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Some Topological Properties of Spaces of Lipschitz Continuous Maps on Quasi-Metric Spaces

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Abstract

Given any quasi-metric space X, d , we can form the space $\mathcal{L}X$ of all lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$, where X is given the d -Scott topology. We give $\mathcal{L}X$ the Scott topology of the pointwise ordering. We can then form the subspace $\mathcal{L}_\alpha(X, d)$ of α -Lipschitz continuous maps from X, d to $\overline{\mathbb{R}}_+$ ($\alpha \in \mathbb{R}_+$). We show that, when X, d is continuous Yoneda-complete, $\mathcal{L}_\alpha(X, d)$ is stably compact, and that its topology coincides with the compact-open topology and with the topology of pointwise convergence. We also show that the space $\mathcal{L}_\infty(X, d)$ of all Lipschitz continuous maps from X, d to $\overline{\mathbb{R}}_+$ has a topology that is determined by the topologies of $\mathcal{L}_\alpha(X, d)$, $\alpha > 0$, if X, d is Lipschitz regular.

1 Introduction

This work is one of several bricks needed in a study of quasi-metrics on spaces of continuous valuations (a notion close to measures) and non-linear extensions of the latter, which should appear in a series of papers¹.

¹An unpublished version of that work is available on arXiv [7]. The present paper covers Sections 6.2, 7, 5, 8, and 9 there, in that order.

Specifically, we are interested in Kantorovich-Rubinshteĭn-like quasi-metrics, where the distance from the continuous valuation μ on the quasi-metric space X, d to ν is obtained as a supremum over all 1-Lipschitz continuous maps $f: X \rightarrow \mathbb{R}_+$ of $\max(\int_{x \in X} f(x)d\mu - \int_{x \in X} f(x)d\nu, 0)$. This requires us to know as much as we can on spaces of 1-Lipschitz continuous maps from X to \mathbb{R} , more generally on spaces of α -Lipschitz continuous maps. It is the purpose of this paper to examine the main topological properties of those spaces of maps.

Outline We recapitulate some preliminary definitions and notions in Section 2. This is probably most needed for the theory of quasi-metric spaces and their d -Scott topology, which is new and not entirely well-known. For example, a Lipschitz map may fail to be continuous with respect to these topologies.

In Section 3, we introduce the spaces of maps that form the subject of this paper. We take the opportunity to state a few new results on the largest α -Lipschitz continuous map $f^{(\alpha)}$ below f not already given in [8]. In particular, we give a simple formula for $f^{(\alpha)}$ when X, d is continuous and f is already α -Lipschitz but not necessarily continuous. We also show how this is related to Scott's formula in domain theory.

We collect some required properties on the space $\mathcal{L}X$ of all lower semi-continuous maps from X (with its d -Scott topology) to $\overline{\mathbb{R}}_+$. Those come from [2], and the most important is that the Scott topology on $\mathcal{L}X$ coincides with the compact-open topology, when X, d is continuous Yoneda-complete.

We make a detour in Section 5 and briefly study the compact saturated subsets of continuous Yoneda-complete quasi-metric spaces.

At this point, we will have enough material to characterize the induced topology of the subspace $\mathcal{L}_\alpha(X, d)$ of α -Lipschitz continuous maps from X to $\overline{\mathbb{R}}_+$ when X, d is continuous Yoneda-complete. As we show in Section 6, this will not only be the compact-open topology, but also the topology of pointwise convergence. This allows to show that $\mathcal{L}_\alpha(X, d)$ is stably compact in Section 7. A number of pleasing continuity properties are obtained from that fact.

Finally, we study the space $\mathcal{L}_\infty(X, d)$ of all Lipschitz continuous maps from X to $\overline{\mathbb{R}}_+$ in Section 8, with no upper bound on the Lipschitz constant. In that short, final section, we show that the topology of $\mathcal{L}_\infty(X, d)$ is determined by the topologies of its subspaces $\mathcal{L}_\alpha(X, d)$, $\alpha > 0$, when X, d is Lipschitz

regular, a notion we introduced in [8]. In that case, the study of $\mathcal{L}_\infty(X, d)$ reduces in a simple way to that of the spaces $\mathcal{L}_\alpha(X, d)$, $\alpha > 0$.

2 Preliminaries

General topology We refer the reader to [6] for basic notions and theorems of topology, domain theory, and in the theory of quasi-metric spaces. The book [5] is the standard reference on domain theory, and I will assume known the notions of directed complete posets (dcpo), Scott-continuous functions, the way-below relation \ll , continuous posets and dcpos, and so on. We write $\uparrow x$ for the set of points y such that $x \ll y$. The Scott topology on a poset consists of the Scott-open subsets, namely the upwards-closed subsets U such that every directed family that has a supremum in U must intersect U . A Scott-continuous map between posets is one that is monotonic and preserves existing directed suprema, and this is equivalent to requiring that it is continuous for the underlying Scott topologies.

The topic of the present paper is on quasi-metric spaces. Chapters 6 and 7 of [6] are a recommended read on that subject. The paper [9] gives additional information on quasi-metric spaces, which we will also rely on.

As far as topology is concerned, compactness does not imply separation. In other words, we call a subset K of a topological space compact if and only if every open cover of K contains a finite subcover. This property is sometimes called quasicompactness.

We shall always write \leq for the specialization preordering of a topological space: $x \leq y$ if and only if every open neighborhood of x is also an open neighborhood of y , if and only if x is in the closure of y . As a result, the closure of a single point y is also its downward closure $\downarrow y$. In general, we write $\downarrow A$ for the downward closure of any set A , $\uparrow A$ for its upward closure, and $\uparrow x = \uparrow\{x\}$. A saturated subset of a topological space is a set that is the intersection of all its open neighborhoods; equivalently, it is an upwards-closed subset in its specialization preordering.

In a topological space, $\uparrow A$ is also equal to the saturation of A , namely the intersection of all the open neighborhoods of A . If K is compact, then so is its saturation $\uparrow K$, and we shall usually use the letter Q for compact *saturated* subsets.

Quasi-metric spaces Let $\overline{\mathbb{R}}_+$ be the set of extended non-negative reals. A *quasi-metric* on a set X is a map $d: X \times X \rightarrow \overline{\mathbb{R}}_+$ satisfying: $d(x, x) = 0$; $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality); $d(x, y) = d(y, x) = 0$ implies $x = y$. The pair X, d is then called a *quasi-metric space*. Given X, d , there is an ordering \leq^d on X given by $x \leq^d y$ if and only if $d(x, y) = 0$.

A trivial example is $\overline{\mathbb{R}}_+$ itself, with the quasi-metric $\mathbf{d}_{\mathbb{R}}$ defined by $\mathbf{d}_{\mathbb{R}}(x, y) = 0$ if $x \leq y$, $\mathbf{d}_{\mathbb{R}}(+\infty, y) = +\infty$ if $y \neq +\infty$, $\mathbf{d}_{\mathbb{R}}(x, y) = x - y$ if $x > y$ and $x \neq +\infty$. Then $\leq^{\mathbf{d}_{\mathbb{R}}}$ is the ordinary ordering \leq .

Naturally, every metric space is a quasi-metric space, but every poset can also naturally be considered as a quasi-metric space, with the quasi-metric d_{\leq} defined by $d_{\leq}(x, y) = 0$ if $x \leq y$, $+\infty$ otherwise.

The space of formal balls $\mathbf{B}(X, d)$ of a quasi-metric space X, d is probably the single most important artifact that has to be considered in the study of quasi-metric spaces [13, 3]. This has a very simple definition: a *formal ball* is syntax for an actual ball, namely a pair (x, r) where $x \in X$ (the *center*) and $r \in \mathbb{R}_+$ (the *radius*). $\mathbf{B}(X, d)$ comes with an ordering \leq^{d^+} defined by $(x, r) \leq^{d^+} (y, s)$ if and only if $d(x, y) \leq r - s$.

Given a quasi-metric space X, d , the *open ball* $B_{x, < r}^d$ with center $x \in X$ and radius $r \in \mathbb{R}_+$ is $\{y \in X \mid d(x, y) < r\}$. The open ball topology is the coarsest containing all open balls, and is the standard topology on metric spaces.

In the realm of quasi-metric spaces, the *d-Scott topology* is the topology we shall always consider, unless specified otherwise. This is defined as follows. We equip $\mathbf{B}(X, d)$ with the Scott topology of \leq^{d^+} . There is an injective map $x \mapsto (x, 0)$ from X to $\mathbf{B}(X, d)$, and the *d-Scott topology* is the coarsest that makes it continuous. This allows us to see X as a topological subspace of $\mathbf{B}(X, d)$.

The specialization ordering of X, d , whether with the open ball topology or with the *d-Scott topology*, is just \leq^d . This does not cause any conflict with the notation \leq^{d^+} for the ordering on $\mathbf{B}(X, d)$, since \leq^{d^+} is in fact the ordering deduced from a quasi-metric d^+ on $\mathbf{B}(X, d)$, defined by $d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0)$.

The *d-Scott topology* coincides with the familiar open ball topology when d is a metric [6, Proposition 7.4.46], or when X, d is Smyth-complete [6, Proposition 7.4.47]. It coincides with the generalized Scott topology of [1] when X, d is an algebraic Yoneda-complete quasi-metric space [6, Exercise 7.4.69]. On $\overline{\mathbb{R}}_+$, the $\mathbf{d}_{\mathbb{R}}$ -Scott topology coincides with the Scott topology of the usual ordering \leq : its non-trivial opens are the open intervals $]r, +\infty]$,

$r \in \mathbb{R}_+$. The d_{\leq} -Scott topology on a poset coincides with the Scott topology of \leq [9, Example 1.8]. The d^+ -Scott topology also coincides with the familiar Scott topology on $\mathbf{B}(X, d)$, see [8, Lemma 3.7] or [6, Exercise 7.4.53].

We say that X, d is *Yoneda-complete* if and only if $\mathbf{B}(X, d)$ is a dcpo. We take this as a definition, although this is really a theorem, due to Kostanek and Waszkiewicz, see [11] and [6, Theorem 7.4.27]. A metric space is Yoneda-complete if and only if it is complete in the usual sense. A poset, seen as a quasi-metric space, is Yoneda-complete if and only if it is a dcpo [9, Example 1.6].

Outside Yoneda-complete spaces, we would like to avoid certain pathologies and we will therefore concentrate on *standard* quasi-metric spaces [9, Section 2]. X, d is standard if and only if, for every directed family of formal balls $(x_i, r_i)_{i \in I}$, for every $s \in \mathbb{R}_+$, $(x_i, r_i)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$ if and only if $(x_i, r_i + s)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$. Writing the supremum of the former as (x, r) , we then have that $r = \inf_{i \in I} r_i$, and that the supremum of the latter is $(x, r + s)$ —this holds not only for $s \in \mathbb{R}_+$, but for every $s \geq -r$. In particular, the radius map $(x, r) \mapsto r$ is Scott-continuous from $\mathbf{B}(X, d)$ to $\overline{\mathbb{R}}_+^{op}$ ($\overline{\mathbb{R}}_+$ with the opposite ordering \geq), and for every $s \in \mathbb{R}_+$, the map $- + s : (x, r) \mapsto (x, r + s)$ is Scott-continuous from $\mathbf{B}(X, d)$ to itself [9, Proposition 2.4]. Moreover, and we will use that fact several times, the set $V_\epsilon = \{(x, r) \in \mathbf{B}(X, d) \mid r < \epsilon\}$ is Scott-open for every $\epsilon > 0$, whenever X, d is standard. This is because V_ϵ is the inverse image of the Scott-open subset $[0, \epsilon[$ of $\overline{\mathbb{R}}_+^{op}$ by the radius map.

All Yoneda-complete quasi-metric spaces are standard. Also, all metric spaces are standard. All posets are standard, too, with the quasi-metric d_{\leq} . Note that $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ is standard, being Yoneda-complete.

Directed complete posets (dcpos) come into further varieties: algebraic dcpos, continuous dcpos, notably, and that extends to algebraic posets and continuous posets. The same happens with quasi-metric spaces. We say that X, d is *continuous Yoneda-complete* if and only if $\mathbf{B}(X, d)$ is a continuous dcpo. This is again originally a theorem, not a definition [11]. More generally, we say that X, d (not necessarily Yoneda-complete) is *continuous* if and only if it is standard and $\mathbf{B}(X, d)$ is a continuous poset [9, Section 3].

A point $z \in X$ is a *center point* if and only if $B_{(z,0),\epsilon}^{d^+}$ is Scott-open in $\mathbf{B}(X, d)$ for every $\epsilon > 0$. For example, the center points of $\overline{\mathbb{R}}_+$ with the $\mathbf{d}_{\mathbb{R}}$ quasi-metric are all points except $+\infty$ [9, Example 5.9]. A standard quasi-metric space X, d is *algebraic* if and only if the open balls $B_{(z,0),\epsilon}^{d^+}$, where z

ranges over the center points and $\epsilon > 0$, form a base of the Scott topology on $\mathbf{B}(X, d)$ [9, Definition 5.11, Theorem 5.16]. Every (standard) algebraic quasi-metric space is continuous. Moreover, for every center point z , we have $(z, t) \ll (y, s)$ if and only if $d(z, y) < t - s$; more generally, $(y, s) \ll (x, r)$ if and only if there is a center point z and some $t \in \mathbb{R}_+$ such that $(y, s) \leq^{d^+} (z, t)$ and $d(z, x) < t - r$ [9, Proposition 5.18].

Given a d -Scott open subset U of X , there is a largest Scott open subset \widehat{U} of $\mathbf{B}(X, d)$ such that $U = \widehat{U} \cap X$. (We silently equate X with a subspace of $\mathbf{B}(X, d)$.) The assignment $U \mapsto \widehat{U}$ is monotonic. Being a right adjoint to the frame homomorphism that maps every open subset V of $\mathbf{B}(X, d)$ to $V \cap X$, it also preserves arbitrary meets, namely interiors of arbitrary intersections; but it satisfies no other remarkable property in general. A quasi-metric space X, d is *Lipschitz regular* if and only if the assignment $U \mapsto \widehat{U}$ is Scott-continuous. This is defined and studied in Section 4 of [8]. A Lipschitz regular quasi-metric space X, d is then *finitarily embedded* in $\mathbf{B}(X, d)$; see [4] for more information on finitary embeddings.

For algebraic Yoneda-complete quasi-metric spaces, Lipschitz regularity is equivalent to having relatively compact balls, a stronger requirement than local compactness. This displays Lipschitz regularity as a rather strong requirement. However, the space of formal balls $\mathbf{B}(X, d)$ of any quasi-metric space X, d , with the d^+ quasi-metric, is Lipschitz regular [8, Theorem 4.13].

Given a map f from a quasi-metric space X, d to a quasi-metric space Y, ∂ , f is α -Lipschitz if and only if $\partial(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. (When $\alpha = 0$ and $d(x, y) = +\infty$, we take the convention that $0 \cdot +\infty = +\infty$.)

For every $\alpha \in \mathbb{R}_+$, and every map $f: X, d \rightarrow Y, \partial$, let $\mathbf{B}^\alpha(f)$ map $(x, r) \in \mathbf{B}(X, d)$ to $(f(x), \alpha r) \in \mathbf{B}(Y, \partial)$. Then f is α -Lipschitz if and only if $\mathbf{B}^\alpha(f)$ is monotonic.

Contrarily to the case of spaces with the open ball topology, a Lipschitz map need not be continuous. The simplest way to appreciate this difficulty is to contemplate the case of posets, seen as quasi-metric spaces: a map between posets is Lipschitz if and only if it is monotonic [9, Example 6.1], and certainly not every monotonic map is continuous with respect to the underlying d_{\leq} -Scott (i.e., Scott) topologies; examples abound on $\overline{\mathbb{R}}_+$.

There is a notion of Lipschitz *Yoneda-continuous* map, characterized as preserving so-called d -limits. When both X, d and Y, ∂ are standard, f is α -Lipschitz Yoneda-continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous [9, Lemma 6.3]. We take the latter as our definition: a map $f: X, d \rightarrow Y, \partial$

between quasi-metric spaces is α -Lipschitz continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous [8, Definition 2.3]. While “ α -Lipschitz continuous” should be thought of as one epithet, not as the conjunction of “ α -Lipschitz” and “continuous” in general, we have the following. When both X, d and Y, ∂ are standard, $f: X \rightarrow Y$ is α -Lipschitz continuous if and only if f is both α -Lipschitz, and continuous from X with its d -Scott topology to Y with its ∂ -Scott topology [8, Proposition 2.5].

We are interested in the α -Lipschitz continuous functions from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$. Given a map $f: X \rightarrow \overline{\mathbb{R}}_+$, where X, d is standard, define $f': \mathbf{B}(X, d) \rightarrow \mathbb{R} \cup \{+\infty\}$ by $f'(x, r) = f(x) - \alpha r$. Then f is α -Lipschitz if and only if f' is monotonic, and f is α -Lipschitz continuous if and only if f' is Scott-continuous [8, Lemma 2.7].

The Lipschitz continuous maps from a standard quasi-metric space to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ enjoy the usual closure properties [9, Proposition 6.7]: if f is β -Lipschitz continuous then αf is $\alpha\beta$ -Lipschitz continuous; if f is α -Lipschitz continuous and g is β -Lipschitz continuous then $f + g$ is $(\alpha + \beta)$ -Lipschitz continuous; if f, g are α -Lipschitz continuous, then so are $\min(f, g)$ and $\max(f, g)$; if $(f_i)_{i \in I}$ is any family of α -Lipschitz continuous maps, then the pointwise supremum $\sup_{i \in I} f_i$ is also α -Lipschitz continuous; if $\alpha \leq \beta$ and f is α -Lipschitz continuous then f is β -Lipschitz continuous; every constant map is α -Lipschitz continuous. Moreover, if $f: X \rightarrow Y$ is α -Lipschitz continuous and $g: Y \rightarrow Z$ is β -Lipschitz continuous, then $g \circ f$ is $\alpha\beta$ -Lipschitz continuous [8, Lemma 2.9].

3 Spaces of Continuous and Lipschitz Maps

We equip $\overline{\mathbb{R}}_+$ with the Scott topology of its ordering \leq , or equivalently, with the $\mathbf{d}_{\mathbb{R}}$ -Scott topology. For a topological space X , the continuous maps from X to $\overline{\mathbb{R}}_+$ are usually called *lower semicontinuous*.

Definition 3.1 ($\mathcal{L}X$) *Let $\mathcal{L}X$ denote the set of lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$. We give it the Scott topology of the pointwise ordering.*

When X, d is standard, every α -Lipschitz continuous map from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ is lower semicontinuous [8, Lemma 5.1].

Definition 3.2 ($\mathcal{L}_\alpha(X, d), \mathcal{L}_\infty(X, d)$) *Let $\mathcal{L}_\alpha(X, d)$ be the set of α -Lipschitz continuous maps from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$, and let $\mathcal{L}_\infty(X, d) = \bigcup_{\alpha \in \mathbb{R}_+} \mathcal{L}_\alpha(X, d)$ be*

the set of all Lipschitz continuous maps from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$. We give those spaces the subspace topology from $\mathcal{L}X$.

We also write $\mathcal{L}_{\infty}X$ for $\mathcal{L}_{\infty}(X, d)$, and $\mathcal{L}_{\alpha}X$ for $\mathcal{L}_{\alpha}(X, d)$.

Beware that there is no reason why the topologies on $\mathcal{L}_{\alpha}(X, d)$ and $\mathcal{L}_{\infty}(X, d)$ would be the Scott topology of the pointwise ordering. When X, d is Lipschitz regular and standard, the (subspace) topology of $\mathcal{L}_{\alpha}(X, d)$ does coincide with the Scott topology, by Proposition 5.6 of [8]. Remark 5.8 of *ibid.* states that, under the same conditions, the subspace topology of the following spaces $\mathcal{L}_{\alpha}^a(X, d)$ are also the same as the Scott topologies. This works for fixed α , not for $\mathcal{L}_{\infty}(X, d)$ or $\mathcal{L}_{\infty}^a(X, d)$.

Definition 3.3 ($\mathcal{L}_{\alpha}^a(X, d), \mathcal{L}_{\infty}^a(X, d)$) *Let \mathcal{L}_{α}^aX or $\mathcal{L}_{\alpha}^a(X, d)$ be the space of all α -Lipschitz continuous maps from X, d to $[0, \alpha a], \mathbf{d}_{\mathbb{R}}$, for $\alpha \in \mathbb{R}_+$, where $a \in \mathbb{R}_+$, $a > 0$. Give it the subspace topology from $\mathcal{L}_{\alpha}X$, or equivalently, from $\mathcal{L}X$.*

Definition 3.4 ($\mathcal{L}_{\infty}^b(X, d)$) *Let $\mathcal{L}_{\infty}^b(X, d)$ be the subspace of all bounded maps in $\mathcal{L}_{\infty}(X, d)$, and $\mathcal{L}_{\alpha}^b(X, d)$ be the corresponding subspace of all bounded maps in $\mathcal{L}_{\alpha}(X, d)$, with the subspace topologies.*

Since the specialization ordering on $\mathcal{L}X$ is the pointwise ordering ($f \leq g$ if and only if for every $x \in X$, $f(x) \leq g(x)$), the same holds for all the spaces defined above as well. We shall always write \leq for that ordering.

We note the following, although we will only use it much later.

Lemma 3.5 *Let X, d be a quasi-metric space. For every $a > 0$, $\mathcal{L}_{\infty}^b(X, d) = \bigcup_{\alpha > 0} \mathcal{L}_{\alpha}^a(X, d)$.*

Proof. Consider any bounded map f from $\mathcal{L}_{\infty}(X, d)$. By definition, $f \leq b \cdot \mathbf{1}$ for some $b \in \mathbb{R}_+$, where $\mathbf{1}$ is the constant map equal to 1, and $f \in \mathcal{L}_{\alpha}(X, d)$ for some $\alpha > 0$. Since $\mathcal{L}_{\alpha}(X, d)$ grows as α increases, we may assume that $\alpha \geq b/a$. Then f is in $\mathcal{L}_{\alpha}^a(X, d)$. The converse inclusion is obvious. \square

Assuming X, d standard, for each $\alpha \in \mathbb{R}_+$, there is a largest α -Lipschitz continuous map $f^{(\alpha)}$ below any lower semicontinuous map $f \in \mathcal{L}X$. Moreover, the family $(f^{(\alpha)})_{\alpha \in \mathbb{R}_+}$ is a chain, and $\sup_{\alpha \in \mathbb{R}_+} f^{(\alpha)} = f$, where suprema are taken pointwise [9, Theorem 6.17].

There is in fact a largest α -Lipschitz continuous map $f^{(\alpha)}$ below any map $f: X \rightarrow \overline{\mathbb{R}}_+$ at all (still assuming X, d standard). Indeed, the constant map

0 is α -Lipschitz continuous, and the (pointwise) supremum of any family of α -Lipschitz continuous maps is again α -Lipschitz continuous [9, Proposition 6.7 (4)]. We make that a definition.

Definition 3.6 *For every map f from a standard quasi-metric space X, d to $\overline{\mathbb{R}}_+$, for every $\alpha \in \mathbb{R}_+$, let $f^{(\alpha)}$ be the largest α -Lipschitz Yoneda-continuous map from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ below f .*

When f is lower semicontinuous, there is a slightly more explicit formula for $f^{(\alpha)}$, see [8, Proposition 5.3]. We are interested in the case where f is not necessarily lower semicontinuous. What the largest α -Lipschitz (not necessarily continuous) map below f looks like is given in the following lemma—that is already well-known in the metric case—and we will give an explicit formula for $f^{(\alpha)}$ when f is already known to be α -Lipschitz in Lemma 3.8, assuming X, d continuous.

Lemma 3.7 *Let X, d be a quasi-metric space, and $\alpha \in \mathbb{R}_+$. The largest α -Lipschitz map below an arbitrary function $f: X \rightarrow \overline{\mathbb{R}}_+$ is given by:*

$$f^\alpha(x) = \inf_{z \in X} (f(z) + \alpha d(x, z)). \quad (1)$$

Proof. For all $x, y \in X$, $f^\alpha(y) + \alpha d(x, y) \geq \inf_{z \in X} (f(z) + \alpha d(x, y) + \alpha d(y, z)) \geq \inf_{z \in X} (f(z) + \alpha d(x, z)) = f^\alpha(x)$, so f^α is α -Lipschitz. It is clear that f^α is below f : take $z = x$ in the infimum defining f^α .

Assume another α -Lipschitz map g below f . For all $x, z \in X$, $g(x) \leq g(z) + \alpha d(x, z) \leq f(z) + \alpha d(x, z)$, hence by taking infima over all $z \in X$, $g(x) \leq f^\alpha(x)$. \square

Every monotonic map g from a space Y to $\overline{\mathbb{R}}_+$ has a lower semicontinuous envelope \overline{g} , defined as the (pointwise) largest map below g that is lower semicontinuous. When Y is a continuous poset, one can define $\overline{g}(y)$ as the directed supremum $\sup_{y' \ll y} g(y')$ (see for example [6, Corollary 5.1.61]). This is sometimes called Scott's formula.

Now assume that $f: X \rightarrow \overline{\mathbb{R}}_+$ is already α -Lipschitz, but not necessarily continuous. There are two ways one can find an α -Lipschitz continuous map below f : either consider $f^{(\alpha)}$, the largest possible such map, or, if X, d is continuous, extend f to $f': (x, r) \mapsto f(x) - \alpha r$, apply Scott's formula to obtain $\overline{f'}$, then restrict the latter to the subspace X of $\mathbf{B}(X, d)$. We show that the two routes lead to the same function.

Lemma 3.8 *Let X, d be a continuous quasi-metric space, and $\alpha \in \mathbb{R}_+$. For every α -Lipschitz map $f: X \rightarrow \overline{\mathbb{R}}_+$, the largest α -Lipschitz continuous map below f , $f^{(\alpha)}$, is given by:*

$$f^{(\alpha)}(x) = \sup_{(y,s) \ll (x,0)} (f(y) - \alpha s). \quad (2)$$

Moreover, $(f^{(\alpha)})'$, defined as mapping (x, r) to $f^{(\alpha)}(x) - \alpha r$, is the largest lower semicontinuous map $\overline{f'}$ from $\mathbf{B}(X, d)$ to $\overline{\mathbb{R}}_+$ below $f': (x, r) \mapsto f(x) - \alpha r$.

Proof. Take $Y = \mathbf{B}(X, d)$. Since f is α -Lipschitz, the map $f': \mathbf{B}(X, d) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by $f'(x, r) = f(x) - \alpha r$ is monotonic, since f is α -Lipschitz. Then $\overline{f'}(x, r) = \sup_{(y,s) \ll (x,r)} (f(y) - \alpha s)$.

By definition, $\overline{f'}$ is Scott-continuous. Note that $\overline{f'}(x, 0)$ is exactly the right-hand side of (2). For clarity, let $g(x) = \overline{f'}(x, 0) = \sup_{(y,s) \ll (x,0)} (f(y) - \alpha s)$.

We check that for every $r \in \mathbb{R}_+$, $\overline{f'}(x, r) = g(x) - \alpha r$. For that, we use the fact that, when X, d is a continuous quasi-metric space, the way-below relation \ll on $\mathbf{B}(X, d)$ is *standard* [9, Proposition 3.6], meaning that, for every $a \in \mathbb{R}_+$, for all formal balls (x, r) and (y, s) , $(y, s) \ll (x, r)$ if and only if $(y, s + a) \ll (x, r + a)$. It follows that $\overline{f'}(x, r) = \sup_{s \geq r, (y,s-r) \ll (x,0)} (f(y) - \alpha s) = \sup_{(y,s') \ll (x,0)} (f(y) - \alpha(s' + r)) = \sup_{(y,s') \ll (x,0)} (f(y) - \alpha s') - \alpha r = g(x) - \alpha r$.

In other words, $\overline{f'} = g'$. Since $\overline{f'}$ is Scott-continuous, so is g' , and therefore g is α -Lipschitz continuous.

We check that g takes its values in $\overline{\mathbb{R}}_+$, not just $\mathbb{R} \cup \{+\infty\}$. We recall that, since X, d is continuous hence standard, the set V_ϵ of all formal balls whose radius is strictly less than ϵ is Scott-open. For every $x \in X$, $(x, 0)$ is in V_ϵ , and since $(x, 0)$ is the supremum of the directed family of all formal balls $(y, s) \ll (x, 0)$, one of them is in V_ϵ ; this implies that $g(x) \geq -\alpha\epsilon$, and as ϵ is arbitrary, that $g(x) \geq 0$.

Also, $g \leq f$, since for every $x \in X$, $g(x) = \overline{f'}(x, 0) \leq f'(x, 0) = f(x)$. Hence g is an α -Lipschitz continuous map from X to $\overline{\mathbb{R}}_+$ below f , from which we deduce that it must also be below the largest such map, $f^{(\alpha)}$.

We show that g is equal to $f^{(\alpha)}$. To that end, we take any α -Lipschitz continuous map $h: X \rightarrow \overline{\mathbb{R}}_+$ below f , and we show that $h \leq g$. Since $h \leq f$, $h' \leq f'$. Since h is α -Lipschitz continuous, h' is Scott-continuous. We use the fact that $\overline{f'}$ is the largest Scott-continuous map below f' to

obtain $h' \leq \overline{f'}$, and we apply both sides of the inequality to $(x, 0)$ to obtain $h(x) = h'(x) \leq \overline{f'}(x, 0) = g(x)$.

Since $g = f^{(\alpha)}$ and $g(x) = \sup_{(y,s) \ll (x,0)} (f(y) - \alpha s)$ by definition, (2) follows.

Finally, we have seen that $\overline{f'} = g'$, namely that $\overline{f'} = (f^{(\alpha)})'$, and that is the final part of the lemma. \square

Corollary 3.9 *Let X, d be a continuous quasi-metric space, and $\alpha, \beta \in \mathbb{R}_+$. For every map $f: X \rightarrow \overline{\mathbb{R}}_+$ that is both α -Lipschitz and β -Lipschitz, $f^{(\alpha)} = f^{(\beta)}$.*

Proof. Since $\mathbf{B}(X, d)$ is a continuous poset, for every $x \in X$, $(x, 0)$ is the supremum of the directed family of formal balls $(y, s) \ll (x, 0)$, and since X, d is standard, the inf of the radii s must be equal to 0; in other words, such formal balls have arbitrarily small radii s .

By (2), $f^{(\alpha)}(x) = \sup_{(y,s) \ll (x,0)} (f(y) - \alpha s)$. For every $(y, s) \ll (x, 0)$, for every $\epsilon > 0$, we can find another formal ball $(z, t) \ll (x, 0)$ such that $t < \epsilon$, by the remark we have just made. Using directedness, we can require (z, t) to be above (y, s) . Then $f(y) - \alpha s \leq f(z) - \alpha t$ since f is α -Lipschitz, and $f(z) - \alpha t = f(z) - \beta t + (\beta - \alpha)t$ is less than or equal to $f^{(\beta)}(x) + |\beta - \alpha|\epsilon$. Indeed, $f(z) - \beta t \leq f^{(\beta)}(x)$ since $(z, t) \ll (x, 0)$ and $(\beta - \alpha)t \leq |\beta - \alpha|\epsilon$ since $t < \epsilon$.

Taking suprema over $(y, s) \ll (x, 0)$, we obtain $f^{(\alpha)}(x) \leq f^{(\beta)}(x) + |\beta - \alpha|\epsilon$. Since ϵ can be made arbitrarily small, $f^{(\alpha)}(x) \leq f^{(\beta)}(x)$. We show $f^{(\beta)}(x) \leq f^{(\alpha)}(x)$ symmetrically, using the fact that f is β -Lipschitz. The equality follows. \square

Lemma 3.10 *Let X, d be a standard algebraic quasi-metric space, and $\alpha \in \mathbb{R}_+$. For any α -Lipschitz map $f: X \rightarrow \overline{\mathbb{R}}_+$, the largest α -Lipschitz continuous map below f , namely $f^{(\alpha)}$, is given by:*

$$f^{(\alpha)}(x) = \sup_{\substack{z \text{ center point} \\ t > d(z,x)}} (f(z) - \alpha t). \quad (3)$$

Moreover, $(f^{(\alpha)})'$, defined as mapping (x, r) to $f^{(\alpha)}(x) - \alpha r$, is the largest lower semicontinuous map $\overline{f'}$ from $\mathbf{B}(X, d)$ to $\overline{\mathbb{R}}_+$ below $f': (x, r) \mapsto f(x) - \alpha r$.

Note that (3) simplifies to $f^{(\alpha)}(x) = \sup_{z \text{ center point}} (f(z) - \alpha d(z, x))$ when $d(z, x) \neq +\infty$ for all center points z , or when $f(z) \neq +\infty$.

Proof. Easy consequence of Lemma 3.8, using the fact that, in a standard algebraic quasi-metric space, $(y, s) \ll (x, r)$ if and only if there is a center point z and some $t \in \mathbb{R}_+$ such that $(y, s) \leq^{d^+} (z, t)$ and $d(z, x) < t - r$. \square

4 Topologies on $\mathcal{L}X$

Every continuous Yoneda-complete quasi-metric space X, d is *domain-complete* in its d -Scott topology [2, Theorem 4.1], meaning that it embeds as a G_δ subset of some continuous dcpo. Explicitly, the continuous dcpo is $\mathbf{B}(X, d)$. The sets V_ϵ , $\epsilon > 0$ of those formal balls (x, r) such that $r < \epsilon$ are Scott-open because X, d is standard (being the inverse image of $[0, \epsilon[$ by the radius map $(x, r) \mapsto r$), and X embeds through $x \mapsto (x, 0)$ as the G_δ set $\bigcap_{n \in \mathbb{N}} V_{1/2^n}$.

Since every continuous dcpo is locally compact and sober, such an X is automatically *LCS-complete*, meaning that it embeds as a G_δ subset of a locally compact sober space [2, Proposition 3.3].

It so turns out that, given any LCS-complete space X , the compact-open topology coincides with the Scott topology on $\mathcal{L}X$ [2, Corollary 13.5]. As a consequence, we obtain:

Proposition 4.1 *Let X, d be a continuous Yoneda-complete quasi-metric space. The compact-open topology on $\mathcal{L}X$ is equal to the Scott topology on $\mathcal{L}X$.* \square

5 Compact Subsets of Continuous Yoneda-Complete Quasi-Metric Spaces

The compact saturated subsets of LCS-complete spaces were characterized in [2, Section 17]. Specializing Theorem 17.4 of *ibid.* to the case where X, d is a continuous Yoneda-complete quasi-metric space in its d -Scott topology, where $Y = \mathbf{B}(X, d)$ and where μ is the radius map, we obtain that the non-empty compact saturated subsets of X are exactly the filtered intersections $\bigcap_{i \in I} \uparrow A_i$ in Y , where each set A_i is a non-empty finite subset of Y , and $\inf_{i \in I} r(A_i) = 0$. Here, for every finite set $A = \{(x_1, r_1), \dots, (x_n, r_n)\}$, the *radius* of A , $r(A)$, is defined as $\max\{r_1, \dots, r_n\}$. Because $\inf_{i \in I} r(A_i) = 0$,

one sees that $\bigcap_{i \in I} \uparrow A_i$ is a subset of X (up to the identification of $x \in X$ with $(x, 0) \in \mathbf{B}(X, d)$).

One may recast this as follows. The *closed ball* $B_{x, \leq r}^d$ of center x and radius r is the set of points $y \in X$, such that $d(x, y) \leq r$. This is merely $\uparrow(x, r) \cap X$. Hence the result we have just stated says that the non-empty compact saturated subsets of X are exactly the filtered intersections of finite unions of closed balls whose radii tend to 0. This is the quasi-metric analogue of the result that, in a complete metric space, the compact subsets are the closed and precompact subsets.

Remark 5.1 *Despite the name, closed balls are not closed in general—except when X, d is metric. Indeed, closed balls are upwards-closed, whereas closed sets must be downwards-closed. Closed balls need not be compact either, despite the fact that they are the intersection of a compact set $\uparrow(x, r)$ of $\mathbf{B}(X, d)$ with X .*

The following is an easy consequence of our previous discussion. We provide an explicit proof for completeness.

Lemma 5.2 *Let X, d be a continuous quasi-metric space. For every compact saturated subset Q of X , for every open neighborhood U of Q , for every $\epsilon > 0$, one can find a finite union A of closed balls $B_{x_i, \leq r_i}^d$ with $r_i < \epsilon$, and such that $Q \subseteq \text{int}(A) \subseteq A \subseteq U$.*

Proof. The image of Q by the embedding $x \mapsto (x, 0)$ of X into $\mathbf{B}(X, d)$ is compact. If we agree to equate x with $(x, 0)$, then Q is compact not only in X , but also in $\mathbf{B}(X, d)$. Recall that \widehat{U} is a Scott-open subset of $\mathbf{B}(X, d)$ and that $\widehat{U} \cap X = U$. By intersecting it with $V_\epsilon = \{(x, r) \mid x \in X, r < \epsilon\}$ (a Scott-open subset, owing to the fact that X, d is standard), we obtain an open neighborhood $\widehat{U} \cap V_\epsilon$ of Q in $\mathbf{B}(X, d)$.

Since $\mathbf{B}(X, d)$ is a continuous poset, $\widehat{U} \cap V_\epsilon$ is the union of all open subsets $\uparrow(x, r)$, $(x, r) \in \widehat{U} \cap V_\epsilon$. By compactness, Q is therefore included in some finite union $\bigcup_{i=1}^n \uparrow(x_i, r_i)$, where every (x_i, r_i) is in $\widehat{U} \cap V_\epsilon$. In particular, $r_i < \epsilon$ for each i . Note also that $\uparrow(x_i, r_i)$ is included in \widehat{U} , since \widehat{U} is upwards-closed, so $\uparrow(x_i, r_i) \cap X = B_{x_i, \leq r_i}^d$ is included in U . Therefore $A = \bigcup_{i=1}^n B_{x_i, \leq r_i}^d$, whose interior contains the open neighborhood $\bigcup_{i=1}^n (\uparrow(x_i, r_i) \cap X)$ of Q , fits. \square

6 Topologies on $\mathcal{L}_\alpha(X, d)$ and $\mathcal{L}_\alpha^a(X, d)$

The topology we have taken on $\mathcal{L}_\alpha(X, d)$ and on $\mathcal{L}_\alpha^a(X, d)$ is the subspace topology from $\mathcal{L}X$ (with its Scott topology). Proposition 4.1 immediately implies that, when X, d is continuous Yoneda-complete, the topologies of $\mathcal{L}_\alpha(X, d)$ and $\mathcal{L}_\alpha^a(X, d)$ are the compact-open topology, i.e., the topology generated by the subsets of the form $\{f \in \mathcal{L}_\alpha X \mid f[Q] \subseteq]r, +\infty]\}$ (resp., $\{f \in \mathcal{L}_\alpha^a X \mid f[Q] \subseteq]r, +\infty]\}$), Q compact saturated in X , and which we write as $[Q > r]$.

The topology of pointwise convergence, on any set of functions from X to Y , is the subspace topology from the ordinary product topology on Y^X . It is always coarser than the compact-open topology. When $Y = \overline{\mathbb{R}}_+$ or $Y = [0, a]$, with its Scott-topology, a subbase for the pointwise topology is given by the subsets $[x > r] = \{f \mid f(x) > r\}$, where $x \in X$ and $r \in \mathbb{R}_+$.

Proposition 6.1 *Let X, d be a continuous Yoneda-complete quasi-metric space. For every $\alpha \in \mathbb{R}_+$, the topology of $\mathcal{L}_\alpha(X, d)$ (resp., $\mathcal{L}_\alpha^a(X, d)$, for any $a > 0$) coincides with the topology of pointwise convergence.*

Proof. We already know that the topology of $\mathcal{L}_\alpha(X, d)$, and of $\mathcal{L}_\alpha^a(X, d)$, is the compact-open topology, as a consequence of Proposition 4.1. It therefore suffices to show that every subset $[Q > r]$ of $\mathcal{L}_\alpha(X, d)$ or $\mathcal{L}_\alpha^a(X, d)$, Q compact saturated in X and $r \in \mathbb{R}_+$, is open in the topology of pointwise convergence. We assume without loss of generality that Q is non-empty. Let $f \in \mathcal{L}_\alpha(X, d)$, resp. $f \in \mathcal{L}_\alpha^a(X, d)$, be an arbitrary element of $[Q > r]$. Since f is lower semicontinuous, i.e., continuous from X to $\overline{\mathbb{R}}_+$ (resp., to $[0, \alpha a]$), the image of Q by f is compact, and its upward closure is compact saturated. The non-empty compact saturated subsets of $\overline{\mathbb{R}}_+$ (resp., $[0, \alpha a]$), in its Scott topology, are the intervals $[b, +\infty]$, $b \in \overline{\mathbb{R}}_+$ (resp., $[b, \alpha a]$, $b \in [0, \alpha a]$); hence the image of Q by f is of this form, and b is necessarily the minimum value attained by f on Q , $\min_{x \in Q} f(x)$. Since $f \in [Q > r]$, we must have $b > r$, and therefore there is an $\eta > 0$ such that $b > r + \eta$.

Let $\epsilon > 0$ be such that $\alpha\epsilon \leq \eta$. Put in a simpler form, let $\epsilon = \eta/\alpha$ if $\alpha \neq 0$, otherwise let $\epsilon > 0$ be arbitrary.

We apply Lemma 5.2, and we obtain finitely many closed balls $B_{x_i, \leq r_i}$, $1 \leq i \leq n$, included in $U = f^{-1}(]r + \eta, +\infty])$, with $r_i < \epsilon$, and whose union contains Q . Now consider $W = \bigcap_{i=1}^n [x_i > r + \eta]$. Since each x_i is in U , f is in W . For every $g \in \mathcal{L}_\alpha(X, d)$ (resp., $g \in \mathcal{L}_\alpha^a(X, d)$) that is in W , we claim that g is in $[Q > r]$. For every $x \in Q$, there is an index i such that $x \in B_{x_i, \leq r_i}$,

and since g is α -Lipschitz, $g(x_i) \leq g(x) + \alpha r_i$. By assumption $g(x_i) > r + \eta$, and $r_i < \epsilon$, so $g(x) > r$, proving the claim. Since W is open in the topology of pointwise convergence, we have proved that every element of $[Q > r]$ is an open neighborhood, for the topology of pointwise convergence, of each of its elements, which proves the result. \square

7 Stable Compactness

We shall now use some results in the theory of stably compact spaces [6, Chapter 9]. A *stably compact* space is a sober, locally compact, compact and coherent space, where coherence means that the intersection of any two compact saturated subsets is compact. For a compact Hausdorff space Z equipped with an ordering \preceq whose graph is closed in Z^2 (a so-called *compact pospace*), the space Z with the upward topology, whose opens are by definition the open subsets of Z that are upwards-closed with respect to \preceq , is stably compact. More: all stably compact spaces can be obtained this way. If X is stably compact, then we may form a second topology, the *cocompact* topology, whose closed subsets are the compact saturated subsets of the original space. With the cocompact topology, we obtain a space called the *de Groot dual* of X , X^d , which is also stably compact. We have $X^{dd} = X$. Also, with the join of the original topology on X and of the cocompact topology, one obtains a compact Hausdorff space X^{patch} , the *patch space* of X . Together with the specialization ordering \leq of X , X^{patch} is then a compact pospace. Moreover, passing from X to its compact pospace, and conversely, are mutually inverse operations.

We require the following auxiliary notion.

Definition 7.1 ($L_\alpha(X, d)$, $L_\alpha^a(X, d)$) *Let $L_\alpha(X, d)$ denote the space of all α -Lipschitz (not necessarily α -Lipschitz continuous) maps from X, d to $\overline{\mathbb{R}}_+$. Let also $L_\alpha^a(X, d)$ be the subspace of all $h \in L_\alpha(X, d)$ such that $h \leq \alpha a$.*

Please pay attention to the change of font compared with $\mathcal{L}_\alpha(X, d)$.

$\overline{\mathbb{R}}_+$ (resp., $[0, \alpha a]$), with its Scott topology, is stably compact, and its patch space $\overline{\mathbb{R}}_+^{\text{patch}}$ (resp., $[0, \alpha a]^{\text{patch}}$) has the usual Hausdorff topology. Since the product of stably compact spaces is stably compact (see [6, Proposition 9.3.1]), $\overline{\mathbb{R}}_+^X$ (resp., $[0, \alpha a]^X$) is stably compact. Moreover, the patch operation commutes with products, so $(\overline{\mathbb{R}}_+^X)^{\text{patch}} = (\overline{\mathbb{R}}_+^{\text{patch}})^X$ (resp., $([0, \alpha a]^X)^{\text{patch}} = ([0, \alpha a]^{\text{patch}})^X$), and the specialization ordering is pointwise.

The subset $L_\alpha(X, d)$ (resp., $L_\alpha^a(X, d)$) is then patch-closed in $\overline{\mathbb{R}}_+^X$ (resp., $[0, \alpha a]^X$), meaning that it is closed in $(\overline{\mathbb{R}}_+^X)^{\text{patch}} = (\overline{\mathbb{R}}_+^{\text{patch}})^X$ (resp., in $([0, \alpha a]^X)^{\text{patch}} = ([0, \alpha a]^{\text{patch}})^X$). Indeed, $L_\alpha(X, d) = \{f \in \overline{\mathbb{R}}_+^X \mid \forall x, y \in X. f(x) \leq f(y) + \alpha d(x, y)\}$ is the intersection of the sets $\{f \in \overline{\mathbb{R}}_+^X \mid f(x) \leq f(y) + \alpha d(x, y)\}$, $x, y \in X$, and each is patch-closed, because the graph of \leq is closed in $(\overline{\mathbb{R}}_+^{\text{patch}})^2$ and the maps $f \mapsto f(x)$ are continuous from $(\overline{\mathbb{R}}_+^{\text{patch}})^X$ to $\overline{\mathbb{R}}_+^{\text{patch}}$ — similarly with $[0, \alpha a]$ in lieu of $\overline{\mathbb{R}}_+$ and $L_\alpha^a(X, d)$ instead of $L_\alpha(X, d)$. It is well-known (Proposition 9.3.4, *ibid.*) that the patch-closed subsets C of stably compact spaces Y are themselves stably compact, and that the topology of C^{patch} is the subspace topology of Y^{patch} . It follows that:

Lemma 7.2 *Let X, d be a continuous quasi-metric space. $L_\alpha(X, d)$ and $L_\alpha^a(X, d)$, for every $a \in \mathbb{R}_+$, $a > 0$, are stably compact in the topology of pointwise convergence. \square*

Lemma 7.3 *Let X, d be a continuous quasi-metric space, $\alpha \in \mathbb{R}_+$, and $a \in \mathbb{R}_+$, $a > 0$. Let ρ_α be the map $f \in L_\alpha(X, d) \mapsto f^{(\alpha)} \in \mathcal{L}_\alpha(X, d)$. Then:*

1. ρ_α is continuous from $L_\alpha(X, d)$ to $\mathcal{L}_\alpha(X, d)$ (resp., from $L_\alpha^a(X, d)$ to $\mathcal{L}_\alpha^a(X, d)$), both spaces being equipped with the topology of pointwise convergence;
2. $\mathcal{L}_\alpha(X, d)$ is a retract of $L_\alpha(X, d)$ (resp., $\mathcal{L}_\alpha^a(X, d)$ is a retract of $L_\alpha^a(X, d)$), with ρ_α the retraction and where inclusion serves as section; again both spaces are assumed to have the topology of pointwise convergence;
3. $\mathcal{L}_\alpha(X, d)$ and $\mathcal{L}_\alpha^a(X, d)$, with the topology of pointwise convergence, are stably compact;
4. If X, d is also Yoneda-complete, then $\mathcal{L}_\alpha(X, d)$ and $\mathcal{L}_\alpha^a(X, d)$ are stably compact (in the subspace topology of the Scott topology on $\mathcal{L}X$).

Proof. Recall from Lemma 3.8 that $f^{(\alpha)}(x) = \sup_{(y,s) \ll (x,0)} (f(y) - \alpha s)$.

(1) The inverse image of $[x > r]$ by ρ_α is the set of α -Lipschitz maps f such that, for some $(y, s) \ll (x, 0)$, $f(y) - \alpha s > r$, hence is the union of the open sets $[y > r + \alpha s]$ over the elements (y, s) way-below $(x, 0)$.

(2) If ι denotes the inclusion of $\mathcal{L}_\alpha(X, d)$ into $L_\alpha(X, d)$ (resp., of $\mathcal{L}_\alpha^a(X, d)$ into $L_\alpha^a(X, d)$), we must show that $\rho_\alpha \circ \iota$ is the identity map. For every

$f \in \mathcal{L}_\alpha(X, d)$, $\rho_\alpha(\iota(f)) = f^{(\alpha)}$. Since f is already α -Lipschitz continuous, $f^{(\alpha)} = f$, and we conclude.

(3) follows from Lemma 7.2, from (2), and the fact that retracts of stably compact spaces are stably compact (see Proposition 9.2.3 in *ibid.*; the result is due to Jimmie Lawson [12, Proposition, bottom of p.153, and subsequent discussion]).

(4) follows from (3) and Proposition 6.1. \square

The proof of (3) above is inspired by a similar argument due to Achim Jung [10].

Stable compactness has a number of pleasing consequences.

Lemma 7.4 *Let X, d be a continuous Yoneda-complete quasi-metric space, and $\alpha \in \mathbb{R}_+$. For every center point x in X and every $b \in \mathbb{R}_+$, the set $[x \geq b]$ of all $f \in \mathcal{L}_\alpha(X, d)$ (resp., $f \in \mathcal{L}_\alpha^a(X, d)$) such that $f(x) \geq b$ is compact saturated in $\mathcal{L}_\alpha(X, d)$ (resp., in $\mathcal{L}_\alpha^a(X, d)$, for every $a \in \mathbb{R}_+$, $a > 0$).*

Proof. We deal with the case of $\mathcal{L}_\alpha(X, d)$. The case of $\mathcal{L}_\alpha^a(X, d)$ is entirely similar.

Let us start with the following observation. The projection map $\pi_x: L_\alpha(X, d) \rightarrow \overline{\mathbb{R}}_+$ that maps f to $f(x)$ is patch-continuous, that is, continuous from $L_\alpha^{\text{patch}}(X, d)$ to $\overline{\mathbb{R}}_+^{\text{patch}}$, where $L_\alpha(X, d)$ has the topology of pointwise convergence. (Beware again: L_α , not \mathcal{L}_α .) The reason was given before Lemma 7.2: the map $f \mapsto f(x)$ is continuous from $(\overline{\mathbb{R}}_+^{\text{patch}})^X$ to $\overline{\mathbb{R}}_+^{\text{patch}}$, and therefore restricts to a continuous map on $L_\alpha^{\text{patch}}(X, d)$, which is a subspace of $(\overline{\mathbb{R}}_+^{\text{patch}})^X$, since $L_\alpha(X, d)$ is patch-closed in the product space $\overline{\mathbb{R}}_+^X$.

Since $[b, +\infty]$ is closed in $\overline{\mathbb{R}}_+^{\text{patch}}$, $\pi_x^{-1}([b, +\infty])$ is also closed in $L_\alpha(X, d)^{\text{patch}}$. It is clearly upwards-closed, and in any stably compact space Y , the closed upwards-closed subsets of Y^{patch} are the compact saturated subsets of Y (see [6, Proposition 9.1.20]): so $\pi_x^{-1}([b, +\infty])$ is compact saturated in $L_\alpha(X, d)$.

Note that $\pi_x^{-1}([b, +\infty])$ is the set of all α -Lipschitz (not necessarily α -Lipschitz continuous) maps f such that $f(x) \geq b$. We claim that its image by ρ_α is exactly $[x \geq b]$. This will prove that the latter is compact as well. The fact that it is upwards-closed (saturated) is obvious.

For every $f \in \pi_x^{-1}([b, +\infty])$, $\rho_\alpha(f)(x)$ is clearly less than or equal to $f(x)$. Because x is a center point, for all y, r , and s , $(x, r) \ll (y, s)$ is equivalent to $d(x, y) < r - s$; in particular, $(x, \epsilon) \ll (x, 0)$. Therefore, $\rho_\alpha(f)(x) \geq f(x) - \alpha\epsilon$. Since ϵ is arbitrary, $\rho_\alpha(f)(x) = f(x)$, and since $f(x) \geq b$, $\rho_\alpha(f)(x) \geq b$. We have proved that $\rho_\alpha(f)$ is in $[x \geq b]$.

Conversely, every $f \in [x \geq b]$ is of the form $\rho_\alpha(g)$ for some $g \in \pi_x^{-1}([b, +\infty])$, namely $g = f$, because ρ_α is a retraction. Hence $[x \geq b]$ is the image of the compact set $\pi_x^{-1}([b, +\infty])$, as claimed.

Our whole argument works provided all considered spaces of functions have the topology of pointwise convergence. We use Proposition 6.1 to conclude. \square

Remark 7.5 *Lemma 7.4 does not hold for non-center points x , as we now illustrate. Let $X = \mathbb{N}_\omega$ be the dcpo obtained by adding a top element ω to \mathbb{N} , with its usual ordering \leq . We consider it as a quasi-metric space with the quasi-metric d_\leq . On formal balls, $(x, r) \leq^{d_\leq} (y, s)$ if and only if $x \leq y$ and $r \geq s$, hence its space of formal balls is isomorphic to the continuous dcpo $\mathbb{N}_\omega \times]-\infty, 0]$, through the map $(x, r) \mapsto (x, -r)$. We consider the point $x = \omega$, and $b = 1$, say. The elements of $\mathcal{L}_\alpha(X, d)$ are exactly the Scott-continuous maps, so $[x \geq b]$ is the set of Scott-continuous maps $f: \mathbb{N}_\omega \rightarrow \overline{\mathbb{R}}_+$ such that $f(\omega) \geq 1$. Let us pick a fixed element f of $[x \geq b]$, for example, the constant function with value $+\infty$. For every $n \in \mathbb{N}$, let $f_n: X \rightarrow \overline{\mathbb{R}}_+$ be defined by $f_n(m) = 0$ if $m \leq n$, $f_n(m) = f(m)$ otherwise. The family $(f_n)_{n \in \mathbb{N}}$ is a decreasing sequence of elements of $[x \geq b]$. If $[x \geq b]$ were compact, then the intersection of the decreasing family of closed sets $\downarrow f_n$ would intersect $[x \geq b]$; but the only element in that intersection is the constant zero map.*

Corollary 7.6 *Let X, d be a continuous Yoneda-complete quasi-metric space, $\alpha \in \mathbb{R}_+$, and $a \in \mathbb{R}_+$, $a > 0$.*

1. *For every center point x in X and every $b \in \mathbb{R}_+$, the set $[x < b]$ of all $f \in \mathcal{L}_\alpha(X, d)$ (resp., $f \in \mathcal{L}_\alpha^a(X, d)$) such that $f(x) < b$ is open and downwards-closed in $\mathcal{L}_\alpha(X, d)^{\text{patch}}$ (resp., in $\mathcal{L}_\alpha^a(X, d)^{\text{patch}}$).*
2. *For every center point $x \in X$, the map $f \mapsto f(x)$ is continuous from $\mathcal{L}_\alpha(X, d)^d$ to $(\overline{\mathbb{R}}_+)^d$, and from $\mathcal{L}_\alpha^a(X, d)^d$ to $[0, \alpha a]^d$.*

Proof.(1) $[x < b]$ is the complement of $[x \geq b]$. Since $[x \geq b]$ is compact saturated (Lemma 7.4), it is closed and upwards-closed in $\mathcal{L}_\alpha(X, d)^{\text{patch}}$ (resp., in $\mathcal{L}_\alpha^a(X, d)^{\text{patch}}$). (2) is just a reformulation of (1). \square

Corollary 7.7 *Let X, d be a continuous Yoneda-complete quasi-metric space, $\alpha \in \mathbb{R}_+$, and $a \in \mathbb{R}_+$, $a > 0$. For every $n \in \mathbb{N}$, for all $a_1, \dots, a_n \in \mathbb{R}_+$ and every n -tuple of center points x_1, \dots, x_n in X, d , the maps:*

1. $f \mapsto \sum_{i=1}^n a_i f(x_i)$,
2. $f \mapsto \max_{i=1}^n f(x_i)$,
3. $f \mapsto \min_{i=1}^n f(x_i)$

are continuous from $\mathcal{L}_\alpha(X, d)^d$ (resp., $\mathcal{L}_\alpha^a(X, d)^d$) to $(\overline{\mathbb{R}}_+)^d$.

Proof. By Corollary 7.6, using the fact that scalar multiplication and addition are continuous on $(\overline{\mathbb{R}}_+)^d$. The latter means that those operations are monotonic and preserve filtered infima. \square

The same argument also shows the following, and an endless variety of similar results.

Corollary 7.8 *Let X, d be a continuous Yoneda-complete quasi-metric space, $\alpha \in \mathbb{R}_+$, and $a \in \mathbb{R}_+$, $a > 0$. For every family of non-negative reals a_{ij} and of center points x_{ij} , $1 \leq i \leq m$, $1 \leq j \leq n_i$, the maps:*

1. $f \mapsto \min_{i=1}^m \sum_{j=1}^{n_i} a_{ij} f(x_{ij})$,
2. $f \mapsto \max_{i=1}^m \sum_{j=1}^{n_i} a_{ij} f(x_{ij})$,

are continuous from $\mathcal{L}_\alpha(X, d)^d$ (resp., $\mathcal{L}_\alpha^a(X, d)^d$) to $(\overline{\mathbb{R}}_+)^d$.

8 Topologies on $\mathcal{L}_\infty(X, d)$ Determined by $\mathcal{L}_\alpha(X, d)$, $\alpha > 0$

Recall that $\mathcal{L}_\infty(X, d) = \bigcup_{\alpha \in \mathbb{R}_+} \mathcal{L}_\alpha(X, d)$. As usual, we equip $\mathcal{L}_\infty(X, d)$ with the subspace topology from $\mathcal{L}X$, the latter having the Scott topology.

Here is a nasty subtle issue. Imagine we have a function $F: \mathcal{L}_\infty(X, d) \rightarrow \overline{\mathbb{R}}_+$, and we wish to show that F is continuous. One might think of showing that the restriction of F to $\mathcal{L}_\alpha(X, d)$ is continuous to that end. This is certainly necessary, but by no means sufficient.

For sufficiency, we would need the topology of $\mathcal{L}_\infty(X, d)$ to be *coherent with*, or *determined by*, the topologies of its subspaces $\mathcal{L}_\alpha(X, d)$, $\alpha \in \mathbb{R}_+$. The name “coherent with” is unfortunate, for a coherent space also means a space in which the intersection of any two compact saturated subsets is compact, as we have already mentioned. We shall therefore prefer “determined by”.

Well-known examples are the topologies of CW-complexes, which are defined as being determined by their cells, and topologies of compactly generated spaces, which are defined as being determined by their compact subspaces.

Let us recall some basic facts about topologies determined by families of subspaces. Fix a topological space Y , and a chain of subspaces Y_i , $i \in I$, where I is equipped with some ordering \leq , in such a way that $i \leq j$ implies $Y_i \subseteq Y_j$. Let also $e_{ij}: Y_i \rightarrow Y_j$ and $e_i: Y_i \rightarrow Y$ be the canonical embeddings, and assume that $Y = \bigcup_{i \in I} Y_i$. Then there is a unique topology \mathcal{O}_c on Y that is determined by the topologies of the subspaces Y_i , $i \in I$.

We call it the *determined topology* on Y , for short. Standard names for such a topology include the *weak topology* or the *inductive topology*. Its open subsets are the subsets A of Y such that $A \cap Y_i$ is open in Y_i for every $i \in I$. In particular, every open subset of Y , with its original topology, is open in \mathcal{O}_c . Categorically, Y with the \mathcal{O}_c topology, together with the map $e_i: Y_i \rightarrow Y$, is the inductive limit, a.k.a., the colimit, of the diagram given by the arrows $Y_i \xrightarrow{e_{ij}} Y_j$, $i \leq j$.

It is known that Y is automatically determined by the topologies of Y_i , $i \in I$, in the following cases: when every Y_i is open; when every Y_i is closed and the cover $(Y_i)_{i \in I}$ is locally finite; when Y has the discrete topology. The case of $\mathcal{L}_\infty(X, d)$ and its subspaces $\mathcal{L}_\alpha(X, d)$ falls into none of those subcases.

Here is another case of a determined topology.

Proposition 8.1 *Let $(Y_i)_{i \in I, \sqsubseteq}$ be a monotone net of subspaces of a topological space Y , forming a cover of Y . Assume there are continuous projections of Y onto Y_i , namely continuous maps $p_i: Y \rightarrow Y_i$ such that $p_i(x) \leq x$ for every $x \in Y$ and $p_i(x) = x$ for every $x \in Y_i$, for every $i \in I$. Then the topology of Y is determined by the topologies of Y_i , $i \in I$.*

What the assumption means is that we do not just have arrows $Y_i \xrightarrow{e_i} Y$, $i \in I$, but embedding-projection pairs $Y_i \xrightleftharpoons[p_i]{e_i} Y$, $i \in I$. This is a standard situation in domain theory. The inequality $e_i \circ p_i \leq \text{id}_Y$ should be read by keeping in mind the specialization preorder of Y .

Proof. Let A be an open subset in the topology on Y determined by the subspaces Y_i . For every $i \in I$, $A \cap Y_i$ is open in Y_i , and since p_i is continuous, $U_i = p_i^{-1}(A \cap Y_i)$ is open in Y .

We claim that $A = \bigcup_{i \in I} U_i$, which will show that A is open in the original topology on Y , allowing us to conclude.

For every $x \in A$, there is an index $i \in I$ such that x is in Y_i , since $(Y_j)_{j \in I, \sqsubseteq}$ is a cover. Then x is in $A \cap Y_i$. Since p_i is a projection, $p_i(x) = x$, so x is in $p_i^{-1}(A \cap Y_i) = U_i$.

Conversely, for every element x of any U_i , $i \in I$, $p_i(x)$ is in the open subset $A \cap Y_i$ of Y_i . Since $(Y_j)_{j \in I, \sqsubseteq}$ is a cover, there is a $j \in I$ such that $x \in Y_j$, and since that is also a monotone net, we may assume that $i \sqsubseteq j$. Hence $Y_i \subseteq Y_j$, so that $p_i(x)$ is also in $A \cap Y_j$. Moreover, $p_i(x) \leq x$. Since the specialization preordering on a subspace (here, Y_j) coincides with the restriction of the specialization preordering on the superspace (here, Y), and $A \cap Y_j$ is open hence upwards-closed in Y_j , x is in $A \cap Y_j$. In particular, x is in A . \square

Corollary 5.5 of [8] states that given $\alpha > 0$ and a Lipschitz regular standard quasi-metric space X, d , the canonical injection $i_\alpha: \mathcal{L}_\alpha(X, d) \rightarrow \mathcal{L}X$ and the map $r_\alpha: f \in \mathcal{L}X \mapsto f^{(\alpha)} \in \mathcal{L}_\alpha(X, d)$ form an embedding-projection pair, viz., r_α and i_α are continuous, $r_\alpha \circ i_\alpha = \text{id}_{\mathcal{L}_\alpha X}$ and $i_\alpha \circ r_\alpha \leq \text{id}_{\mathcal{L}X}$.

Using the fact that $\mathcal{L}_\infty(X, d)$ has the subspace topology from $\mathcal{L}X$, that implies that there is an embedding-projection pair i_α, r_α between $\mathcal{L}_\alpha(X, d)$ and $\mathcal{L}_\infty(X, d)$. When $\alpha > 0$ varies, the spaces $\mathcal{L}_\alpha(X, d)$ form a cover of $\mathcal{L}_\infty(X, d)$. Hence we can apply Proposition 8.1, and we obtain:

Proposition 8.2 *Let X, d be a Lipschitz regular standard quasi-metric space. The topology of $\mathcal{L}_\infty(X, d)$ is determined by those of the subspaces $\mathcal{L}_\alpha(X, d)$, $\alpha > 0$.* \square

We have a similar result for bounded maps. Relying on Lemma 3.5 to ensure that the spaces $\mathcal{L}_\alpha^a(X, d)$, $\alpha > 0$, form a cover of $\mathcal{L}_\infty^b(X, d)$, we obtain:

Proposition 8.3 *Let X, d be a Lipschitz regular standard quasi-metric space. Fix $a > 0$. Then the topology of $\mathcal{L}_\infty^b(X, d)$ is determined by those of the subspaces $\mathcal{L}_\alpha^a(X, d)$, $\alpha > 0$.* \square

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