Computing State Invariants Using Point-Wise Integral Quadratic Constraints and the S-procedure

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***Abstract*— This paper deals with the problem of computing ellipsoidal state invariant sets for uncertain systems that consist of a nominal part and an uncertainty feedback operator. The set of allowable uncertainty operators is characterized using point-wise integral quadratic constraints (IQCs). The proposed solution methodology combines the S-procedure and the notion of point-wise IQCs in a novel way. The approach allows for a point-wise characterization of the disturbance inputs and involves solving a grid of convex optimization problems. An illustrative example is provided to show how to apply the proposed results.**

I. Introduction

This paper deals with the problem of finding state invariant sets for uncertain systems that are formed by the interconnection of a discrete-time linear time-invariant (LTI) nominal system and an uncertainty operator that is assumed to lie within a prespecified set. The set of allowable uncertainty operators is described using point-wise integral quadratic constraints (IQCs). Namely, a point-wise IQC for an uncertainty class is defined similarly to the standard IQC for that class, but further requires each term in the infinite summation to be nonnegative. The reader is referred to [1], [2] for examples of common uncertainty classes characterized using IQCs, and to [3] for the use of point-wise IQCs in the analysis of the convergence of optimization algorithms. The advantage of the IQC framework is that it allows for formulating the system to be analyzed as the interconnection of a linear system, namely, the nominal part of the plant or controller to be analyzed, and a feedback operator that lumps the system's parametric uncertainties, nonlinearities, and time-varying coefficients. The condition on the allowable feedback operators is that they lie within a prespecified set that is characterized using an IQC (herein, a point-wise IQC).

The present work builds on the work of [4], which deals with the problem of finding state invariant sets for discrete-time LTI systems. As with the work of [4], the present work makes extensive use of the S-procedure [5] to derive conditions that allow for the computation of the sought invariant sets. The novelty herein resides in fusing the IQC approach for characterizing uncertainty sets with the S-procedure. In particular, tools and concepts from IQC theory, e.g., the concept of point-wise IQCs, are leveraged and combined with the S-procedure to derive conditions that allow for computing ellipsoidal invariant sets for uncertain systems.

If an uncertainty set satisfies a given point-wise IQC, then it automatically satisfies the corresponding IQC. However, the converse is not necessarily true. Thus, the notion of a point-wise IQC is, by definition, more restrictive than that of an IQC, which limits the applicability of the results derived in this paper. In Section III-B, some examples of uncertainty sets that admit point-wise IQC characterizations are given.

An interesting feature of the proposed results is that the disturbance inputs affecting the uncertain system are characterized using point-wise bounds. In other words, it is assumed that, at every time-step, the disturbance input lies within a predefined set. This characterization of the disturbance input is in contrast with the classical ℓ_2-norm-bound placed on disturbance inputs in robustness analysis using the IQC framework and the characterization of the disturbance inputs using signal IQCs [6], [7]. In particular, it is assumed herein that the disturbance input lies in a convex, closed, and bounded polytope at each time-step. In this case, the problem of computing the desired invariant sets is formulated as a finite dimensional nonconvex problem in which the relevant constraints are only imposed at the vertices (extreme points) of the polytope. Specifically, the nonconvex problem of interest is tackled by solving a grid of convex optimization problems, where gridding is adopted to avoid the nonconvexity introduced by one decision variable. Clearly, this problem formulation suffers from scalability issues. For instance, a set of ℓ_∞-norm-bounded vectors in $\mathbb{R}^n$ has 2^n vertices. Nonetheless, since an array of convex optimization problems of equal sizes is to be solved, it is possible to leverage modern numerical tools/software for convex optimization, such as YALMIP [8] and SDPT3 [9], to efficiently address the problem at hand as long as the number of vertices of the aforementioned disturbance polytope (and hence, the number of constraints in each convex optimization problem) remains reasonably small.

There are related works in the literature that focus on finite horizon analysis. The case of nominal LTI systems was addressed by Chernousko [10], where time-dependent ellipsoids are computed as a solution of an initial value problem; this work was later extended to linear time-varying (LTV) systems with IQC constraints [11]. Other related works include [12]–[16], wherein various IQC approaches

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are followed to compute state and output bounds that depend on the ℓ_2-norm-bound placed on the disturbance inputs.

The paper is structured as follows. Section II defines the adopted notation and gives the S-procedure. Section III formulates the problem under consideration, with Section III-B giving some examples of uncertainty sets that admit point-wise IQC characterizations. Section IV derives the main results of the paper. Section V gives an illustrative example. The paper concludes with Section VI.

II. Notation and Preliminary Results

The sets of nonnegative integers, $n \times m$ real-valued matrices, and real-valued vectors of dimension n are denoted by $\mathbb{N}_0$, $\mathbb{R}^{n\times m}$, and $\mathbb{R}^n$, respectively. The transpose, inverse, pseudo-inverse, and adjoint (conjugate transpose) of a matrix X are denoted by X^T, X^{-1}, $X^{\dagger}$, and X^*, respectively. The zero matrix of dimensions $n \times m$ is denoted by $0_{n\times m}$, whereas the identity matrix of dimensions $n\times n$ is denoted by I_n. The subscripts are dropped when the dimensions are clear from context. Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, and denote by a_{ij} the (i, j)-th entry of matrix A. Then, the Kronecker product $A \otimes B \in \mathbb{R}^{(n\times p)\times(m\times q)}$ is defined as $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}$, and $\mathrm{blkdiag}(A, B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the block-diagonal augmentation of A and B.

The set $\mathbb{S}^n$ is defined as $\mathbb{S}^n = \{X \in \mathbb{R}^{n\times n} \,|\, X = X^T\}$. A matrix $X \in \mathbb{S}^n$ is said to be positive semidefinite, or $X \succeq 0$ for short, if $z^T Xz \geq 0$ for all $z \in \mathbb{R}^n$, and positive definite, or $X \succ 0$ for short, if $z^T Xz > 0$ for all $z \neq 0$. The set of positive semidefinite matrices is denoted by $\mathbb{S}^n_+$ and that of positive definite matrices is denoted by $\mathbb{S}^n_{++}$. Given a matrix $X \in \mathbb{S}^n_{++}$, we define the ellipsoid $\mathcal{E}_X = \{x \in \mathbb{R}^n \,|\, x^T Xx \leq 1\}$.

$\mathbb{RH}^{m\times n}_\infty$ denotes the space of real, rational, $m\times n$ matrix-valued functions that have no poles outside the open unit disk in the complex plane. Let ℓ^n_{2e}, or ℓ_{2e} for simplicity, denote the space of all real, vector-valued sequences $w = (w(0), w(1), \ldots)$, where $w(k) \in \mathbb{R}^n$ for all $k \in \mathbb{N}_0$. The Hilbert space ℓ_2 is the subspace of all sequences w in ℓ_{2e} that have a finite ℓ_2-norm defined as $\|w\| = \sqrt{\sum_{k=0}^{\infty} w(k)^T w(k)} < \infty$.

The following result, known as the S-procedure, is used in the derivation of the results of this paper. The proof of the S-procedure can be found in [5] and the references therein.

Lemma 1. *Consider the following quadratic functionals defined over $\mathbb{R}^m$: $\sigma_i(y) = y^T Q_i y + 2s_i^T y + r_i$ for $i = 0, 1, \ldots, N$, where $Q_i \in \mathbb{S}^m$, $s_i \in \mathbb{R}^m$, and $r_i \in \mathbb{R}$, and consider the following two conditions:*

S_1: *$\sigma_0(y) \geq 0$ for all $y \in \mathbb{R}^m$ such that $\sigma_i(y) \geq 0$ for all $i = 1, \ldots, N$;*

S_2: *there exist $\tau_i \geq 0$ for $i = 1, \ldots, N$ such that $\sigma_0(y) - \sum_{i=1}^{N} \tau_i \sigma_i(y) \geq 0$ for all $y \in \mathbb{R}^m$.*

Then, S_2 implies S_1. Moreover, if $N = 1$, then S_1 also implies S_2, in which case the S-procedure is said to be lossless.

If $N = 2$, the S-procedure is, in general, not lossless. But even if the conditions S_1 and S_2 are not equivalent, the S-procedure remains useful since verifying S_2 amounts to verifying that there exist $\tau_i \geq 0$ for $i = 1, \ldots, N$ such that

$$\begin{bmatrix} Q_0 & s_0 \\ s_0^T & r_0 \end{bmatrix} - \sum_{i=1}^{N} \tau_i \begin{bmatrix} Q_i & s_i \\ s_i^T & r_i \end{bmatrix} \succeq 0.$$

III. Problem Statement

A. Uncertain System Equations

Let $k \in \mathbb{N}_0$ denote discrete time, and consider the uncertain system $(G, \mathbf{\Delta})$ described by the following equations:

$$\begin{aligned} x_G(k+1) &= A_G x_G(k) + B_{G1}\vartheta(k) + B_{G2}d(k), \\ \varphi(k) &= C_G x_G(k) + D_{G1}\vartheta(k) + D_{G2}d(k), \\ \vartheta &= \Delta(\varphi), \end{aligned} \tag{1}$$

for all $k \in \mathbb{N}_0$. $x_G(k) \in \mathbb{R}^{n_G}$ is the state vector, $d(k)$ is the external disturbance affecting the uncertain system, and $\Delta : \ell^{n_\varphi}_{2e} \to \ell^{n_\varphi}_{2e}$ is the uncertainty operator affecting the nominal system. Δ is assumed to be a causal and bounded operator on ℓ_2 that lies within the pre-specified set $\mathbf{\Delta}$. The nominal system G is assumed to be a stable LTI system. The disturbance input $d(k)$ is assumed to lie within a convex, closed, and bounded polytope Γ for all $k \in \mathbb{N}_0$. For convenience, we define the set $\mathbf{D}_\Gamma = \{d \in \ell_{2e} \,|\, d(k) \in \Gamma \text{ for all } k \in \mathbb{N}_0\}$. The purpose of this paper is to compute an ellipsoidal invariant set for x_G in (1) for all $\Delta \in \mathbf{\Delta}$ and disturbance inputs $d \in \mathbf{D}_\Gamma$. Namely, the problem is to find a matrix $Q \in \mathbb{S}^{n_G}_{++}$ such that if $x_G(0) \in \mathcal{E}_Q$, then $x_G(k) \in \mathcal{E}_Q$ for all $k \in \mathbb{N}_0$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$.

It is assumed throughout that the uncertain system $(G, \mathbf{\Delta})$ is well-posed and robustly stable. That is, for all $\Delta \in \mathbf{\Delta}$ and $d \in \ell_{2e}$, there exist unique solutions to (1) that causally depend on d. Moreover, if $d \in \ell_2$, these solutions are in ℓ_2, and the system equations define a bounded causal mapping from d to x_G. As will become apparent in Section IV, there is a connection between the derived results herein and standard robust stability results from IQC theory. An investigation of this connection will be the topic of future work.

B. Point-Wise Integral Quadratic Constraints

The IQC framework is used to model the operators in the uncertainty set $\mathbf{\Delta}$. Let Ψ be a stable LTI system with realization $(A_\Psi, \begin{bmatrix} B_{\Psi1} & B_{\Psi2} \end{bmatrix}, C_\Psi, \begin{bmatrix} D_{\Psi1} & D_{\Psi2} \end{bmatrix})$ and S be a symmetric matrix of appropriate dimensions. Then, the operator Δ is said to satisfy the IQC defined by (Ψ, S) if $\sum_{k=0}^{\infty} r^T(k)Sr(k) \geq 0$ for all $\varphi \in \ell_2$ and $\vartheta = \Delta(\varphi)$, where

$$\begin{aligned} \xi(k+1) &= A_\Psi\xi(k) + B_{\Psi1}\varphi(k) + B_{\Psi2}\vartheta(k), \quad \xi(k) \in \mathbb{R}^{n_\xi}, \\ r(k) &= C_\Psi\xi(k) + D_{\Psi1}\varphi(k) + D_{\Psi2}\vartheta(k), \quad \xi(0) = 0. \end{aligned} \tag{2}$$

The set $\mathbf{\Delta}$ is said to satisfy the IQC defined by (Ψ, S), or $\mathbf{\Delta} \in \mathrm{IQC}(\Psi, S)$ for short, if all Δ in $\mathbf{\Delta}$ satisfy the IQC defined by (Ψ, S). In this work, a more restricted constraint, termed point-wise IQC, on the output r of the filter Ψ driven by (φ, ϑ) is used to derive the results. Namely, the operator Δ is said to satisfy the point-wise IQC defined by (Ψ, S) if $r^T(k)Sr(k) \geq 0$ for all $k \in \mathbb{N}_0$, $\varphi \in \ell_{2e}$, and $\vartheta = \Delta(\varphi)$, where r is defined in (2). Furthermore, the set $\mathbf{\Delta}$ is said to satisfy the point-wise IQC defined by (Ψ, S), or $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$ for short, if all Δ in $\mathbf{\Delta}$ satisfy the point-wise IQC defined by (Ψ, S). Clearly, requiring the quantity $r^T(k)Sr(k)$ to be nonnegative for all $k \in \mathbb{N}_0$ is more restrictive than requiring $\sum_{k=0}^{\infty} r^T(k)Sr(k) \geq 0$. In other words, if $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$, then $\mathbf{\Delta} \in \mathrm{IQC}(\Psi, S)$. However, the converse is not necessarily true. While requiring point-wise IQCs restricts the applicability of the proposed results, it is illustrated next that some common uncertainty sets admit point-wise IQC characterizations.

Proposition 1. *Let $\mathbf{\Delta}$ denote the set of static LTI operators Δ that represent the multiplication in the time domain by an uncertain parameter δ such that $|\delta| \leq \alpha$, i.e., $\vartheta = \Delta(\varphi)$ and $\Delta \in \mathbf{\Delta}$ mean that $\vartheta(k) = \delta I_{n_\varphi}\varphi(k)$ for all $k \in \mathbb{N}_0$. Then, $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$, where*

$$\Psi(z) = \begin{bmatrix} \mathcal{B}(z) & 0 \\ 0 & \mathcal{B}(z) \end{bmatrix} \otimes I_{n_\varphi}, \quad S = \begin{bmatrix} \alpha^2 X & Y \\ Y^T & -X \end{bmatrix}, \quad (3)$$

$X = X^T \in \mathbb{S}_+^{(n_b \times n_\varphi)}$, $Y = -Y^T$ in $\mathbb{R}^{(n_b \times n_\varphi)\times(n_b \times n_\varphi)}$, and $\mathcal{B} \in \mathbb{RH}_\infty^{n_b \times 1}$.

Proposition 2. *Let $\mathbf{\Delta}$ denote the set of static LTV operators Δ that represent the multiplication in the time domain by a time-varying parameter $\delta(k)$ such that $|\delta(k)| \leq \alpha$ for all $k \in \mathbb{N}_0$, i.e., $\vartheta = \Delta(\varphi)$ and $\Delta \in \mathbf{\Delta}$ mean that $\vartheta(k) = \delta(k) I_{n_\varphi}\varphi(k)$ for all $k \in \mathbb{N}_0$. Then, $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$, where*

$$\Psi(z) = \begin{bmatrix} I_{n_\varphi} & 0 \\ 0 & I_{n_\varphi} \end{bmatrix}, \qquad S = \begin{bmatrix} \alpha^2 X & Y \\ Y^T & -X \end{bmatrix}, \quad (4)$$

$X = X^T \in \mathbb{S}_+^{n_\varphi}$ and $Y = -Y^T$ in $\mathbb{R}^{n_\varphi \times n_\varphi}$.

Proposition 3. *Consider the function $\phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ such that $\alpha\varphi^2 \leq \phi(\varphi, k)\varphi \leq \beta\varphi^2$ for all $\varphi \in \mathbb{R}$ and $k \geq 0$. Let $\mathbf{\Delta}$ denote the set of memoryless sector bounded nonlinearities Δ such that $\vartheta = \Delta(\varphi)$ means that $\vartheta(k) = (\Delta(\varphi))(k) = \phi(\varphi(k), k)$ for all $k \in \mathbb{N}_0$. Then, $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$, where*

$$\Psi(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix}.$$

The proofs of Propositions 1-3 are omitted here for brevity. The factors (Ψ, S) (along with the imposed conditions) in Propositions 2 and 3 correspond to the standard IQC factors for the respective uncertainty sets. That is, in addition to satisfying $\mathbf{\Delta} \in \mathrm{IQC}(\Psi, S)$, these sets satisfy $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$ for the same standard factors (Ψ, S). As for Proposition 1, the imposed condition $X \succeq 0$ is more restrictive than the standard condition $\left(\mathcal{B}(e^{j\omega}) \otimes I_{n_\varphi}\right)^* X \left(\mathcal{B}(e^{j\omega}) \otimes I_{n_\varphi}\right) \succeq 0$ for all $\omega \in [-\pi, \pi]$. However, in the IQC analysis performed in [17] on flight controllers of fixed-wing unmanned aircraft systems, it is reported that imposing the latter (less restrictive) constraint did not significantly improve the results. This observation from a practical example motivates imposing the condition $X \succeq 0$ in Proposition 1, thereby rendering the IQC point-wise.

IV. Main Results

In this section, the S-procedure is leveraged to compute an ellipsoidal invariant set for uncertain systems in which the uncertainty set is described by point-wise IQCs. Consider the uncertain system $(G, \mathbf{\Delta})$ defined in (1), and assume that $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$. The nominal system G maps (ϑ, d) to φ, and the IQC filter Ψ maps (φ, ϑ) to r. To derive our results, an augmented system H that maps (ϑ, d) to r is formed by combining the equations of G and Ψ. Let $x_H = (x_G, \xi)$. Then, for all $k \in \mathbb{N}_0$, the equations of system H are given by

$$\begin{aligned} x_H(k+1) &= A_H x_H(k) + B_{H1}\vartheta(k) + B_{H2}d(k), \\ r(k) &= C_H x_H(k) + D_{H1}\vartheta(k) + D_{H2}d(k), \end{aligned} \quad (5)$$

$$\begin{aligned} A_H &= \begin{bmatrix} A_G & 0 \\ B_{\Psi 1}C_G & A_\Psi \end{bmatrix}, \quad B_{H1} = \begin{bmatrix} B_{G1} \\ B_{\Psi 1}D_{G1} + B_{\Psi 2} \end{bmatrix}, \\ C_H &= \begin{bmatrix} D_{\Psi 1}C_G & C_\Psi \end{bmatrix}, \quad B_{H2} = \begin{bmatrix} B_{G2} \\ B_{\Psi 1}D_{G2} \end{bmatrix}, \\ D_{H1} &= D_{\Psi 1}D_{G1} + D_{\Psi 2}, \quad D_{H2} = D_{\Psi 1}D_{G2}. \end{aligned}$$

Using the S-procedure, it is shown how to compute $P \in \mathbb{S}_{++}^{n_G+n_\xi}$ such that, for all $k \in \mathbb{N}_0$, $x_H^T(k)Px_H(k) \leq 1$ and $r^T(k)Sr(k) \geq 0$ imply that $x_H^T(k+1)Px_H(k+1) \leq 1$ for all $d(k) \in \Gamma$. If the IQC filter Ψ is a static operator, i.e., $x_H = x_G$, then the matrix P defines the sought ellipsoidal invariant set, since in this case $x_G^T(k)Px_G(k) \leq 1$ implies that $x_G^T(k+1)Px_G(k+1) \leq 1$ for all $k \in \mathbb{N}_0$, $\Delta \in \mathbf{\Delta}$, and $d(k) \in \Gamma$. Thus, if $x_G(0) \in \mathcal{E}_P$, it follows that $x_G(k) \in \mathcal{E}_P$ for all $k \in \mathbb{N}_0$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$. On the other hand, if the IQC filter Ψ is a dynamic operator, which is a flexibility permitted by the proposed approach, then the matrix P defines an ellipsoidal invariant set for the state vector of the augmented system H. Namely, if $x_H(0) \in \mathcal{E}_P$, then $x_H(k) \in \mathcal{E}_P$ for all $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$, where x_H is defined in (5) and φ is defined in (1). Nonetheless, the S-procedure can be applied a second time to compute a matrix $Q \in \mathbb{S}_{++}^{n_G}$ such that $x_H^T(k)Px_H(k) \leq 1$ implies that $x_G^T(k)Qx_G(k) \leq 1$ for all $k \in \mathbb{N}_0$. Then, if $x_H(0) \in \mathcal{E}_P$, it follows that $x_G(k) \in \mathcal{E}_Q$ for all $k \in \mathbb{N}_0$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$. This conclusion can be further strengthened since $\xi(0) = 0$ in (2), i.e., $x_H(0) = (x_G(0), 0)$. Namely, by partitioning P conformably with x_H as in $P = \begin{bmatrix} P_{ij} \end{bmatrix}_{i,j=1,2}$, where $P_{11} \in \mathbb{S}_{++}^{n_G}$, it can be seen that $x_H^T(0)Px_H(0) = x_G^T(0)P_{11}x_G(0)$, and so $x_H(0) \in \mathcal{E}_P$ is equivalent to $x_G(0) \in \mathcal{E}_{P_{11}}$.

Theorem 1. *Consider the uncertain system $(G, \mathbf{\Delta})$ defined in (1), where $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$, and the equations of the IQC filter Ψ are given in (2). Let H be the system defined in (5). If there exist $P \in \mathbb{S}_{++}^{n_G+n_\xi}$, $\tau_1 \geq 0$, and $\tau_2 \geq 0$ such* *that*

$$\begin{bmatrix} -\begin{bmatrix} A_H^T \\ B_{H1}^T \end{bmatrix} P \begin{bmatrix} A_H & B_{H1} \end{bmatrix} & -\begin{bmatrix} A_H^T \\ B_{H1}^T \end{bmatrix} P B_{H2} \hat{d} \\ -\hat{d}^T B_{H2}^T P \begin{bmatrix} A_H & B_{H1} \end{bmatrix} & 1 - \hat{d}^T B_{H2}^T P B_{H2} \hat{d} \end{bmatrix}$$
$$- \tau_1 \text{ blkdiag} \left(-\begin{bmatrix} I \\ 0 \end{bmatrix} P \begin{bmatrix} I & 0 \end{bmatrix}, 1 \right)$$
$$- \tau_2 \begin{bmatrix} \begin{bmatrix} C_H^T \\ D_{H1}^T \end{bmatrix} S \begin{bmatrix} C_H & D_{H1} \end{bmatrix} & \begin{bmatrix} C_H^T \\ D_{H1}^T \end{bmatrix} S D_{H2} \hat{d} \\ \hat{d}^T D_{H2}^T S \begin{bmatrix} C_H & D_{H1} \end{bmatrix} & \hat{d}^T D_{H2}^T S D_{H2} \hat{d} \end{bmatrix} \succeq 0, \tag{6}$$

for all $\hat{d} \in \Gamma$*, then* $x_H^T(0) P x_H(0) \leq 1$ *implies that* $x_H^T(k) P x_H(k) \leq 1$ *for all* $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, *and* $d \in \mathbf{D}_\Gamma$.

Proof: We prove the result by showing that, for all $k \in \mathbb{N}_0$, $x_H^T(k) P x_H(k) \leq 1$ implies that $x_H^T(k+1) P x_H(k+1) \leq 1$ for all $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d(k) \in \Gamma$. Since $\mathbf{\Delta} \in \text{pwIQC}(\Psi, S)$, then $r^T(k) S r(k) \geq 0$ for all $k \in \mathbb{N}_0$, $d \in \ell_{2e}$, $\vartheta = \Delta(\varphi)$, and $\Delta \in \mathbf{\Delta}$, where r is defined in (5) (or (2)). For a given $\hat{d} \in \Gamma$, define the quadratic function

$$\sigma_{\hat{d},2}(x) = x^T \begin{bmatrix} C_H^T \\ D_{H1}^T \end{bmatrix} S \begin{bmatrix} C_H & D_{H1} \end{bmatrix} x + x^T \begin{bmatrix} C_H^T \\ D_{H1}^T \end{bmatrix} S D_{H2} \hat{d} + \hat{d}^T D_{H2}^T S \begin{bmatrix} C_H & D_{H1} \end{bmatrix} x + \hat{d}^T D_{H2}^T S D_{H2} \hat{d}.$$

Thus, $r^T(k) S r(k) = \sigma_{d(k),2}\big((x_H(k), \vartheta(k))\big)$. Similarly, by defining $\sigma_1(x) = -x^T \begin{bmatrix} I \\ 0 \end{bmatrix} P \begin{bmatrix} I & 0 \end{bmatrix} x + 1$, the inequality $x_H^T(k) P x_H(k) \leq 1$ can be expressed as $\sigma_1\big((x_H(k), \vartheta(k))\big) \geq 0$. Finally, for a given $\hat{d} \in \Gamma$, define the quadratic function

$$\sigma_{\hat{d},0}(x) = -x^T \begin{bmatrix} A_H^T \\ B_{H1}^T \end{bmatrix} P \begin{bmatrix} A_H & B_{H1} \end{bmatrix} x - x^T \begin{bmatrix} A_H^T \\ B_{H1}^T \end{bmatrix} P B_{H2} \hat{d} - \hat{d}^T B_{H2}^T P \begin{bmatrix} A_H & B_{H1} \end{bmatrix} x + 1 - \hat{d}^T B_{H2}^T P B_{H2} \hat{d},$$

and rewrite the inequality $x_H^T(k+1) P x_H(k+1) \leq 1$ as $\sigma_{d(k),0}\big((x_H(k), \vartheta(k))\big) \geq 0$. Thus, the proof boils down to verifying that, for all $d(k) \in \Gamma$, $\sigma_{d(k),0}\big((x_H(k), \vartheta(k))\big) \geq 0$ whenever $\sigma_1\big((x_H(k), \vartheta(k))\big) \geq 0$ and $\sigma_{d(k),2}\big((x_H(k), \vartheta(k))\big) \geq 0$. Indeed, this follows by the S-procedure (Lemma 1) since (6) holds for all $\hat{d} \in \Gamma$.

When solving standard IQC analysis problems, the state-space realization of the IQC filter Ψ is usually fixed a priori, and the variables in S are found by solving a convex optimization problem. However, in (6), the multiplication between τ_2 and the term that contains S is one source of non-convexity. To remedy this issue, $\tau_2 \geq 0$ is lumped into the variables in S. For example, let $S = \begin{bmatrix} \alpha^2 X & Y \\ Y^T & -X \end{bmatrix}$, with $X \succeq 0$ and $Y = -Y^T$. In this case, τ_2 is lumped into S by defining $\hat{X} = \tau_2 X \succeq 0$ and $\hat{Y} = \tau_2 Y = -\hat{Y}^T$. Thus, one defines $\tau_2 S = \begin{bmatrix} \alpha^2 \hat{X} & \hat{Y} \\ \hat{Y}^T & -\hat{X} \end{bmatrix}$ and solves for $\hat{X} \succeq 0$ and $\hat{Y} = -\hat{Y}^T$. In fact, if $\mathbf{\Delta} \in \text{IQC}(\Psi, S)$ (respectively, $\mathbf{\Delta} \in \text{pwIQC}(\Psi, S)$), then $\mathbf{\Delta} \in \text{IQC}(\Psi, \tau_2 S)$ (respectively, $\mathbf{\Delta} \in \text{pwIQC}(\Psi, \tau_2 S)$) for all $\tau_2 \geq 0$. Thus, if the variables in S appear linearly, it is possible to perform a change of variables similar to the one here to lump the variable $\tau_2 \geq 0$ with the variables in S.

In Theorem 1, inequality (6) must hold for all $\hat{d} \in \Gamma$. The polytope Γ can be defined as the convex hull of the set of vertices $\hat{d}_i$ for $i = 1, \ldots, p$, where p denotes the number of vertices (extreme points) of Γ. Next, a sufficient condition is derived allowing to impose (6) only at the vertices $\hat{d}_i$ for $i = 1, \ldots, p$. To simplify the discussion, inequality (6) is compactly rewritten as

$$\tilde{M}(\hat{d}) = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \hat{d} \\ \hat{d}^T \tilde{M}_{12}^T & \tilde{m}_{22} - \hat{d}^T \tilde{M}_{22} \hat{d} \end{bmatrix} \succeq 0 \tag{7}$$

by appropriately defining the terms $\tilde{M}_{11}$, $\tilde{M}_{12}$, $\tilde{M}_{22}$, and $\tilde{m}_{22}$, e.g., $\tilde{m}_{22} = 1 - \tau_1$ and $\tilde{M}_{22} = B_{H2}^T P B_{H2} + \tau_2 D_{H2}^T S D_{H2}$. As the vectors $\hat{d} \in \Gamma$ can be expressed as $\hat{d} = \sum_{i=1}^p c_i \hat{d}_i$ for some coefficients $c_i \geq 0$ such that $\sum_{i=1}^p c_i = 1$, it is desired to determine a sufficient condition such that if $\tilde{M}(\hat{d}_i) \succeq 0$ for $i = 1, \ldots, p$, then $\tilde{M}(\hat{d}) \succeq 0$ for all $\hat{d} \in \Gamma$. In this case, inequality (6) in Theorem 1 only needs to be imposed at the vertices $\hat{d}_i$ of Γ. Assume that $\tilde{M}(\hat{d}_i) \succeq 0$ for $i = 1, \ldots, p$ and that $\hat{d} \in \Gamma$ is expressed as $\hat{d} = \sum_{i=1}^p c_i \hat{d}_i$, where $c_i \geq 0$ and $\sum_{i=1}^p c_i = 1$, then it is not difficult to see that

$$\sum_{i=1}^p c_i \tilde{M}(\hat{d}_i) = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \hat{d} \\ \hat{d}^T \tilde{M}_{12}^T & \tilde{m}_{22} - \sum_{i=1}^p c_i \hat{d}_i^T \tilde{M}_{22} \hat{d}_i \end{bmatrix} \succeq 0.$$

If $\tilde{M}_{22} \succeq 0$, i.e., $f(x) = x^T \tilde{M}_{22} x$ is a convex function, then it follows by Jensen's inequality [18] that $0 \leq \hat{d}^T \tilde{M}_{22} \hat{d} \leq \sum_{i=1}^p c_i \hat{d}_i^T \tilde{M}_{22} \hat{d}_i$, and so, $\tilde{m}_{22} - \hat{d}^T \tilde{M}_{22} \hat{d} \geq \tilde{m}_{22} - \sum_{i=1}^p c_i \hat{d}_i^T \tilde{M}_{22} \hat{d}_i$ and $\tilde{M}(\hat{d}) \succeq 0$. In conclusion, if

$$\tilde{M}_{22} = B_{H2}^T P B_{H2} + \tau_2 D_{H2}^T S D_{H2} \succeq 0, \tag{8}$$

then it suffices to impose (6) in Theorem 1 at the vertices $\hat{d}_i$ for $i = 1, \ldots, p$.

Furthermore, if the polytope Γ is symmetric, i.e., $-\Gamma = \{-d \,|\, d \in \Gamma\} = \Gamma$, then it is possible to eliminate half of the vertices $\hat{d}_i$ at which inequality (6) needs to be imposed. To see this, we note that if $\hat{d}_i = -\hat{d}_j$, then $\tilde{M}(\hat{d}_i) \succeq 0$ if and only if $\tilde{M}(\hat{d}_j) \succeq 0$. This follows from a Schur complement result [18] that states that $\tilde{M}(\hat{d}) \succeq 0$ for a given vector $\hat{d}$ if and only if $\tilde{M}_{11} \succeq 0$, $(I - \tilde{M}_{11} \tilde{M}_{11}^\dagger) \tilde{M}_{12} \hat{d} = 0$, and $\tilde{m}_{22} - \hat{d}^T \tilde{M}_{22} \hat{d} - \hat{d}^T \tilde{M}_{12}^T \tilde{M}_{11}^\dagger M_{12} \hat{d} \succeq 0$.

The second source of nonconvexity in (6) is the multiplication between the variables τ_1 and P. For a fixed τ_1, (6) is a linear matrix inequality (LMI) in P and S (assuming that τ_2 is lumped in S). Therefore, to solve inequality (6), we resort to gridding, wherein at each grid point, τ_1 is fixed and the resulting LMI is solved. It can be seen from the $(2,2)$-block of (7) that $\tilde{m}_{22} - \hat{d}^T \tilde{M}_{22} \hat{d} = 1 - \tau_1 - \hat{d}^T \tilde{M}_{22} \hat{d} \geq 0$. If $\tilde{M}_{22} \succeq 0$ as per (8), then $0 \leq \tau_1 \leq 1$. Therefore, in this case, a suitable search space for the variable τ_1 is the interval $[0, 1]$.

Theorem 1 allows to compute an ellipsoidal invariant set $\mathcal{E}_P$ such that if $x_H(0) \in \mathcal{E}_P$, then $x_H(k) \in \mathcal{E}_P$ for all $k \in$

$\mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$. Theorem 2 strengthens the conclusion of Theorem 1 by introducing a new matrix variable and imposing an additional constraint.

Theorem 2. *Consider the uncertain system $(G, \mathbf{\Delta})$ defined in (1), where $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$, and the equations of the IQC filter Ψ are given in (2). Let H be the system defined in (5). If there exist $P \in \mathbb{S}_{++}^{n_G+n_\xi}$, $Q \in \mathbb{S}_{++}^{n_G}$, $\tau_1 \geq 0$, $\tau_2 \geq 0$, and $\tau_3 \geq 0$ such that (6) holds for all $\hat{d} \in \Gamma$ and*

$$\begin{bmatrix} -E^T Q E & 0 \\ 0 & 1 \end{bmatrix} - \tau_3 \begin{bmatrix} -P & 0 \\ 0 & 1 \end{bmatrix} \succeq 0, \tag{9}$$

where $E = \begin{bmatrix} I_{n_G} & 0_{n_G \times n_\xi} \end{bmatrix}$, then $x_H^T(0) P x_H(0) \leq 1$ implies that $x_G^T(k) Q x_G(k) \leq 1$ for all $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$.

Proof: From Theorem 1, it can be concluded that $x_H^T(0) P x_H(0) \leq 1$ implies that $x_H^T(k) P x_H(k) \leq 1$ for all $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$. Therefore, to prove Theorem 2, it suffices to show that if (9) holds, then $x_H^T(k) P x_H(k) \leq 1$ implies that $x_G^T(k) Q x_G(k) \leq 1$ for all $k \in \mathbb{N}_0$. To show this, the quadratic functions $\hat{\sigma}_0(x) = 1 - x^T E^T Q E x$ and $\hat{\sigma}_1(x) = 1 - x^T P x$ are defined. It is thus desired to prove that $\hat{\sigma}_0(x_H(k)) \geq 0$ whenever $\hat{\sigma}_1(x_H(k)) \geq 0$, since $x_G(k) = E x_H(k)$. Indeed, by the S-procedure (Lemma 1), the existence of $\tau_3 \geq 0$ such that (9) holds is equivalent to $\hat{\sigma}_0(x_H(k)) \geq 0$ whenever $\hat{\sigma}_1(x_H(k)) \geq 0$, which concludes the proof.

For $\tau_3 > 0$, the inequality in (9) can be transformed into the following equivalent LMI in P, $\hat{Q} \in \mathbb{S}_{++}^{n_G}$, and $\frac{1}{\tau_3} > 0$:

$$\mathrm{blkdiag}\left(-E^T \hat{Q} E, 1/\tau_3\right) - \mathrm{blkdiag}\left(-P, 1\right) \succeq 0.$$

The sought Q is then determined from $Q = \hat{Q}/(1/\tau_3) = \tau_3 \hat{Q}$. The above LMI can be decoupled into two LMIs, namely,

$$E^T \hat{Q} E \preceq P, \qquad \frac{1}{\tau_3} - 1 \geq 0. \tag{10}$$

Theorem 2 allows to compute ellipsoidal sets $\mathcal{E}_P$ and $\mathcal{E}_Q$ such that if $x_H(0) \in \mathcal{E}_P$, then $x_G(k) \in \mathcal{E}_Q$ for all $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$. As $\xi(0) = 0$ and $x_H(0) = (x_G(0), 0)$, by partitioning $P = \begin{bmatrix} P_{ij} \end{bmatrix}_{i,j=1,2}$, where $P_{11} \in \mathbb{S}_{++}^{n_G}$, it follows from Theorem 2 that if $x_G(0) \in \mathcal{E}_{P_{11}}$, then $x_G(k) \in \mathcal{E}_Q$ for all $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$.

Alternatively, one can use results from [19] on linear transformations and projections of ellipsoids to find a solution $Q \succ 0$ to (9) given a solution $P \succ 0$ to (6). Namely, if we set $\mathcal{E}_Q = \{y \in \mathbb{R}^{n_G} \,|\, y = Ex, \, x \in \mathcal{E}_P\}$, then $Q = (E P^{-1} E^T)^{-1} = ((P^{-1})_{11})^{-1} = P_{11} - P_{12} P_{22}^{-1} P_{12}^T \succ 0$, where the $(1,1)$-block of P^{-1}, i.e., $(P^{-1})_{11}$, is given by $(P_{11} - P_{12} P_{22}^{-1} P_{12}^T)^{-1}$ by the properties of the Schur complement [18]. Verifying that this choice of Q satisfies (9) for a given P and $\tau_3 = 1$ reduces to verifying the following inequality, which immediately follows from the Schur complement formula since $P_{22} \succ 0$:

$$\begin{bmatrix} P_{12} P_{22}^{-1} P_{12}^T & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \succeq 0.$$

Nonetheless, solving (6) and (10) simultaneously allows for flexibility in computing the matrix $Q \in \mathbb{S}_{++}^{n_G}$. If it is desired to find the minimum volume ellipsoid $\mathcal{E}_Q$, then the objective function $\log\det Q^{-1} = -\log\det Q = -\log\det \tau_3 \hat{Q} = -n_G \log \tau_3 - \log\det \hat{Q}$ is minimized. In this case, $\tau_3 = 1$, $Q = \hat{Q}$, and the objective function becomes $-\log\det \hat{Q}$.

If $Q = P_{11}$ and since $\xi(0) = 0$, the conclusion from Theorem 2 becomes: if $x_G(0) \in \mathcal{E}_{P_{11}}$, then $x_G(k) \in \mathcal{E}_{P_{11}}$ for all $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$. For $Q = P_{11}$ to be a solution to (9), we must have $\tau_3 = 1$ and $P = \mathrm{blkdiag}\,(P_{11}, P_{22})$. This can be seen by substituting $Q = P_{11}$ in (9) and referring to the Schur complement formula, while noting that $0 \leq \tau_3 \leq 1$, $P_{11} \succ 0$, and $P_{22} \succ 0$. That is, in this case, we find a block-diagonal matrix $P \succ 0$ that satisfies (6), and set $Q = P_{11}$. If imposing a block-diagonal structure on P renders the problem infeasible, the condition $Q = P_{11}$ is approximated by maximizing $\lambda \in [0, 1]$ such that $\lambda P_{11} = \lambda E P E^T \preceq Q$. To avoid adding a nonconvex inequality to the optimization problem at hand, the maximum possible value of λ is found using the bisection method.

V. Illustrative Example

Consider the uncertain system $(G, \mathbf{\Delta})$, where the state-space matrices of the nominal system G are given by

$$A_G = \begin{bmatrix} 0.05 & -0.2 & 0.3 \\ 0.1 & 0.8 & 0.2 \\ -0.2 & 0.5 & -0.1 \end{bmatrix}, \quad B_{G1} = \begin{bmatrix} 0.2 & 0.1 \\ 0.5 & -0.4 \\ -0.3 & 0.2 \end{bmatrix},$$

$$C_G = \begin{bmatrix} 1 & -0.5 & 0.3 \\ 0.9 & 0.2 & -0.5 \end{bmatrix}, \quad B_{G2} = \begin{bmatrix} 0.5 & 0.1 \\ -0.3 & -0.7 \\ 0.5 & -0.2 \end{bmatrix},$$

$$D_{G1} = \begin{bmatrix} 0.1 & 0.6 \\ 0.6 & -0.9 \end{bmatrix}, \quad D_{G2} = \begin{bmatrix} -0.5 & 0.4 \\ 0.3 & 0.1 \end{bmatrix}.$$

$\mathbf{\Delta}$ is the set of static LTI operators Δ that represent the multiplication in the time domain by an uncertain parameter δ such that $|\delta| \leq 0.3$. The uncertain system $(G, \mathbf{\Delta})$ can be shown to be well-posed and robustly stable. The set $\mathbf{\Delta}$ satisfies $\mathbf{\Delta} \in \mathrm{pwIQC}(\Psi, S)$, where Ψ and S are given in (3). $\mathcal{B}(z)$ is chosen as $\mathcal{B}(z) = \begin{bmatrix} 1 & \frac{1}{z-0.5} & \frac{1}{(z-0.5)^2} \end{bmatrix}^T$. Thus, $n_\varphi = 2$ and $n_b = 3$, and so $X \in \mathbb{S}_+^6$ and $Y = -Y^T \in \mathbb{R}^{6 \times 6}$. Let $\left(A_\Psi, \begin{bmatrix} B_{\Psi 1} & B_{\Psi 2} \end{bmatrix}, C_\Psi, \begin{bmatrix} D_{\Psi 1} & D_{\Psi 2} \end{bmatrix}\right)$ be a minimal realization of the IQC filter $\Psi(z)$ with $n_\xi = 8$. The set of exogenous disturbances $\Gamma = \{d \in \mathbb{R}^2 \,|\, -0.5 \leq d_i \leq 0.5, \; i = 1, 2\}$. By imposing the constraint in (8), it becomes sufficient to impose (6) at the vertices $\hat{d}_1 = (0.5, 0.5)$, $\hat{d}_2 = (0.5, -0.5)$, $\hat{d}_3 = (-0.5, 0.5)$, and $\hat{d}_4 = (-0.5, -0.5)$ of Γ. Also, since Γ is a symmetric set, (6) holds at $\hat{d}_1$ (respectively, $\hat{d}_2$) if and only if it holds at $\hat{d}_4$ (respectively, $\hat{d}_3$). Thus, (6) needs to be imposed only at the vertices $\hat{d}_1$ and $\hat{d}_2$.

Theorem 2 is applied to compute ellipsoidal sets $\mathcal{E}_P$ and $\mathcal{E}_Q$ such that if $x_H^T(0) P x_H(0) \leq 1$, then $x_H^T(k) P x_H(k) \leq 1$ and $x_G^T(k) Q x_G(k) \leq 1$ for all $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$. In particular, we solve for $P = \mathrm{blkdiag}\,(P_{11}, P_{22})$, and set $Q = P_{11}$. Since $\xi(0) = 0$, we further conclude from Theorem 2 that if $x_G(0) \in \mathcal{E}_{P_{11}}$, then

$x_G(k) \in \mathcal{E}_{P_{11}}$ for all $k \in \mathbb{N}_0$, $\vartheta = \Delta(\varphi)$, $\Delta \in \mathbf{\Delta}$, and $d \in \mathbf{D}_\Gamma$.

To find the minimum volume invariant ellipsoid $\mathcal{E}_Q$, we minimize the objective function $\log\det(Q^{-1}) = -\log\det P_{11}$. Namely, the following semidefinite program (SDP) is solved for various values of $\tau_1 \in [0,1]$:

$$\begin{aligned} &\text{minimize} && -\log\det P_{11} \\ &\text{subject to} && \text{(6) imposed at } \hat{d}_1 \text{ and } \hat{d}_2, \text{(8)}, \\ & && P = \text{blkdiag}(P_{11}, P_{22}) \succ 0,\ X \succeq 0,\ Y = -Y^T. \end{aligned}$$

The SDPs in this example are solved in MATLAB using the parser Yalmip [8] and solver SDPT3 [9]. For the above SDP, the number of constraints is 78, the dimension of the SDP variable is 50, and the number of SDP blocks is 6. A typical total solution time is 1.60s with total CPU time of 0.79s. It is found that among the tested values of τ_1, $\tau_1 = 0.935$ yields the smallest $\log\det Q^{-1} = 8.3756$. The ellipsoids corresponding to this value of τ_1 are used in the simulations reported below. For the tested values of $\tau_1 \leq 0.90$, numerical problems start to appear and the solver no longer outputs feasible solutions.

In the following, we illustrate via sample simulations that the obtained ellipsoidal sets $\mathcal{E}_P$ and $\mathcal{E}_Q$ indeed define invariant sets for x_H and x_G, respectively. From the equation of φ in (1) and noting that $\vartheta(k) = \delta I_{n_\varphi}\varphi(k)$ for all $k \in \mathbb{N}_0$, we get the following system of equations for all $k \in \mathbb{N}_0$:

$$\begin{aligned} \vartheta(k) &= (I - \delta D_{G1})^{-1}\delta(C_G x_G(k) + D_{G2} d(k)), \\ x_G(k+1) &= A_G x_G(k) + B_{G1}\vartheta(k) + B_{G2} d(k). \end{aligned}$$

To simulate these equations, it is required to specify the initial condition $x_G(0)$, the uncertain parameter δ, and the disturbance input $d(k)$ for all $k \in \mathbb{N}_0$. The goal is to illustrate that if $x_G(0) \in \mathcal{E}_Q$, then $x_G(k)$ remains in $\mathcal{E}_Q$ for all $k \in \mathbb{N}_0$, $|\delta| \leq 0.3$, and $d \in \mathbf{D}_\Gamma$. The state vector of the augmented system H is also computed by computing the state vector $\xi(k)$ of the IQC filter using (2) and appending it to $x_G(k)$ as in $x_H(k) = \begin{bmatrix} x_G^T(k) & \xi^T(k) \end{bmatrix}^T$ for all $k \in \mathbb{N}_0$. Since $\xi(0) = 0$, then $x_H(0) \in \mathcal{E}_P$ whenever $x_G(0) \in \mathcal{E}_Q = \mathcal{E}_{P_{11}}$. Thus, the simulations also illustrate that if $x_H(0) \in \mathcal{E}_P$, then $x_H(k)$ remains in $\mathcal{E}_P$ for all $k \in \mathbb{N}_0$, $|\delta| \leq 0.3$, and $d \in \mathbf{D}_\Gamma$. Two sets of simulations are considered in which multiple initial conditions are chosen on the boundary of $\mathcal{E}_Q$. In the first set of simulations, δ is randomly generated between $[-0.3, 0.3]$ and the disturbance input $d(k)$ is computed from $d(k) = \sum_{i=1}^4 c_i(k)\hat{d}_i$, where $\hat{d}_i$ are the vertices of the set Γ and $c_i(k)$ are randomly generated such that $c_i(k) \geq 0$ and $\sum_i^4 c_i(k) = 1$. In the second set of simulations, the uncertain parameter δ is chosen as $\delta = -0.3$ and $d(k) = \hat{d}_4 = (-0.5, -0.5)$ for all $k \in \mathbb{N}_0$. The goal is to show the behavior of the system under more extreme uncertainty and disturbance scenarios. The results from both sets of simulations, namely, the evolution of the quantities $x_H^T(k)Px_H(k)$ and $x_G^T(k)Qx_G(k)$ as functions of k, are shown in Figure 1. As expected from Theorem 2, the conditions $x_H(k)^T P x_H(k) \leq 1$ and $x_G^T(k)Qx_G \leq 1$ are satisfied for all $k \in \mathbb{N}_0$ in all the simulations considered.

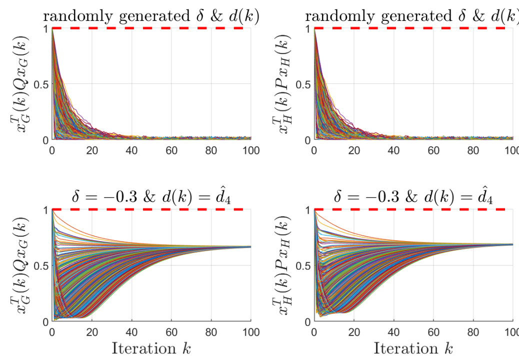

Fig. 1. Illustration of the ellipsoidal invariant sets $\mathcal{E}_Q$ and $\mathcal{E}_P$.

VI. Conclusion and Future Work

This paper shows how to combine the IQC framework with the S-procedure to compute ellipsoidal invariants for uncertain systems. To compute the matrices defining the invariant ellipsoids, it is required to solve a nonconvex optimization problem, which is handled by gridding over one decision variable and solving the ensuing convex optimization problems. Future work will apply the proposed approach to a practical example. It is also of interest to address the nonlinearity in a more systematic way, using different numerical approaches: BMI solvers capable of solving bilinear matrix inequality constraints, SOS optimization, or policy iteration.

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