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Hereditary rigidity, separation and density

In memory of Professor I.G. Rosenberg.

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Abstract—We continue the investigation of systems of hereditarily rigid relations started in Couceiro, Haddad, Pouzet and Schölzel [1]. We observe that on a set V with m elements, there is a hereditarily rigid set \mathcal{R} made of n tournaments if and only if $m(m-1) \leq 2^n$. We ask if the same inequality holds when the tournaments are replaced by linear orders. This problem has an equivalent formulation in terms of separation of linear orders. Let $h_{\text{Lin}}(m)$ be the least cardinal n such that there is a family \mathcal{R} of n linear orders on an m -element set V such that any two distinct ordered pairs of distinct elements of V are separated by some member of \mathcal{R} , then $\lceil \log_2(m(m-1)) \rceil \leq h_{\text{Lin}}(m)$ with equality if $m \leq 7$. We ask whether the equality holds for every m . We prove that $h_{\text{Lin}}(m+1) \leq h_{\text{Lin}}(m) + 1$. If V is infinite, we show that $h_{\text{Lin}}(m) = \aleph_0$ for $m \leq 2^{\aleph_0}$. More generally, we prove that the two equalities $h_{\text{Lin}}(m) = \log_2(m) = d(\text{Lin}(V))$ hold, where $\log_2(m)$ is the least cardinal μ such that $m \leq 2^\mu$, and $d(\text{Lin}(V))$ is the topological density of the set $\text{Lin}(V)$ of linear orders on V (viewed as a subset of the power set $\mathcal{P}(V \times V)$ equipped with the product topology). These equalities follow from the Generalized Continuum Hypothesis, but we do not know whether they hold without any set theoretical hypothesis.

I. INTRODUCTION

The motivation for this paper is a question which can be better formulated in terms of Social Choice Theory. Let us consider a committee of n members c_1, \dots, c_n having to express its preferences among m candidates. Each member c_k writes his own preferences among the m candidates in a linearly ordered list ℓ_k of the candidates. The *profile* of an ordered pair (x, y) of two different candidates x and y is the 0-1 list $(\ell_k(x, y))_{1 \leq k \leq n}$, where $\ell_k(x, y) = 1$ if x is preferred to y , and $\ell_k(x, y) = 0$ otherwise. As the profiles of the $m(m-1)$ ordered pairs belong to $\{0, 1\}^n$, if $m(m-1) > 2^n$, there are two distinct ordered pairs with the same profile. The question is: does the converse hold? That is, if $m(m-1) \leq 2^n$, are there n lists $(\ell_k)_{1 \leq k \leq n}$ yielding $m(m-1)$ distinct profiles? As we will see, the answer is positive for $m \leq 7$. For other integers we do not know.

Tackling this question, we do not limit ourselves to finite sets. Considering a set V of cardinality m , let $h_{\text{Lin}}(m)$ be the least cardinal n such that there is a family \mathcal{R} of n linear orders on V such that any distinct ordered pairs (x, y) and (x', y') of distinct elements of V yield distinct profiles. This parameter plays a role in the investigation of systems of hereditarily rigid relations started in Couceiro, Haddad, Pouzet and Schölzel [1]. An h -ary relation ρ on a set V is said to be *hereditarily rigid* if the unary partial functions on V that preserve ρ are the subfunctions of the identity map or of constant maps. A family of relations \mathcal{R} is said to be *hereditarily rigid* if the unary partial functions on V that preserve every $\rho \in \mathcal{R}$ are the subfunctions of the identity map or of constant maps. As it turns out, a family of tournaments \mathcal{R} is hereditarily rigid if and only if any two distinct ordered pairs (x, y) and (x', y') of distinct elements of V yield distinct profiles of tournaments. We note that for $m < \aleph_0$ we may find such a family \mathcal{R} made of n tournaments if and only if $m(m-1) \leq 2^n$, that is $\log_2(m(m-1)) \leq n$. We ask if the same inequality holds when tournaments are replaced by linear orders, that is, whether $h_{\text{Lin}}(m) = \lceil \log_2(m(m-1)) \rceil$.

We show that $h_{\text{Lin}}(m) = \aleph_0$ if $\aleph_0 \leq m \leq 2^{\aleph_0}$. We show more generally that $h_{\text{Lin}}(m) = \log_2(m) = d(\text{Lin}(V))$, where $\log_2(m)$ is the least cardinal μ such that $m \leq 2^\mu$ and $d(\text{Lin}(V))$ is the topological density of the set $\text{Lin}(V)$ of linear orders on V (viewed as a subset of the power set $\mathcal{P}(V \times V)$ equipped with the product topology). The last set of equalities follows from GCH (Generalized Continuum Hypothesis); we do not know if it holds without any set theoretical hypothesis. The finite case is more substantial, but apparently more difficult. In that direction, we verify that $h_{\text{Lin}}(m) = \lceil \log_2(m(m-1)) \rceil$ for $m \leq 7$ and prove that

$h_{\text{Lin}}(m) \leq h_{\text{Lin}}(m+1) \leq h_{\text{Lin}}(m) + 1$ for all $m < \aleph_0$.

Notations in this paper are quite elementary. The *diagonal* of a set X is the set $\Delta_X := \{(x, x) : x \in X\}$. We denote by $\mathcal{P}(X)$ the collection of subsets of X , by X^m the set of m -tuples (x_1, \dots, x_m) of X , by $\binom{X}{m}$ the subset of $\mathcal{P}(X)$ made of m -element subsets of X , and by $[X]^{<\omega}$ the collection of finite subsets of X . The cardinality of X is denoted by $|X|$. If κ denotes a cardinal, 2^κ is the cardinality of the power set $\mathcal{P}(X)$ of a set X of cardinality κ ; we denote by $2^{<\kappa}$ the supremum of 2^μ for $\mu < \kappa$. If κ is an infinite cardinal, we set $\log_2(\kappa)$ for the least cardinal μ such that $\kappa \leq 2^\mu$. If κ is an integer, we use $\log_2(\kappa)$ in the ordinary sense, hence the least integer μ such that $\kappa \leq 2^\mu$ is $\lceil \log_2 \kappa \rceil$. We denote by \aleph_0 the first infinite cardinal. We refer the reader to [4] for further background about axioms of set theory if needed. The proof of one of our results (Theorem 5) relies on the famous theorem of Sperner (see [2]). To state it, we recall that an *antichain* of subsets of a set X is a collection of subsets such that none is contained in another.

Theorem 1. *Let n be a non-negative integer. The largest size of an antichain family of subsets of an n -element set X is $\binom{n}{\lfloor n/2 \rfloor}$. It is only realized by $\binom{X}{\lfloor n/2 \rfloor}$ and $\binom{X}{\lceil n/2 \rceil}$.*

Let $2 := \{0, 1\}$ be ordered with $0 < 1$. The poset 2^X equipped with the product order is isomorphic to the powerset $\mathcal{P}(X)$ ordered by inclusion. Also note that if Y is any set, then the posets $(2 \times 2)^Y$, $2^Y \times 2^Y$ and $2^{Y \times 2}$, all equipped with the product order, are isomorphic. If $|Y| = m$, $m \in \mathbb{N}$, Sperner's theorem asserts that the maximum sized antichain in these posets, once identified to 0-1-sequences, is made of sequences containing roughly as many 0 as 1. This is the key for proving Theorem 5.

We present first the rigidity notions, then the case of tournaments and linear orders and we conclude with density properties.

II. HEREDITARY RIGIDITY

In [1], Couceiro *et al.* studied a general notion of rigidity for relations and sets of relations w.r.t. partial operations. They show a noticeable difference between rigidity w.r.t. to unary operations and rigidity w.r.t. to operations of arity at least two. Here we consider the rigidity notion w.r.t. unary operations, mostly when the relations are binary. Considering hereditarily rigid sets of binary relations, we give an exact upper bound on the size of their domain (Theorem 5).

Let V be a set. A *partial function* on V is a map f from a subset of V , its *domain*, denoted by $\text{dom}(f)$, to another, possibly different subset, its *image*, denoted by $\text{im}(f)$. A partial function f is *constant* if it does not have two distinct values. If A is a subset of V , the *restriction* of f to A ,

denoted by $f_{\upharpoonright A}$, is the map induced by f on $A \cap \text{dom}(f)$. A *subfunction* of f is any restriction of f to a subset of its domain.

Let $h \geq 1$ be an integer, an *h -ary relation* on V is a subset ρ of V^h . Sometimes, we identify ρ with its characteristic function, that is, we write $\rho(v_1, \dots, v_h) = 1$ if $(v_1, \dots, v_h) \in \rho$ and 0 otherwise. If A is a subset of V , the restriction of ρ to A , denoted by $\rho_{\upharpoonright A}$, is $\rho \cap A^h$. If \mathcal{R} is a set of relations on V , we set $\mathcal{R}_{\upharpoonright A} := \{\rho_{\upharpoonright A} : \rho \in \mathcal{R}\}$.

We say that a partial function f *preserves* the h -ary relation ρ , or ρ is *invariant under* f , if for every h -tuple (v_1, \dots, v_h) belonging to $(\text{dom}(f))^h \cap \rho$, its image $(f(v_1), \dots, f(v_h))$ belongs to ρ . We say that a partial function f *preserves* a family of relations \mathcal{R} on V if it preserves each $\rho \in \mathcal{R}$. If $A := \text{dom}(f)$, we also say that f is a *homomorphism* of $\mathcal{R}_{\upharpoonright A}$ in \mathcal{R} .

A h -ary relation ρ on a set V is said to be *rigid* if the identity map is the only unary function on V that preserves ρ . The relation ρ is *semirigid* if every unary function that preserves ρ is the identity map or a constant map. It is *hereditarily semirigid* if the unary partial functions on V that preserve ρ are the subfunctions of the identity map or of constant maps. A family of relations \mathcal{R} on V is said to be *hereditarily semirigid* if the unary partial functions on V that preserve every $\rho \in \mathcal{R}$ are the subfunctions of the identity map or of constant maps. In order to agree with [1], we delete the prefix "semi" in the sequel. Rigid binary relations are introduced in [12], semirigid relations in [3], [5], [8], [13].

Proposition 2. *For a family of relations \mathcal{R} on a set V the following properties are equivalent:*

- (i) \mathcal{R} is hereditarily rigid;
- (ii) For every 2-element subset A of V , every homomorphism f with domain A of $\mathcal{R}_{\upharpoonright A}$ in \mathcal{R} is either constant or a subfunction of the identity map;
- (iii) For every 2-element subset A of V , every 1-1 homomorphism f with domain f of $\mathcal{R}_{\upharpoonright A}$ in \mathcal{R} is a subfunction of the identity map.

Proof: Implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are immediate. We prove that implication (iii) \Rightarrow (i) holds. Let $V' \subseteq V$ and f be a homomorphism with domain V' of $\mathcal{R}_{\upharpoonright V'}$ in \mathcal{R} . We need to prove that f is either constant or a subfunction of the identity. We may suppose that V' is not a singleton, otherwise, f is constant. Suppose that f is 1-1. Then (iii) asserts that for every two-element subset A of V' , $f_{\upharpoonright A}$ is a subfunction of the identity map. It follows that f is a subfunction of the identity and (i) holds. If f is not 1-1 then there is some $x \in V'$ such that $X := f^{-1}(f(x))$ has at least two elements. We may suppose that $f(x) \neq x$. If there is $y \in V' \setminus X$ then $f_{\upharpoonright \{x, y\}}$ is 1-1 (indeed, $f(x) = f(y)$ amounts

to $y \in X$ which is excluded) and not the restriction of the identity since $f(x) \neq x$. Hence, $X = V'$ and f is constant. ■
The above result has a particularly simpler form if the relations are binary and each one is either reflexive or irreflexive. To do this translation, we view such binary relations on V as maps from $V \times V \setminus \Delta_V$ to $2 := \{0, 1\}$.

Definition 3. Let \mathcal{R} be a set of binary relations on V , each $\rho \in \mathcal{R}$ being either reflexive or irreflexive. Let $(x, y) \in V \times V \setminus \Delta_V$. The profile of (x, y) with respect to \mathcal{R} is $p_{\mathcal{R}}(x, y) := (\rho(x, y))_{\rho \in \mathcal{R}}$. The profile of \mathcal{R} is the map $p_{\mathcal{R}}$ from $V \times V \setminus \Delta_V$ to $2^{\mathcal{R}}$, associating $p_{\mathcal{R}}(x, y)$ to each (x, y) . The double profile is the map $\tilde{p}_{\mathcal{R}}$ associating the element $(\rho(x, y), \rho(y, x))_{\rho \in \mathcal{R}}$ of $(2 \times 2)^{\mathcal{R}}$ to each ordered pair $(x, y) \in V \times V \setminus \Delta_V$.

Let $-$ be the involution defined on $V \times V \setminus \Delta_V$ by $\overline{(x, y)} := (y, x)$ for every $(x, y) \in V \times V \setminus \Delta_V$. Similarly, let $-$ be the involution on 2×2 defined by $\bar{u} := (\beta, \alpha)$ for every $u := (\alpha, \beta) \in 2 \times 2$. If θ is any map from a set X to 2×2 , let $\bar{\theta}$ be the composition of θ and $-$, that is $\bar{\theta}(\rho) := \overline{\theta(\rho)}$ for every $\rho \in X$. We say that a map $\varphi : V \times V \setminus \Delta_V \rightarrow (2 \times 2)^X$ is *self-dual* if $\varphi(\overline{(x, y)}) = \overline{\varphi(x, y)}$ for all $(x, y) \in \text{dom}(\varphi)$.

Lemma 4. Let V be a set.

- 1) A set \mathcal{R} of binary relations on V , each one being either reflexive or irreflexive, is hereditarily rigid if and only if $\tilde{p}_{\mathcal{R}}$, the double profile of \mathcal{R} , is 1-1 and its range is an antichain of $(2 \times 2)^{\mathcal{R}}$.
- 2) Let X be a set. If φ is any 1-1 self-dual map from $V \times V \setminus \Delta_V$ to $(2 \times 2)^X$ whose range is an antichain, then there is a map θ from X onto a hereditary rigid set \mathcal{R} of irreflexive binary relations on V such that the natural map $\hat{\theta} : (2 \times 2)^{\mathcal{R}} \rightarrow (2 \times 2)^X$ defined by $\hat{\theta}(\psi) := \psi \circ \theta$ satisfies $\hat{\theta} \circ \tilde{p}_{\mathcal{R}} = \varphi$.

Proof: 1) Observe that if (x, y) and (x', y') are in $V \times V \setminus \Delta_V$, the map transforming x to x' and y to y' is a homomorphism of $\mathcal{R}_{\{x, y\}}$ to \mathcal{R} if and only if $(\rho(x, y), \rho(y, x))_{\rho \in \mathcal{R}} \leq (\rho(x', y'), \rho(y', x'))_{\rho \in \mathcal{R}}$. Hence, the above condition on $\tilde{p}_{\mathcal{R}}$ amounts to (iii) of Proposition 2.

2) Let $p_1 : 2 \times 2 \rightarrow 2$ be the first projection, let $\theta : X \rightarrow 2^{V \times V \setminus \Delta_V}$ defined by $\theta(u)(x, y) := p_1(\varphi(x, y)(u))$ for $u \in X$, $(x, y) \in V \times V \setminus \Delta_V$ and let \mathcal{R} be the range of θ . ■

Theorem 5. There is a hereditarily rigid set \mathcal{R} of κ binary relations, each one reflexive or irreflexive on a set V of cardinality μ if and only if $\mu(\mu - 1) \leq \binom{2\kappa}{\kappa}$ if κ is finite and $\mu \leq 2^\kappa$ otherwise.

Proof: According to 1) of Lemma 4 and Sperner's Theorem the first inequality is satisfied. If κ is infinite, we get the second. For the converse, we define a 1-1 self dual map φ from $V \times V \setminus \Delta_V$ to $(2 \times 2)^X$, where $|X| = \kappa$, whose range

is an antichain and apply 2) of Lemma 4. For that, let ℓ be a tournament on V . Due to Sperner's Theorem, we may choose a 1-1 map φ' from ℓ to the middle level of $(2 \times 2)^X$. Next, select an involution σ on this middle level with no fixed point (e.g. associate to each 0-1-sequence the sequence obtained by exchanging the 0 and 1). Then, set $\varphi(x, y) := \varphi'(x, y)$ for $(x, y) \in \ell$ and $\varphi(x, y) := \sigma(\varphi'(y, x))$ otherwise. This map is self-dual. ■

We examine the case of tournaments and linear orders in the next two sections.

III. SEPARATION AND HEREDITARY RIGIDITY OF TOURNAMENTS

Let V be a set. A *tournament* on V is an irreflexive binary relation τ on V such that for every ordered pair (x, y) either $(x, y) \in \tau$ or $(y, x) \in \tau$, but not both.

Let $\text{Tour}(V)$ be the set of tournaments on V . We say that a tournament τ *separates* two distinct ordered pairs $(x, y), (x', y') \in V \times V \setminus \Delta_V$ if $\tau(x, y) \neq \tau(x', y')$.

Despite that fact that linear orders are reflexive, and tournaments are not, we may view linear orders as tournaments and apply to them what follows.

Lemma 6. Let V be a set; then two distinct pairs $(x, y), (x', y') \in V \times V \setminus \Delta_V$ are always separated by some linear order.

Proof: Indeed, if $(x, y) = (y', x')$ any linear order containing (x, y) will do. If not then the reflexive transitive closure of $\{(x, y), (y', x')\}$ is an order. Any linear extension of that order will do. ■

Lemma 7. Let \mathcal{R} be a family of tournaments on a set V . The following properties are equivalent:

- (i) For all distinct ordered pairs $(x, y), (x', y') \in V \times V \setminus \Delta_V$ there is always some member of $\rho \in \mathcal{R}$ that separates them;
- (ii) The family \mathcal{R} is hereditarily rigid.

Proof: (i) \Rightarrow (ii). Let U be a two-element subset of V and f be a partial homomorphism of \mathcal{R} defined on U . Supposing f non constant, we prove that f is the identity. Let x, y be the two elements of U , let $x' := f(x)$, $y' := f(y)$. If the ordered pairs (x, y) and (x', y') are distinct, they are separated by some $\ell \in \mathcal{R}$, i.e., verifying $\ell(x, y) \neq \ell(x', y')$. Since f is an endomorphism, if $\ell(x, y) = 1$ then $\ell(x', y') = 1$. Thus $\ell(x, y) = 0$. Since ℓ is a tournament, $\ell(y, x) = 1$ and since f is an endomorphism, $\ell(y', x') = 1$, but then $\ell(x, y) = \ell(x', y') = 0$, contradicting the fact that ℓ separates (x, y) and (x', y') .

(ii) \Rightarrow (i). Let $(x, y), (x', y')$ be two distinct irreflexive ordered pairs. The 1-1 map f defined on $U := \{x, y\}$

such that $f(x) := x', f(y) := y'$ is not the identity, hence it cannot be a homomorphism; so there is some $\ell \in \mathcal{R}$ which is not preserved by f , meaning that there is some $(u, v) \in \ell$ such that $(f(u), f(v)) \notin \ell$. If $(u, v) = (x, y)$ then $1 = \ell(x, y) \neq \ell(x', y') = 0$ while if $(u, v) = (y, x)$, then since ℓ is a tournament, $0 = \ell(x, y) \neq \ell(x', y') = 1$, proving that ℓ separates these two ordered pairs. ■

Definitions 8. Let V be a set, κ be its cardinality (possibly infinite) and \mathcal{R} be a set of tournaments on V satisfying one of the equivalent conditions of Lemma 7. We define $h_{\mathcal{R}}(\kappa)$ as the least cardinal μ such that there is some subset X of \mathcal{R} of cardinality μ such that all distinct ordered pairs $(x, y), (x', y') \in V \times V \setminus \Delta_V$ are always separated by some member of X . Let $X \subseteq \mathcal{R}$. Fix $\ell \in X$. The profile of X with respect to ℓ is the family $p_{\ell}(X) := \{p_X(x, y) : (x, y) \in \ell\}$. This profile is minimal if $p_X(x, y) \neq p_X(x', y')$ for any two distinct ordered pairs $(x, y), (x', y') \in \ell$.

Lemma 9. Let $\ell \in X$ and $p_{\ell}(X)$ be the profile of X with respect to ℓ . Then $p_{\ell}(X)$ is minimal if and only if all distinct ordered pairs $(x, y), (x', y') \in V \times V \setminus \Delta_V$ are always separated by some member of X .

Proof: Suppose that $p_{\ell}(X)$ is minimal. Let $(x, y), (x', y') \in V \times V \setminus \Delta_V$ be two distinct ordered pairs. If ℓ separates these pairs, we are done. Otherwise $\ell(x, y) = \ell(x', y')$. If the common value is 1, then since $p_X(x, y) \neq p_X(x', y')$ there is some $\ell' \in X \setminus \{\ell\}$ such that $\ell'(x, y) \neq \ell'(x', y')$. If the common value is 0, then $(y, x), (y', x') \in \ell$ and the previous reasoning yields the same conclusion. Suppose that the separation property holds. Then two distinct ordered pairs $(x, y), (x', y') \in \ell$ are separated by some member of X , thus $p_X(x, y) \neq p_X(x', y')$. ■

An immediate corollary is the following.

Corollary 10. If the profile of X with respect to $\ell \in X$ is minimal, then its profile with respect to any other $\ell' \in X$ is minimal too.

Another straightforward consequence is the following result.

Proposition 11. Let V be a set of cardinality κ , \mathcal{R} be a set of tournaments on V satisfying one of the equivalent conditions of Lemma 7. Then $h_{\mathcal{R}}(\kappa)$ is the minimum of the cardinality of a subset X of \mathcal{R} such that its profile with respect to some tournament $\ell \in X$ is minimal.

Lemma 12. Under the conditions of Definition 8 the following inequality holds: $h_{\mathcal{R}}(\kappa) \geq \log_2(\kappa \cdot (\kappa - 1))$.

Proof: Let V be a set, X be a subset of $\text{Tour}(V)$. Suppose that $p_{\ell}(X)$ is minimal. Associate to each $(x, y) \in \ell$ the profile of $X \setminus \{\ell\}$, that is $p_{X \setminus \{\ell\}}(x, y)$. This defines a map

from ℓ into $2^{X \setminus \{\ell\}}$. This map being 1-1, we have $|\ell| \leq 2^{|X|-1}$, that is $\frac{\kappa \cdot (\kappa - 1)}{2} \leq 2^{|X|-1}$. This amounts to $\kappa \cdot (\kappa - 1) \leq 2^{|X|}$, that is $\log_2(\kappa \cdot (\kappa - 1)) \leq |X|$. ■

We show in Theorem 13 below that the equality holds for $\mathcal{R} = \text{Tour}(V)$ but for $\mathcal{R} := \text{Lin}(V)$ the exact value of $h_{\mathcal{R}}(n)$ for $n \in \mathbb{N}$ eludes us.

Theorem 13. $h_{\text{Tour}}(\kappa) = \log_2(\kappa)$ if κ is an infinite cardinal and $h_{\text{Tour}}(\kappa) = \lceil \log_2(\kappa \cdot (\kappa - 1)) \rceil$ if κ is a non negative integer.

Proof: From the lemma above we have $h_{\text{Tour}}(\kappa) \geq \log_2(\kappa \cdot (\kappa - 1))$. For the reverse inequality, let Z be a subset of cardinality $\lceil \log_2(\kappa \cdot (\kappa - 1)) \rceil$ of $\text{Tour}(V)$. Fix $\ell \in Z$. Choose a 1-1 map φ from ℓ into $2^{Z \setminus \{\ell\}}$. For $k \in Z \setminus \{\ell\}$ set $\ell_k := \{(x, y) \in \ell : \varphi(x, y)(k) = 1\} \cup \{(x, y) : (y, x) \in \ell \text{ and } \varphi(y, x)(k) = 0\}$. It is straightforward to check that $X := \{\ell_k : k \in Z \setminus \{\ell\}\} \cup \{\ell\}$ is a separating family of tournaments. Inequality $h_{\text{Tour}}(\kappa) \leq \log_2(\kappa \cdot (\kappa - 1))$ follows. ■

Remark 14. We could choose $Z \subseteq \text{Lin}(V)$ in the proof above. However, there is a priori no much relationship between Z and X .

IV. THE CASE OF LINEAR ORDERS

On $\{1, \dots, m\}$ the m cyclic permutations of the natural order form a separating family, hence $h_{\text{Lin}}(m) \leq m$ for every integer m . Due to the minoration of $h_{\text{Lin}}(m)$ by $\lceil \log_2(m \cdot (m - 1)) \rceil$ we get the equality for $3 \leq m \leq 5$. Here is an example of a separating family of 5 linear orders on a 6-element set proving that $h_{\text{Lin}}(6) = 5$. We give these orders by the five following strings: 123456; 136542; 216543; 432165; 532146.

With the proposition below we get $h_{\text{Lin}}(7) = 6$. We do not know if $h_{\text{Lin}}(8) = 6$.

Proposition 15. Let $m \in \mathbb{N}$. Then $h_{\text{Lin}}(m) \leq h_{\text{Lin}}(m + 1) \leq h_{\text{Lin}}(m) + 1$.

Proof: The first inequality is trivial. For the second, let $V := \{1, \dots, m\}$. Let $\mathcal{R} := (\leq_k)_{1 \leq k \leq n}$, with $n := h_{\text{Lin}}(m)$, be a separating family of linear orders \leq_k on V . Our aim is to extend these linear orders on $V \cup \{m + 1\}$ and, with an extra linear order, obtain a separating family. We suppose that \leq_1 is the natural order on V and we add $m + 1$ just after m in \leq_1 . For each k , $2 \leq k \leq n$, we insert $m + 1$ just before or just after m in \leq_k . These choices are decided by a 0-1-sequence $s(m, m + 1)$ of length $n - 1$ that we are going to define. For $1 \leq i < j \leq m$, let $s(i, j) := (s_k(i, j))_{2 \leq k \leq n}$, where $s_k(i, j) = 1$ if $i <_k j$ and 0 otherwise. Since \mathcal{R} is separating, $m(m - 1) \leq 2^n$, hence $2^{n-1} - \frac{m(m+1)}{2} \geq p := 2^{\frac{n}{2}-1} \geq 1$. Hence, there are at least p 0-1-sequences of length $n - 1$, which are distinct of all the $\frac{m(m-1)}{2}$ sequences $s(i, j)$. Let $s(m, m + 1)$ be such a sequence. Extend the orders as said, and add a new linear order, say \leq_{n+1} ,

for which $m+1$ is the least element and m the last one. We show that all new profiles, still denoted $s(i, j)$, are distinct, thus proving that the new family of orders is separating. The claimed inequality follows from the following observations.

- 1) $s(i, m) \neq s(i, m+1)$ for $i = 2, m-1$ since $s_{n+1}(i, m+1) = 0 \neq 1 = s_{n+1}(i, m)$.
- 2) $s(m, m+1) \neq s(i, j)$ for $i < j \leq m$. This is just the choice of $s(m, m+1)$.
- 3) $s(i, m+1) \neq s(j, m+1)$ for $i < j < m+1$. Indeed, since for every k , $2 \leq k \leq n$, $m+1$ is immediately before or after m in \leq_k we have $s_k(h, m) = s_k(h, m+1)$ for all $h < m$. Hence $s(h, m) = s(h, m+1)$, and in particular $s(i, m) = s(i, m+1)$ and $s(j, m) = s(j, m+1)$. Since \mathcal{R} is separating, we have $s(i, m) \neq s(j, m)$, hence $s(i, m+1) \neq s(j, m+1)$. ■

V. DENSITY

Let T be a topological space and D be a subset of T . An element x of T is *adherent* to D if every open set containing x meets D . The *topological closure* (or the *adherence*) of D is the set of elements of T adherent to D , denoted by \overline{D} ; this is the least closed set containing D . The subset D is *dense* in T if $\overline{D} = T$. The *density character* of T , denoted by $d(T)$, is the minimum cardinality of a dense subset. Let V be a set; if ℓ is a linear order on V , we may equip V with the *interval topology* whose open sets are generated by the *open intervals* of this order, i.e., sets of the form $]a, b[$ with $a <_\ell b$. We will denote by $d(V, \ell)$ the density of this space. We may note that the density of any subset with the interval topology is at most the density of (V, ℓ) . We may equip the power set $\mathcal{P}(V)$ with the product topology, a basis of open sets being made of sets of the form $O(F, G) := \{X \in \mathcal{P}(V) : F \subseteq X \subseteq V \setminus G\}$ where F, G are finite subsets of V . This is the well known Cantor space. A basic result in topology due to Hausdorff (1936) (see [11] in the Handbook on Boolean Algebras, vol. 2 p. 465) asserts that $d(\mathcal{P}(V)) = \log_2(|V|)$ provided that V is infinite.

Let m be an integer and \mathcal{R} be a set of m -ary relations on a set V ; each $\rho \in \mathcal{R}$ can be viewed as a map from V^m to $\{0, 1\}$ as well as a subset of V^m . Viewing \mathcal{R} as a subset of 2^{V^m} , we may equip \mathcal{R} of the topology induced by the product topology on 2^{V^m} . Let D be a subset of \mathcal{R} and k be an integer. A relation $\rho \in \mathcal{R}$ is *k-adherent* to D if on every k -element subset F of V^m , ρ coincides with some $\rho' \in D$. The *k-adherence* of D in \mathcal{R} is the set of $\rho \in \mathcal{R}$ that are k -adherent to D . If this set is \mathcal{R} , D is said *k-dense*. The relation ρ is adherent to D in the topological sense if it is k -adherent for every integer k . The adherence of D is often called the *local closure* of D . The topological density of several sets of relations can be computed. For example, the following sets of relations on a set of cardinality κ have density $\log_2(\kappa)$: the collection of n -ary relations, of binary relations, of directed graphs without

loops, of undirected graphs, of tournaments. Indeed, all of these sets are homeomorphic to the powerset of some set of cardinality κ . We just illustrate this fact with the collection of tournaments. Fix a tournament $\tau := (V, E)$ on a set V of size κ . To each map $f : E \rightarrow 2$ associate the tournament τ_f whose arc set is $E_f := f^{-1}(1) \cup \{(u, v) \in V^2 \setminus \Delta_V : f(v, u) = 0\}$. This defines a homeomorphism from 2^E onto the set of tournaments.

If X is a set of binary relations ρ on V , we set $X^{-1} := \{\rho^{-1} : \rho \in X\}$.

Lemma 16. *Let \mathcal{R} be a collection of tournaments on V such that two distinct ordered pairs $(x, y), (x', y') \in V \times V \setminus \Delta_V$ are always separated by some member of \mathcal{R} . The following properties are equivalent for a non-empty subset X of \mathcal{R} :*

- (i) *All distinct ordered pairs $(x, y), (x', y') \in V \times V \setminus \Delta_V$ can be always separated by some member of X ;*
- (ii) *$X \cup X^{-1}$ is 2-dense in \mathcal{R} .*

Proof: (i) \Rightarrow (ii) Let $\rho \in \mathcal{R}$. Let U be a two element subset of $V \times V \setminus \Delta_V$. We need to prove that some $\tau \in X \cup X^{-1}$ coincides with ρ on U . Let (x, y) and (x', y') be two distinct ordered pairs such that $U = \{(x, y), (x', y')\}$. Let $\alpha := \rho(x, y)$ and $\beta := \rho(x', y')$. If $(\alpha, \beta) = (0, 0)$, let τ be a tournament separating (x, y) and (y', x') . Then τ or τ^{-1} coincides with ρ on U . If $(\alpha, \beta) = (0, 1)$, let τ separating (x, y) and (x', y') . Then τ or τ^{-1} coincides with ρ . The two other cases are similar.

(ii) \Rightarrow (i) Let (x, y) and (x', y') be two distinct ordered pairs. These pairs are separated by some member ρ of A . Since $X \cup X^{-1}$ is 2-dense in A there is some $\tau \in X \cup X^{-1}$ which coincides with ρ on $U := \{(x, y), (x', y')\}$. Hence τ separates (x, y) and (x', y') . ■

Let $\text{Lin}(V)$ be the set of linear orders on an infinite set V of cardinality κ . We may identify each linear order with a subset of $V \times V \setminus \Delta_V$. By definition a subset X of $\text{Lin}(V)$ is dense if every nonempty open set of $\text{Lin}(V)$ meets X . This amounts to the fact that for every $\rho \in \text{Lin}(V)$, every finite subsets F, G of $V \times V \setminus \Delta_V$, every set $O(F, G) := \{\tau \in \mathcal{P}(V \times V \setminus \Delta_V) : F \subseteq \tau \subseteq V \times V \setminus G\}$ containing ρ meets X .

According to Lemma 6, we may apply Lemma 16. This yields:

Lemma 17. *If X is dense in $\text{Lin}(V)$, then X separates all distinct ordered pairs. Hence: $h_{\text{Lin}(V)}(|V|) \leq d(\text{Lin}(V))$.*

An alternative condition to density is this:

Lemma 18. *X is dense in $\text{Lin}(V)$ if and only if for every non-negative integer m and every m -tuple (a_1, \dots, a_m) of distinct elements of V there is some $\ell \in X$ such that $a_1 <_\ell \dots <_\ell a_m$.*

Proof: Suppose that X is dense. Let $m \in \mathbb{N}$ and a m -tuple

$(a_1, \dots, a_m) \in V^m$ with distinct entries. Let $\rho \in \text{Lin}(V)$ such that $a_1 <_\rho \dots <_\rho a_m$. Set $F := \{(a_i, a_{i+1}) : 1 \leq i < m\}$ and $G := \emptyset$. Then $\rho \in O(F, G)$ thus there is some $\ell \in X \cap O(F, G)$ that is $a_1 <_\ell \dots <_\ell a_m$. Conversely, let $\rho \in O(F, G) \cap \text{Lin}(V)$. We prove that $O(F, G)$ meets X . Let A be a finite subset of V such that $\bigcup\{(x, y) : (x, y) \in F \cup G\} \subseteq A$. Let (a_1, \dots, a_m) be an enumeration of elements of A so that $a_1 <_\rho \dots <_\rho a_m$. Then there is $\ell \in X$ such that the ordering on A coincides with this of ρ . It follows that $\ell \in O(F, G) \cap X$. ■

It is easy to show that $d(\text{Lin}(V)) = \log_2(|V|)$ if $\aleph_0 \leq |V| \leq 2^{\aleph_0}$ (e.g., see the proof of Theorem 19 below). Provided that GCH holds, the equality $d(\text{Lin}(V)) = \log_2(|V|)$ holds for every infinite set; but we do not know if this holds without any set theoretical hypothesis.

To prove that this equality holds, we introduce the following related parameters. For a cardinal κ , let $\delta(\kappa)$ be the least cardinal μ such that there is a linear order ℓ on a set of cardinality κ admitting a dense subset of cardinality μ and for a cardinal μ , let $\text{ded}(\mu)$ be the supremum of cardinals κ such that there is a chain of cardinality κ and density at most μ . Recall that if GCH holds then $2^{<\nu} = \nu$ for every infinite cardinal ν . Next, under this cardinality condition, $\delta(\kappa) = \log_2(\kappa)$ and $\text{ded}(\mu) = 2^\mu$ for every infinite κ and μ . (hint: let W be a well ordered chain of cardinality μ and 2^W be the power set lexicographically ordered. The density of this chain, as any subchain, is at most $2^{<\mu}$. By definition, if $\mu := \log_2(\kappa)$, $2^{<\mu} \leq \kappa \leq 2^\mu$, thus $\delta(\kappa) \leq 2^{<\mu}$. If $2^{<\mu} = \mu$ then $\delta(\kappa) \leq \mu$ and hence $\delta(\kappa) = \log_2(\kappa)$. The proof of the second equality is similar, and shorter). Mitchell [7] showed that it is consistent with ZFC that for uncountable regular μ , $\text{ded}(\mu) < 2^\mu$ [7] and in fact that no chain of cardinality 2^κ has a dense subchain of cardinality κ . In particular, $\delta(\kappa) \neq \log_2(\kappa)$. Now we prove:

Theorem 19. *For every infinite cardinal κ and set V of cardinality κ , we have $\log_2(\kappa) \leq h_{\text{Lin}}(|V|) \leq d(\text{Lin}(V)) \leq \delta(\kappa)$.*

Proof: The first inequality is Lemma 12 and the second is Lemma 17. We prove that the third inequality holds. For that, let $\ell := (V, \leq)$ be a linear order having a dense set D of cardinality $\delta(\kappa)$. For each finite subset F of D , write its elements in an increasing order, $F := d_1 < \dots < d_{m-1}$ with $m = |F| + 1$ and decompose ℓ into the m intervals $\ell_1 :=]-\infty, d_1[, \dots, \ell_{i+1} := [d_i, d_{i+1}[, \dots, \ell_m := [d_{m-1}, +\infty[$. Let \mathfrak{S}_m be the set of permutations of $\{1, \dots, m\}$. Each permutation $\sigma \in \mathfrak{S}_m$ induces a permutation of the intervals ℓ_1, \dots, ℓ_m . The lexicographical sum $\ell_\sigma := \sum_{1 \leq i \leq m} \ell_{\sigma(i)}$ yields a linear order \leq_σ on V .

Claim. The set $\mathcal{C} := \{\leq_\sigma : F \in D^{<\omega} \text{ and } \sigma \in \mathfrak{S}_{|F|+1}\}$ is dense in $\text{Lin}(V)$.

Since the number of pairs (F, σ) where $F \in D^{<\omega}$ and $\sigma \in \mathfrak{S}_{|F|+1}$ is $|D|$ this claim suffices to prove the inequality.

Proof of the Claim. According to Proposition 18 this amounts to prove that for every integer m and every m -tuple (a_1, \dots, a_m) of distinct elements of V there is some $\leq_\sigma \in \mathcal{C}$ such that $a_1 <_\sigma \dots <_\sigma a_m$. Let $\sigma \in \mathfrak{S}_m$ such that $a_{\sigma^{-1}(1)} < \dots < a_{\sigma^{-1}(m)}$ in the chain ℓ . Due to the density of D , there are $d_i \in [a_{\sigma^{-1}(i)}, a_{\sigma^{-1}(i+1)}[\cap D$ for $1 \leq i \leq m-1$. Let $\ell_1 :=]-\infty, d_1[, \dots, \ell_{i+1} := [d_i, d_{i+1}[, \dots, \ell_m := [d_{m-1}, +\infty[$. Permute these intervals according to σ . The resulting chain ℓ_σ reorders the m -tuple $a_{\sigma^{-1}(1)} < \dots < a_{\sigma^{-1}(m)}$ as $a_1 <_\sigma \dots <_\sigma a_m$. This proves our claim. ■

Corollary 20. $\log_2(\kappa) = h_{\text{Lin}}(|V|) = d(\text{Lin}(V)) = \delta(\kappa)$ iff $\log_2(\kappa) = \delta(\kappa)$.

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