \phi) + 2\zeta \frac{\omega}{\omega_0} X \cos(\omega t - \phi) = \frac{F_0 \sin \omega t}{m \omega_0^2}. \quad (17.23)$$

This can be rewritten as

$$\begin{aligned} &\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) X (\sin \omega t \cos \phi - \cos \omega t \sin \phi) \\ &+ 2\zeta \frac{\omega}{\omega_0} X (\cos \omega t \cos \phi + \sin \omega t \sin \phi) = \frac{F_0 \sin \omega t}{m \omega_0^2} \end{aligned} \quad (17.24)$$

or

$$\begin{aligned} &\left[- \left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) X \sin \phi + 2\zeta \frac{\omega}{\omega_0} X \cos \phi \right] \cos \omega t \\ &+ \left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) X \cos \phi + 2\zeta \frac{\omega}{\omega_0} X \sin \phi \right] \sin \omega t = \frac{F_0 \sin \omega t}{m \omega_0^2}. \end{aligned} \quad (17.25)$$

This can be split up into two equations:

$$\left[- \left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) X \sin \phi + 2\zeta \frac{\omega}{\omega_0} X \cos \phi \right] \cos \omega t = 0 \quad (17.26)$$

and

$$\left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) X \cos \phi + 2\zeta \frac{\omega}{\omega_0} X \sin \phi \right] \sin \omega t = \frac{F_0 \sin \omega t}{m \omega_0^2}. \quad (17.27)$$

From (17.26), it follows that

$$\phi = \arctan \left(\frac{2\zeta \frac{\omega}{\omega_0}}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \right). \quad (17.28)$$

After dividing through by $\sin \omega t$, (17.27) is simplified to

$$\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) X \cos \phi + 2\zeta \frac{\omega}{\omega_0} X \sin \phi = \frac{F_0}{m \omega_0^2} = \frac{F_0}{s}. \quad (17.29)$$

Combining (17.28) and (17.29) yields

$$X = \frac{\frac{F_0}{s}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}}. \quad (17.30)$$

Hence, the magnification factor

$$M = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}}. \quad (17.31)$$

Similarly, for a damped base-forced mass-spring-dashpot system, the steady-state excursion amplitude is given by

$$X = \frac{b\left(\frac{\omega}{\omega_0}\right)^2}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}}. \quad (17.32)$$

A

List of symbols

M L T Θ dimensions

\vec{a}	acceleration	$L T^{-2}$
\dot{a}	jerk	$L T^{-3}$
\ddot{a}	jounce	$L T^{-4}$
a_c	central acceleration	$L T^{-2}$
a_i	acceleration of point i	$L T^{-2}$
a_t	tangential acceleration	$L T^{-2}$
A	oscillation amplitude	L
b	base-forced amplitude	L
B	oscillation amplitude	L
c	constant	$L^2 T^{-2}$
c	constant	T
C	oscillation amplitude	L
C_R	coefficient of restitution	
E_K	kinetic energy	$M L^2 T^{-2}$
E_P	potential energy	$M L^2 T^{-2}$
f_0	resonance frequency	T^{-1}
\vec{F}	force	$M L T^{-2}$
F_0	driving force amplitude	$M L T^{-2}$

F_c	central force	$M L T^{-2}$
F_g	gravitational force	$M L T^{-2}$
F_n	normal force	$M L T^{-2}$
F_r	reaction force	$M L T^{-2}$
F_{\parallel}	force parallel to a plane	$M L T^{-2}$
F_{\perp}	force perpendicular to a plane	$M L T^{-2}$
g	gravity acceleration	$L T^{-2}$
G	gravitational constant	$M^{-1} L^3 T^{-2}$
\vec{G}	momentum	$M L T^{-1}$
\vec{G}_i	momentum of element i	$M L T^{-1}$
h	height	L
I	moment of inertia	$M L^2$
I_C	moment of inertia about C	$M L^2$
I_G	moment of inertia about G	$M L^2$
I_i	moment of inertia about axis i	$M L^2$
j	complex number with the property $j^2 = -1$	
\vec{J}	impulse	$M L T^{-1}$
L	length	L
m	mass	M
m_i	elemental mass i	M
\vec{M}	moment	$M L^2 T^{-2}$
M_{\oplus}	mass of the earth	M
M	magnification factor	
n	number of elements	
N_i	number of teeth of gear wheel i	
\vec{r}	radius	L
r_i	radius of wheel i	L
\vec{r}_i	position of element i	L
$(\vec{r}_i)_G$	position of element i relative to centroid G	L
\vec{r}_G	distance from centroid G	L
R	radius	L
s	spring stiffness	$M T^{-2}$
t	time	T
T	period	T
T_0	resonance period	T
T_d	damping period	T
u_i	velocity of element i	$L T^{-1}$
v	velocity	$L T^{-1}$

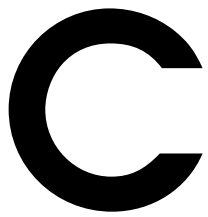
\vec{v}	directed velocity	$L T^{-1}$
\dot{v}	acceleration	$L T^{-2}$
\ddot{v}	jerk	$L T^{-3}$
\dddot{v}	jounce	$L T^{-4}$
v_0	initial velocity	$L T^{-1}$
v_i	velocity of point i	$L T^{-1}$
x	position	L
Δx	step size	L
\hat{x}	unit vector in x -direction	
\dot{x}	velocity in x -direction	$L T^{-1}$
\ddot{x}	acceleration	$L T^{-2}$
\ddot{x}	jerk	$L T^{-3}$
\dddot{x}	jounce	$L T^{-4}$
x_0	initial position	L
\dot{x}_0	initial velocity	$L T^{-1}$
x_b	base excursion	L
\bar{x}_i	x -distance to to element i	L
X	excursion amplitude	L
y	position	L
\hat{y}	unit vector in y -direction	
\dot{y}	velocity in y -direction	$L T^{-1}$
\ddot{y}	acceleration in y -direction	$L T^{-2}$
\bar{y}_i	y -distance to to element i	L
z	position	L
α	angular acceleration	T^{-2}
β	mechanical resistance	$M T^{-1}$
δ	excursion amplitude decay	
δ_s	static deflection	L
ζ	damping coefficient	
θ	angle	
$\dot{\theta}$	angular velocity	T^{-1}
$\ddot{\theta}$	angular acceleration	T^{-2}
θ_i	angle between $(\vec{r}_i)_G$ and $\overrightarrow{(r_G + x_i)}$	
ϑ	azimuth	
λ	$\omega_0 \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right)$	T^{-1}
μ	coefficient of friction	
ρ	density	$M L^{-3}$

ϕ	phase	
φ	zenith	
Φ	phase	
ω	angular velocity	T^{-1}
$\dot{\omega}$	angular acceleration	T^{-2}
ω_0	angular resonance frequency	T^{-1}
ω_d	damped natural frequency	T^{-1}
ω_i	angular frequency of wheel i	T^{-1}
$\langle \vec{r}, \vec{v} \rangle$	inner product of vectors \vec{r}, \vec{v}	$\text{L}^2 \text{T}^{-1}$

B

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Biographies

Keith Attenborough, B.Sc., Ph.D., CEng, FIOA, FASA, holds a B.Sc.(Hons) (UCL) in physics and a Ph.D. (Leeds) in applied science. He is Emeritus Professor in Engineering at The University of Hull and Research Professor at the Open University. He has carried out studies on the linear and nonlinear acoustical characteristics of porous surfaces, acoustic-to-seismic coupling, sound propagation through suspensions and emulsions, acoustical methods for surveying soils, outdoor sound propagation, blast noise reduction, sonic boom propagation, passive acoustic tracking and ranging, nonlinear acoustic detection of buried objects, and on noise levels in the UK.

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