



HAL
open science

Randomizations of groups are connected

Itai Ben Yaacov, Jorge Muñoz Carvajal

► **To cite this version:**

Itai Ben Yaacov, Jorge Muñoz Carvajal. Randomizations of groups are connected. 2021. hal-03187898v2

HAL Id: hal-03187898

<https://hal.science/hal-03187898v2>

Preprint submitted on 18 Sep 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RANDOMIZATIONS OF GROUPS ARE CONNECTED

ITAI BEN YAACOV, JORGE MUÑOZ CARVAJAL

ABSTRACT. Randomizations are continuous structures consisting of random variables with values on a given family of structures. We show that if G is a definable group on a continuous structure then G^R , the randomization of the group, defined as the set of variables whose values belong to the group, is definably connected, i.e. it has no proper invariant subgroups of bounded index.

INTRODUCTION

Given a structure \mathfrak{M} , continuous or classical, a randomization of \mathfrak{M} is a new structure consisting of random functions from a probability space into M . For classical structures, they were first introduced by Keisler on [Kei99] and were then formalized in the continuous setting by Ben Yaacov and Keisler on [BYK09]. They are defined as two-sorted continuous structures, consisting of a sort of random elements and a sort of events.

Moreover, from a classical theory T we can obtain a continuous theory T^R , known as the randomization of T , defined as the common theory of randomizations of models T . In [BYK09], an axiomatisation of T^R is given and several preservation results are proven, such as completeness, ω -categoricity and stability. This framework proved to be the adequate one for the study of randomizations, and subsequent works of Andrews, Goldbring and Keisler were developed in this context (see [AGK15], [AGK19], [AK15]).

Finally, Ben Yaacov in [BY13] generalized the randomization construction to continuous structures. In this slightly different set-up, although equivalent in the classical case, the sort of events is replaced by a sort of $[0, 1]$ -valued random variables and the language is simplified. As before, for a continuous theory T we can construct T^R and the same preservation results will hold.

This paper, as the title suggests, deals with the notion of connectedness in randomizations of groups. Let \mathfrak{M} be a continuous structure and G be a definable group over \emptyset , we have that G^R , defined as the set of variables whose values belong to the group, is a definable group. The second author together with Berenstein proved in [BM21], via a study of generic types, that if G is a classical definably amenable NIP group then G^R is definably connected, i.e. it has no proper type-definable subgroups of bounded index (when looked inside a monster model). In this paper, we show that in general, without any model-theoretical assumptions on the model, G^R is definably connected. Moreover, we prove that it has no proper invariant subgroups of bounded index. For the proof, we use the construction of a homogeneous space. We use the presentation given in [GN08], that allows us to embed the group G^R in the group of automorphisms of a certain structure and then use the fact that in randomizations all types are Lascar strong.

The paper is organized as follows, in Section 1 we introduce the construction of the homogeneous space. In Section 2 we define randomizations, give some results on definability in continuous logic and proof the main result.

1. THE HOMOGENEOUS SPACE CONSTRUCTION

Suppose G is an \emptyset -definable group in a structure \mathfrak{M} . The homogeneous space structure consists of \mathfrak{M} together with a new sort X , corresponding to an affine copy of G , and a symbol for the action $\cdot : G \times X \rightarrow X$. This is a classical construction and has been considered several times. For instance, in [Zie02, Section 7] it is used to prove an unpublished result of Bouscaren, Lascar and Pillay concerning compact Lie groups. It was later used in [GN08] to describe groups of automorphisms of the homogeneous space in terms of groups of automorphisms of the original structure and groups related to G . This section is based on the latter, we recreate in the continuous setting the construction given in Section 3 therein.

Authors supported by ANR project AGRUME (ANR-17-CE40-0026).

As previously stated, we let X be an affine copy of G . We also add G itself as a sort, so the group operation can be represented by a symbol in the language and we can avoid dealing with partially defined functions. However, this step is not strictly necessary and the same results can be obtained by just adding the affine copy. Let us denote this new structure by $\tilde{\mathfrak{M}}$. As mentioned before, the main goal of the construction is to describe some subgroups $\text{Aut}(\tilde{\mathfrak{M}})$ in terms of subgroups of $\text{Aut}(\mathfrak{M})$ and subgroups of G . Towards this end, we must embed G as a normal subgroup of $\text{Aut}(\tilde{\mathfrak{M}})$. To make this task easier, we will suppose that the distance on G is bi-invariant, which is always true in the classical context. Nevertheless, this does not impose us a limitation, since the group always admits a definable bi-invariant metric, see Fact 2.14 below.

Let \mathcal{L} be a continuous language and let \mathfrak{M} be an \mathcal{L} -structure. Suppose G is a definable group over \emptyset whose metric is bi-invariant. We will denote by S the sort in which G is defined. Let $\tilde{\mathcal{L}}$ be the many-sorted language obtained by adjoining two sorts S_1, S_2 and the following function symbols to \mathcal{L} :

- A unary function symbol $\iota : S_1 \rightarrow S$.
- A binary function symbol $\cdot_G : S_1 \times S_1 \rightarrow S_1$
- A binary function symbol $\cdot : S_1 \times S_2 \rightarrow S_2$.

We require the new unary function symbol to be 1-Lipschitz. To both binary function symbols we assign the same modulus of uniform continuity of the group multiplication. We define $\tilde{\mathfrak{M}} = (M, G(M), X(M))$ as the $\tilde{\mathcal{L}}$ -structure where

- (1) The sort M has the original structure of \mathfrak{M} .
- (2) $\iota : G(M) \rightarrow M$ is the inclusion map.
- (3) $\cdot_G : G(M) \times G(M) \rightarrow G(M)$ is the group operation.
- (4) $\cdot : G(M) \times X(M) \rightarrow X(M)$ is a regular (free and transitive) action of $G(M)$ on $X(M)$.

We endow $G(M)$ and $X(M)$ with the same metric, the one they inherit from M . So, $G(M)$ acts by isometries on $X(M)$, and for all $g, h \in G(M)$ and $x \in X(M)$, $d_X(g \cdot x, h \cdot x) = d_G(g, h)$. We will simply write \cdot instead of \cdot_G , the distinction with the action being clear from the context.

Fix $\tilde{\mathfrak{C}} = (\mathfrak{C}, G, X)$ a monster model extending $\tilde{\mathfrak{M}}$. In particular, \mathfrak{C} is a monster model of $\text{Th}(\mathfrak{M})$. A set is said to be *bounded*, or *small*, if its cardinality is less than the saturation of our monster. As usual, sets of parameters are supposed to be small.

Remark 1.1. The structure $\tilde{\mathfrak{C}}$ is interpretable in \mathfrak{C} (for more details on interpretation in continuous logic see [BYK16]). Indeed, fix $x_0 \in X$, the map that corresponds to the identity on the sort \mathfrak{C} , the inclusion on the sort G and on the sort X is given by $h \cdot x_0 \mapsto h$ defines an interpretation of $\tilde{\mathfrak{C}}$ on \mathfrak{C} .

In other words, we add two interpretable sorts, namely two copies of G interacting with each other, and in one of them we forget the inner group structure. We now characterize the elementary substructures of $\tilde{\mathfrak{C}}$.

Proposition 1.2. *Every elementary substructure of $\tilde{\mathfrak{C}}$ is of the form $(\mathfrak{N}, G(\mathfrak{N}), G(\mathfrak{N}) \cdot x_0)$ for some $\mathfrak{N} \leq \mathfrak{C}$ and $x_0 \in X$. Conversely, if $\mathfrak{N} \leq \mathfrak{C}$ then for every $x \in X$, $(\mathfrak{N}, G(\mathfrak{N}), G(\mathfrak{N}) \cdot x)$ is an elementary substructure of $\tilde{\mathfrak{C}}$.*

Proof. Let $\tilde{\mathfrak{N}} = (\mathfrak{N}, G_0, X_0)$ be an elementary substructure of $\tilde{\mathfrak{C}}$. It follows that $\mathfrak{N} \leq \mathfrak{C}$, which implies $G_0 = G(\mathfrak{N})$. Pick any $x_0 \in X_0$, since the action $G_0 \curvearrowright X_0$ is regular, we have $X_0 = G_0 \cdot x_0 = G(\mathfrak{N}) \cdot x_0$. We conclude that $\tilde{\mathfrak{N}} = (\mathfrak{N}, G(\mathfrak{N}), G(\mathfrak{N}) \cdot x_0)$. For the converse, suppose $\mathfrak{N} \leq \mathfrak{C}$, pick $x \in X$ and define $\tilde{\mathfrak{N}} = (\mathfrak{N}, G(\mathfrak{N}), G(\mathfrak{N}) \cdot x)$. Since $\tilde{\mathfrak{C}}$ is interpretable in \mathfrak{C} , for every $\tilde{\mathcal{L}}$ -formula $\varphi(u, y, z)$, where u, y and z are tuple of variables in the sorts S, S_1 and S_2 , respectively, there is a definable predicate $P_\varphi(u, v, w)$ in \mathcal{L} such that for every $a \in \mathfrak{C}, g, h \in G$

$$\varphi^{\tilde{\mathfrak{C}}}(a, g, h \cdot x) = P_\varphi^{\mathfrak{C}}(a, g, h).$$

Restricting to $\tilde{\mathfrak{N}}$, we obtain an interpretation of $\tilde{\mathfrak{N}}$ in \mathfrak{N} and an analogous equality for every $a \in \mathfrak{N}, g, h \in G(\mathfrak{N})$. Let w denote a variable in any of the sorts of $\tilde{\mathcal{L}}$, then for any $\tilde{\mathcal{L}}$ -formula $\varphi(u, y, z) = \inf_w \psi(u, y, z, w)$, and tuples $a \in \mathfrak{N}$ and $g, h \in G(\mathfrak{N})$

$$\inf_{w \in \tilde{\mathfrak{N}}} \psi^{\tilde{\mathfrak{N}}}(a, g, h \cdot x, w) = P_\varphi^{\mathfrak{N}}(a, g, h) = P_\varphi^{\mathfrak{C}}(a, g, h) = \inf_{w \in \tilde{\mathfrak{C}}} \psi^{\tilde{\mathfrak{C}}}(a, g, h \cdot x, w).$$

The result follows from the Tarski-Vaught test. □

Now, take any $x_0 \in X$, then $X = G \cdot x_0$. For $g \in G$ we define $\bar{g} : \tilde{\mathfrak{C}} \rightarrow \tilde{\mathfrak{C}}$ by

$$\bar{g} \upharpoonright_{\mathfrak{C}} = \text{id}_{\mathfrak{C}}, \quad \bar{g} \upharpoonright_G = \text{id}_G, \quad \bar{g}(h \cdot x_0) = (hg^{-1}) \cdot x_0.$$

Claim: For any $g \in G$, \bar{g} is an automorphism of $\tilde{\mathfrak{C}}$.

Proof. It is immediate that \bar{g} is a bijection that respects the structure of \mathfrak{C} , the group operation and that commutes with ι . Now, for $k, h \in G$

$$\bar{g}(k \cdot (h \cdot x_0)) = \bar{g}(kh \cdot x_0) = (khg^{-1}) \cdot x_0 = \bar{g}(k) \cdot \bar{g}(h \cdot x_0),$$

and

$$d_X(\bar{g}(h \cdot x_0), \bar{g}(k \cdot x_0)) = d_X(hg^{-1} \cdot x_0, kg^{-1} \cdot x_0) = d_G(h, k) = d_X(h \cdot x_0, k \cdot x_0). \quad \square$$

On the other hand, suppose $f \in \text{Aut}(\mathfrak{C})$, since G is definable, $f(G) = G$. So we can define $\bar{f} : \tilde{\mathfrak{C}} \rightarrow \tilde{\mathfrak{C}}$ by

$$\bar{f} \upharpoonright_{\mathfrak{C}} = f, \quad \bar{f} \upharpoonright_G = f \upharpoonright_G, \quad \bar{f}(h \cdot x_0) = f(h) \cdot x_0.$$

Claim: For any $f \in \text{Aut}(\mathfrak{C})$, \bar{f} is an automorphism of $\tilde{\mathfrak{C}}$.

Proof. By definition \bar{f} is a bijection that respects the structure of \mathfrak{C} , the group operation and commutes with ι . Now, for $k, h \in G$

$$\bar{f}(k \cdot (h \cdot x_0)) = \bar{f}(kh \cdot x_0) = f(kh) \cdot x_0 = f(k)f(h) \cdot x_0 = \bar{f}(k) \cdot \bar{f}(h \cdot x_0).$$

And

$$d_X(\bar{f}(h \cdot x_0), \bar{f}(k \cdot x_0)) = d_X(f(h) \cdot x_0, f(k) \cdot x_0) = d_G(f(h), f(k)) = d_G(h, k) = d_X(h \cdot x_0, k \cdot x_0). \quad \square$$

In this way, we have embeddings $\bar{\cdot} : \text{Aut}(\mathfrak{C}) \hookrightarrow \text{Aut}(\tilde{\mathfrak{C}})$ and $\bar{\cdot} : G \hookrightarrow \text{Aut}(\tilde{\mathfrak{C}})$. Note that for $f \in \text{Aut}(\tilde{\mathfrak{C}})$ and $g \in G$, $f \circ \bar{g} \circ f^{-1} = \bar{k}^{-1} \bar{f}(g) \bar{k}$ where $k \in G$ satisfies $f(x_0) = k \cdot x_0$. Thus, G is a normal subgroup of $\text{Aut}(\tilde{\mathfrak{C}})$. We now define the notion of Lascar types.

Definition 1.3. Let A be a set of parameters and, a and b be tuples (possibly infinite) of the same length in \mathfrak{C} . We say that $d_A(a, b) \leq 1$ if there exists $\mathfrak{N} \preceq \mathfrak{C}$ containing A such that $\text{tp}(a/\mathfrak{N}) = \text{tp}(b/\mathfrak{N})$. We say that a and b have the same Lascar type over A , and we write $\text{Lstp}(a/A) = \text{Lstp}(b/A)$, if there is a finite sequence of tuples of the same length $a_0 = a, a_1, \dots, a_n = b$ such that $d_A(a_i, a_{i+1}) \leq 1$.

An equivalence relation is *bounded* if the set of equivalence classes is bounded. It is a well-known fact that having the same Lascar type over A is the smallest bounded A -invariant equivalence relation. The group of *Lascar strong automorphisms over A* is the group generated by all $\text{Aut}(\mathfrak{C}/M)$ where M ranges over all models containing A , and is denoted by $\text{Autf}_L(\mathfrak{C}/A)$. It follows from the definition that every automorphism in $\text{Autf}_L(\mathfrak{C}/A)$ preserves Lascar strong types. Moreover, if $\text{Lstp}(a/A) = \text{Lstp}(b/A)$ then there is $f \in \text{Autf}_L(\mathfrak{C}/A)$, such that $f(a) = b$.

Definition 1.4. Given $A \subseteq \mathfrak{C}$, we define G_A^∞ to be the smallest A -invariant subgroup of G of bounded index.

Proposition 1.5. For $A \subseteq \mathfrak{C}$, $G_A^\infty = \langle g^{-1}h ; g, h \in G \text{ and } \text{Lstp}(g/A) = \text{Lstp}(h/A) \rangle$.

Proof. Denote $H = \langle g^{-1}h ; g, h \in G \text{ and } \text{Lstp}(g/A) = \text{Lstp}(h/A) \rangle$. The relation $E(g, h)$ given by $g^{-1}h \in G_A^\infty$ is A -invariant and bounded. Therefore, if $g, h \in G$ and $\text{Lstp}(g/A) = \text{Lstp}(h/A)$ then $g^{-1}h \in G_A^\infty$. Hence, $H \subseteq G_A^\infty$. For the other inclusion, clearly H is A -invariant. Let λ be the index of H in G and suppose that $\{g_\alpha H ; \alpha \in \lambda\}$ is an enumeration of the cosets of H in G . Take $\alpha, \beta \in \lambda$ with $\alpha \neq \beta$, then $g_\alpha^{-1}g_\beta \notin H$. This implies that $\text{Lstp}(g_\alpha/A) \neq \text{Lstp}(g_\beta/A)$. Since there are boundedly many Lascar types over A , then the index of H in G is bounded. Therefore, $G_A^\infty \subseteq H$ and the equality follows. \square

The following proposition corresponds to Propositions 3.3 and 3.4 in [GN08]. Even though they are proved in the classical setting, the proofs readily adapts to continuous structures.

Proposition 1.6. (1) $\text{Aut}(\tilde{\mathfrak{C}}) = G \rtimes \text{Aut}(\mathfrak{C})$. More precisely, for $F \in \text{Aut}(\tilde{\mathfrak{C}})$, $F = \bar{g} \circ \bar{f}$ where $f = F \upharpoonright_{\mathfrak{C}}$ and $F(x_0) = g^{-1} \cdot x_0$.
(2) If $\tilde{\mathfrak{M}}_0 \preceq \tilde{\mathfrak{C}}$ with $\tilde{\mathfrak{M}}_0 = (\mathfrak{M}_0, G_0, X_0)$ and $X_0 = G_0 \cdot (h_0 \cdot x_0)$ for some $h_0 \cdot x_0 \in X_0$. Then $F \in \text{Aut}(\tilde{\mathfrak{C}}/\tilde{\mathfrak{M}}_0)$ if and only if there is $f \in \text{Aut}(\mathfrak{C}/\mathfrak{M}_0)$ such that $F = \bar{f}^{\bar{h}_0}$.

- (3) For any $A \subseteq \mathfrak{C}$, $\text{Autf}_L(\widetilde{\mathfrak{C}}/A) = G_A^\infty \rtimes \text{Autf}_L(\mathfrak{C}/A)$.
- (4) $G_A^\infty = G \cap \text{Autf}_L(\widetilde{\mathfrak{C}}/A)$.

2. RANDOMIZATION OF THE HOMOGENEOUS SPACE

2.1. Randomizations.

Here we give some basic definitions and results on randomizations, for a more detailed treatment of the subject we refer the reader to [BY13]. Let \mathcal{L} be a continuous language, we define the *randomization language*, denoted \mathcal{L}^R , as a continuous language consisting of:

- (1) A sort in \mathcal{L}^R for each sort in \mathcal{L} . These sorts will be referred as the *main sorts*.
- (2) A new sort, which will be called the *auxiliary sort*, equipped with the language of random variables \mathcal{L}_{RV} .
- (3) For each function symbol in \mathcal{L} a function symbol in \mathcal{L}^R between the corresponding main sorts, equipped with the same modulus of uniform continuity.
- (4) For each predicate symbol P in \mathcal{L} , a function symbol $\llbracket P(\dots) \rrbracket$ from the corresponding main sorts into the auxiliary sort, equipped with the same modulus of uniform continuity.

Definition 2.1. Let $(\Omega, \mathcal{B}, \mu)$ be an atomless probability space and let $\mathcal{M}_\Omega = \{\mathcal{M}_\omega; \omega \in \Omega\}$ be a family of \mathcal{L} -structures. A *randomization based on \mathcal{M}_Ω* , is an \mathcal{L}^R -prestructure $\mathbf{M} = (\mathcal{K}, \mathcal{A})$ such that:

- (1) \mathcal{A} corresponds to $L^1((\Omega, \mathcal{B}, \mu), [0, 1])$.
- (2) \mathcal{K} is a subset of $\prod_{\omega \in \Omega} \mathcal{M}_\omega$.
- (3) For $F \in \mathcal{L}$, n -ary function symbol, and $a_1, \dots, a_n \in \mathcal{K}$, the mapping $\omega \mapsto F^{\mathcal{M}_\omega}(a_1(\omega), \dots, a_n(\omega))$ belongs to \mathcal{K} , and we define $F^{\mathbf{M}}(a_1, \dots, a_n)$ as this map.
- (4) For $P \in \mathcal{L}$, n -ary predicate symbol, and $a_1, \dots, a_n \in \mathcal{K}$, the mapping $\omega \mapsto P^{\mathcal{M}_\omega}(a_1(\omega), \dots, a_n(\omega))$ belongs to \mathcal{A} , and we define $\llbracket P^{\mathbf{M}}(a_1, \dots, a_n) \rrbracket$ as this map.
- (5) For every $a, b \in \mathbf{M}$ and $A \in \mathcal{A}$ there is $c \in \mathbf{M}$ such that

$$c(\omega) = \begin{cases} a(\omega) & \text{if } A(\omega) = 1 \\ b(\omega) & \text{if } A(\omega) = 0 \\ \text{anything} & \text{otherwise.} \end{cases}$$

- (6) For every $a, b \in \mathcal{K}$, the mapping $\llbracket d(a, b) \rrbracket \in \mathcal{A}$ and $d_{\mathcal{K}}(a, b) = E\llbracket d(a, b) \rrbracket$.

In case \mathcal{M}_ω is equal to a single structure \mathfrak{M} for every $\omega \in \Omega$, we say that \mathbf{M} is a randomization of \mathfrak{M} . Given an \mathcal{L} -theory T , we define the *randomization theory*, denoted T^R , as the common theory of randomizations based on families of models of T .

Let \mathbf{M} be a randomization based on \mathcal{M}_Ω . Proceeding by induction, for every \mathcal{L} -formula $\varphi(x)$, we have a definable map $\llbracket \varphi(\cdot) \rrbracket : \mathcal{K}^n \rightarrow \mathcal{A}$ defined in the obvious way. In particular, if φ is an \mathcal{L} -sentence, then $\llbracket \varphi \rrbracket \in \mathcal{A}$. So, if $\mathcal{M}_\omega \models \varphi = 0$ for every $\omega \in \Omega$ then $\mathbf{M} \models \llbracket \varphi \rrbracket = 0$. Hence, if T is an \mathcal{L} -theory, and \mathbf{M} is a randomization based on models of T , then $\mathbf{M} \models \llbracket \varphi \rrbracket = 0$ for every condition $(\varphi = 0) \in T$. We thus obtain *Keisler's transference axiom*:

$$T^R \models \llbracket \varphi \rrbracket = 0 \text{ for every condition } (\varphi = 0) \in T.$$

Conversely, if \mathbf{M} is a model of T^R , then we can construct an atomless probability space $(\Omega, \mathcal{B}, \mu)$, a family of \mathcal{L} -structures \mathcal{M}_Ω and a randomization based on \mathcal{M}_Ω , $\widetilde{\mathbf{M}}$, such that \mathbf{M} is isomorphic to $\widetilde{\mathbf{M}}$ and for every formula $\varphi(\bar{x})$ and every tuple \bar{a} in $\widetilde{\mathbf{M}}$, $\llbracket \varphi(\bar{a}) \rrbracket^{\widetilde{\mathbf{M}}}(\omega) = \varphi^{\mathcal{M}_\omega}(\bar{a}(\omega))$ (see the discussion after Definition 3.9 and Theorem 3.14 in [BY13]). This randomization is known as the *canonical representation* and we will identify a randomization with its canonical representation. In particular, each \mathcal{M}_ω is a model of T . In this way, we have a correspondence between models of T^R and randomizations based on models of T .

An explicit axiomatization of T^R is given in [BY13]. This theory has many desirable properties, in particular it has quantifier elimination in the main sorts down to formulas of the form $E\llbracket \varphi(\bar{x}) \rrbracket$ (see Theorem 3.32 in [BY13]). However, it does not preserve simplicity, if T has IP then T^R has TP_2 (see Theorem 4.13 in [BY13]).

2.2. Definable groups.

Fix a continuous language \mathcal{L} , \mathfrak{M} an \mathcal{L} -structure and G an \emptyset -definable group. Let $(\mathcal{K}, \mathcal{A})$ be a monster model of $\text{Th}(\mathfrak{M})^R$, which we identify with its canonical representation. By taking \mathfrak{C} to be a sufficiently saturated elementary extension of \mathfrak{M} , we can assume that $(\mathcal{K}, \mathcal{A})$ corresponds to a randomization of \mathfrak{C} and we will denote it \mathfrak{C}^R . The randomization of G will correspond to the set of functions in \mathcal{K} whose image is contained in G . We will show that this is in fact a definable group in \mathfrak{C}^R . First a technical result.

Fact 2.2 (Theorem 9.12 in [BYBHU08]). *Let \mathcal{M} be a continuous structure. Given a definable predicate $P : M \rightarrow [0, 1]$, we define the two conditions*

$$E_1 : \sup_x \inf_y P(y) \vee |P(x) - d(x, y)| = 0$$

$$E_2 : \sup_x |P(x) - \inf_y (P(y) + d(x, y))| = 0.$$

Then, $P(x)$ corresponds to the distance to the zeroset of P if and only if \mathcal{M} satisfies E_1 and E_2 .

Proposition 2.3. *If D is definable subset of \mathfrak{C}^n over \emptyset then*

$$D^R = \{a \in \mathcal{K}^n ; a(\omega) \in D \text{ for all } \omega \in \Omega\}.$$

is definable over \emptyset .

Proof. Suppose that $d(x, D) = P(x)$ for some definable predicate $P : \mathfrak{C}^n \rightarrow [0, 1]$. Then D^R corresponds to the zeroset of $E[P(x)]$, we will use the previous proposition to show that this predicate corresponds in fact to the distance predicate to D^R . By the transfer axiom applied to E_1 , $\sup_x \inf_y (\llbracket P(y) \rrbracket \vee \llbracket |P(x) - d(x, y)| \rrbracket) = 0$. This implies that, $\sup_x \inf_y (E[\llbracket P(y) \rrbracket \vee E[\llbracket |P(x) - d(x, y)| \rrbracket]]) = 0$. Hence, $\sup_x \inf_y (E[\llbracket P(y) \rrbracket \vee |E[\llbracket P(x) \rrbracket] - E[\llbracket d(x, y) \rrbracket]|]) = 0$. This means that \mathfrak{C}^R satisfies E_1 for $E[\llbracket P(x) \rrbracket]$. Analogously we obtain that $\sup_x |E[\llbracket P(x) \rrbracket] - \inf_y (E[\llbracket P(y) \rrbracket] + E[\llbracket d(x, y) \rrbracket])| = 0$, i.e. \mathfrak{C}^R satisfies E_2 for $E[\llbracket P(x) \rrbracket]$. \square

For the proof of the next lemma we will use the following fact, which is the heart of the proof of Theorem 9.17 in [BYBHU08].

Fact 2.4. *Let \mathcal{M} be a continuous structure. If $D \subseteq M^n$ is definable and $P : M^m \times M^n \rightarrow [0, 1]$ is a definable predicate. Then, there is a continuous function $\alpha : [0, 1] \rightarrow [0, 1]$ such that for any $x \in M^m$:*

- $|P(x, y) - P(x, z)| \leq \alpha(d(y, z))$ for every $y, z \in M^n$.
- $\inf_{y \in D} P(x, y) = \inf_y (P(x, y) + \alpha(d(y, D)))$.

Lemma 2.5. *Let D be an \emptyset -definable subset of \mathfrak{C}^n and let $P : \mathfrak{C}^m \times \mathfrak{C}^n \rightarrow [0, 1]$ be an \emptyset -definable predicate. Then,*

$$\llbracket \inf_{y \in D} P(x, y) \rrbracket = \inf_{y \in D^R} \llbracket P(x, y) \rrbracket.$$

Proof. Fix $x \in \mathcal{K}^m$, then for any $y_0 \in D^R$, $\llbracket P(x, y_0) \rrbracket \geq \llbracket \inf_{y \in D} P(x, y) \rrbracket$. By taking the infimum over $y_0 \in D^R$ we conclude that $\inf_{y \in D^R} \llbracket P(x, y) \rrbracket \geq \llbracket \inf_{y \in D} P(x, y) \rrbracket$. For the other inequality, take $\alpha : [0, 1] \rightarrow [0, 1]$ as in the previous fact. Given $\epsilon > 0$, take $y_0 \in \mathcal{K}^n$ satisfying

$$\llbracket P(x, y_0) + \alpha(d(y_0, D)) \rrbracket \leq \llbracket \inf_{y \in D} P(x, y) \rrbracket + \epsilon.$$

Let $z_0 \in D^R$ be such that $\llbracket d(y_0, z_0) \rrbracket = \llbracket d(y_0, D) \rrbracket$. Then,

$$\inf_{y \in D^R} \llbracket P(x, y) \rrbracket \leq \llbracket P(x, z_0) \rrbracket \leq \llbracket P(x, y_0) + \alpha(d(y_0, z_0)) \rrbracket \leq \llbracket \inf_{y \in D} P(x, y) \rrbracket + \epsilon.$$

Since ϵ was arbitrary, we get the desired inequality. \square

Proposition 2.6. *G^R is an \emptyset -definable subgroup of \mathfrak{C}^R with the group operation defined pointwise, i.e. for $f_1, f_2 \in G^R$, $(f_1 \cdot f_2)(\omega) = f_1(\omega)f_2(\omega)$ for every $\omega \in \Omega$.*

Proof. By Proposition 2.3, G^R is an \emptyset -definable subset of \mathfrak{C}^R . Let $P : \mathfrak{C}^3 \rightarrow [0, 1]$ be the \emptyset -definable predicate such that $P(x, y, z) = d(x \cdot y, z)$ when restricted to $G^2 \times \mathfrak{C}$. We have that $\mathfrak{C} \models \sup_{x \in G} \sup_{y \in G} \inf_{z \in G} P(x, y, z) = 0$.

0. By transference and the previous Lemma,

$$\mathfrak{C}^R \models \sup_{x \in G^R} \sup_{y \in G^R} \inf_{z \in G^R} \llbracket P(x, y, z) \rrbracket = 0.$$

This implies that for every $x, y \in G^R$ there is $z \in G^R$ such that $x \cdot y = z$. So, G^R is closed under pointwise multiplication. Clearly, for $x, y \in G^R$ and $z \in \mathcal{K}$, $d(x \cdot y, z) = E \llbracket P(x, y, z) \rrbracket$. Thus, the group operation is definable. It remains to show that G^R is in fact a group. It is immediate that the operation is associative. Again by transference and the previous lemma, $\mathfrak{C}^R \models \inf_{x \in G^R} \sup_{y \in G^R} \llbracket P(x, y, y) \vee P(y, x, y) \rrbracket = 0$

and $\mathfrak{C}^R \models \sup_{x \in G^R} \inf_{y \in G^R} \sup_{z \in G^R} \llbracket P(x \cdot y, z, z) \vee P(y \cdot x, z, z) \rrbracket = 0$. So, G^R has a neutral element and inverses. \square

2.3. The main result.

We first assume that the metric in G^R is bi-invariant. Then, we can construct the $(\widetilde{\mathcal{L}}^R)$ -structure $\widetilde{\mathfrak{C}}^R = (\mathcal{K}, \mathcal{A}, G^R, X^R)$. By reordering the sorts of $(\widetilde{\mathcal{L}}^R)$, we see that $(\widetilde{\mathcal{L}}^R) \subset (\widetilde{L})^R$. The inclusion being strict due to the fact that the mappings $\llbracket d(-, -) \rrbracket$ from the sorts of G^R and X^R do not belong to $(\widetilde{\mathcal{L}}^R)$. However, if $x, y \in G^R$ (or X^R) then they can be identified with elements in \mathcal{K} , so we can define these maps in the obvious way, namely:

$$\llbracket d(x, y) \rrbracket(\omega) = d(x(\omega), y(\omega)) \text{ for every } \omega \in \Omega.$$

Hence, $(\mathcal{K}, G^R, X^R, \mathcal{A})$ admits a natural interpretation as an $(\widetilde{L})^R$ -structure. The next result is clear.

Proposition 2.7. *With the maps $\llbracket d(-, -) \rrbracket$ defined as before, $\widetilde{\mathfrak{C}}^R$ is a randomization of $\widetilde{\mathfrak{C}}$.*

In this way, when we consider $(\mathcal{K}, G^R, X^R, \mathcal{A})$ as an $(\widetilde{L})^R$ -structure we will write $\widetilde{\mathfrak{C}}^R$.

Proposition 2.8. *The maps $\llbracket d(-, -) \rrbracket : (G^R)^2 \rightarrow \mathcal{A}$ and $\llbracket d(-, -) \rrbracket : (X^R)^2 \rightarrow \mathcal{A}$ are \emptyset -definable in $\widetilde{\mathfrak{C}}^R$.*

Proof. For any $g, h \in G^R$ we have that $\llbracket d(g, h) \rrbracket = \llbracket d(i(g), i(h)) \rrbracket$, a definable map. For the other map, first note that $e \in G^R$ is definable, the distance predicate being given by $d_{\mathcal{K}}(x, e) = d_{\mathcal{K}}(x^2, x)$. Now, given $x, y \in X^R$ define $\chi(x, y)$ to be the unique $g \in G^R$ such that $g \cdot x = y$. The map $\chi : (X^R)^2 \rightarrow G^R$ is definable over \emptyset . Indeed, given $h \in G^R$, $d_{G^R}(h, \chi(x, y)) = d_{X^R}(h \cdot x, y)$. Hence, $\llbracket d(x, y) \rrbracket = \llbracket d(\chi(x, y), e) \rrbracket$, which is a definable map. \square

Corollary 2.9. *Given $A \subseteq \mathcal{K}$, $\text{Aut}(\widetilde{\mathfrak{C}}^R/A) = \text{Aut}(\widetilde{\mathfrak{C}}^R/A)$.*

Proof. Immediate from the previous result. \square

The next result says that in randomizations all types are Lascar strong.

Theorem 2.10 (Theorem 5.8 in [BY13]). *Let T be any theory, $(\mathcal{K}, \mathcal{B}) \models T^R$, $a, b \in \mathcal{K}$ and $A \subseteq \mathcal{K}$ any set of parameters. Then the following are equivalent:*

- (1) $\text{Lstp}(a/A) = \text{Lstp}(b/A)$.
- (2) $\text{tp}(a/A) = \text{tp}(b/A)$.

Corollary 2.11. *Given $A \subseteq \mathcal{K}$, $\text{Aut}_L(\widetilde{\mathfrak{C}}^R/A) = \text{Aut}(\widetilde{\mathfrak{C}}^R/A)$*

Proposition 2.12. *Given $A \subseteq \mathcal{K}$, $\text{Aut}_L(\widetilde{\mathfrak{C}}^R/A) = \text{Aut}_L(\widetilde{\mathfrak{C}}^R/A)$.*

Proof. \subseteq) It follows directly from Corollaries 2.9 and 2.11.

\supseteq) Take $\mathfrak{N} \leq \widetilde{\mathfrak{C}}^R$ with $A \subseteq \mathfrak{N}$ and $f \in \text{Aut}(\widetilde{\mathfrak{C}}^R/\mathfrak{N})$. Then \mathfrak{N}_0 , the reduct of \mathfrak{N} to a $(\widetilde{\mathcal{L}}^R)$ -structure, is an elementary substructure of $\widetilde{\mathfrak{C}}^R$. Hence, $f \in \text{Aut}(\widetilde{\mathfrak{C}}^R/\mathfrak{N}_0)$. So $\text{Aut}(\widetilde{\mathfrak{C}}^R/\mathfrak{N}) \subseteq \text{Aut}_L(\widetilde{\mathfrak{C}}^R/A)$. Since \mathfrak{N} was arbitrary, the result follows. \square

Corollary 2.13. *For any $A \subseteq \mathcal{K}$, $(G^R)_A^\infty = G^R$. Thus, G^R is definably connected.*

Proof. By 4. of Proposition 1.6, $(G^R)_A^\infty = G^R \cap \text{Aut}_L(\widetilde{\mathfrak{C}}^R/A)$. Using Proposition 2.12 we get $(G^R)_A^\infty = G^R \cap \text{Aut}_L(\widetilde{\mathfrak{C}}^R/A)$. Now, by Corollary 2.11, $(G^R)_A^\infty = G^R \cap \text{Aut}(\widetilde{\mathfrak{C}}^R/A)$. Finally, since $G^R \subseteq \text{Aut}(\widetilde{\mathfrak{C}}^R/A)$, then $(G^R)_A^\infty = G^R$. \square

In order to prove this corollary we assume that that the metric on the group is bi-invariant. However, this requirement can be removed, as we now show.

Fact 2.14 (Proposition 3.13 in [BY10]). *Let \mathcal{M} be a continuous structure and let G be an \emptyset -definable group. Then there is an \emptyset -definable metric on \mathcal{M} which is bi-invariant on G .*

Theorem 2.15 (Main Theorem). *Let G be an \emptyset -definable group in a continuous theory. Then, for any $A \subseteq \mathcal{K}$, $(G^R)_A^\infty = G^R$.*

Proof. The case where the metric in G^R is bi-invariant corresponds to Corollary 2.13. Suppose now that the metric in G^R is not bi-invariant. Let d_1 be a metric on \mathcal{K} which is bi-invariant on G^R , as per the previous fact. We define \mathfrak{C}_1^R as the structure having the same universe that \mathfrak{C}^R but whose distance predicate on \mathcal{K} is given by d_1 and the original metric will correspond to a new predicate d_2 . So,

$$d_{\mathcal{K}}^{\mathfrak{C}_1^R} = d_1^{\mathfrak{C}^R} \text{ and } d_2^{\mathfrak{C}_1^R} = d_{\mathcal{K}}^{\mathfrak{C}^R}.$$

Since \mathfrak{C}^R and \mathfrak{C}_1^R have the same definable predicates, then for any $A \subseteq \mathcal{K}$, $\text{Aut}(\tilde{\mathfrak{C}}^R/A) = \text{Aut}(\tilde{\mathfrak{C}}_1^R/A)$. Furthermore, if $\mathfrak{N} \leq \tilde{\mathfrak{C}}$ then the structure \mathfrak{N}_1 , defined in the same way, is an elementary substructure of $\tilde{\mathfrak{C}}_1$. Hence, for any $A \subseteq \mathcal{K}$, $\text{Aut}_L(\tilde{\mathfrak{C}}^R/A) = \text{Aut}_L(\tilde{\mathfrak{C}}_1^R/A)$. The metric on G^R , viewed now as a definable group on $\tilde{\mathfrak{C}}_1^R$, is bi-invariant. By Corollary 2.13, G^R does not have non-trivial A -invariant subgroups of bounded index in the new language. As we have the same definable predicates, the same is true for G^R in the original structure. \square

REFERENCES

- [AGK15] Uri Andrews, Isaac Goldbring, and H. Jerome Keisler. Definable closure in randomizations. *Annals of Pure and Applied Logic*, 166(3):325–342, 2015.
- [AGK19] Uri Andrews, Isaac Goldbring, and H. Jerome Keisler. Independence in randomizations. *Journal of Mathematical Logic*, 19(1), 2019.
- [AK15] Uri Andrews and H. Jerome Keisler. Separable models of randomizations. *Journal of Symbolic Logic*, 80(4):1149–1181, 2015.
- [BM21] Alexander Berenstein and Jorge Daniel Muñoz. Definable connectedness of randomizations of groups. *Archive for Mathematical Logic*, 2021. doi:10.1007/s00153-021-00773-8.
- [BY10] Itai Ben Yaacov. Definability of groups in \aleph_0 -stable metric structures. *Journal of Symbolic Logic*, 75(3):817–840, 2010.
- [BY13] Itai Ben Yaacov. On theories of random variables. *Israel Journal of Mathematics*, 194(2):957–1012, 2013.
- [BYBHU08] Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov. Model theory for metric structures. In Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, editors, *Model Theory with Applications to Algebra and Analysis. Vol. 2*, volume 350 of *London Math Society Lecture Note Series*, pages 315–427. Cambridge University Press, Cambridge, 2008.
- [BYK09] Itai Ben Yaacov and H. Jerome Keisler. Randomizations of models as metric structures. *Confluentes Mathematici*, 1(2):197–223, 2009.
- [BYK16] Itai Ben Yaacov and Adriane Kaïchouh. Reconstruction of separably categorical metric structures. *Journal of Symbolic Logic*, 81(1):216–224, 2016.
- [GN08] Jakub Gismatullin and Ludomir Newelski. g -compactness and groups. *Archive for Mathematical Logic*, 47(5):479–501, 2008.
- [Kei99] H. Jerome Keisler. Randomizing a model. *Advances in Mathematics*, 143(1):124–158, 1999.
- [Zie02] Martin Ziegler. Introduction to the lascar group. In Katrin Tent, editor, *Tits Buildings and the Model Theory of Groups*, London Mathematical Society Lecture Note Series, page 279–298. Cambridge University Press, 2002. doi:10.1017/CB09780511549786.013.

ITAÏ BEN YAACOV, UNIVERSITÉ CLAUDE BERNARD - LYON 1, INSTITUT CAMILLE JORDAN, CNRS UMR 5208, 43 BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE
 URL: <http://math.univ-lyon1.fr/~begnac/>

JORGE MUÑOZ CARVAJAL, UNIVERSITÉ CLAUDE BERNARD - LYON 1, INSTITUT CAMILLE JORDAN, CNRS UMR 5208, 43 BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE
 Email address: muno@math.univ-lyon1.fr