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SCHRÖDINGER EQUATION ON NONCOMPACT SYMMETRIC SPACES

JEAN-PHILIPPE ANKER, STEFANO MEDA, VITTORIA PIERFELICE,
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ABSTRACT. We establish sharp-in-time pointwise kernel estimates for the Schrödinger equation on noncompact symmetric spaces of general rank. A well-known difficulty in higher rank analysis, namely the fact that the Plancherel density is not a differential symbol in general, is overcome by using a spectral decomposition introduced recently by two of the authors in the study of the wave equation. We deduce the dispersive property of the Schrödinger propagator and prove global-in-time Strichartz inequalities for a large family of admissible pairs. As consequences, we extend the global well-posedness and small data scattering results previously obtained on real hyperbolic spaces to general Riemannian symmetric spaces of noncompact type.

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1. INTRODUCTION

In this paper, we establish sharp-in-time kernel estimates and dispersive properties for the Schrödinger equation on general Riemannian symmetric spaces of noncompact type. Such estimates allow us to prove a global-in-time Strichartz inequality for a large family of admissible couples. This inequality has two important applications in the study of nonlinear Schrödinger equations. On the one hand, it serves as a tool for finding minimal regularity conditions on the initial data ensuring well-posedness. On the other hand, it is one of the key ingredient to prove scattering results.

The Schrödinger equation was studied extensively in the Euclidean setting. We refer to [Caz03; Tao06] and the references therein for more details. On a compact or nonnegatively curved manifold, one gets, in general, a Strichartz inequality with some loss of derivatives, which implies weaker well-posedness and scattering results than in the Euclidean case, see for instance [Bou93; BGT04; GePi10; Zha20].

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We consider in this paper noncompact symmetric spaces of general rank. The interesting feature about such spaces is their exponential growth at infinity and the fact that dispersive properties of PDE are more pronounced. The Schrödinger equation was understood progressively on real hyperbolic spaces, first under radial assumptions [Pie06; Ban07; BCS08] and next for general functions [AnPi09; IoSt09]. These results were further extended to Damek-Ricci spaces, which are not always symmetric, but contain all noncompact symmetric spaces of rank one [APV11].

The analysis carried out in these papers works in rank one but not in higher rank. One of the main reason is the well-known problem that the Plancherel density, which occurs in spherical Fourier analysis on symmetric spaces, is not a differential symbol in general. As for the wave equation in [AnZh20], we achieve in this paper the analysis of the Schrödinger equation in higher rank by using on the one hand the Hadamard parametrix and, on the other hand, a spectral barycentric decomposition introduced in [AnZh20].

Let us state the main two results obtained in this paper. Consider the Schrödinger propagator $e^{it\Delta}$ associated to the Laplace-Beltrami operator Δ defined on a d -dimensional noncompact symmetric space $\mathbb{X} = G/K$ and let us denote by s_t its K -bi-invariant convolution kernel.

Theorem 1.1 (Pointwise kernel estimates). *There exist $C > 0$ and $N > 0$ such that the following estimates hold, for all $t \in \mathbb{R}^*$ and $x \in \mathbb{X}$:*

$$|s_t(x)| \leq C (1 + |x|)^N e^{-\langle \rho, x^+ \rangle} \begin{cases} |t|^{-\frac{d}{2}} & \text{if } 0 < |t| < 1, \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1, \end{cases}$$

where $x^+ \in \overline{\mathfrak{a}^+}$ denotes the radial component of x in the Cartan decomposition, $\rho \in \mathfrak{a}^+$ the half sum of all positive roots and $D = \ell + 2|\Sigma_r^+|$ the so-called dimension at infinity or pseudo-dimension of \mathbb{X} .

Remark 1.2. *Contrarily to the Euclidean setting, the Schrödinger kernel satisfies no rescaling and behaves differently for small and large times. Notice that the large polynomial factor $(1 + |x|)^N$ is harmless for the dispersive estimates because of the exponential factor $e^{-\langle \rho, x^+ \rangle}$.*

Theorem 1.3 (Dispersive properties). *Let $2 < q, \tilde{q} < +\infty$. Then there exists a constant $C > 0$ such that following dispersive estimates hold for all $t \in \mathbb{R}^*$:*

$$\|e^{it\Delta}\|_{L^{\tilde{q}'}(\mathbb{X}) \rightarrow L^q(\mathbb{X})} \leq C \begin{cases} |t|^{-\max(\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\tilde{q}})d} & \text{if } 0 < |t| < 1, \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1, \end{cases}$$

where \tilde{q} and \tilde{q}' are dual indices in the sense that $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$.

Remark 1.4. *At the endpoint $q = \tilde{q} = 2$, $t \mapsto e^{it\Delta}$ is a one-parameter group of unitary operators on $L^2(\mathbb{X})$. [Theorem 1.1](#) and [Theorem 1.3](#) extend earlier results obtained in rank one (where $D = 3$) to higher rank. Notice that we obtain stronger dispersive properties in our setting than in Euclidean spaces, especially in large time, where we have a fast time decay independent of q or \tilde{q} .*

This paper is organized as follows. After reviewing spherical Fourier analysis on noncompact symmetric spaces and the barycentric decomposition of the Weyl chamber in [Sect. 2](#), we prove in [Sect. 3](#) our main result, namely the pointwise kernel estimates. In [Sect. 3](#), we deduce the dispersive properties and the Strichartz inequalities for a large family of admissible pairs, by adapting straightforwardly the method carried out in rank one. Such stronger estimates imply better well-posedness and scattering results for nonlinear Schrödinger equations, which are listed in [Sect. 5](#).

Throughout this paper, the symbol $A \lesssim B$ between two positive expressions means that there is a constant $C > 0$ such that $A \leq CB$. The symbol $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$.

2. PRELIMINARIES

In this section we review briefly harmonic analysis on Riemannian symmetric spaces of noncompact type. We adopt the standard notation and refer to [Hel78; Hel00; GaVa88] for more details.

2.1. Noncompact symmetric spaces. Let G be a semisimple Lie group, connected, noncompact, with finite center, and K be a maximal compact subgroup of G . The homogeneous space $\mathbb{X} = G/K$ is a Riemannian symmetric space of noncompact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G , the Killing form of \mathfrak{g} induces a K -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{p} , hence a G -invariant Riemannian metric on \mathbb{X} . Fix a maximal abelian subspace \mathfrak{a} in \mathfrak{p} . The rank of \mathbb{X} is the dimension ℓ of \mathfrak{a} . We identify \mathfrak{a} with its dual \mathfrak{a}^* by means of the inner product inherited from \mathfrak{p} . Let $\Sigma \subset \mathfrak{a}$ be the root system of $(\mathfrak{g}, \mathfrak{a})$ and denote by W the Weyl group associated to Σ . Once a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ has been selected, Σ^+ (resp. Σ_r^+ or Σ_s^+) denotes the corresponding set of positive roots (resp. positive reduced roots or simple roots). Let d be the dimension of \mathbb{X} and let D be the so-called dimension at infinity or pseudo-dimension of \mathbb{X} :

$$d = \ell + \sum_{\alpha \in \Sigma^+} m_\alpha \quad \text{and} \quad D = \ell + 2|\Sigma_r^+| \quad (2.1)$$

where m_α is the dimension of the positive root subspace \mathfrak{g}_α . Notice that these two dimensions behave quite differently. For example, $D = 3$ while $d \geq 2$ is arbitrary in rank one, $D = d$ if G is complex, and $D > d$ (actually $D = 2d - \ell$) if G is split.

Let \mathfrak{n} be the nilpotent Lie subalgebra of \mathfrak{g} associated to Σ^+ and let $N = \exp \mathfrak{n}$ be the corresponding Lie subgroup of G . We have the decompositions

$$\begin{cases} G = N(\exp \mathfrak{a})K & \text{(Iwasawa),} \\ G = K(\exp \overline{\mathfrak{a}^+})K & \text{(Cartan).} \end{cases}$$

In the Cartan decomposition, the Haar measure on G writes

$$\int_G dx f(x) = \text{const.} \int_K dk_1 \int_{\mathfrak{a}^+} dx^+ \delta(x^+) \int_K dk_2 f(k_1(\exp x^+)k_2),$$

with density

$$\delta(x^+) = \prod_{\alpha \in \Sigma^+} (\sinh \langle \alpha, x^+ \rangle)^{m_\alpha} \asymp \prod_{\alpha \in \Sigma^+} \left\{ \frac{\langle \alpha, x^+ \rangle}{1 + \langle \alpha, x^+ \rangle} \right\}^{m_\alpha} e^{2\langle \rho, x^+ \rangle} \quad \forall x^+ \in \overline{\mathfrak{a}^+}.$$

Here $\rho \in \mathfrak{a}^+$ denotes the half sum of all positive roots $\alpha \in \Sigma^+$ counted with their multiplicities m_α :

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

2.2. Spherical Fourier analysis on symmetric spaces. Let $\mathcal{S}(K \backslash G / K)$ be the Schwartz space of K -bi-invariant functions on G . The spherical Fourier transform \mathcal{H} is defined by

$$\mathcal{H}f(\lambda) = \int_G dx \varphi_{-\lambda}(x) f(x) \quad \forall \lambda \in \mathfrak{a}, \forall f \in \mathcal{S}(K \backslash G / K),$$

where $\varphi_\lambda \in \mathcal{C}^\infty(K \backslash G / K)$ denotes the spherical function of index $\lambda \in \mathfrak{a}$, which is a smooth K -bi-invariant eigenfunction of all invariant differential operators on \mathbb{X} , in particular of the Laplace-Beltrami operator:

$$-\Delta \varphi_\lambda(x) = (|\lambda|^2 + |\rho|^2) \varphi_\lambda(x).$$

The spherical functions have the following integral representation

$$\varphi_\lambda(x) = \int_K dk e^{\langle i\lambda + \rho, A(kx) \rangle} \quad \forall \lambda \in \mathfrak{a}, \quad (2.2)$$

where $A(kx)$ denotes the \mathfrak{a} -component in the Iwasawa decomposition of kx . They satisfy the basic estimate

$$|\varphi_\lambda(x)| \leq \varphi_0(x) \quad \forall \lambda \in \mathfrak{a}, \forall x \in G,$$

where

$$\varphi_0(x) \asymp \left\{ \prod_{\alpha \in \Sigma_r^+} (1 + \langle \alpha, x^+ \rangle) \right\} e^{-\langle \rho, x^+ \rangle} \quad \forall x \in G.$$

Denote by $\mathcal{S}(\mathfrak{a})^W$ the subspace of W -invariant functions in the Schwartz space $\mathcal{S}(\mathfrak{a})$. Then \mathcal{H} is an isomorphism between $\mathcal{S}(K \backslash G / K)$ and $\mathcal{S}(\mathfrak{a})^W$. The inverse spherical Fourier transform is given by

$$f(x) = C_0 \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) \mathcal{H}f(\lambda) \quad \forall x \in G, \forall f \in \mathcal{S}(\mathfrak{a})^W, \quad (2.3)$$

where $C_0 > 0$ is a constant depending only on the geometry of \mathbb{X} . By using the Gindikin-Karpelevič formula of the Harish-Chandra \mathbf{c} -function (see [Hel00] or [GaVa88]), we can write the Plancherel density as

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha \in \Sigma_r^+} |\mathbf{c}_\alpha(\langle \alpha, \lambda \rangle)|^{-2}, \quad (2.4)$$

with

$$c_\alpha(v) = \frac{\overbrace{\Gamma(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{2}m_\alpha)}^{C_\alpha} \Gamma(\frac{1}{2}\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha})}{\Gamma(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle}) \Gamma(\frac{1}{2}\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \alpha \rangle} + \frac{1}{4}m_\alpha)} \frac{\Gamma(iv)}{\Gamma(iv + \frac{1}{2}m_\alpha)} \frac{\Gamma(\frac{i}{2}v + \frac{1}{4}m_\alpha)}{\Gamma(\frac{i}{2}v + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha})}.$$

Notice that $|\mathbf{c}_\alpha|^{-2}$ is a homogeneous differential symbol on \mathbb{R} of order $m_\alpha + m_{2\alpha}$, for every $\alpha \in \Sigma_r^+$. Hence $|\mathbf{c}(\lambda)|^{-2}$ is a product of one-dimensional symbols, but not a symbol on \mathfrak{a} in general. It has the following behavior

$$|\mathbf{c}(\lambda)|^{-2} \asymp \prod_{\alpha \in \Sigma_r^+} \langle \alpha, \lambda \rangle^2 (1 + |\langle \alpha, \lambda \rangle|)^{m_\alpha + m_{2\alpha} - 2}$$

and satisfies

$$|\mathbf{c}(\lambda)|^{-2} \lesssim \begin{cases} |\lambda|^{D-\ell} & \text{if } |\lambda| \leq 1, \\ |\lambda|^{d-\ell} & \text{if } |\lambda| \geq 1, \end{cases} \quad (2.5)$$

together with all its derivatives.

2.3. Barycentric decomposition of the Weyl chamber. This tool plays an important role in spherical Fourier analysis in higher rank. It allows us to overcome a well-known difficulty, namely the fact that the Plancherel density is not a symbol in general. We list here only some useful properties and refer to [AnZh20, Subsection 2.2] for more details about this decomposition.

Let $\Sigma_s^+ = \{\alpha_1, \dots, \alpha_\ell\}$ be the set of positive simple roots, and let $\{\Lambda_1, \dots, \Lambda_\ell\}$ be the dual basis of \mathfrak{a} , which is defined by

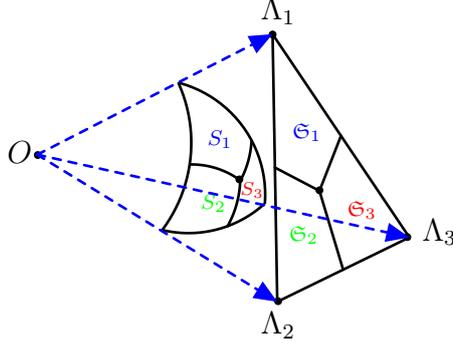
$$\langle \alpha_j, \Lambda_k \rangle = \delta_{jk} \quad \forall 1 \leq j, k \leq \ell.$$

Denote by \mathfrak{B} the convex hull of $W.\Lambda_1 \sqcup \dots \sqcup W.\Lambda_\ell$, and by \mathfrak{S} its polyhedral boundary. Notice that $\mathfrak{B} \cap \mathfrak{a}^+$ is the ℓ -simplex with vertices $0, \Lambda_1, \dots, \Lambda_\ell$, and $\mathfrak{S} \cap \mathfrak{a}^+$ is the $(\ell - 1)$ -simplex with vertices $\Lambda_1, \dots, \Lambda_\ell$. The following tiling is obtained by regrouping the barycentric subdivisions of the simplices $\mathfrak{S} \cap w.\mathfrak{a}^+$:

$$\mathfrak{S} = \bigcup_{w \in W} \bigcup_{1 \leq j \leq \ell} w.\mathfrak{S}_j$$

where

$$\mathfrak{S}_j = \{\lambda \in \mathfrak{S} \cap \mathfrak{a}^+ \mid \langle \alpha_j, \lambda \rangle = \max_{1 \leq k \leq \ell} \langle \alpha_k, \lambda \rangle\}.$$


 FIGURE 1. Example of barycentric subdivisions in A_3

We project these subdivisions \mathfrak{S}_j onto the unit sphere and denote these projections by S_j . We establish next a smooth version of the partition of unity

$$\sum_{w \in W} \sum_{1 \leq j \leq \ell} \mathbf{1}_{w.S_j} \left(\frac{\lambda}{|\lambda|} \right) = 1 \quad \text{a.e..}$$

Let $\chi_{C_1} : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function such that $\chi_{C_1}(r) = 1$ when $r \geq 0$ and $\chi_{C_1}(r) = 0$ when $r \leq -C_1$, where $C_1 > 0$ is a suitable constant (see [AnZh20, p.8]). For every $w \in W$ and $1 \leq j \leq \ell$, we define

$$\tilde{\chi}_{w.S_j}(\lambda) = \prod_{1 \leq k \leq \ell, k \neq j} \chi_{C_1} \left(\frac{\langle w, \alpha_k, \lambda \rangle}{|\lambda|} \right) \chi_{C_1} \left(\frac{\langle w, \alpha_j, \lambda \rangle - \langle w, \alpha_k, \lambda \rangle}{|\lambda|} \right) \quad \forall \lambda \in \mathfrak{a} \setminus \{0\},$$

and

$$\chi_{w.S_j} = \frac{\tilde{\chi}_{w.S_j}}{\sum_{w \in W} \sum_{1 \leq j \leq \ell} \tilde{\chi}_{w.S_j}}.$$

Then, $\chi_{w.S_j}$ is a homogeneous symbol of order 0 and

$$\sum_{w \in W} \sum_{1 \leq j \leq \ell} \chi_{w.S_j} = 1 \quad \text{on} \quad \mathfrak{a} \setminus \{0\}. \quad (2.6)$$

In addition, the following properties hold for the support of $\chi_{w.S_j}$

Lemma 2.1. ([AnZh20, Proposition 2.6]) *Let $w \in W$ and $1 \leq j \leq \ell$. Then a root $\alpha \in \Sigma$ satisfies either $\langle \alpha, w, \Lambda_j \rangle = 0$ or*

$$|\langle \alpha, \lambda \rangle| \asymp |\lambda| \quad \forall \lambda \in \text{supp } \chi_{w.S_j}. \quad (2.7)$$

Moreover,

$$|\langle w, \Lambda_j, \lambda \rangle| \asymp |\lambda| \quad \forall \lambda \in \text{supp } \chi_{w.S_j}. \quad (2.8)$$

3. POINTWISE ESTIMATES OF THE SCHRÖDINGER KERNEL

For simplicity, we consider in this section the shifted Schrödinger propagator $e^{-it\mathbf{D}^2}$ with $\mathbf{D} = \sqrt{-\Delta - |\rho|^2}$, and denote still by s_t its K -bi-invariant convolution kernel. By using the inverse formula of the spherical Fourier transform, we have

$$s_t(x) = C_0 \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) e^{-it|\lambda|^2} \quad \forall t \in \mathbb{R}^*, \forall x \in \mathbb{X}. \quad (3.1)$$

As usual, such an oscillatory integral makes sense by applying standard procedures (as a limit of convergent integrals and/or after performing several integrals by parts). We will study (3.1)

differently, depending whether $|t|$ is large or small. Let us begin with the easier case where $|t|$ is large.

3.1. Large time kernel estimate. Assume that $|t| \geq 1$. In this case, we establish the following pointwise kernel estimate, by using the standard stationary phase method and the elementary estimate (2.5) about the Plancherel density and its derivatives.

Theorem 3.1. *There exist an integer $N > \max\{d, D\}$ and a constant $C > 0$ such that*

$$|s_t(x)| \leq C |t|^{-\frac{D}{2}} (1 + |x|)^N \varphi_0(x) \quad \forall |t| \geq 1, \forall x \in \mathbb{X}.$$

Proof. By using the integral expression (2.2) of the spherical function, we write

$$s_t(x) = C_0 \int_K dk e^{\langle \rho, A(kx) \rangle} \underbrace{\int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} e^{-it|\lambda|^2} e^{i\langle \lambda, A(kx) \rangle}}_{I(t, A(kx))} \quad (3.2)$$

where $A(kx)$ denotes the \mathfrak{a} -component in the Iwasawa decomposition of kx , which satisfies $|A(kx)| \leq |x|$ and which we abbreviate A in the sequel. **Theorem 3.1** will follow from

$$|I(t, A)| \lesssim |t|^{-\frac{D}{2}} (1 + |A|)^N. \quad (3.3)$$

Let us split up

$$I(t, A) = I_0(t, A) + I_\infty(t, A) = \int_{\mathfrak{a}} d\lambda \chi_t^0(\lambda) \dots + \int_{\mathfrak{a}} d\lambda \chi_t^\infty(\lambda) \dots$$

where $\chi_t^0(\lambda) = \chi(\sqrt{|t|}|\lambda|)$ is a radial cut-off function such that $\text{supp } \chi_t^0 \subset B(0, 2|t|^{-\frac{1}{2}})$, $\chi_t^0 = 1$ on $B(0, |t|^{-\frac{1}{2}})$ and $\chi_t^\infty = 1 - \chi_t^0$. On the one hand, by using (2.5), we easily estimate

$$|I_0(t, A)| \lesssim \int_{|\lambda| \lesssim |t|^{-\frac{1}{2}}} d\lambda |\mathbf{c}(\lambda)|^{-2} \lesssim |t|^{-\frac{D}{2}}. \quad (3.4)$$

On the other hand, after performing N integrations by parts based on

$$e^{-it|\lambda|^2} = -\frac{1}{2it} \sum_{j=1}^{\ell} \frac{\lambda_j}{|\lambda|^2} \frac{\partial}{\partial \lambda_j} e^{-it|\lambda|^2}, \quad (3.5)$$

we obtain

$$I_\infty(t, A) = (2it)^{-N} \int_{\mathfrak{a}} d\lambda e^{-it|\lambda|^2} \left\{ \sum_{j=1}^{\ell} \frac{\partial}{\partial \lambda_j} \circ \frac{\lambda_j}{|\lambda|^2} \right\}^N \left\{ \chi_t^\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} e^{i\langle \lambda, A \rangle} \right\}. \quad (3.6)$$

Assume that

- N_0 derivatives are applied to the cut-off function $\chi_t^0(\lambda)$, which produces $O(|t|^{\frac{N_0}{2}})$,
- N_1 derivatives are applied to the factors $\frac{\lambda_j}{|\lambda|^2}$, which produces $O(|\lambda|^{-N-N_1})$,
- N_2 derivatives are applied to the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$, which is not a symbol in general and which produces

$$\begin{cases} O(|\lambda|^{D-\ell}) & \text{if } |\lambda| \leq 1, \\ O(|\lambda|^{d-\ell}) & \text{if } |\lambda| \geq 1, \end{cases}$$

- N_3 derivatives are applied to the exponential factor $e^{i\langle \lambda, A \rangle}$, which produces $O(|A|^{N_3})$,

with $N_0 + N_1 + N_2 + N_3 = N$. If some derivatives hit the cut-off function $\chi_t^0(\lambda)$, i.e., if $N_0 \geq 1$, then the integral reduces to a spherical shell where $|\lambda| \asymp |t|^{-\frac{1}{2}}$, and the contribution to (3.6) is estimated by

$$|t|^{-\frac{N}{2}} |t|^{\frac{N_0}{2}} |t|^{\frac{N_1}{2}} |t|^{-\frac{D}{2}} |A|^{N_3} \lesssim |t|^{-\frac{D}{2}} (1 + |A|)^N, \quad (3.7)$$

since $|t| \geq 1$. If $N_0 = 0$, then

$$\begin{aligned}
 |I_\infty(t, A)| &\lesssim |t|^{-N} \int_{|\lambda| \gtrsim |t|^{-\frac{1}{2}}} d\lambda |\nabla_\lambda^{N_2} |\mathbf{c}(\lambda)|^{-2}| |\lambda|^{-N-N_1} |A|^{N_3} \\
 &\lesssim |t|^{-N} |A|^{N_3} \left\{ \int_{|t|^{-\frac{1}{2}} \lesssim |\lambda| \leq 1} d\lambda |\lambda|^{D-\ell-N-N_1} + \int_{|\lambda| \geq 1} d\lambda |\lambda|^{d-\ell-N-N_1} \right\} \\
 &\lesssim |t|^{-\frac{D}{2}} (1 + |A|)^N + |t|^{-N} (1 + |A|)^N \lesssim |t|^{-\frac{D}{2}} (1 + |A|)^N
 \end{aligned} \tag{3.8}$$

provided that $N > d$ and $N \geq \frac{D}{2}$. In conclusion, (3.3) follows from (3.4), (3.7) and (3.8). \square

Remark 3.2. *The analysis carried out in the proof of Theorem 3.1 yields at best the following small time estimate*

$$|s_t(x)| \lesssim |t|^{-d} (1 + |x|)^d \varphi_0(x) \quad \forall 0 < |t| < 1, \forall x \in \mathbb{X}. \tag{3.9}$$

3.2. Small time kernel estimate. Assume that $0 < |t| < 1$. Our aim is to reduce the negative power $|t|^{-d}$ in (3.9) to $|t|^{-\frac{d}{2}}$. We shall use different tools, depending on the size of $\frac{|x|}{\sqrt{|t|}}$. If $\frac{|x|}{\sqrt{|t|}}$ is small, we decompose the Weyl chamber into several subcones according to the barycentric decompositions described in Sect. 2.3, and perform in each subcone several integrations by parts along a well chosen direction. If $\frac{|x|}{\sqrt{|t|}}$ is large, we express in addition the Schrödinger propagator in terms of the wave propagator and use the Hadamard parametrix.

Theorem 3.3. *The following estimate holds, for $0 < |t| < 1$ and $|x| \leq \sqrt{|t|}$:*

$$|s_t(x)| \lesssim |t|^{-\frac{d}{2}} \varphi_0(x).$$

Proof. By resuming the notation in the proof of Theorem 3.1, we have

$$s_t(x) = C_0 \int_K dk e^{\langle \rho, A(kx) \rangle} (I_0(t, A) + I_\infty(t, A)).$$

Clearly,

$$|I_0(t, A)| = \left| \int_{\mathfrak{a}} d\lambda \chi_t^0(\lambda) |\mathbf{c}(\lambda)|^{-2} e^{-it|\lambda|^2} e^{i\langle \lambda, A \rangle} \right| \lesssim \int_{|\lambda| \lesssim |t|^{-\frac{1}{2}}} d\lambda |\mathbf{c}(\lambda)|^{-2} \lesssim |t|^{-\frac{d}{2}}. \tag{3.10}$$

In order to estimate

$$I_\infty(t, A) = \int_{\mathfrak{a}} d\lambda \chi_t^\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} e^{-it|\lambda|^2} e^{i\langle \lambda, A \rangle},$$

we split up

$$I_\infty(t, A) = \sum_{w \in W} \sum_{1 \leq j \leq \ell} \underbrace{\int_{\mathfrak{a}} d\lambda \chi_{w.S_j}(\lambda) \chi_t^\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} e^{-it|\lambda|^2} e^{i\langle \lambda, A \rangle}}_{I_{w.S_j}(t, x)}.$$

according to the barycentric decomposition (2.6). Next, we study $I_{w.S_j}(t, x)$ by performing N integrations by parts based on

$$e^{-it|\lambda|^2} = -\frac{1}{2it} \frac{1}{\langle w, \Lambda_j, \lambda \rangle} \partial_{w, \Lambda_j} e^{-it|\lambda|^2},$$

which yields

$$I_{w.S_j}(t, x) = (2it)^{-N} \int_{\mathfrak{a}} d\lambda e^{-it|\lambda|^2} \left\{ \partial_{w, \Lambda_j} \circ \frac{1}{\langle w, \Lambda_j, \lambda \rangle} \right\}^N \left\{ \chi_{w.S_j}(\lambda) \chi_t^\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} e^{i\langle \lambda, A \rangle} \right\}.$$

As in the the proof of Theorem 3.1, we assume that

- N_0 derivatives are applied to the cut-off function $\chi_t^\infty(\lambda)$:

$$\partial_{w.\Lambda_j}^{N_0} \chi_t^\infty(\lambda) = O(|t|^{-\frac{N_0}{2}}),$$

- N_1 derivatives are applied to the factors $\frac{1}{\langle w.\Lambda_j, \lambda \rangle}$, which produces $O(|\lambda|^{-N-N_1})$,
- N_2 derivatives are applied to the factor $\chi_{w.S_j}(\lambda)$, which is a homogeneous symbol of order 0:

$$\partial_{w.\Lambda_j}^{N_2} \chi_{w.S_j}(\lambda) = O(|\lambda|^{-N_2}),$$

- N_3 derivatives are applied to the factor $e^{i\langle \lambda, A \rangle}$:

$$\partial_{w.\Lambda_j}^{N_3} e^{i\langle \lambda, A \rangle} = O(|A|^{N_3}),$$

- N_4 derivatives are applied to the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$,

with $N_0 + N_1 + N_2 + N_3 + N_4 = N$. According to [Lemma 2.1](#), any root $\alpha \in \Sigma$ satisfies either $\langle \alpha, w.\Lambda_j \rangle = 0$ or $|\langle \alpha, \lambda \rangle| \asymp |\lambda|$, for every $\lambda \in \text{supp } \chi_{w.S_j}$. On the one hand, if $\langle \alpha, w.\Lambda_j \rangle = 0$, all derivatives

$$\partial_{w.\Lambda_j}^k |\mathbf{c}_\alpha(\langle \alpha, \lambda \rangle)|^{-2} \quad \forall k \in \mathbb{N}^*$$

vanish. On the other hand, if $\langle \alpha, w.\Lambda_j \rangle \neq 0$, we have

$$|\partial_{w.\Lambda_j}^k |\mathbf{c}_\alpha(\langle \alpha, \lambda \rangle)|^{-2}| \lesssim |\langle \alpha, \lambda \rangle|^{m_\alpha + m_{2\alpha} - k} \asymp |\lambda|^{m_\alpha + m_{2\alpha} - k}$$

for any $k \in \mathbb{N}$ and for every $\lambda \in (\text{supp } \chi_{w.S_j}) \cap (\text{supp } \chi_t^\infty)$. Here we have used [\(2.7\)](#) and the fact that $|\mathbf{c}_\alpha|^{-2}$ is an inhomogeneous symbol of order $m_\alpha + m_{2\alpha}$ on \mathbb{R} . Hence

$$\partial_{w.\Lambda_j}^{N_4} |\mathbf{c}(\lambda)|^{-2} = O(|\lambda|^{d-\ell-N_4}) \quad \forall \lambda \in (\text{supp } \chi_{w.S_j}) \cap (\text{supp } \chi_t^\infty).$$

Therefore, if no derivative hits the cut-off function $\chi_t^\infty(\lambda)$, i.e., if $N_0 = 0$, then

$$\begin{aligned} |I_{w.S_j}(t, x)| &\lesssim |t|^{-N} \int_{|\lambda| \gtrsim |t|^{-\frac{1}{2}}} d\lambda |\lambda|^{-N-N_1-N_2+d-\ell-N_4} |A|^{N_3} \\ &\lesssim |t|^{-N} |t|^{-\frac{d}{2} + \frac{N}{2} + \frac{N_1}{2} + \frac{N_2}{2} + \frac{N_4}{2}} |t|^{\frac{N_3}{2}} \leq |t|^{-\frac{d}{2}} \end{aligned}$$

provided that $N > d$. If $N_0 \geq 1$, then the integral is reduced to a spherical shell where $|\lambda| \asymp |t|^{-\frac{1}{2}}$, and hence

$$|I_{w.S_j}(t, x)| \lesssim |t|^{-N} |t|^{\frac{N_0}{2}} |t|^{-\frac{d}{2} + \frac{N}{2} + \frac{N_1}{2} + \frac{N_2}{2} + \frac{N_4}{2}} |A|^{N_3} \leq |t|^{-\frac{d}{2}},$$

since $|A| \leq |x| \leq \sqrt{|t|}$. Together with [\(3.10\)](#), we conclude that $|I(t, A)| \lesssim |t|^{-\frac{d}{2}}$ and

$$|s_t(x)| \lesssim |t|^{-\frac{d}{2}} \varphi_0(x),$$

for all $0 < |t| < 1$ and $x \in \mathbb{X}$ such that $|x| \leq \sqrt{|t|}$. \square

The above proof shows that, for every $\lambda \in (\text{supp } \chi_{w.S_j}) \cap (\text{supp } \chi_t^\infty)$, the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$ behaves like an inhomogeneous symbol of order $d - \ell$ if we differentiate it along the direction $w.\Lambda_j$. When $|x| > \sqrt{|t|}$, we combine this argument with the Hadamard parametrix, which was used in the study of other equations on symmetric spaces, see for instance [\[CGM01; AnZh20\]](#).

Let us express the Schrödinger propagator

$$e^{-it\mathbf{D}^2} = \underbrace{\pi^{-\frac{1}{2}} e^{-i\frac{\pi}{4} \text{sign}(t)}}_{C_2} |t|^{-\frac{1}{2}} \int_0^{+\infty} ds e^{\frac{i}{4t} s^2} \cos(s\mathbf{D})$$

in terms of the wave propagator and correspondingly

$$s_t(x) = C_2 |t|^{-\frac{1}{2}} \int_0^{+\infty} ds e^{\frac{i}{4t}s^2} \Phi_s(x) \quad (3.11)$$

for their K -bi-invariant convolution kernels. On the one hand, by finite propagation speed,

$$\Phi_s(x) = 0 \quad \text{if } |x| > |s|. \quad (3.12)$$

On the other hand, recall the Hadamard parametrix

$$\Phi_s(\exp H) = J(H)^{-\frac{1}{2}} |s| \sum_{k=0}^{+\infty} 4^{-k} U_k(H) R_+^{k-\frac{d-1}{2}} (s^2 - |H|^2) \quad \forall s \in \mathbb{R}^*, \forall H \in \mathfrak{p}, \quad (3.13)$$

where J denotes the Jacobian of the exponential map $\mathfrak{p} \rightarrow G/K$, which is given by

$$J(H) = \prod_{\alpha \in \Sigma^+} \left(\frac{\sinh\langle \alpha, H \rangle}{\langle \alpha, H \rangle} \right)^{m_\alpha} \quad \forall H \in \mathfrak{a}^+,$$

and $\{R_+^z \mid z \in \mathbb{C}\}$ denotes the analytic family of Riesz distributions on \mathbb{R} , which is defined by

$$R_+^z(r) = \begin{cases} \Gamma(z)^{-1} r^{z-1} & \text{if } r > 0 \\ 0 & \text{if } r \leq 0 \end{cases} \quad \forall \operatorname{Re} z > 0.$$

This parametrix was constructed and used in various settings, see for instance [Ber77; Hor94; CGM01]. We refer to [AnZh20, Appendix B] for details about the wave propagator $\cos(t\sqrt{-\Delta})$ associated to the unshifted Laplacian Δ on noncompact symmetric spaces and notice that the same results hold for $\cos(t\mathbf{D})$. Specifically (3.13) is an asymptotic expansion

$$\Phi_s(\exp H) = J(H)^{-\frac{1}{2}} |s| \sum_{k=0}^{[d/2]} 4^{-k} U_k(H) R_+^{k-\frac{d-1}{2}} (s^2 - |H|^2) + E(s, H) \quad (3.14)$$

where the coefficients U_k are Ad K -invariant smooth functions on \mathfrak{p} , which are bounded together with their derivatives, while the remainder satisfies

$$|E(s, H)| \lesssim (1 + |s|)^{3(\frac{d}{2}+1)} e^{-\langle \rho, H \rangle} \quad \forall s \in \mathbb{R}^*, \forall H \in \overline{\mathfrak{a}^+}. \quad (3.15)$$

Let us split up

$$\int_0^{+\infty} ds = \int_0^{+\infty} ds \chi_0\left(\frac{s}{|x|}\right) + \int_0^{+\infty} ds \chi_1\left(\frac{s}{|x|}\right) + \int_0^{+\infty} ds \chi_\infty\left(\frac{s}{|x|}\right)$$

in (3.11) by means of a smooth partition of unity $1 = \chi_0 + \chi_1 + \chi_\infty$ on \mathbb{R} such that

$$\begin{cases} \operatorname{supp} \chi_0 \subset (-1, 1), \\ \operatorname{supp} \chi_1 \subset (-2C_3, -\frac{1}{2}) \cup (\frac{1}{2}, 2C_3), \\ \operatorname{supp} \chi_\infty \subset (-\infty, -C_3) \cup (C_3, +\infty) \end{cases}$$

where the choice of $C_3 > 1$ will be specified later. Then the contribution of the first integral vanishes according to (3.12) and we are left with

$$s_t(x) = \underbrace{C_2 |t|^{-\frac{1}{2}} \int_0^{+\infty} ds \chi_1\left(\frac{s}{|x|}\right) e^{\frac{i}{4t}s^2} \Phi_s(x)}_{s_t^1(x)} + \underbrace{C_2 |t|^{-\frac{1}{2}} \int_0^{+\infty} ds \chi_\infty\left(\frac{s}{|x|}\right) e^{\frac{i}{4t}s^2} \Phi_s(x)}_{s_t^\infty(x)}$$

where $s_t^1(x)$ and $s_t^\infty(x)$ are K -bi-invariant. Let us first study $s_t^\infty(x)$ by using again the barycentric decomposition. In comparison with the proof of Theorem 3.3, we have now $|x| > \sqrt{|t|}$ and there is an additional integral over $s \in (1, \infty)$ to control. Let us state the theorem.

Theorem 3.4. *The following estimate holds, for all $0 < |t| < 1$ and $|x| > \sqrt{|t|}$:*

$$|s_t^\infty(x)| \lesssim |t|^{-\frac{d}{2}} \varphi_0(x).$$

Proof. We express

$$s_t^\infty(x) = \frac{1}{2} C_0 C_2 |t|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} ds \chi_\infty\left(\frac{s}{|x|}\right) e^{\frac{i}{4t}s^2} \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) e^{-is|\lambda|}$$

by evenness and by expressing the wave kernel Φ_s by means of the inverse spherical Fourier transform. Let us split up $s_t^\infty = \frac{1}{2} C_0 C_2 (s_t^{\infty,0} + s_t^{\infty,\infty})$, where

$$s_t^{\infty,0}(x) = |t|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} ds \chi_\infty\left(\frac{s}{|x|}\right) e^{\frac{i}{4t}s^2} \underbrace{\int_{\mathfrak{a}} d\lambda \chi_t^0(\lambda) |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) e^{-is|\lambda|}}_{I_0(s,t,x)}$$

and

$$s_t^{\infty,\infty}(x) = |t|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} ds \chi_\infty\left(\frac{s}{|x|}\right) e^{\frac{i}{4t}s^2} \underbrace{\int_{\mathfrak{a}} d\lambda \chi_t^\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) e^{-is|\lambda|}}_{I_\infty(s,t,x)}.$$

Recall that $\chi_t^0(\lambda) = \chi(\sqrt{|t||\lambda|})$ is a radial cut-off function such that $\text{supp } \chi_t^0 \subset B(0, 2|t|^{-\frac{1}{2}})$, $\chi_t^0 = 1$ on $B(0, |t|^{-\frac{1}{2}})$ and $\chi_t^\infty = 1 - \chi_t^0$.

Estimate of $s_t^{\infty,0}$. Notice that the obvious estimate $|I_0(s,t,x)| \lesssim |t|^{-\frac{d}{2}} |\varphi_0(x)|$ is not enough for our purpose. We need indeed to compensate on the one hand the factor $|t|^{-\frac{1}{2}}$ and to get on the other hand enough decay in $|s|$ to ensure the convergence of the external integral. To this end, we perform two integrations by parts based on

$$e^{\frac{i}{4t}s^2} = -\frac{2it}{s} \frac{\partial}{\partial s} e^{\frac{i}{4t}s^2} \quad (3.16)$$

and obtain this way

$$s_t^{\infty,0}(x) = -4|t|^{\frac{3}{2}} \int_{-\infty}^{+\infty} ds e^{\frac{i}{4t}s^2} \frac{\partial}{\partial s} \left(\frac{1}{s} \frac{\partial}{\partial s} \right) \left\{ \frac{1}{s} \chi_\infty\left(\frac{s}{|x|}\right) I_0(s,t,x) \right\}. \quad (3.17)$$

Notice that

$$\left| \left(\frac{\partial}{\partial s} \right)^k I_0(s,t,x) \right| \lesssim |t|^{-\frac{d+k}{2}} \varphi_0(x) \quad \forall k \in \mathbb{N}.$$

If any derivative hits $\chi_\infty\left(\frac{s}{|x|}\right)$ in (3.17), the integral reduces to two intervals where $|s| \asymp |x|$, and the corresponding contribution is estimated by

$$|t|^{-\frac{d-3}{2}} \varphi_0(x) \int_{|s| \asymp |x|} ds \{ s^{-2} |x|^{-2} + s^{-3} |x|^{-1} + s^{-2} |x|^{-1} |t|^{-\frac{1}{2}} \} \lesssim |t|^{-\frac{d}{2}} \varphi_0(x),$$

since $|t|^{\frac{1}{2}} < |x|$. Otherwise we end up with the estimate

$$|t|^{-\frac{d-3}{2}} \varphi_0(x) \int_{|s| \gtrsim |x|} ds \{ s^{-4} + s^{-3} |t|^{-\frac{1}{2}} + s^{-2} |t|^{-1} \} \lesssim |t|^{-\frac{d}{2}} \varphi_0(x).$$

In conclusion,

$$|s_t^{\infty,0}(x)| \lesssim |t|^{-\frac{d}{2}} \varphi_0(x),$$

for all $0 < |t| < 1$ and $x \in \mathbb{X}$ such that $|x| > \sqrt{|t|}$.

Estimate of $s_t^{\infty,\infty}$. Let us turn to

$$s_t^{\infty,\infty}(x) = \int_{-\infty}^{+\infty} ds \chi_\infty\left(\frac{s}{|x|}\right) e^{\frac{i}{4t}s^2} I_\infty(s,t,x).$$

We will prove the following estimate, for any integer $N > d$,

$$|I_\infty(s, t, x)| \lesssim |s|^{-N} |t|^{-\frac{d}{2} + \frac{N}{2}} \varphi_0(x) \quad \forall |s| \geq C_3|x|. \quad (3.18)$$

Then, as $|x| > \sqrt{|t|}$, we conclude easily that

$$|s_t^{\infty, \infty}(x)| \lesssim |t|^{-\frac{d}{2} + \frac{N}{2} - \frac{1}{2}} \varphi_0(x) \int_{|s| \gtrsim |x|} ds |s|^{-N} \lesssim |t|^{-\frac{d}{2}} \left(\frac{\sqrt{|t|}}{|x|}\right)^{N-1} \varphi_0(x) \lesssim |t|^{-\frac{d}{2}} \varphi_0(x). \quad (3.19)$$

In order to establish (3.18), we express

$$\begin{aligned} I_\infty(s, t, x) &= \int_{\mathfrak{a}} d\lambda \chi_t^\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) e^{-is|\lambda|} \\ &= \int_K dk e^{-\langle \rho, A \rangle} \sum_{w \in W} \sum_{1 \leq j \leq \ell} \underbrace{\int_{\mathfrak{a}} d\lambda \chi_{w.S_j}(\lambda) \chi_t^\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} e^{-i(s|\lambda| - \langle \lambda, A \rangle)}}_{I_{w.S_j}(s, t, A)} \end{aligned}$$

by using again the integral formula (2.2) and the barycentric decomposition (2.6). According to (2.8), we can choose $C_3 > 0$ such that, if $|s| \geq C_3|x|$, then

$$\begin{aligned} |\partial_{w.\Lambda_j}(s|\lambda| - \langle \lambda, A \rangle)| &= \left| s \frac{\langle w.\Lambda_j, \lambda \rangle}{|\lambda|} - \langle w.\Lambda_j, A \rangle \right| \\ &\geq |s| \underbrace{\frac{|\langle w.\Lambda_j, \lambda \rangle|}{|\lambda|}}_{\gtrsim 1} - \underbrace{|\langle w.\Lambda_j, A \rangle|}_{\lesssim |x|} \gtrsim |s|, \end{aligned}$$

for every $\lambda \in (\text{supp } \chi_t^\infty) \cap (\text{supp } \chi_{w.S_j})$. Under these assumptions, the phase function $\lambda \mapsto s|\lambda| - \langle \lambda, A \rangle$ has no critical point along the direction $w.\Lambda_j$. By performing N integrations by parts based on

$$e^{-i(s|\lambda| - \langle \lambda, A \rangle)} = \frac{i}{\partial_{w.\Lambda_j}(s|\lambda| - \langle \lambda, A \rangle)} \partial_{w.\Lambda_j} e^{-i(s|\lambda| - \langle \lambda, A \rangle)},$$

we write

$$\begin{aligned} I_{w.S_j}(s, t, A) &= (is)^{-N} \int_{\mathfrak{a}} d\lambda e^{-i(s|\lambda| - \langle \lambda, A \rangle)} \\ &\quad \left\{ \partial_{w.\Lambda_j} \circ \frac{s}{\partial_{w.\Lambda_j}(s|\lambda| - \langle \lambda, A \rangle)} \right\}^N \left\{ \chi_{w.S_j}(\lambda) \chi_t^\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} \right\}. \end{aligned}$$

Assume that

- N_0 derivatives are applied to the cut-off function $\chi_t^\infty(\lambda)$, which produces $O(|t|^{\frac{N_0}{2}})$,
- N_1 derivatives are applied to the factors $\frac{s}{\partial_{w.\Lambda_j}(s|\lambda| - \langle \lambda, A \rangle)}$, which produces $O(|\lambda|^{-N_1})$,
- N_2 derivatives are applied to the cut-off functions $\chi_{w.S_j}(\lambda)$, which produces $O(|\lambda|^{-N_2})$,
- N_3 derivatives are applied to the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$, which produces $O(|\lambda|^{d-\ell-N_3})$,

with $N_0 + N_1 + N_2 + N_3 = N$. Again, if some derivatives hit $\chi_t^\infty(\lambda)$, i.e., if $N_0 \geq 1$, then the integral reduces to a spherical shell where $|\lambda| \asymp |t|^{-\frac{1}{2}}$, and its contribution is estimated by

$$|s|^{-N} |t|^{-\frac{\ell}{2}} |t|^{\frac{N_0}{2}} |t|^{\frac{N_1}{2}} |t|^{\frac{N_2}{2}} |t|^{-\frac{d}{2} + \frac{\ell}{2} + \frac{N_3}{2}} = |s|^{-N} |t|^{-\frac{d}{2} + \frac{N}{2}}.$$

If $N_0 = 0$, then

$$|I_{w.S_j}(s, t, A)| \lesssim |s|^{-N} \int_{|\lambda| \gtrsim |t|^{-\frac{1}{2}}} d\lambda |\lambda|^{-N_1} |\lambda|^{-N_2} |\lambda|^{d-\ell-N_3} \lesssim |s|^{-N} |t|^{-\frac{d}{2} + \frac{N}{2}}$$

provided that $N > d$. This proves (3.18) and hence (3.19). \square

Theorem 3.5. *The following estimate holds, for all $0 < |t| < 1$ and $|x| > \sqrt{|t|}$:*

$$|s_t^1(x)| \lesssim |t|^{-\frac{d}{2}} (1 + |x|)^{\frac{3}{2}d+4} e^{-\langle \rho, x^+ \rangle}.$$

Proof. Since s_t^1 is K -bi-invariant, we have

$$\begin{aligned} s_t^1(x) &= \frac{C_2}{2} |t|^{-\frac{1}{2}} J(x^+)^{-\frac{1}{2}} \sum_{k=0}^{[d/2]} 4^{-k} U_k(x^+) \overbrace{\int_0^{+\infty} d(s^2) \chi_1\left(\frac{s}{|x|}\right) e^{\frac{i}{4t}s^2} R_+^{k-\frac{d-1}{2}}(s^2 - |x|^2)}^{I_k(t, |x|)} \\ &\quad + \frac{C_2}{2} |t|^{-\frac{1}{2}} \underbrace{\int_0^{+\infty} ds \chi_1\left(\frac{s}{|x|}\right) e^{\frac{i}{4t}s^2} E(s, x^+)}_{\tilde{E}(t, |x|)} \end{aligned}$$

according to (3.14). On the one hand, the remainder estimate

$$|\tilde{E}(t, |x|)| \lesssim |x| (1 + |x|)^{3(\frac{d}{2}+1)} e^{-\langle \rho, x^+ \rangle} \quad (3.20)$$

follows from (3.15). On the other hand, we claim that

$$|I_k(t, |x|)| \lesssim |t|^{k-\frac{d-1}{2}} \quad (3.21)$$

if $|x| > \sqrt{|t|} > 0$. Let us first prove (3.21) when d is odd. By a change of variables and by using the fact that $R_+^{k-\frac{d-1}{2}}(s-1) = \left(\frac{\partial}{\partial s}\right)^{\frac{d-1}{2}-k} R_+^0(s-1)$, we obtain

$$\begin{aligned} |I_k(t, |x|)| &= |x|^{2k-d+1} \int_0^{+\infty} ds \chi_1(\sqrt{s}) e^{\frac{i|x|^2}{4t}s} R_+^{k-\frac{d-1}{2}}(s-1) \\ &= |x|^{2k-d+1} \int_0^{+\infty} ds R_+^0(s-1) \left(-\frac{\partial}{\partial s}\right)^{\frac{d-1}{2}-k} \{\chi_1(\sqrt{s}) e^{\frac{i|x|^2}{4t}s}\}. \end{aligned}$$

As $R_+^0(s-1)$ is the Dirac measure at $s=1$, we conclude that

$$I_k(t, |x|) = |x|^{2k-d+1} \left(-\frac{i|x|^2}{4t}\right)^{\frac{d-1}{2}-k} = O(|t|^{k-\frac{d-1}{2}}).$$

When d is even, we obtain similarly

$$|I_k(t, |x|)| = \pi^{-\frac{1}{2}} |x|^{2k-d+1} \int_1^{+\infty} \frac{ds}{\sqrt{s-1}} \left(-\frac{\partial}{\partial s}\right)^{\frac{d}{2}-k} \{\chi_1(\sqrt{s}) e^{\frac{i|x|^2}{4t}s}\},$$

which is a linear combination of expressions

$$t^{j+k-\frac{d}{2}} |x|^{1-2j} \underbrace{\int_1^{+\infty} \frac{ds}{\sqrt{s-1}} \theta_j(s) e^{\frac{i|x|^2}{4t}s}}_{J_j(t, |x|)}$$

where $0 \leq j \leq \frac{d}{2} - k$ and $\theta_j \in C_c^\infty(\mathbb{R})$ with $\text{supp } \theta_j \subset (-4C_3^2, 4C_3^2)$. Notice that the elementary estimate $J_j(t, |x|) = O(1)$, together with the assumption $|x| > \sqrt{|t|}$ implies that

$$|I_k(t, |x|)| \lesssim |x| |t|^{-\frac{d}{2}+k},$$

which might be enough for our purpose as long as $k > 0$. The case $k = 0$ requires actually a more careful analysis. Let us show that

$$J_j(t, |x|) \lesssim \frac{\sqrt{|t|}}{|x|}$$

by splitting up

$$\int_1^{+\infty} ds = \int_1^{1+\frac{|t|}{|x|^2}} ds + \int_{1+\frac{|t|}{|x|^2}}^{+\infty} ds$$

in the definition of $J_j(t, |x|)$. The contribution of the first integral is easily estimated by

$$\int_1^{1+\frac{|t|}{|x|^2}} ds \frac{ds}{\sqrt{s-1}} = 2\sqrt{s-1} \Big|_{s=1}^{s=1+\frac{|t|}{|x|^2}} = 2\frac{\sqrt{|t|}}{|x|}.$$

After performing an integration by parts based on

$$e^{\frac{i|x|^2}{4t}s} = -i \frac{4t}{|x|^2} \frac{\partial}{\partial s} e^{\frac{i|x|^2}{4t}s},$$

the contribution of the second integral is also estimated by

$$\frac{|t|}{|x|^2} \int_{1+\frac{|t|}{|x|^2}}^{4C_3^2} ds \{(s-1)^{-\frac{1}{2}} + (s-1)^{-\frac{3}{2}}\} \lesssim \frac{\sqrt{|t|}}{|x|}$$

under the assumption $|x| > \sqrt{|t|}$. Thus (3.21) holds as well when d is even. In conclusion,

$$\begin{aligned} |s_t^1(x)| &\lesssim |t|^{-\frac{d}{2}} J(x^+)^{-\frac{1}{2}} + |t|^{-\frac{1}{2}} |x| (1+|x|)^{3(\frac{d}{2}+1)} e^{-\langle \rho, x^+ \rangle} \\ &\lesssim |t|^{-\frac{d}{2}} (1+|x|)^{\frac{3}{2}d+4} e^{-\langle \rho, x^+ \rangle} \end{aligned}$$

when $0 < |t| < 1$ and $x \in \mathbb{X}$ satisfies $|x| > \sqrt{|t|}$. \square

In summary, we have divided our kernel analysis into three parts and deduced [Theorem 1.1](#) from [Theorem 3.1](#), [Theorem 3.3](#) and [Theorem 3.5](#). Notice that the method used to prove small time kernel estimates can be also used for large time.

4. DISPERSIVE ESTIMATES AND STRICHARTZ INEQUALITIES ON SYMMETRIC SPACES

Once pointwise kernel estimates are available on symmetric spaces, one can deduce dispersive properties for the corresponding propagator, by using an interpolation argument based on the Kunze-Stein phenomenon, more precisely the following K -bi-invariant version, which is a straightforward generalization of [\[APV11, Theorem 4.2\]](#).

Lemma 4.1. *Let κ be a reasonable K -bi-invariant function on G . Then*

$$\|\cdot * \kappa\|_{L^{q'}(\mathbb{X}) \rightarrow L^q(\mathbb{X})} \leq \left\{ \int_G dx \varphi_0(x) |\kappa(x)|^{\frac{q}{2}} \right\}^{\frac{2}{q}}$$

for every $q \in [2, +\infty)$. In the limit case $q = \infty$,

$$\|\cdot * \kappa\|_{L^1(\mathbb{X}) \rightarrow L^\infty(\mathbb{X})} = \sup_{x \in G} |\kappa(x)|.$$

The following dispersive property generalizes the results previously obtained on hyperbolic spaces. Its proof is adapted straightforwardly from the rank one case [\[AnPi09; APV11\]](#) and is therefore omitted.

Theorem 4.2 (Dispersive property). *Let $2 < q, \tilde{q} \leq +\infty$. There exists a constant $C > 0$ such that, for every $t \in \mathbb{R}^*$,*

$$\|e^{it\Delta}\|_{L^{\tilde{q}'}(\mathbb{X}) \rightarrow L^q(\mathbb{X})} \leq C \begin{cases} |t|^{-\max\{\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\tilde{q}}\}d} & \text{if } 0 < |t| < 1, \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1 \end{cases}$$

As already observed on hyperbolic spaces [\[Ban07; AnPi09\]](#), the large scale dispersive effects in negative curvature imply stronger dispersion properties than in the Euclidean setting. Notice that $D = 3$ in rank one and that D is larger in higher rank. Thus we obtain a fast decay for large time, which is independent of q and \tilde{q} .

By using the classical TT^* method (see [\[GiVe95\]](#) and [\[KeTa98\]](#) for the endpoint), one can deduce the Strichartz inequality from the dispersive property. This space-time estimate serves

as an important tool for finding minimal regularity conditions on the initial data ensuring well-posedness of nonlinear Schrödinger equations.

Consider the inhomogeneous linear Schrödinger equation on \mathbb{X} :

$$\begin{cases} i\partial_t u(t, x) + \Delta_x u(t, x) = F(t, x), \\ u(0, x) = f(x), \end{cases} \quad (\text{LS})$$

whose solution is given by Duhamel's formula:

$$u(t, x) = e^{it\Delta} f(x) - i \int_0^t ds e^{i(t-s)\Delta} F(s, x).$$

A couple (p, q) is called *admissible* if $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle

$$\{(\frac{1}{p}, \frac{1}{q}) \in (0, \frac{1}{2}] \times (0, \frac{1}{2}) \mid \frac{2}{p} + \frac{d}{q} \geq \frac{d}{2}\} \cup \{(0, \frac{1}{2})\}$$

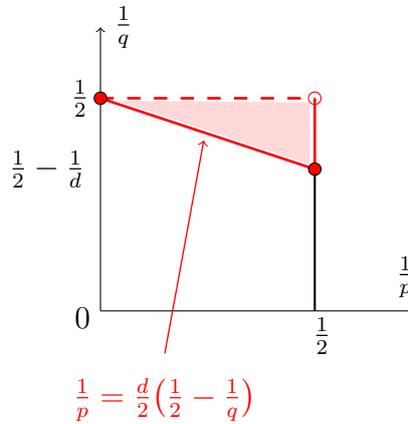


FIGURE 2. Admissibility in dimension $d \geq 3$.

Theorem 4.3. *Let (p, q) and (\tilde{p}, \tilde{q}) be two admissible couples and $I \subseteq \mathbb{R}$ be any bounded or unbounded time interval. Then there exists a constant $C > 0$ such that the following estimate holds for all solutions to the Cauchy problem (LS):*

$$\|u\|_{L^p(I; L^q(\mathbb{X}))} \leq C \{ \|f\|_{L^2(\mathbb{X})} + \|F\|_{L^{\tilde{p}'}(I; L^{\tilde{q}'}(\mathbb{X}))} \}.$$

This inequality was established progressively in rank one, see for instance [Pie06; BCS08; AnPi09; IoSt09] on real hyperbolic spaces and [Pie08; APV11] on Damek-Ricci spaces. In higher rank, the admissible set for \mathbb{X} is again much larger than the admissible set for \mathbb{R}^d , which corresponds only to the lower edge of the admissible triangle. Notice however that we don't take full advantage of the large time decay $|t|^{-\frac{d}{2}}$ in Theorem 4.2. The proof of Theorem 4.3 requires indeed only $|t|^{-1}$.

5. WELL-POSEDNESS AND SCATTERING FOR THE SEMILINEAR SCHRÖDINGER EQUATION

Strichartz inequalities for linear inhomogeneous equations are used to prove well-posedness and scattering results for nonlinear perturbations. We list in this section some of these results in higher rank. The proofs are adapted straightforwardly from the rank one case considered in [AnPi09; APV11] and are therefore omitted.

Consider the nonlinear Schrödinger equation on \mathbb{X} :

$$\begin{cases} i\partial_t u(t, x) + \Delta_x u(t, x) = F(u(t, x)), \\ u(0, x) = f(x), \end{cases} \quad (\text{NLS})$$

where $F = F(u(t, x))$ is a *power-like* nonlinearity of order $\gamma > 1$ in the sense that

$$|F(u)| \lesssim |u|^\gamma \quad \text{and} \quad |F(u) - F(v)| \lesssim (|u|^{\gamma-1} + |v|^{\gamma-1}) |u - v|.$$

Recall some classical definitions in the analysis of PDE. The nonlinearity F is called *gauge invariant* if $\text{Im}\{F(u)\bar{u}\} = 0$. It is well known that gauge invariance implies L^2 conservation of mass. If there exists a nonnegative \mathcal{C}^1 function G such that $F(u) = G'(|u|^2)u$, the nonlinearity F is called *defocusing* and we have H^1 conservation of energy. Here the Sobolev space $H^1(\mathbb{X})$ is defined as the image of $L^2(\mathbb{X})$ under the operator $(-\Delta)^{-\frac{1}{2}}$. Let us state our well-posedness and scattering results.

Well-posedness for nonlinear Schrödinger equation on \mathbb{X} :

- If $1 < \gamma \leq 1 + \frac{4}{d}$, the Cauchy problem (NLS) is globally well-posed for small L^2 data.
- If $1 < \gamma < 1 + \frac{4}{d}$, the Cauchy problem (NLS) is locally well-posed for arbitrary L^2 data. Moreover, if F is in addition gauge invariant, then the L^2 conservation of mass implies the global well-posedness for arbitrary L^2 data in this subcritical case.
- If $1 < \gamma \leq 1 + \frac{4}{d-2}$, the Cauchy problem (NLS) is globally well-posed for small H^1 data.
- If $1 < \gamma < 1 + \frac{4}{d-2}$, the Cauchy problem (NLS) is locally well-posed for arbitrary H^1 data. Moreover, if F is in addition defocusing, then the H^1 conservation of energy implies the global well-posedness for arbitrary H^1 data in this subcritical case.

As for real hyperbolic spaces, the stronger Strichartz inequality obtained on noncompact symmetric spaces implies better well-posedness results than in Euclidean spaces. For instance, small L^2 data global well-posedness holds for any exponent $1 < \gamma \leq 1 + \frac{4}{d}$ on \mathbb{X} , while on \mathbb{R}^d one must assume in addition gauge invariance. However, under this condition, one can handle arbitrary L^2 data by using conservation laws as in Euclidean spaces.

Scattering for nonlinear Schrödinger equation on \mathbb{X} :

- If $1 < \gamma \leq 1 + \frac{4}{d}$, then global solutions $u(t, x)$ to the Cauchy problem (NLS) corresponding to small L^2 data have the following scattering property: there exists $u_\pm \in L^2$ such that

$$\|u(t, \cdot) - e^{it\Delta} u_\pm\|_{L^2(\mathbb{X})} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty.$$

- If $1 < \gamma \leq 1 + \frac{4}{d-2}$, then global solutions $u(t, x)$ to the Cauchy problem (NLS) corresponding to small H^1 data have the following scattering property: there exists $u_\pm \in H^1$ such that

$$\|u(t, \cdot) - e^{it\Delta} u_\pm\|_{H^1(\mathbb{X})} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty.$$

On Euclidean spaces, scattering to linear solutions is known to fail in general for small exponents $\gamma \in (1, 1 + \frac{2}{d}]$. On noncompact symmetric spaces, the stronger Strichartz inequality allows us again to obtain scattering for small data in the full range.

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