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Geometry on the Wasserstein space over a compact Riemannian manifold

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Abstract

We will revisit the intrinsic differential geometry of the Wasserstein space over a Riemannian manifold, due to a series of papers by Otto, Otto-Villani, Lott, Ambrosio-Gigli-Savaré and so on.

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1 Introduction

For the sake of simplicity, we will consider in this paper a connected compact Riemannian manifold M of dimension m . We denote by d_M the Riemannian distance and dx the Riemannian measure on M such that $\int_M dx = 1$. Since the diameter of M is finite, any probability measure μ on M is such that $\int_M d_M^2(x_0, x) d\mu(x) < +\infty$, where x_0 is a fixed point of M . As usual, we denote by $\mathbb{P}_2(M)$ the space of probability measures on M , endowed with the Wasserstein distance W_2 defined by

$$W_2^2(\mu_1, \mu_2) = \inf \left\{ \int_{M \times M} d_M^2(x, y) \pi(dx, dy), \quad \pi \in \mathcal{C}(\mu_1, \mu_2) \right\},$$

where $\mathcal{C}(\mu_1, \mu_2)$ is the set of probability measures π on $M \times M$, having μ_1, μ_2 as two marginal laws. It is well known that $\mathbb{P}_2(M)$ endowed with W_2 is a Polish space. In this compact case, the weak convergence for probability measures is metrized by W_2 ; therefore $(\mathbb{P}_2(M), W_2)$ is a compact Polish space.

The introduction of tangent spaces of $\mathbb{P}_2(M)$ can go back to the early work [19], as well as in [18]. A more rigorous treatment was given in [2]. In differential geometry, for a smooth curve $\{c(t); t \in [0, 1]\}$ on a manifold M , the derivative $c'(t)$ with respect to the time t is in the tangent space : $c'(t) \in T_{c(t)}M$. A classical result says that for an absolutely continuous curve $\{c(t); t \in [0, 1]\}$ on M , the derivative $c'(t) \in T_{c(t)}M$ exists for almost all $t \in [0, 1]$.

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Following [2], we say that a curve $\{c(t); t \in [0, 1]\}$ on $\mathbb{P}_2(M)$ is absolutely continuous in L^2 if there exists $k \in L^2([0, 1])$ such that

$$W_2(c(t_1), c(t_2)) \leq \int_{t_1}^{t_2} k(s) ds, \quad t_1 < t_2.$$

The following result is our starting point:

Theorem 1.1 (see [2], Theorem 8.3.1). *Let $\{c_t; t \in [0, 1]\}$ be an absolutely continuous curve on $\mathbb{P}_2(M)$ in L^2 , then there exists a Borel vector field Z_t on M such that*

$$\int_{[0,1]} \left[\int_M |Z_t(x)|_{T_x M}^2 dc_t(x) \right] dt < +\infty$$

and the following continuity equation

$$\frac{dc_t}{dt} + \nabla \cdot (Z_t c_t) = 0, \quad (1.1)$$

holds in the sense of distribution. Uniqueness to (1.1) holds if moreover Z_t is imposed to be in

$$\overline{\{\nabla \psi, \psi \in C^\infty(M)\}}^{L^2(c_t)}.$$

In this work, we define the tangent space $\bar{\mathbf{T}}_\mu$ of $\mathbb{P}_2(M)$ at μ by

$$\bar{\mathbf{T}}_\mu = \overline{\{\nabla \psi, \psi \in C^\infty(M)\}}^{L^2(\mu)}, \quad (1.2)$$

the closure of gradients of smooth functions in the space $L^2(\mu)$. Equation (1.1) implies that for almost all $t \in [0, 1]$,

$$\frac{d}{dt} \int_M f(x) dc_t(x) = \int_M \langle \nabla f(x), Z_t(x) \rangle_{T_x M} dc_t(x), \quad f \in C^1(M). \quad (1.3)$$

We will say that Z_t is the intrinsic derivative of c_t and use the notation

$$\frac{d^I c_t}{dt} = Z_t \in \bar{\mathbf{T}}_{c_t}.$$

In what follows, we will describe the tangent space $\bar{\mathbf{T}}_\mu$ with the least conditions as possible on the measure μ . Consider the quadratic form defined by

$$\mathcal{E}(\psi) = \int_M |\nabla \psi(x)|^2 d\mu(x), \quad \psi \in C^1(M).$$

We assume that there is a constant $C_\mu > 0$ such that

$$\int_M (\psi - \langle \psi \rangle)^2 d\mu \leq C_\mu \int_M |\nabla \psi|^2 d\mu, \quad (1.4)$$

where $\langle \psi \rangle = \int_M \psi(x) dx$. The condition (1.4) is satisfied if μ admits a positive density $\rho > 0$: $d\mu = \rho dx$. In fact, let

$$\beta_1 = \inf_{x \in M} \rho(x) > 0, \quad \beta_2 = \sup_{x \in M} \rho(x) < +\infty.$$

Since M is compact, the following Poincaré inequality holds :

$$\int_M (\psi - \langle \psi \rangle)^2 dx \leq C \int_M |\nabla \psi|^2 dx,$$

then

$$\int_M (\psi - \langle \psi \rangle)^2 d\mu \leq \frac{C\beta_2}{\beta_1} \int_M |\nabla \psi|^2 d\mu.$$

Now let $Z \in \bar{\mathbf{T}}_\mu$; there is a sequence of functions $\psi_n \in C^\infty(M)$ such that $Z = \lim_{n \rightarrow +\infty} \nabla \psi_n$ in $L^2(\mu)$. By changing ψ_n to $\psi_n - \langle \psi_n \rangle$ and by condition (1.4), $\{\psi_n; n \geq 1\}$ is a Cauchy sequence in $L^2(\mu)$. If the quadratic form $\mathcal{E}(\psi)$ is closable in $L^2(\mu)$, then there exists a function φ_μ in the Sobolev space $\mathbb{D}_1^2(\mu)$ such that $Z = \nabla \varphi_\mu$, where $\mathbb{D}_1^2(\mu)$ is the closure of $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{\mathbb{D}_1^2(\mu)}^2 := \int_M |\varphi(x)|^2 d\mu(x) + \int_M |\nabla \varphi(x)|^2 d\mu(x).$$

A sufficient condition to insure the closability for \mathcal{E} is that the formula of integration by parts holds for μ ; more precisely, for any C^1 vector field Z on M , there exists a function denoted by $\operatorname{div}_\mu(Z) \in L^2(\mu)$ such that

$$\int_M \langle \nabla f(x), Z(x) \rangle_{T_x M} d\mu(x) = - \int_M f(x) \operatorname{div}_\mu(Z)(x), \quad f \in C^1(M). \quad (1.5)$$

Definition 1.2. We say that the measure μ is a measure having divergence if $\operatorname{div}_\mu(Z) \in L^2(\mu)$ exists. We will use the notation

$$\mathbb{P}_{\operatorname{div}}(M)$$

to denote the set of probability measures on M having strictly positive continuous density and satisfying conditions (1.5).

Proposition 1.3. For a measure $\mu \in \mathbb{P}_{\operatorname{div}}(M)$, we have

$$\bar{\mathbf{T}}_\mu = \{\nabla \psi; \psi \in \mathbb{D}_1^2(\mu)\}.$$

The inconvenient for (1.3) is the existence of derivative for almost all $t \in [0, 1]$. In what follows, we will present two typical classes of absolutely continuous curves in $\mathbb{P}_2(M)$.

1.1 Constant vector fields on $\mathbb{P}_2(M)$

For any gradient vector field $\nabla \psi$ on M with $\psi \in C^\infty(M)$, consider the ordinary differential equation (ODE):

$$\frac{d}{dt} U_t(x) = \nabla \psi(U_t(x)), \quad U_0(x) = x \in M.$$

Then $x \rightarrow U_t(x)$ is a flow of diffeomorphisms on M . Let $\mu \in \mathbb{P}_2(M)$, consider $c_t = (U_t)_\# \mu$. It is easy to see that the curve $\{c_t; t \in [0, 1]\}$ is absolutely continuous in L^2 and for $f \in C^1(M)$,

$$\frac{d}{dt} \int_M f(x) dc_t(x) = \frac{d}{dt} \int_M f(U_t(x)) d\mu(x) = \int_M \langle \nabla f(U_t(x)), \nabla \psi(U_t(x)) \rangle d\mu(x),$$

which is equal to, for any $t \in [0, 1]$,

$$\int_M \langle \nabla f, \nabla \psi \rangle dc_t.$$

In other term, c_t is a solution to the following continuity equation:

$$\frac{dc_t}{dt} + \nabla \cdot (\nabla \psi c_t) = 0.$$

According to above definition, we see that for each $t \in [0, 1]$,

$$\frac{d^l c_t}{dt} = \nabla \psi.$$

It is why we call $\nabla \psi$ a constant vector field on $\mathbb{P}_2(M)$. In order to make clearly different roles played by $\nabla \psi$, we will use notation

$$V_\psi$$

when it is seen as a constant vector field on $\mathbb{P}_2(M)$.

Remark 1.4. In section 3 below, we will compute Lie brackets of two constant vector fields on $\mathbb{P}_2(M)$ without explicitly using the existence of density of measure, the Lie bracket of two constant vector fields is NOT a constant vector field.

1.2 Geodesics with constant speed

It is easy to introduce geodesics with constant speed when the base space is a flat space \mathbb{R}^m . A probability measure μ on \mathbb{R}^m is in $\mathbb{P}_2(\mathbb{R}^m)$ if $\int_{\mathbb{R}^m} |x|^2 d\mu(x) < +\infty$. Let $c_0, c_1 \in \mathbb{P}_2(\mathbb{R}^m)$, there is an optimal coupling plan $\gamma \in \mathcal{C}(c_0, c_1)$ such that

$$W_2^2(c_0, c_1) = \int_{\mathbb{R}^m \times \mathbb{R}^m} |x - y|^2 d\gamma(x, y).$$

For each $t \in [0, 1]$, define $c_t \in \mathbb{P}_2(\mathbb{R}^m)$ by

$$\int_{\mathbb{R}^m} f(x) dc_t(x) = \int_{\mathbb{R}^m \times \mathbb{R}^m} f(u_t(x, y)) d\gamma(x, y),$$

where $u_t(x, y) = (1 - t)x + ty$. For $0 \leq s < t \leq 1$, define $\pi_{s,t} \in \mathcal{C}(c_s, c_t)$ by

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} g(x, y) d\pi_{s,t}(x, y) = \int_{\mathbb{R}^m \times \mathbb{R}^m} g(u_s(x, y), u_t(x, y)) d\gamma(x, y).$$

Then

$$W_2^2(c_s, c_t) \leq \int_{\mathbb{R}^m \times \mathbb{R}^m} |u_t(x, y) - u_s(x, y)|^2 d\gamma(x, y) = (t - s)^2 W_2^2(c_0, c_1)^2.$$

It follows that $W_2(c_s, c_t) \leq (t - s)W_2(c_0, c_1)$. Combing with triangulaire inequality,

$$\begin{aligned} W_2(c_0, c_1) &\leq W_2(c_0, c_s) + W_2(c_s, c_t) + W_2(c_t, c_1) \\ &\leq sW_2(c_0, c_1) + (t - s)W_2(c_0, c_1) + (1 - t)W_2(c_0, c_1) = W_2(c_0, c_1), \end{aligned}$$

we get the property of geodesic with constant speed:

$$W_2(c_s, c_t) = |t - s| W_2(c_0, c_1).$$

According to Theorem 1.1, there is $Z_t \in \bar{\mathbf{T}}_{c_t}$ such that, for $f \in C_c^1(\mathbb{R}^d)$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^m} f(x) dc_t(x) &= \int_{\mathbb{R}^m} \langle \nabla f(u_t(x, y)), y - x \rangle_{\mathbb{R}^m} d\gamma(x, y) \\ &= \int_{\mathbb{R}^d} \langle \nabla f(x), Z_t(x) \rangle_{\mathbb{R}^m} dc_t(x) \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^m}$ is the canonical inner product of \mathbb{R}^m . We heuristically look for Z_t such that

$$Z_t(u_t(x, y)) = y - x.$$

Taking the derivative with respect to t yields

$$\left(\frac{d}{dt} Z_t\right)(u_t(x, y)) + \langle \nabla Z_t(u_t(x, y)), y - x \rangle = 0.$$

It follows that

$$\left(\frac{d}{dt} Z_t\right) + \nabla Z_t(Z_t) = 0.$$

In the case where $Z_t = \nabla \psi_t$, we have

$$\left(\frac{d}{dt} \nabla \psi_t\right) + \nabla^2 \psi_t(\nabla \psi_t) = 0.$$

We remark that $\{\nabla \psi_t, t \in]0, 1[\}$ satisfies heuristically the equation of Riemannian geodesic obtained in [14] or heuristically obtained in [19], in which the authors showed that the convexity of entropy functional along these geodesics is equivalent to Bakry-Emery's curvature condition [3] (see also [12], [21, 20]).

In the case of Riemannian manifold M , it is a bit complicated. We follow the exposition of [10]. Let TM be the tangent bundle of M and $\pi : TM \rightarrow M$ the natural projection. For each $\mu \in \mathbb{P}_2(M)$, we consider the set

$$\Gamma_\mu = \left\{ \gamma \text{ probability measure on } TM; \pi_{\#} \gamma = \mu, \int_{TM} |v|_{T_x M}^2 d\gamma(x, v) < +\infty \right\}.$$

The set Γ_μ is obviously non empty. Let $\gamma \in \Gamma_\mu$, we consider $\nu = \exp_{\#} \gamma$, that is,

$$\int_M f(x) d\nu(x) = \int_{TM} f(\exp_x(v)) d\gamma(x, v),$$

where $\exp_x : T_x M \rightarrow M$ is the exponential map induced by geodesics on M . The map

$$TM \rightarrow M \times M, \quad (x, v) \rightarrow (x, \exp_x(v))$$

sends γ to a coupling plan $\tilde{\gamma} \in \mathcal{C}(\mu, \nu)$. We have

$$W_2^2(\mu, \nu) \leq \int_{TM} d_M^2(x, \exp_x(v)) d\gamma(x, v) \leq \int_{TM} |v|_{T_x M}^2 d\gamma(x, v).$$

In order to construct geodesics $\{c_t; t \in [0, 1]\}$ connecting μ and ν , we need to find $\gamma_0 \in \Gamma_\mu$ such that

$$W_2^2(\mu, \nu) = \int_{TM} |v|_{T_x M}^2 d\gamma_0(x, v). \quad (1.6)$$

As M is connected, let $x \in M$, for each y , there is a minimizing geodesic $\{\xi(t), t \in [0, 1]\}$ connecting x and y . Let $v_{x,y} = \xi'(0) \in T_x M$, then

$$y = \exp_x(v_{x,y}) \text{ and } d_M(x, y) = |v_{x,y}|_{T_x M}.$$

Take a Borel version Ξ of such a map $(x, y) \rightarrow (x, v_{x,y})$ from $M \times M$ to TM . Let $\tilde{\gamma}_0 \in \mathcal{C}(\mu, \nu)$ be an optimal coupling plan; define $\gamma_0 \in \Gamma_\mu$ by

$$\int_{TM} g(x, v) d\gamma_0(x, v) = \int_{M \times M} g(x, \Xi(x, y)) d\tilde{\gamma}_0(x, y).$$

Therefore

$$\begin{aligned} \int_{TM} |v|_{T_x M}^2 d\gamma_0(x, v) &= \int_{M \times M} |\Xi(x, y)|^2 d\tilde{\gamma}_0(x, y) \\ &= \int_{M \times M} d_M(x, y)^2 d\tilde{\gamma}_0(x, y) = W_2^2(\mu, \nu). \end{aligned}$$

Now we define the curve $\{c_t; t \in [0, 1]\}$ on $\mathbb{P}_2(M)$ by

$$\int_M f(x) dc_t(x) = \int_{TM} f(\exp_x(tv)) d\gamma_0(x, v).$$

Similarly we check that

$$W_2(c_s, c_t) = |t - s| W_2(c_0, c_1).$$

The organization of the paper is as follows. In Section 2, we consider ordinary equations on $\mathbb{P}_2(M)$, a Cauchy-Peano's type theorem is established, also McKean-Vlasov equation involved. In Section 3, we emphasize that the suitable class of probability measures for developing the differential geometry is one having divergence and the strictly positive density with certain regularity. The Levi-Civita connection is introduced and the formula for the covariant derivative of a general but smooth enough vector field is obtained. In section 4, we precise results on the derivability of the Wasserstein distance on $\mathbb{P}_2(M)$, which enable us to obtain the extension of a vector field along a quite good curve on $\mathbb{P}_2(M)$ in Section 5 as in differentiable geometry; the parallel translation along such a good curve on $\mathbb{P}_2(M)$ is naturally and rigorously introduced. The existence for parallel translations is established for a curve whose intrinsic derivative gives rise a good enough vector field on $\mathbb{P}_2(M)$.

2 Ordinary differential equations on $\mathbb{P}_2(M)$

Let $\varphi \in C^1(M)$, consider the function F_φ on $\mathbb{P}_2(M)$ defined by

$$F_\varphi(\mu) = \int_M \varphi(x) d\mu(x). \tag{2.1}$$

A function F on $\mathbb{P}_2(M)$ is said to be a polynomial if there exists a finite number of functions $\varphi_1, \dots, \varphi_k$ in $C^1(M)$ such that $F = F_{\varphi_1} \cdots F_{\varphi_k}$. Let $Z = V_\psi$ be a constant vector field on $\mathbb{P}_2(M)$ with $\psi \in C^\infty(M)$, and U_t the flow on M associated to $\nabla\psi$. For $\mu_0 \in \mathbb{P}_2(M)$, we set $\mu_t = (U_t)_\# \mu_0$. Then we have seen in section 1.1,

$$\left\{ \frac{d}{dt} F_\varphi(\mu_t) \right\}_{|t=0} = \int_M \langle \nabla\varphi(x), \nabla\psi(x) \rangle d\mu_0(x) = \langle V_\varphi, V_\psi \rangle_{\mathbb{T}\mu_0}.$$

The left hand side of above equality is the derivative of F_φ along V_ψ . More generally, for a function F on $\mathbb{P}_2(M)$, we say that F is derivable at μ_0 along V_ψ , if

$$(\bar{D}_{V_\psi} F)(\mu_0) = \left\{ \frac{d}{dt} F(\mu_t) \right\}_{|_{t=0}} \text{ exists.}$$

We say that the gradient $\bar{\nabla} F(\mu_0) \in \bar{\mathbf{T}}_{\mu_0}$ exists if for each $\psi \in C^\infty(M)$, $(\bar{D}_{V_\psi} F)(\mu_0)$ exists and

$$\bar{D}_{V_\psi} F(\mu_0) = \langle \bar{\nabla} F, V_\psi \rangle_{\bar{\mathbf{T}}_{\mu_0}}. \quad (2.2)$$

Note that for $\varphi \in C^1(M)$, there is a sequence of $\psi_n \in C^\infty(M)$ such that $\nabla \psi_n$ converge uniformly to $\nabla \varphi$ so that $V_\varphi \in \bar{\mathbf{T}}_\mu$ for any $\mu \in \mathbb{P}_2(M)$. It is obvious that $\bar{\nabla} F_\varphi = V_\varphi$. For the polynomial $F = \prod_{i=1}^k F_{\varphi_i}$, we have

$$\bar{\nabla} F = \sum_{i=1}^k \left(\prod_{j \neq i} F_{\varphi_j} \right) V_{\varphi_i}.$$

Note that the family $\{F_\varphi, \varphi \in C^1(M)\}$ separates the point of $\mathbb{P}_2(M)$. By Stone-Weierstrauss theorem, the space of polynomials is dense in the space of continuous functions on $\mathbb{P}_2(M)$.

Convention of notations: We will use ∇ to denote the gradient operator on the base space M , and $\bar{\nabla}$ to denote the gradient operator on the Wasserstein space $(\mathbb{P}_2(M), W_2)$. For example, if $(\mu, x) \rightarrow \Phi(\mu, x)$ is a function on $\mathbb{P}_2(M) \times M$, then $\nabla \Phi(\mu, x)$ is the gradient with respect to x , while $\bar{\nabla} \Phi(\mu, x)$ is the gradient with respect to μ .

Definition 2.1. We will say that Z is a vector field on $\mathbb{P}_2(M)$ if there exists a Borel map $\Phi : \mathbb{P}_2(M) \times M \rightarrow \mathbb{R}$ such that for any $\mu \in \mathbb{P}_2(M)$, $x \rightarrow \Phi(\mu, x)$ is C^1 and $Z(\mu) = V_{\Phi(\mu, \cdot)}$.

A class of test vector fields on $\mathbb{P}_2(M)$ is

$$\chi(\mathbb{P}) = \left\{ \sum_{\text{finite}} \alpha_i V_{\psi_i}, \quad \alpha_i \text{ polynomial, } \psi_i \in C^\infty(M) \right\}. \quad (2.3)$$

Let Z be a vector field on $\mathbb{P}_2(M)$, how to construct a solution $\mu_t \in \mathbb{P}_2(M)$ to the following ODE

$$\frac{d^I \mu_t}{dt} = Z(\mu_t)?$$

Theorem 2.2. Let Z be a vector field on $\mathbb{P}_2(M)$ given by Φ . Assume that $(\mu, x) \rightarrow \nabla \Phi(\mu, x)$ is continuous, then for any $\mu_0 \in \mathbb{P}_2(M)$, there is an absolutely curve $\{\mu_t; t \in [0, 1]\}$ on $\mathbb{P}_2(M)$ such that

$$\frac{d^I \mu_t}{dt} = Z(\mu_t), \quad \mu_{|_{t=0}} = \mu_0. \quad (2.4)$$

If moreover, for any $\mu \in \mathbb{P}_2(M)$, $x \rightarrow \Phi(\mu, x)$ is C^2 and

$$C_2 := \sup_{\mu \in \mathbb{P}_2(M)} \sup_{x \in M} \|\nabla^2 \Phi(\mu, x)\| < +\infty, \quad (2.5)$$

then there is a flow of continuous maps $(t, x) \rightarrow U_t(x)$ on M , solution to the following McKean-Vlasov equation

$$\frac{d}{dt} U_t(x) = \nabla \Phi(\mu_t, U_t(x)), \quad \mu_t = (U_t)_\# \mu_0. \quad (2.6)$$

Proof. We use the Euler approximation to construct a solution. We first note that

$$C_1 := \sup_{(\mu, x) \in \mathbb{P}_2(M) \times M} |\nabla \Phi(\mu, x)| < +\infty. \quad (2.7)$$

Let $P_t = e^{t\Delta_M}$ be the heat semi-group associated to the Laplace operator Δ_M on functions, and $\mathbf{T}_t = e^{-t\Box}$ the heat semigroup on differential forms, with de Rham-Hodge operator \Box . It is well-known that

$$|\mathbf{T}_t(\nabla\varphi)| \leq e^{-t\kappa/2} P_t |\nabla\varphi|, \quad \varphi \in C^1(M)$$

where κ is lower bound of Ricci tensor on M . As $t \rightarrow 0$, $\mathbf{T}_t(\nabla\varphi)$ converges to $\nabla\varphi$ uniformly. For $n \geq 1$, let

$$Z_n(\mu, x) = (\mathbf{T}_{1/n} \nabla \Phi(\mu, \cdot))(x).$$

According to (2.7) and above estimate, for n big enough,

$$\sup_{(\mu, x) \in \mathbb{P}_2(M) \times M} |Z_n(\mu, x)| \leq 2C_1. \quad (2.8)$$

Now let $t_k = k2^{-n}$ for $k = 1, \dots, 2^n$ and

$$[t] = t_k \quad \text{if } t \in [t_k, t_{k+1}[.$$

On the intervall $[t_0, t_1]$, consider the ODE on M :

$$\frac{dU_t^{(n)}}{dt} = Z_n(\mu_0, U_t^{(n)}), \quad U_0^{(n)}(x) = x, \quad (2.9)$$

and $\mu_t^{(n)} = (U_t^{(n)})_{\#} \mu_0$ for $t \in [t_0, t_1]$; inductively, on $[t_k, t_{k+1}]$, we consider

$$\frac{dU_t^{(n)}}{dt} = Z_n(\mu_{t_k}^{(n)}, U_t^{(n)}), \quad U_{|t=t_k}^{(n)}(x) = U_{t_k}^{(n)}(x), \quad (2.10)$$

and for $t \in [t_k, t_{k+1}]$,

$$\mu_t^{(n)} = (U_t^{(n)})_{\#} \mu_{t_k}^{(n)} \quad (2.11)$$

and so on, we get a curve $\{\mu_t^{(n)}; t \in [0, 1]\}$ on $\mathbb{P}_2(M)$. We now prove that this family is equicontinuous in $C([0, 1], \mathbb{P}_2(M))$. Let $0 \leq s < t \leq 1$, define $\gamma(\theta) = U_{(1-\theta)s+\theta t}^{(n)}$, then

$$\frac{d\gamma(\theta)}{d\theta} = (t-s)Z_n(\mu_{[(1-\theta)s+\theta t]}^{(n)}, U_{(1-\theta)s+\theta t}^{(n)}).$$

We have, according to (2.8),

$$d_M(U_t^{(n)}(x), U_s^{(n)}(x)) \leq \int_0^1 \left| \frac{d\gamma(\theta)}{d\theta} \right| d\theta \leq 2C_1(t-s).$$

Define a probability measure π on $M \times M$ by

$$\int_{M \times M} g(x, y) \pi(dx, dy) = \int_M g(U_t^{(n)}(x), U_s^{(n)}(x)) d\mu_0(x).$$

Then $\pi \in \mathcal{C}(\mu_t^{(n)}, \mu_s^{(n)})$, we have

$$W_2^2(\mu_t^{(n)}, \mu_s^{(n)}) \leq \int_M d_M^2(U_t^{(n)}(x), U_s^{(n)}(x)) d\mu_0(x) \leq 4C_1^2(t-s)^2.$$

By Ascoli theorem, up to a subsequence, $\mu_t^{(n)}$ converges in $C([0, 1], \mathbb{P}_2(M))$ to a continuous curve $\{\mu_t; t \in [0, 1]\}$ such that $W_2(\mu_t, \mu_s) \leq 2C_1(t-s)$.

For proving that $\{\mu_t; t \in [0, 1]\}$ is a solution to ODE (2.4), we need the following preparation:

Lemma 2.3. *Set $\Phi_\mu(x) = \Phi(\mu, x)$, then*

$$\sup_{(\mu, x) \in \mathbb{P}_2(M) \times M} |(\mathbf{T}_t \nabla \Phi_\mu)(x) - \nabla \Phi(x)|_{T_x M} \rightarrow 0, \quad \text{as } t \rightarrow 0. \quad (2.12)$$

Proof. We use $\|\cdot\|_\infty$ to denote the uniform norm on M . Let $\varepsilon > 0$, for $\mu \in \mathbb{P}_2(M)$, there is $\hat{t}_\mu > 0$ such that

$$\sup_{t \leq \hat{t}_\mu} \|\mathbf{T}_t \nabla \Phi_\mu - \nabla \Phi_\mu\|_\infty < \varepsilon.$$

Since $(\mu, t) \rightarrow \|\mathbf{T}_t \nabla \Phi_\mu - \nabla \Phi_\mu\|_\infty$ is continuous, there is $\delta_\mu > 0$ such that for $t \leq \hat{t}_\mu$,

$$W_2(\mu, \nu) < \delta_\mu \Rightarrow \|\mathbf{T}_t \nabla \Phi_\nu - \nabla \Phi_\nu\|_\infty < \varepsilon.$$

Let $B(\mu, \delta)$ be the open ball in $(\mathbb{P}_2(M), W_2)$ centered at μ , of radius δ . We have

$$\mathbb{P}_2(M) = \cup_{\mu \in \mathbb{P}_2(M)} B(\mu, \delta_\mu);$$

so there is a finite number of $\{\mu_1, \dots, \mu_K\}$ such that

$$\mathbb{P}_2(M) = \cup_{i=1}^K B(\mu_i, \delta_{\mu_i}).$$

Let $\hat{t} = \min\{\hat{t}_{\mu_i}, i = 1, \dots, K\} > 0$. Then for $0 < t < \hat{t}$,

$$\sup_{\mu \in \mathbb{P}_2(M)} \|\mathbf{T}_t \nabla \Phi_\mu - \nabla \Phi_\mu\|_\infty \leq \varepsilon.$$

So we get (2.12). □

End of the proof of theorem : $\{\mu_t^{(n)}; t \in [0, 1]\}$ satisfies the following continuity equation

$$\begin{aligned} & \int_{[0,1] \times M} \alpha'(t) f(x) d\mu_t^{(n)}(x) dt \\ &= \alpha(0) \int_M f(x) d\mu_0(x) + \int_{[0,1] \times M} \alpha(t) \langle \nabla f(x), Z_n(\mu_{[t]}^{(n)}, x) \rangle d\mu_t^{(n)}(x) dt, \end{aligned} \quad (2.13)$$

for all $\alpha \in C_c^1([0, 1])$ and $f \in C^1(M)$. We have

$$\begin{aligned} & \int_{[0,1] \times M} \alpha(t) \langle \nabla f(x), Z_n(\mu_{[t]}^{(n)}, x) \rangle d\mu_t^{(n)} dt - \int_{[0,1] \times M} \alpha(t) \langle \nabla f(x), \nabla \Phi(\mu_t, x) \rangle d\mu_t dt \\ &= \int_{[0,1] \times M} \alpha(t) \langle \nabla f(x), Z_n(\mu_{[t]}^{(n)}, x) - \nabla \Phi(\mu_t, x) \rangle d\mu_t^{(n)} dt \\ &+ \int_{[0,1] \times M} \alpha(t) \langle \nabla f(x), \nabla \Phi(\mu_t, x) \rangle d\mu_t^{(n)} dt - \int_{[0,1] \times M} \alpha(t) \langle \nabla f(x), \nabla \Phi(\mu_t, x) \rangle d\mu_t dt. \end{aligned}$$

It is obvious that the sum of two last terms converge to 0 as $n \rightarrow +\infty$. Let I_n be the first term on the right side, then

$$|I_n| \leq \|\nabla f\|_\infty \int_0^1 |\alpha(t)| \|\mathbf{T}_{1/n} \nabla \Phi_{\mu_{[t]}^{(n)}} - \nabla \Phi_{\mu_t}\|_\infty dt$$

Note that

$$\|\mathbf{T}_{1/n} \nabla \Phi_{\mu_{[t]}^{(n)}} - \nabla \Phi_{\mu_t}\|_\infty \leq \|\mathbf{T}_{1/n} \nabla \Phi_{\mu_{[t]}^{(n)}} - \nabla \Phi_{\mu_{[t]}^{(n)}}\|_\infty + \|\nabla \Phi_{\mu_{[t]}^{(n)}} - \nabla \Phi_{\mu_t}\|_\infty.$$

The term $\|\mathbf{T}_{1/n} \nabla \Phi_{\mu_{[t]}^{(n)}} - \nabla \Phi_{\mu_{[t]}^{(n)}}\|_\infty \rightarrow 0$ is due to above lemma. As $n \rightarrow +\infty$, $\mu_{[t]}^{(n)}$ converges to μ_t . By continuity of $(\mu, x) \rightarrow \nabla \Phi(\mu, x)$, the last term tends to 0. Letting $n \rightarrow +\infty$ in (2.13) yields

$$\begin{aligned} & \int_{[0,1] \times M} \alpha'(t) f(x) d\mu_t(x) dt \\ &= \alpha(0) \int_M f(x) d\mu_0(x) + \int_{[0,1] \times M} \alpha(t) \langle \nabla f(x), \nabla \Phi(\mu_t, x) \rangle d\mu_t(x) dt, \end{aligned}$$

which is the meaning of Equation (2.4) in distribution sense.

For the proof of second part, since $x \rightarrow \Phi(\mu, x)$ is C^2 , we can directly use $\nabla \Phi(\mu, \cdot)$ instead of Z_n in (2.9), (2.10), (2.11).

On the interval $[t_0, t_1]$, consider the ODE on M :

$$\frac{dU_t^{(n)}}{dt} = \nabla \Phi(\mu_0, U_t^{(n)}), \quad U_0^{(n)}(x) = x, \quad (2.14)$$

and $\mu_t^{(n)} = (U_t^{(n)})_{\#} \mu_0$ for $t \in [t_0, t_1]$; inductively, on $[t_k, t_{k+1}]$, we consider

$$\frac{dU_t^{(n)}}{dt} = \nabla \Phi(\mu_{t_k}^{(n)}, U_t^{(n)}), \quad U_{|t=t_k}^{(n)}(x) = U_{t_k}^{(n)}(x), \quad (2.15)$$

and for $t \in [t_k, t_{k+1}]$,

$$\mu_t^{(n)} = (U_t^{(n)})_{\#} \mu_{t_k}^{(n)}. \quad (2.16)$$

By above result, up to a subsequence, $\{\mu_t^{(n)}, t \in [0, 1]\}$ converges to $\{\mu_t, t \in [0, 1]\}$ in $C([0, 1], \mathbb{P}_2(M))$. We use this subsequence to prove the convergence of $\{U_t^{(n)}(x), t \in [0, 1]\}$. Now we prove that, under Condition (2.7),

$$d_M(U_t^{(n)}(x), U_t^{(n)}(y)) \leq e^{C_2 t} d_M(x, y), \quad x, y \in M. \quad (2.17)$$

For $x, y \in M$ given, there is a minimizing geodesic $\{\xi_s, s \in [0, 1]\}$ connecting x and y such that $d_M(x, y) = \int_0^1 |\xi'_s| ds$. Set

$$\sigma(t, s) = U_t^{(n)}(\xi_s).$$

Since the torsion is free, we have the relation:

$$\frac{D}{ds} \frac{d}{dt} \sigma(t, s) = \frac{D}{dt} \frac{d}{ds} \sigma(t, s), \quad (2.18)$$

where $\frac{D}{ds}$ denotes the covariant derivative. We have

$$\frac{d}{dt}U_t^{(n)}(\xi_s) = \nabla\Phi\left(\mu_{[t]}^{(n)}, U_t^{(n)}(\xi_s)\right).$$

Taking the derivative with respect to s , we get

$$\frac{D}{ds}\frac{d}{dt}U_t^{(n)}(\xi_s) = \nabla^2\Phi\left(\mu_{[t]}^{(n)}, U_t^{(n)}(\xi_s)\right) \cdot \frac{d}{ds}U_t^{(n)}(\xi_s).$$

Combining with (2.18) yields

$$\frac{D}{ds}\frac{d}{dt}U_t^{(n)}(\xi_s) = \nabla^2\Phi\left(\mu_{[t]}^{(n)}, U_t^{(n)}(\xi_s)\right) \cdot \frac{d}{ds}U_t^{(n)}(\xi_s).$$

Now,

$$\frac{d}{dt}\left|\frac{d}{ds}U_t^{(n)}(\xi_s)\right|^2 = 2\left\langle\nabla^2\Phi\left(\mu_{[t]}^{(n)}, U_t^{(n)}(\xi_s)\right) \cdot \frac{d}{ds}U_t^{(n)}(\xi_s), \frac{d}{ds}U_t^{(n)}(\xi_s)\right\rangle,$$

which is, by Condition (2.7), less than

$$2C_2\left|\frac{d}{ds}U_t^{(n)}(\xi_s)\right|^2.$$

By Gronwall lemma,

$$\left|\frac{d}{ds}U_t^{(n)}(\xi_s)\right| \leq e^{C_2t}|\xi'_s|,$$

which implies that

$$d_M\left(U_t^{(n)}(x), U_t^{(n)}(y)\right) \leq e^{C_2t}d_M(x, y).$$

Therefore the family $\{(t, x) \rightarrow U_t^{(n)}(x); n \geq 1\}$ is equicontinuous in $C([0, 1] \times M)$. By Ascoli theorem, up to a subsequence, $U_t^{(n)}(x)$ converges to $U_t(x)$ uniformly in $(t, x) \in [0, 1] \times M$. It is obvious to see that (U_t, μ_t) solves McKean-Vlasov equation (2.6). \square

Remark 2.4. Comparing to [5], as well to [24], we did not suppose the Lipschitz continuity with respect to μ ; in counterpart, we have no uniqueness of solutions of (2.6).

Remark 2.5. Many interesting PDE can be interpreted as gradient flows on the Wasserstein space $\mathbb{P}_2(M)$ (see [2], [22],[23], [9]). The interpolation between geodesic flows and gradient flows were realized using Langevin's deformation in [12, 13].

3 Levi-Civita connection on $\mathbb{P}_2(M)$

In this section, we will revisit the paper by J. Lott [14]: we try to reformulate conditions given there as weak as possible, also to expose some of them in an intrinsic way, avoiding the use of density. In order to obtain good pictures on the geometry of $\mathbb{P}_2(M)$, the suitable class of probability measures should be the class $\mathbb{P}_{\text{div}}(M)$ of probability measures on M having divergence (see Definition 1.2).

For convenience of readers, we will briefly prepare materials needed for our exposition. For a measure $\mu \in \mathbb{P}_2(M)$, for any C^1 vector field A on M , the divergence $\text{div}_\mu(A) \in L^2(M, \mu)$ is such that

$$\int_M \langle \nabla\phi(x), A(x) \rangle_{T_x M} d\mu(x) = - \int_M \phi(x) \text{div}_\mu(A)(x) d\mu(x)$$

for any $\phi \in C^1(M)$. It is easy to see that $\operatorname{div}_\mu(fA) = f \operatorname{div}_\mu(A) + \langle \nabla f, A \rangle$ for $f \in C^1(M)$. If $d\mu = \rho dx$ has a density $\rho > 0$ in the space $C^1(M)$, we have

$$\int_M \langle \nabla \phi, A \rangle d\mu = \int_M \langle \nabla \phi, \rho A \rangle dx = - \int_M \phi \operatorname{div}(\rho A) dx = - \int_M \phi \operatorname{div}(\rho A) \rho^{-1} d\mu,$$

It follows that

$$\operatorname{div}_\mu(A) = \rho^{-1} \operatorname{div}(\rho A) = \operatorname{div}(A) + \langle \nabla(\log \rho), A \rangle. \quad (3.1)$$

For $\mu \in \mathbb{P}_{\operatorname{div}}(M)$ and $\phi \in C^2(M)$, we denote $\mathcal{L}^\mu(\phi) \in L^2(\mu)$ such that

$$\int_M \langle \nabla f, \nabla \phi \rangle d\mu = - \int_M f \mathcal{L}^\mu \phi d\mu, \quad \text{for any } f \in C^1(M), \quad (3.2)$$

where $\mathcal{L}^\mu \phi = \operatorname{div}_\mu(\nabla \phi)$ is a negative operator.

Let $\psi \in C^3(M)$, consider the ODE

$$\frac{dU_t}{dt} = \nabla \psi(U_t), \quad U_0(x) = x.$$

Proposition 3.1. *Let $d\mu = \rho dx$ be a probability measure in $\mathbb{P}_{\operatorname{div}}(M)$ with a strictly positive density ρ in $C^1(M)$ and $\psi \in C^3(M)$. Then for each $t \in [0, 1]$, $\mu_t := (U_t)_\# \mu \in \mathbb{P}_{\operatorname{div}}(M)$.*

Proof. By Kunita [11] (see also [7], [17]), the push-forward measure $(U_t^{-1})_\# \mu$ by inverse map of U_t admits a density \tilde{K}_t with respect to μ , having the following explicit expression

$$\tilde{K}_t = \exp\left(- \int_0^t \operatorname{div}_\mu(\nabla \psi)(U_s(x)) ds\right).$$

It follows that the density K_t of μ_t with respect to μ has the expression

$$K_t = \exp\left(\int_0^t \operatorname{div}_\mu(\nabla \psi)(U_{-s}(x)) ds\right).$$

According to (3.1), $x \rightarrow \operatorname{div}_\mu(\nabla \psi(x))$ is C^1 . Therefore the condition in [7]

$$\int_M \exp(\lambda \operatorname{div}_\mu(\nabla \psi(x))) d\mu(x) < +\infty, \quad \text{for all } \lambda > 0$$

is automatically satisfied. Again by (3.1), $x \rightarrow K_t(x)$ is in C^1 . Now let A be a C^1 vector field on M and $f \in C^1(M)$, we have

$$\int_M \langle \nabla f(x), A(x) \rangle_{T_x M} d\mu_t(x) = \int_M \langle \nabla f, A \rangle_{T_x M} K_t(x) d\mu(x) = - \int_M f \operatorname{div}_\mu(K_t Z) d\mu.$$

It follows that

$$\operatorname{div}_{\mu_t}(A) = \operatorname{div}_\mu(K_t A) K_t^{-1}.$$

□

For $\psi_1, \psi_2 \in C^2(M)$, we denote by V_{ψ_1}, V_{ψ_2} the associated constant vector fields on $\mathbb{P}_2(M)$. In what follows, we will compute the Lie bracket $[V_{\psi_1}, V_{\psi_2}]$.

For $f \in C^1(M)$, we set $F_f(\mu) = \int_M f d\mu$. According to preparations given at the beginning of Section 2,

$$(\bar{D}_{V_{\psi_2}} F_f)(\mu) = \int_M \langle \nabla \psi_2, \nabla f \rangle d\mu = F_{\langle \nabla \psi_2, \nabla f \rangle}(\mu).$$

Using again above formula, we have

$$(\bar{D}_{V_{\psi_1}} \bar{D}_{V_{\psi_2}} F_f)(\mu) = \int_M \langle \nabla \psi_1, \nabla \langle \nabla \psi_2, \nabla f \rangle \rangle d\mu = - \int_M \mathcal{L}^\mu \psi_1 \langle \nabla \psi_2, \nabla f \rangle d\mu.$$

Therefore

$$\begin{aligned} [V_{\psi_2}, V_{\psi_1}] F_f &= \bar{D}_{V_{\psi_2}} \bar{D}_{V_{\psi_1}} F_f - \bar{D}_{V_{\psi_1}} \bar{D}_{V_{\psi_2}} F_f \\ &= \int_M \langle (\mathcal{L}^\mu \psi_1 \nabla \psi_2 - \mathcal{L}^\mu \psi_2 \nabla \psi_1), \nabla f \rangle d\mu. \end{aligned}$$

Let

$$\mathcal{C}_{\psi_1, \psi_2}(\mu) = \mathcal{L}^\mu \psi_1 \nabla \psi_2 - \mathcal{L}^\mu \psi_2 \nabla \psi_1. \quad (3.3)$$

Note that $\mathcal{C}_{\psi_1, \psi_2}(\mu)$ is in $L^2(M, TM; \mu)$, not in $\bar{\mathbf{T}}_\mu$. Consider the orthogonal projection:

$$\Pi_\mu : L^2(M, TM; \mu) \rightarrow \bar{\mathbf{T}}_\mu.$$

As $\mu \in \mathbb{P}_{div}(M)$ and by Proposition 1.3, there exists $\tilde{\Phi}_\mu \in \mathbb{D}_1^2(\mu)$ such that

$$\Pi_\mu(\mathcal{C}_{\psi_1, \psi_2}(\mu)) = \nabla \tilde{\Phi}_\mu. \quad (3.4)$$

Then we have

$$[V_{\psi_2}, V_{\psi_1}] F_f = \int_M \langle \nabla \tilde{\Phi}_\mu, \nabla f \rangle d\mu = (\bar{D}_{V_{\tilde{\Phi}_\mu}} F_f)(\mu). \quad (3.5)$$

Above equality can be extended to the class of polynomials on $\mathbb{P}_2(M)$, that is to say that

$$[V_{\psi_2}, V_{\psi_1}]_\mu = V_{\tilde{\Phi}_\mu} \quad \text{on polynomials,} \quad (3.6)$$

We emphasize that Lie bracket of two constant vector fields is no more a constant vector field.

Proposition 3.2. *Let $\psi_1, \psi_2 \in C^3(M)$, for $d\mu = \rho dx$ with $\rho > 0$ and $\rho \in C^2(M)$, the function $\tilde{\Phi}_\mu$ obtained in (3.4) has the following expression :*

$$\tilde{\Phi}_\mu = (\mathcal{L}^\mu)^{-1} \operatorname{div}_\mu(\mathcal{C}_{\psi_1, \psi_2}(\mu)). \quad (3.7)$$

Proof. By (3.1),

$$\mathcal{L}^\mu \psi = \Delta_M \psi + \langle \nabla \log \rho, \nabla \psi \rangle,$$

where Δ_M denotes the Laplace operator on M . It is well-known that \mathcal{L}^μ has a spectral gap if $\log \rho \in C^2(M)$. In [14], the Lie bracket $[V_{\psi_2}, V_{\psi_1}]$ was expressed using Hodge decomposition for vector fields in $L^2(\mu)$. For $\psi_1, \psi_2 \in C^3(M)$, we have

$$\operatorname{div}_\mu(\mathcal{C}_{\psi_1, \psi_2}(\mu)) = \langle \nabla \mathcal{L}^\mu \psi_1, \nabla \psi_2 \rangle - \langle \nabla \mathcal{L}^\mu \psi_2, \nabla \psi_1 \rangle.$$

By Hodge decomposition, $\mathcal{C}_{\psi_1, \psi_2}(\mu)$ admits the decomposition

$$\mathcal{C}_{\psi_1, \psi_2}(\mu) = d_\mu^* \omega + \nabla f + h,$$

where ω is a differential 2-form on M , d_μ^* is adjoint operator of exterior derivative in $L^2(\mu)$, h is harmonic form : $(d_\mu^* d + d d_\mu^*) h = 0$. Taking the divergence div_μ on the two sides of above equality, we see that f is a solution the following equation

$$\mathcal{L}^\mu f = \operatorname{div}_\mu(\mathcal{C}_{\psi_1, \psi_2}(\mu)).$$

It follows that $\tilde{\Phi}_\mu$ has the expression (3.7). □

Now we introduce the covariant derivative $\bar{\nabla}_{V_{\psi_1}} V_{\psi_2}$ associated to the Levi-Civita connection on $\mathbb{P}_2(M)$ by

$$2\langle \bar{\nabla}_{V_{\psi_1}} V_{\psi_2}, V_{\psi_3} \rangle = \bar{D}_{V_{\psi_1}} \langle V_{\psi_2}, V_{\psi_3} \rangle + \bar{D}_{V_{\psi_2}} \langle V_{\psi_3}, V_{\psi_1} \rangle - \bar{D}_{V_{\psi_3}} \langle V_{\psi_1}, V_{\psi_2} \rangle \\ + \langle V_{\psi_3}, [V_{\psi_1}, V_{\psi_2}] \rangle - \langle V_{\psi_2}, [V_{\psi_1}, V_{\psi_3}] \rangle - \langle V_{\psi_1}, [V_{\psi_2}, V_{\psi_3}] \rangle.$$

We have $\langle V_{\psi_2}, V_{\psi_3} \rangle = \int_M \langle \nabla \psi_2, \nabla \psi_3 \rangle d\mu = F_{\langle \nabla \psi_2, \nabla \psi_3 \rangle}$. Then

$$\bar{D}_{V_{\psi_1}} \langle V_{\psi_2}, V_{\psi_3} \rangle = \int_M \langle \nabla \psi_1, \nabla \langle \nabla \psi_2, \nabla \psi_3 \rangle \rangle d\mu = - \int_M \langle \mathcal{L}^\mu \psi_1 \nabla \psi_2, \nabla \psi_3 \rangle d\mu.$$

Replacing ψ_1 by ψ_2 , ψ_2 by ψ_3 and ψ_3 by ψ_1 , we get

$$\bar{D}_{V_{\psi_2}} \langle V_{\psi_3}, V_{\psi_1} \rangle = - \int_M \langle \mathcal{L}^\mu \psi_2 \nabla \psi_1, \nabla \psi_3 \rangle d\mu.$$

We have, in the same way

$$\bar{D}_{V_{\psi_3}} \langle V_{\psi_1}, V_{\psi_2} \rangle = - \int_M \langle \mathcal{L}^\mu \psi_3 \nabla \psi_1, \nabla \psi_2 \rangle d\mu.$$

Now using expression of $[V_{\psi_1}, V_{\psi_2}]$, we have

$$\langle V_{\psi_3}, [V_{\psi_1}, V_{\psi_2}] \rangle = \int_M \langle -\mathcal{L}^\mu \psi_1 \nabla \psi_2 + \mathcal{L}^\mu \psi_2 \nabla \psi_1, \nabla \psi_3 \rangle d\mu.$$

In the same way, we get

$$\langle V_{\psi_2}, [V_{\psi_1}, V_{\psi_3}] \rangle = \int_M \langle -\mathcal{L}^\mu \psi_1 \nabla \psi_3 + \mathcal{L}^\mu \psi_3 \nabla \psi_1, \nabla \psi_2 \rangle d\mu$$

and

$$\langle V_{\psi_1}, [V_{\psi_2}, V_{\psi_3}] \rangle = \int_M \langle -\mathcal{L}^\mu \psi_2 \nabla \psi_3 + \mathcal{L}^\mu \psi_3 \nabla \psi_2, \nabla \psi_1 \rangle d\mu.$$

Combining all these terms, we finally get

$$2\langle \bar{\nabla}_{V_{\psi_1}} V_{\psi_2}, V_{\psi_3} \rangle = \int_M \langle \nabla \langle \nabla \psi_1, \nabla \psi_2 \rangle, \nabla \psi_3 \rangle d\mu + \int_M \langle \mathcal{L}^\mu \psi_2 \nabla \psi_1 - \mathcal{L}^\mu \psi_1 \nabla \psi_2, \nabla \psi_3 \rangle d\mu.$$

Theorem 3.3. *For two constant vector fields V_{ψ_1}, V_{ψ_2} , we have*

$$\bar{\nabla}_{V_{\psi_1}} V_{\psi_2} = \frac{1}{2} V_{\langle \nabla \psi_1, \nabla \psi_2 \rangle} + \frac{1}{2} [V_{\psi_1}, V_{\psi_2}]. \quad (3.8)$$

Moreover, for any constant vector field V_{ψ_3} ,

$$\langle \bar{\nabla}_{V_{\psi_1}} V_{\psi_2}, V_{\psi_3} \rangle_{\bar{\mathbf{T}}_\mu} = \int_M \langle \nabla^2 \psi_2, \nabla \psi_1 \otimes \nabla \psi_3 \rangle d\mu. \quad (3.9)$$

Proof. It is enough to prove (3.9). We have

$$\begin{aligned}
\langle V_{\psi_3}, [V_{\psi_1}, V_{\psi_2}] \rangle_{\bar{\Gamma}_\mu} &= \int_M \langle -\mathcal{L}^\mu \psi_1 \nabla \psi_2 + \mathcal{L}^\mu \psi_2 \nabla \psi_1, \nabla \psi_3 \rangle d\mu \\
&= \int_M \langle \nabla \psi_1, \nabla \langle \nabla \psi_2, \nabla \psi_3 \rangle \rangle d\mu - \int_M \langle \nabla \psi_2, \nabla \langle \nabla \psi_1, \nabla \psi_3 \rangle \rangle d\mu \\
&= \int_M \left(\langle \nabla^2 \psi_2, \nabla \psi_1 \otimes \nabla \psi_3 \rangle + \langle \nabla^2 \psi_3, \nabla \psi_1 \otimes \nabla \psi_2 \rangle \right) d\mu \\
&\quad - \int_M \left(\langle \nabla^2 \psi_1, \nabla \psi_2 \otimes \nabla \psi_3 \rangle + \langle \nabla^2 \psi_3, \nabla \psi_2 \otimes \nabla \psi_1 \rangle \right) d\mu \\
&= \int_M \left(\langle \nabla^2 \psi_2, \nabla \psi_1 \otimes \nabla \psi_3 \rangle - \langle \nabla^2 \psi_1, \nabla \psi_2 \otimes \nabla \psi_3 \rangle \right) d\mu,
\end{aligned}$$

due to the symmetry of the Hessian $\nabla^2 \psi_3$. On the other hand,

$$\langle V_{\psi_3}, V_{\langle \nabla \psi_1, \nabla \psi_2 \rangle} \rangle_{\bar{\Gamma}_\mu} = \int_M \left(\langle \nabla^2 \psi_2, \nabla \psi_3 \otimes \nabla \psi_1 \rangle + \langle \nabla^2 \psi_1, \nabla \psi_3 \otimes \nabla \psi_2 \rangle \right) d\mu.$$

Summing these last two equalities yields (3.9). \square

Remark 3.4. By (3.8), for two constant vector fields V_{ψ_1}, V_{ψ_2} , the covariant derivative $\bar{\nabla}_{V_{\psi_1}} V_{\psi_2}$ is not a constant vector field on $\mathbb{P}_2(M)$ if $\psi_1 \neq \psi_2$.

Let $\alpha : \mathbb{P}_2(M) \rightarrow \mathbb{R}$ be a differentiable function, we define

$$\bar{\nabla}_{V_{\psi_1}} (\alpha V_{\psi_2}) = \bar{D}_{V_{\psi_1}} \alpha \cdot V_{\psi_2} + \alpha \bar{\nabla}_{V_{\psi_1}} V_{\psi_2}. \quad (3.10)$$

Proposition 3.5. Let Z be a vector field on $\mathbb{P}_2(M)$ in the test space $\chi(\mathbb{P})$, that is, $Z = \sum_{i=1}^k \alpha_i V_{\psi_i}$ with α_i polynomials. Then $\bar{\nabla}_Z Z$ still is in the test space; moreover

$$\bar{\nabla}_Z Z = V_{\Phi_1} + \frac{1}{2} V_{|\nabla \Phi_2|^2},$$

where

$$\Phi_1 = \sum_{j=1}^k \left(\sum_{i=1}^k \alpha_i \bar{D}_{V_{\psi_i}} \alpha_j \right) \psi_j, \quad \Phi_2 = \sum_{i=1}^k \alpha_i \psi_i.$$

Proof. Using the rule concerning covariant derivatives, $\bar{\nabla}_Z Z$ is equal to

$$\sum_{i,j=1}^k \alpha_i (\bar{D}_{V_{\psi_i}} \alpha_j) V_{\psi_j} + \frac{1}{2} \sum_{i,j=1}^k \alpha_i \alpha_j V_{\langle \nabla \psi_i, \nabla \psi_j \rangle} + \frac{1}{2} \sum_{i,j=1}^k \alpha_i \alpha_j [V_{\psi_i}, V_{\psi_j}].$$

The last sum is equal to 0 due to the skew-symmetry of $[V_{\psi_i}, V_{\psi_j}]$, the first one gives rise to Φ_1 and the second one gives rise to Φ_2 . \square

In what follows, we will extend the definition of covariant derivative (3.10) for a general vector field Z on $\mathbb{P}_2(M)$. Let Δ be the Laplace operator on M , let $\{\varphi_n, n \geq 0\}$ be the eigenfunctions of Δ :

$$-\Delta \varphi_n = \lambda_n \varphi_n.$$

We have $\lambda_0 = 0$ and $\varphi_0 = 1$. It is well-known, by Weyl's result, that

$$\lambda_n \sim n^{2/m}, \quad n \rightarrow +\infty$$

where m is the dimension of M . The functions $\{\varphi_n; n \in \mathbb{N}\}$ are smooth, chosen to form an orthonormal basis of $L^2(M, dx)$. A function f on M is said to be in $H^k(M)$ for $k \in \mathbb{N}$, if

$$\|f\|_{H^k}^2 = \int_M |(I - \Delta)^{k/2} f|^2 dx < +\infty.$$

By Sobolev embedding inequality, for $k > \frac{m}{2} + q$,

$$\|f\|_{C^q} \leq C \|f\|_{H^k}.$$

For $f \in H^k(M)$, put $f = \sum_{n \geq 0} a_n \varphi_n$ which holds in $L^2(M, dx)$ with

$$a_n = \int_M f(x) \varphi_n(x) dx.$$

We have :

$$\|f\|_{H^k}^2 = \sum_{n \geq 0} a_n^2 (1 + \lambda_n)^k.$$

The system $\left\{ \frac{\nabla \varphi_n}{\sqrt{\lambda_n}}; n \geq 1 \right\}$ is orthonormal. Let $V_n = V_{\varphi_n / \sqrt{\lambda_n}}$, then $\{V_n; n \geq 1\}$ is an orthonormal basis of $\bar{\mathbf{T}}_{dx}$.

Let Z be a vector field on $\mathbb{P}_2(M)$ given by $Z(\mu) = V_{\Phi(\mu, \cdot)}$ or $Z(\mu) = \nabla \Phi(\mu, \cdot)$. In the sequel, we denote: $\Phi_\mu(x) = \Phi(\mu, x)$, $\Phi^x(\mu) = \Phi(\mu, x)$. Then, if $x \rightarrow \nabla \Phi_\mu(x)$ is continuous,

$$\nabla \Phi_\mu = \sum_{n \geq 1} \left(\int_M \langle \nabla \Phi_\mu, \frac{\nabla \varphi_n}{\sqrt{\lambda_n}} \rangle dx \right) \frac{\nabla \varphi_n}{\sqrt{\lambda_n}} = \sum_{n \geq 1} \left(\int_M \Phi_\mu \varphi_n dx \right) \nabla \varphi_n,$$

which converges in $L^2(M, dx)$. Let $\mu \in \mathbb{P}_{\text{div}}(M)$, the above series converges also in $\bar{\mathbf{T}}_\mu$. Let

$$a_n(\mu) = \int_M \Phi_\mu(x) \varphi_n(x) dx. \quad (3.11)$$

Let V_ψ be a constant vector field on $\mathbb{P}_2(M)$ with $\psi \in C^\infty(M)$. For $q \geq p \geq 1$, set

$$S_{p,q} = \sum_{n=p}^q \left(\bar{D}_{V_\psi} a_n V_{\varphi_n} + a_n \bar{\nabla}_{V_\psi} V_{\varphi_n} \right) = S_{p,q}^1 + S_{p,q}^2 \quad (3.12)$$

respectively. Let $\phi \in C^\infty(M)$, according to (3.9), we have

$$\langle S_{p,q}^2, V_\phi \rangle_{\bar{\mathbf{T}}_\mu} = \int_M \left(\sum_{n=p}^q a_n(\mu) \nabla^2 \varphi_n \right) (\nabla \psi(x), \nabla \phi(x)) d\mu(x).$$

It follows that

$$|\langle S_{p,q}^2, V_\phi \rangle_{\bar{\mathbf{T}}_\mu}| \leq \left\| \sum_{n=p}^q a_n(\mu) \nabla^2 \varphi_n \right\|_\infty |V_\psi|_{\bar{\mathbf{T}}_\mu} |V_\phi|_{\bar{\mathbf{T}}_\mu},$$

therefore

$$|S_{p,q}^2|_{\bar{\mathbf{T}}_\mu} \leq \left\| \sum_{n=p}^q a_n(\mu) \nabla^2 \varphi_n \right\|_\infty |V_\psi|_{\bar{\mathbf{T}}_\mu}.$$

We have

$$\begin{aligned} & \left\| \sum_{n=p}^q a_n(\mu) (I - \Delta)^{k/2} \varphi_n \right\|_{L^2(dx)}^2 = \sum_{n=p}^q a_n(\mu)^2 (1 + \lambda_n)^k \\ & = \sum_{n=p}^q \left(\int_M (I - \Delta)^{k/2} \Phi_\mu \varphi_n dx \right)^2 \rightarrow 0 \end{aligned}$$

as $p, q \rightarrow +\infty$ if $\Phi_\mu \in H^k(M)$. On the other hand, we have

$$(\bar{D}_{V_\psi} a_n)(\mu) = \int_M (\bar{D}_{V_\psi} \Phi^x)(\mu) \varphi_n(x) dx = \int_M \left\langle \nabla \bar{D}_{V_\psi} \Phi^x, \frac{\nabla \varphi_n}{\sqrt{\lambda_n}} \right\rangle \frac{dx}{\sqrt{\lambda_n}},$$

then

$$S_{p,q}^1 = \sum_{n=p}^q \left(\int_M \left\langle \nabla \bar{D}_{V_\psi} \Phi^x, \frac{\nabla \varphi_n}{\sqrt{\lambda_n}} \right\rangle dx \right) \frac{\nabla \varphi_n}{\sqrt{\lambda_n}}$$

and

$$\int_M |S_{p,q}^1|^2 dx = \sum_{n=p}^q \left(\int_M \left\langle \nabla \bar{D}_{V_\psi} \Phi^x, \frac{\nabla \varphi_n}{\sqrt{\lambda_n}} \right\rangle dx \right)^2 \rightarrow 0$$

as $p, q \rightarrow +\infty$ if

$$\int_M |\nabla \bar{D}_{V_\psi} \Phi^x|^2 dx < +\infty.$$

Therefore for $d\mu = \rho dx$ with $\mu \in \mathbb{P}_{\text{div}}(M)$, as $p, q \rightarrow \infty$,

$$|S_{p,q}^1|_{\bar{\mathbf{T}}_\mu}^2 \leq \|\rho\|_\infty \int_M |S_{p,q}^1|^2 dx \rightarrow 0.$$

We get the following result

Theorem 3.6. *Let Z be a vector field on $\mathbb{P}_2(M)$ given by $\Phi : \mathbb{P}_2(M) \times M \rightarrow \mathbb{R}$. Assume that*

(i) *for any $\mu \in \mathbb{P}_2(M)$, $\Phi_\mu \in H^k(M)$ with $k > \frac{n}{2} + 2$,*

(ii) *for any $x \in M$, $\bar{D}_{V_\psi} \Phi^x$ exists and $\nabla \bar{D}_{V_\psi} \Phi^x \in L^2(M, dx)$.*

Then the covariant derivative $\bar{\nabla}_{V_\psi} Z$ is well defined at $\mu \in \mathbb{P}_{\text{div}}(M)$ and for $\phi \in C^\infty(M)$,

$$\langle \bar{\nabla}_{V_\psi} Z, V_\phi \rangle_{\bar{\mathbf{T}}_\mu} = \int_M \langle (\nabla \bar{D}_{V_\psi} \Phi^x), \nabla \phi \rangle d\mu + \int_M \nabla^2 \Phi_\mu (\nabla \psi, \nabla \phi) d\mu. \quad (3.13)$$

Proof. Let $Z_q = \sum_{n=1}^q a_n V_{\varphi_n}$. Then

$$\bar{\nabla}_{V_\psi} Z_q = S_{1,q}.$$

Letting $q \rightarrow +\infty$ yields the result. \square

4 Derivability of the square of the Wasserstein distance

Let $\{c_t; t \in [0, 1]\}$ be an absolutely continuous curve on $\mathbb{P}_2(M)$, for $\sigma \in \mathbb{P}_2(M)$ given, the derivability of $t \rightarrow W_2^2(\sigma, c_t)$ was established in chapter 8 of [2], as well as in [22] (see pages 636-649); however they hold true only for almost all $t \in [0, 1]$. The derivability at $t = 0$ was proved in Theorem 8.13 of [23] if σ and c_0 have a density with respect to dx . When $\{c_t\}$ is a geodesic of constant speed, the derivability at $t = 0$ was given in theorem 4.2 of [10] where the property of semi concavity was used. In what follows, we will use constant vector fields on $\mathbb{P}_2(M)$.

Before stating our result, we recall some well-known facts concerning optimal transport maps (see [4, 6, 16, 2, 22]). Let $\sigma \in \mathbb{P}_{2,ac}(M)$ be absolutely continuous with respect to dx and $\mu \in \mathbb{P}_2(M)$, then there is an unique Borel map $\phi \in \mathbb{D}_1^2(\sigma)$ such that

$$\int_M |\nabla\phi(x)|^2 d\sigma(x) = W_2^2(\sigma, \mu)$$

and $x \rightarrow T(x) = \exp_x(\nabla\phi(x))$ pushes σ forward to μ . If μ is also in $\mathbb{P}_{2,ac}(M)$, the map $T : M \rightarrow M$ is invertible and its inverse map T^{-1} is given by $y \rightarrow \exp_y(\nabla\tilde{\phi}(y))$ with some function $\tilde{\phi}$ such that $\int_M |\nabla\tilde{\phi}|^2 d\mu < +\infty$. We need also the following result

Lemma 4.1. *Let $x, y \in M$ and $\{\xi(t); t \in [0, 1]\}$ be a minimizing geodesic connecting x and y , given by $\xi(t) = \exp_x(tu)$ with some $u \in T_x M$. Then*

$$d_M^2(\exp_y(v), x) - d_M^2(y, x) \leq 2\langle v, \xi'(1) \rangle_{T_y M} + o(|v|) \quad \text{as } |v| \rightarrow 0. \quad (4.1)$$

Proof. See [16], page 10. □

Theorem 4.2. *Assume that $\sigma \in \mathbb{P}_{2,ac}(M)$ is absolutely continuous with respect to dx , then $\mu \rightarrow \chi(\mu) := W_2^2(\sigma, \mu)$ is derivable along each constant vector field V_ψ at any $\mu \in \mathbb{P}_2(M)$. If $\mu \in \mathbb{P}_{2,ac}(M)$, the gradient $\nabla\chi$ exists and admits the expression :*

$$\nabla\chi(\mu) = \nabla\tilde{\phi}. \quad (4.2)$$

Proof. Let $\psi \in C^\infty(M)$ and $(U_t)_{t \in \mathbb{R}}$ be the associated flow of diffeomorphisms of M :

$$\frac{dU_t(x)}{dt} = \nabla\psi(U_t(x)), \quad x \in M. \quad (4.3)$$

The inverse map U_t^{-1} of U_t satisfies the ODE

$$\frac{dU_t^{-1}(x)}{dt} = -\nabla\psi(U_t^{-1}(x)), \quad x \in M. \quad (4.4)$$

Set $\mu_t = (U_t)_\# \mu$, then $\mu = (U_t^{-1})_\# \mu_t$. Let $\gamma \in \mathcal{C}_o(\sigma, \mu)$ be the optimal coupling plan such that

$$W_2^2(\sigma, \mu) = \int_{M \times M} d_M^2(x, y) d\gamma(x, y).$$

The map $(x, y) \rightarrow (x, U_t(y))$ pushes γ forward to a coupling plan $\gamma_t \in \mathcal{C}(\sigma, \mu_t)$. Then for $t > 0$,

$$\begin{aligned} \frac{1}{t} \left[W_2^2(\sigma, \mu_t) - W_2^2(\sigma, \mu) \right] &\leq \frac{1}{t} \int_{M \times M} \left(d_M^2(x, U_t(y)) - d_M^2(x, y) \right) d\gamma(x, y) \\ &= \frac{1}{t} \int_{M \times M} \left(d_M^2(x, U_t(y)) - d_M^2(x, \exp_y(t\nabla\psi(y))) \right) d\gamma(x, y) \\ &+ \frac{1}{t} \int_{M \times M} \left(d_M^2(x, \exp_y(t\nabla\psi(y))) - d_M^2(x, y) \right) d\gamma(x, y) = I_1(t) + I_2(t) \end{aligned}$$

respectively. Let $\xi(t) = \exp_x(t\nabla\phi(x))$, by [16], ξ is a minimizing geodesic connecting x and $y = T(x)$. By Lemma 4.1, we have

$$d_M^2(x, \exp_y(t\nabla\psi(y))) - d_M^2(y, x) \leq 2t\langle\nabla\psi(y), \xi'(1)\rangle_{T_y M} + o(|t|) \quad \text{as } t \rightarrow 0.$$

On other hand,

$$\xi'(1) = d\exp_x(\nabla\phi(x)) \cdot \nabla\phi(x) = //_1^\xi \nabla\phi(x),$$

where $//_1^\xi$ denotes the parallel translation along the geodesic ξ . Hence $|\xi'(1)| = |\nabla\phi(x)|$. Therefore

$$I_2(t) \leq 2 \int_M \langle\nabla\psi(T(x)), d\exp_x(\nabla\phi(x)) \cdot \nabla\phi(x)\rangle d\sigma(x) + o(1)$$

To justify the passage of limit through the integral, we note that for $t > 0$,

$$\begin{aligned} & \frac{1}{t} \left| d_M^2(x, \exp_y(t\nabla\psi(y))) - d_M^2(x, y) \right| \\ & \leq \frac{2}{t} \text{diam}(M) d_M(y, \exp_y(t\nabla\psi(y))) \leq 2 \text{diam}(M) |\nabla\psi(y)|. \end{aligned}$$

Then

$$\overline{\lim}_{t \downarrow 0} I_2(t) \leq 2 \int_M \langle\nabla\psi(T(x)), d\exp_x(\nabla\phi(x)) \cdot \nabla\phi(x)\rangle d\sigma(x).$$

For estimating $I_1(t)$, it is obvious that

$$\lim_{t \downarrow 0} \frac{1}{t} \sup_{y \in M} d_M(U_t(y), \exp_y(t\nabla\psi(y))) = 0. \quad (4.5)$$

Then $\lim_{t \downarrow 0} I_1(t) = 0$. In conclusion:

$$\overline{\lim}_{t \downarrow 0} \frac{1}{t} \left[W_2^2(\sigma, \mu_t) - W_2^2(\sigma, \mu) \right] \leq 2 \int_M \langle\nabla\psi(T(x)), d\exp_x(\nabla\phi(x)) \cdot \nabla\phi(x)\rangle d\sigma(x). \quad (4.6)$$

For obtaining the minoration, we use as in [23] the fact that $\overline{\lim}_{t \downarrow 0}(-a_t) = -\underline{\lim}_{t \downarrow 0} a_t$.

Let $\tilde{\gamma}_t \in \mathcal{C}_o(\sigma, \mu_t)$ be the optimal coupling plan:

$$W_2^2(\sigma, \mu_t) = \int_{M \times M} d_M^2(x, y) d\tilde{\gamma}_t(x, y).$$

Let $\eta_t \in \mathcal{C}(\sigma, \mu_t)$ be defined by

$$\int_{M \times M} f(x, y) d\eta_t(x, y) = \int_{M \times M} f(x, U_t^{-1}(y)) d\tilde{\gamma}_t(x, y).$$

Then for $t > 0$,

$$\frac{1}{t} \left[W_2^2(\sigma, \mu) - W_2^2(\sigma, \mu_t) \right] \leq \frac{1}{t} \int_{M \times M} \left(d_M^2(x, U_t^{-1}(y)) - d_M^2(x, y) \right) d\tilde{\gamma}_t(x, y).$$

Let $T_t : M \rightarrow M$ be the optimal transport map which pushes σ forward to μ_t , with $T_t(x) = \exp_x(\nabla\phi_t(x))$. As $t \downarrow 0$, the map T_t converges in measure to T (see for example [23], page

265). We have

$$\begin{aligned}
& \frac{1}{t} \int_{M \times M} \left(d_M^2(x, U_t^{-1}(y)) - d_M^2(x, y) \right) d\tilde{\gamma}_t(x, y) \\
&= \frac{1}{t} \int_M \left(d_M^2(x, U_t^{-1}(T_t(x))) - d_M^2(x, T_t(x)) \right) d\sigma(x) \\
&= \frac{1}{t} \int_M \left(d_M^2(x, U_t^{-1}(T_t(x))) - d_M^2(x, \exp_{T_t(x)}(-t\nabla\psi(T_t(x)))) \right) d\sigma(x) \\
&+ \frac{1}{t} \int_M \left(d_M^2(x, \exp_{T_t(x)}(-t\nabla\psi(T_t(x)))) - d_M^2(x, T_t(x)) \right) d\sigma(x) = J_1(t) + J_2(t)
\end{aligned}$$

respectively. According to (4.5), $\lim_{t \downarrow 0} J_1(t) = 0$. Concerning $J_2(t)$, we note as above,

$$\begin{aligned}
& \frac{1}{t} \left| d_M^2(x, \exp_{T_t(x)}(-t\nabla\psi(T_t(x))) - d_M^2(x, T_t(x)) \right| \\
&\leq \frac{2}{t} \text{diam}(M) d_M(T_t(x), \exp_{T_t(x)}(-t\nabla\psi(T_t(x)))) \\
&\leq 2 \text{diam}(M) |\nabla\psi(T_t(x))| \leq 2 \text{diam}(M) \|\nabla\psi\|_\infty.
\end{aligned}$$

Then by Lemma 4.1,

$$J_2(t) \leq -2 \int_M \langle \nabla\psi(T_t(x)), d \exp_x(\nabla\phi_t(x)) \cdot \nabla\phi_t(x) \rangle d\sigma(x) + o(1)$$

Therefore

$$\overline{\lim}_{t \downarrow 0} \frac{1}{t} \left[W_2^2(\sigma, \mu) - W_2^2(\sigma, \mu_t) \right] \leq -2 \int_M \langle \nabla\psi(T(x)), d \exp_x(\nabla\phi(x)) \cdot \nabla\phi(x) \rangle d\sigma(x). \quad (4.7)$$

Combining (4.6) and (4.7), we finally get

$$\lim_{t \downarrow 0} \frac{1}{t} \left[W_2^2(\sigma, \mu_t) - W_2^2(\sigma, \mu) \right] = 2 \int_M \langle \nabla\psi(T(x)), d \exp_x(\nabla\phi(x)) \cdot \nabla\phi(x) \rangle d\sigma(x). \quad (4.8)$$

Now if $\mu \in \mathbb{P}_{2,ac}(M)$ and the map $y \rightarrow \exp_y(\nabla\tilde{\phi}(y))$ is the optimal transport map which pushes μ to σ . Consider the minimizing geodesic

$$\xi(t) = \exp_y((1-t)\nabla\tilde{\phi}(y)),$$

which connects x and y . We have $\xi'(1) = \nabla\tilde{\phi}(y)$. In this case, replacing $d \exp_x(\nabla\phi(x)) \cdot \nabla\phi(x)$ in (4.8) by $\nabla\tilde{\phi}(y)$, we obtain

$$\begin{aligned}
\lim_{t \downarrow 0} \frac{1}{t} \left[W_2^2(\sigma, \mu_t) - W_2^2(\sigma, \mu) \right] &= 2 \int_M \langle \nabla\psi(T(x)), \nabla\tilde{\phi}(T(x)) \rangle d\sigma(x) \\
&= 2 \int_M \langle \nabla\psi(y), \nabla\tilde{\phi}(y) \rangle d\mu(y),
\end{aligned}$$

from which we get (4.2). The proof is complete. \square

5 Parallel translations

Before introducing parallel translations on the space $\mathbb{P}_{div}(M)$, let's give a brief review on the definition of parallel translations on the manifold M , endowed with an affine connection. Let $\{\gamma(t); t \in [0, 1]\}$ be a smooth curve on M , and $\{Y_t; t \in [0, 1]\}$ a family vector fields along γ : $Y_t \in T_{\gamma(t)}M$. Then there exist vector fields X and Y on M such that

$$X(\gamma(t)) = \dot{\gamma}(t), \quad Y(\gamma(t)) = Y_t.$$

Y_t is said to be parallel along $\{\gamma(t); t \in [0, 1]\}$ if

$$(\nabla_X Y)(\gamma(t)) = 0, \quad t \in [0, 1].$$

Now let $\{c_t; t \in [0, 1]\}$ be an absolutely curve on $\mathbb{P}_{div}(M)$ such that

$$\frac{d^I c_t}{dt} = V_{\Phi_t}, \quad \text{with } \Phi_t \in \mathbb{D}_1^2(c_t). \quad (5.1)$$

Let $\{Y_t; t \in [0, 1]\}$ be a vector field along $\{c_t; t \in [0, 1]\}$, that is, $Y_t \in \bar{\mathbf{T}}_{c_t}$ given by $Y_t = V_{\Psi_t}$ with $\Psi_t \in \mathbb{D}_1^2(c_t)$.

Theorem 5.1. *Assume that $t \rightarrow c_t$ is C^1 in the sense that for any $f \in C^1(M)$, $t \rightarrow F_f(c_t)$ is C^1 and for $t \in [0, 1]$, $x \rightarrow \Phi_t(x)$ is C^1 . If for each $t \in [0, 1]$,*

$$|V_{\Phi_t}|_{\bar{\mathbf{T}}_{c_t}}^2 = \int_M |\nabla \Phi_t(x)|^2 dc_t(x) > 0, \quad (5.2)$$

then there are functions $(\mu, x) \rightarrow \tilde{\Phi}(\mu, x)$ and $(\mu, x) \rightarrow \tilde{\Psi}(\mu, x)$ on $\mathbb{P}_2(M) \times M$ such that

$$\tilde{\Phi}(c_t, x) = \Phi_t(x), \quad \tilde{\Psi}(c_t, x) = \Psi_t(x); \quad (5.3)$$

moreover for $x \in M$, $\mu \rightarrow \tilde{\Phi}(\mu, x)$ and $\mu \rightarrow \tilde{\Psi}(\mu, x)$ are derivable on $\mathbb{P}_2(M)$ along any constant vector fields V_ψ , their gradients exist on $\mathbb{P}_{2,ac}(M)$.

Proof. Fix $t_0 \in [0, 1]$; consider $\alpha(t) = F_{\Phi_{t_0}}(c_t)$. Then

$$\alpha'(t) = \frac{d}{dt} F_{\Phi_{t_0}}(c_t) = \int_M \langle \nabla \Phi_{t_0}, \nabla \Phi_t \rangle dc_t,$$

which is > 0 at $t = t_0$. Therefore there is an open interval $I(t_0)$ of t_0 such that $t \rightarrow \alpha(t)$ is a C^1 diffeomorphism from $I(t_0)$ onto an interval $J(t_0)$ containing $\alpha(t_0)$. Let $\beta : J(t_0) \rightarrow I(t_0)$ be the inverse map of α . We have

$$F_{\Phi_{t_0}}(c_t) \in J(t_0) \quad \text{for } t \in I(t_0).$$

Let

$$U(t_0) = \{\mu \in \mathbb{P}_2(M); F_{\Phi_{t_0}}(\mu) \in J(t_0)\},$$

which is an open set in $\mathbb{P}_2(M)$. Let $r > 0$ and $\nu \in \mathbb{P}_2(M)$, we denote by $B(\nu, r)$ the open ball in $\mathbb{P}_2(M)$ centered at ν of radius r . Take $r_0 > 0$ small enough such that

$$B(c_{t_0}, r_0) \subset U(t_0).$$

We define, for $\mu \in B(c_{t_0}, r_0)$,

$$\tilde{\Phi}_{t_0}(\mu) = \Phi_{\beta(F_{\Phi_{t_0}}(\mu))}, \quad \tilde{\Psi}_{t_0}(\mu) = \Psi_{\beta(F_{\Phi_{t_0}}(\mu))}. \quad (5.4)$$

We remark that for $t \in [0, 1]$ such that $c_t \in U(t_0)$, we have: $\beta(F_{\Phi_{t_0}}(c_t)) = t$. Note that $\{c_t; t \in [0, 1]\}$ is a compact set of $\mathbb{P}_2(M)$ and

$$\{c_t; t \in [0, 1]\} \subset \cup_{t_0 \in [0, 1]} B(c_{t_0}, r_0).$$

There exists a finite number of $t_1, \dots, t_k \in [0, 1]$ such that

$$\{c_t; t \in [0, 1]\} \subset \cup_{i=1}^k B(c_{t_i}, r_i).$$

Set $U = \cup_{i=1}^k B(c_{t_i}, r_i)$. Let $\mu \in U$, then $\mu \in B(c_{t_i}, r_i)$; according to (5.4), we define,

$$\tilde{\Phi}_{t_i}(\mu) = \Phi_{\beta_i(F_{\Phi_{t_i}}(\mu))}, \quad \tilde{\Psi}_{t_i}(\mu) = \Psi_{\beta_i(F_{\Phi_{t_i}}(\mu))}.$$

Then for $t \in [0, 1]$ such that $c_t \in B(c_{t_i}, r_i)$, $\tilde{\Phi}_{t_i}(c_t) = \Phi_t$ and $\tilde{\Psi}_{t_i}(c_t) = \Psi_t$. Now for $r > 0$ and $\nu \in \mathbb{P}_2(M)$, we define

$$g_{r,\nu}(\mu) = \exp\left(\frac{1}{W_2^2(\nu, \mu) - r^2}\right), \quad \text{if } W_2(\nu, \mu) < r, \quad (5.5)$$

and $g_{r,\nu}(\mu) = 0$ otherwise. Then $g_{r,\nu}(\mu) > 0$ if and only if $\mu \in B(\nu, r)$. By Theorem 4.2, if $\nu \in \mathbb{P}_{\text{div}}$, $\mu \rightarrow g_{r,\nu}(\mu)$ is derivable along any constant vector field V_ψ . Remark that

$$\sum_{i=1}^k g_{r_i, c_{t_i}} > 0 \quad \text{on } U.$$

Let

$$\alpha_i = \frac{g_{r_i, c_{t_i}}}{\sum_{i=1}^k g_{r_i, c_{t_i}}} \quad \text{for } \mu \in U, \quad \text{and } \alpha_i = 0 \text{ otherwise.}$$

Now define

$$\Phi(\mu) = \sum_{i=1}^k \alpha_i(\mu) \tilde{\Phi}_{t_i}(\mu), \quad \Psi(\mu) = \sum_{i=1}^k \alpha_i(\mu) \tilde{\Psi}_{t_i}(\mu).$$

We have

$$\Phi(c_t) = \sum_{i=1}^k \alpha_i(c_t) \tilde{\Phi}_{t_i}(c_t).$$

Note that $\alpha_i(c_t) > 0$ if and only if $c_t \in B(c_{t_i}, r_i)$, which implies that $\tilde{\Phi}_{t_i}(c_t) = \Phi_t$ and $\Phi(c_t) = \sum_{i=1}^k \alpha_i(c_t) \Phi_t = \Phi_t$. It is the same for Ψ . The proof is completed. \square

Notice that for such a curve $\{c_t; t \in [0, 1]\}$ given in Theorem 5.1, and $\{Y_t; t \in [0, 1]\}$ a vector field along $\{c_t; t \in [0, 1]\}$ given by Ψ_t . If furthermore for any $t \in [0, 1]$, $\Psi_t \in H^k(M)$ with $k > \frac{m}{2} + 2$, then the extension obtained $\tilde{\Psi}$ obtained in Theorem 5.1 satisfies conditions in Theorem 3.6.

Definition 5.2. We say that $\{Y_t; t \in [0, 1]\}$ is parallel along $\{c_t; t \in [0, 1]\}$ if

$$(\bar{\nabla}_{\frac{d^j c_t}{dt}} V_{\tilde{\Psi}})(c_t) = 0, \quad t \in [0, 1].$$

Theorem 5.3. *Keeping same notations in Theorem 5.1, if $\{Y_t; t \in [0, 1]\}$ is parallel along $\{c_t, t \in [0, 1]\}$, the following equation holds*

$$\int_M \left\langle \nabla \left(\frac{d\Psi_t}{dt} \right) + \nabla_{\nabla\Phi_t} \nabla\Psi_t, \nabla\phi \right\rangle dc_t = 0, \quad \phi \in C^\infty(M). \quad (5.6)$$

Proof. Note that

$$(\bar{D}_{\frac{dI_{c_t}}{dt}} \tilde{\Psi})(c_t) = \frac{d}{dt} \tilde{\Psi}(c_t) = \frac{d\Psi_t}{dt} \text{ and } \nabla \tilde{\Psi}(c_t, \cdot) = \nabla\Psi_t.$$

Then (5.6) follows from (3.13). \square

When $\nabla \left(\frac{d\Psi_t}{dt} \right) = \frac{d\nabla\Psi_t}{dt}$, it is more convenient to put Equation (5.6) in the following form :

$$\Pi_{c_t} \left(\frac{d}{dt} \nabla\Psi_t + \nabla_{\nabla\Phi_t} \nabla\Psi_t \right) = 0, \quad (5.7)$$

or

$$\frac{d}{dt} \nabla\Psi_t + \Pi_{c_t} \left(\nabla_{\nabla\Phi_t} \nabla\Psi_t \right) = 0, \quad (5.8)$$

where Π_{c_t} the orthogonal projection from $L^2(M, TM, c_t)$ onto $\bar{\mathbf{T}}_{c_t}$. By arguments in the proof of Proposition 3.2, when $dc_t = \rho_t dx$ with $\rho_t \in C^2(M)$ and $\rho_t > 0$, Π_{c_t} admits the expression

$$\Pi_{c_t} u = (\nabla \mathcal{L}_{c_t}^{-1} \operatorname{div}_{c_t})(u), \quad u \in L^2(M, TM, c_t).$$

The price for this pointwise formulation of (5.7) as well as of (5.8) is the involvement of second order derivative of Ψ .

Remark 5.4. Let $s \rightarrow \xi(s)$ is a smooth curve on M such that $\xi(0) = x$ and $\xi'(0) = \nabla\Phi_t(x)$, then

$$\frac{d}{dt} \nabla\Psi_t + \nabla_{\nabla\Phi_t} \nabla\Psi_t = \lim_{\varepsilon \rightarrow 0} \frac{\tau_\varepsilon^{-1} \nabla\Psi_{t+\varepsilon}(\xi(\varepsilon)) - \nabla\Psi_t(x)}{\varepsilon}, \quad (5.9)$$

where τ_s is the parallel translation along $s \rightarrow \xi(s)$. We remind the similar expression of parallel translations given in [1].

Proposition 5.5. *Assume that the curve $\{c_t; t \in [0, 1]\}$ is induced by a flow of diffeomorphisms Φ_t , that is, there is a $C^{1,2}$ function $(t, x) \rightarrow \Phi_t(x)$ such that*

$$\begin{cases} \frac{dU_{s,t}(x)}{dt} = \nabla\Phi_t(U_{s,t}(x)), & U_{s,s}(x) = x, \\ c_t = (U_{0,t})_{\#} c_0. \end{cases}$$

Then for any $u_0 = \nabla\Psi_0 \in \bar{\mathbf{T}}_{c_0}$, there is a unique vector field $u_t = \nabla\Psi_t \in \bar{\mathbf{T}}_{c_t}$ along $\{c_t; t \in [0, 1]\}$ such that

$$\Pi_{c_t} \left(\lim_{\varepsilon \rightarrow 0} \frac{\tau_\varepsilon^{-1} \nabla\Psi_{t+\varepsilon}(U_{t,t+\varepsilon}(x)) - \nabla\Psi_t(x)}{\varepsilon} \right) = 0 \quad (5.10)$$

holds in $L^2(c_t)$, where τ_ε is the parallel translation along $\{s \rightarrow U_{t,t+s}(x), s \in [0, \varepsilon]\}$.

Proof. Following Section 5 of [1], for $s \leq t$, we define

$$\mathcal{P}_{t,s} : \bar{\mathbf{T}}_{c_s} \rightarrow \bar{\mathbf{T}}_{c_t}, \quad u_s \rightarrow \Pi_{c_t}(\tau_{t-s} u_s \circ U_{s,t}^{-1}).$$

For a subdivision $\mathcal{D} = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ of $[0, 1]$, we define

$$\mathcal{P}_{\mathcal{D}} : \bar{\mathbf{T}}_{c_0} \rightarrow \bar{\mathbf{T}}_{c_1}, \quad u_0 \rightarrow (\mathcal{P}_{1,t_{n-1}} \circ \dots \circ \mathcal{P}_{t_1,0})(u_0).$$

Under the assumption of Theorem, we have the uniform bound

$$\sup_{(t,x) \in [0,1] \times M} \|\nabla^2 \Phi_t(x)\| < +\infty,$$

which allows us to mimic the construction of section 5 in [1], so that we get that $\mathcal{P}_{\mathcal{D}}$ converges as \mathcal{D} becomes finer and finer, with $|\mathcal{D}| = \max_i |t_i - t_{i-1}| \rightarrow 0$. \square

As a result of (5.10), we have as in [1] the following property:

Proposition 5.6. *Let $\{\nabla \Psi_t; t \in [0, 1]\}$ be given in Proposition 5.5, then*

$$\frac{d}{dt} \|\nabla \Psi_t\|_{c_t}^2 = 0. \quad (5.11)$$

Proof. We have $c_{t+\varepsilon} = (U_{t,t+\varepsilon})_{\#} c_t$, and

$$\int_M |\nabla \Psi_{t+\varepsilon}(x)|^2 dc_{t+\varepsilon}(x) = \int_M |\nabla \Psi_{t+\varepsilon}(U_{t,t+\varepsilon}(x))|^2 dc_t(x).$$

Therefore

$$\begin{aligned} \|u_{t+\varepsilon}\|_{\bar{\mathbf{T}}_{t+\varepsilon}}^2 - \|u_t\|_{\bar{\mathbf{T}}_{c_t}}^2 &= \int_M \left[|\tau_{\varepsilon}^{-1} \nabla \Psi_{t+\varepsilon}(U_{t,t+\varepsilon}(x))|^2 - |\nabla \Psi_t(x)|^2 \right] dc_t(x) \\ &= \int_M \left\langle \tau_{\varepsilon}^{-1} \nabla \Psi_{t+\varepsilon}(U_{t,t+\varepsilon}(x)) - \nabla \Psi_t(x), \tau_{\varepsilon}^{-1} \nabla \Psi_{t+\varepsilon}(U_{t,t+\varepsilon}(x)) \right\rangle dc_t(x) \\ &\quad + \int_M \left\langle \nabla \Psi_t(x), \tau_{\varepsilon}^{-1} \nabla \Psi_{t+\varepsilon}(U_{t,t+\varepsilon}(x)) - \nabla \Psi_t(x) \right\rangle dc_t(x). \end{aligned}$$

It follows that

$$\frac{d}{dt} \|\nabla \Psi_t\|_{c_t}^2 = 2 \int_M \left\langle \lim_{\varepsilon \rightarrow 0} \frac{\tau_{\varepsilon}^{-1} \nabla \Psi_{t+\varepsilon}(U_{t,t+\varepsilon}(x)) - \nabla \Psi_t(x)}{\varepsilon}, \nabla \Psi_t(x) \right\rangle dc_t(x) = 0. \quad \square$$

In what follows, we will relax a bit conditions in Proposition 5.5. We return to the situation in Theorem 5.1. Let $\{c_t; t \in [0, 1]\}$ be an absolutely curve in $\mathbb{P}_{\text{div}}(M)$ satisfying conditions in Theorem 5.1, set

$$\frac{d^I c_t}{dt} = V_{\Phi_t}.$$

If furthermore $(t, x) \rightarrow \nabla^2 \Phi_t(x)$ is continuous, according to the the construction, the extension $\tilde{\Phi}(\mu, x)$ of $(t, x) \rightarrow \nabla^2 \Phi_t(x)$ obtained in (5.3) satisfies $(\mu, x) \rightarrow \nabla^2 \tilde{\Phi}(\mu, x)$ is continuous. In particular, the condition (2.5)

$$\sup_{(\mu, x) \in \mathbb{P}_2(M) \times M} \|\nabla^2 \tilde{\Phi}(\mu, x)\| < +\infty,$$

holds. By theorem 2.2, there exists a solution (μ_t, U_t) to the following McKean-Vlasov equation

$$\frac{dU_t(x)}{dt} = \nabla \tilde{\Phi}(\mu_t, U_t(x)), \quad U_0(x) = x,$$

with $\mu_t = (U_t)_\# c_0$ which solves the ODE on $\mathbb{P}_2(M)$:

$$\frac{d^I \mu_t}{dt} = V_{\tilde{\Phi}(\mu_t, \cdot)}. \quad (5.12)$$

Theorem 5.7. *If the ODE (5.12) has the unique solution, then for each $V_{\Psi_0} \in \bar{\mathbf{T}}_{c_0}$, there is a vector field $\{V_{\Psi_t} \in \bar{\mathbf{T}}_{c_t}; t \in [0, 1]\}$ along $\{c_t; t \in [0, 1]\}$ such that*

$$\frac{d}{dt} \|\nabla \Psi_t\|_{c_t}^2 = 0.$$

holds in $L^2(c_t)$.

Proof. Note that $\nabla \tilde{\Phi}(c_t, x) = \nabla \Phi_t(x)$, then $V_{\tilde{\Phi}(c_t, \cdot)} = V_{\Phi_t}$. The curve $\{c_t; t \in [0, 1]\}$ is therefore a solution to

$$\frac{d^I c_t}{dt} = V_{\Phi_t} = V_{\tilde{\Phi}(c_t, \cdot)}.$$

Under the assumption of uniqueness of solution to (5.12), we get that $c_t = \mu_t$ for $t \in [0, 1]$. Now by arguments in the proof of Propositions 5.5 and 5.6, we obtain the result. \square

Remark 5.8. *The parallel translations along diffusion paths on the Wasserstein space are discussed in a forthcoming paper [8].*

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