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Toward a new fully algebraic preconditioner for symmetric positive definite problems

Nicole Spillane

Abstract A new domain decomposition preconditioner is introduced for efficiently solving linear systems $\mathbf{Ax} = \mathbf{b}$ with a symmetric positive definite matrix \mathbf{A} . The particularity of the new preconditioner is that it is not necessary to have access to the so-called *Neumann* matrices (*i.e.*: the matrices that result from assembling the variational problem underlying \mathbf{A} restricted to each subdomain). All the components in the preconditioner can be computed with the knowledge only of \mathbf{A} (and this is the meaning given here to the word *algebraic*). The new preconditioner relies on the GenEO coarse space for a matrix that is a low-rank modification of \mathbf{A} and on the Woodbury matrix identity. The idea underlying the new preconditioner is introduced here for the first time with a first version of the preconditioner. Some numerical illustrations are presented. A more efficient version will be proposed in a full-length article.

1 Introduction

We set out to solve the linear system $\mathbf{Ax}_* = \mathbf{b}$, for a given symmetric positive definite (spd) matrix \mathbf{A} . Let n be the order of the matrix. Assume that there are N subdomains defined by a family of full-rank boolean interpolation matrices $\mathbf{R}^{s\top} : \mathbb{R}^{n^s} \rightarrow \mathbb{R}^n$ (for $s \in \llbracket 1, N \rrbracket$) that satisfy $\mathbb{R}^n = \sum_{s=1}^N \text{range}(\mathbf{R}^{s\top})$.

There exist a variety of two-level methods for which convergence is guaranteed without making assumptions on the number of subdomains, their shape, or the distribution of the coefficients in the underlying PDE (see *e.g.* [11, 5, 14, 15, 7, 10, 12, 6, 17, 16, 3]). These methods have in common to select vectors for the coarse space by computing low- or high-frequency eigenvectors of well chosen generalized eigenvalue problems posed in the subdomains. To the best of the author's knowledge,

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none of these methods can be applied if the so called local *Neumann* matrices are not known. Specifically, the pencils of the generalized eigenvalue problems that enter into the definition of the coarse space make use of a family of symmetric positive semi-definite (spsd) matrices \mathbf{N}^s that satisfy the assumption

$$\exists C > 0, \text{ such that } \sum_{s=1}^N \mathbf{x}^\top \mathbf{R}^{s\top} \mathbf{N}^s \mathbf{R}^s \mathbf{x} \leq C \mathbf{x}^\top \mathbf{A} \mathbf{x}; \forall \mathbf{x} \in \mathbf{R}^n. \quad (1)$$

The Neumann matrices are a natural choice for \mathbf{N}^s and the above estimate then holds with constant C equal to the maximal multiplicity of a mesh element. This limitation is very well known (and stated clearly in *e.g.*, [1, 2]).

In this work, it is proposed to relax the assumptions on the family of matrices \mathbf{N}^s in (1) by allowing the matrices to be symmetric (but not necessarily positive semi-definite (psd)). Such matrices \mathbf{N}^s can always be defined based on the knowledge only of the coefficients in \mathbf{A} and the choice of the interpolation operators $\mathbf{R}^{s\top}$. Special treatment will need to be applied to the non-positive part of \mathbf{N}^s and this will be reflected in the cost of setting up and applying the preconditioner. In Section 2 the new preconditioner is defined and the result on the condition number is given. In Section 3 some preliminary numerical illustrations are provided. Finally, Section 4 offers up some conclusive remarks about the new preconditioner, as well as some of its current limitations that will be addressed in the upcoming full length article.

2 Definition of the new preconditioner and theory

In this section we first introduce a splitting of \mathbf{A} (in the sense of (1)) that can be computed algebraically. The matrices \mathbf{N}^s in this splitting are symmetric (but not psd). Then, an auxiliary matrix \mathbf{A}_+ is defined. It is a low-rank modification of \mathbf{A} for which a splitting with spsd matrices is known. This auxiliary matrix satisfies the assumptions for the abstract GenEO coarse space theory [13]. Consequently, a family of two-level preconditioner $\mathbf{H}_+(\tau)$ parametrized by a threshold τ can be defined in such a way that the preconditioned operator $(\mathbf{H}_+(\tau)\mathbf{A}_+)$ is *well* conditioned. Finally, a preconditioner for the original operator \mathbf{A} is defined by adding to $\mathbf{H}_+(\tau)$ a term coming from the Woodbury matrix identity.

2.1 Splitting of \mathbf{A} with symmetric matrices \mathbf{N}^s

The following theorem is not groundbreaking, but it is new so a proof is given. The proof provides one way of choosing symmetric matrices \mathbf{B}^s that satisfy (1). The choice is not unique.

Theorem 1 *Let \mathbf{A} be an order n spd matrix. Let $\mathbf{R}^{s\top} : \mathbb{R}^{n^s} \rightarrow \mathbb{R}^n$ for $s \in \llbracket 1, N \rrbracket$ be a family of full-rank boolean matrices that satisfy $\mathbb{R}^n = \sum_{s=1}^N \text{range}(\mathbf{R}^{s\top})$. A family*

of symmetric matrices $\mathbf{B}^s \in \mathbb{R}^{n^s \times n^s}$ for $s = 1, \dots, N$ such that

$$\mathbf{A} = \sum_{s=1}^N \mathbf{R}^s \mathbf{B}^s \mathbf{R}^{s\top}, \quad (2)$$

can be computed with no further knowledge.

Proof We provide a constructive proof. Let $\mathbf{spy}(\mathbf{A})$ be the boolean matrix that has the same sparsity pattern as \mathbf{A} . For any $s \in \llbracket 1, N \rrbracket$, let $\mathbf{spy}(\mathbf{A}^s)$ be the boolean matrix that has the same sparsity pattern as $\mathbf{A}^s := \mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top}$. Then, the multiplicity of each entry in \mathbf{A} can be defined and stored in the matrix $\mathbf{mult} := \sum_{s=1}^N \mathbf{R}^s \mathbf{spy}(\mathbf{A}^s) \mathbf{R}^s$.

A matrix partition of unity is given by the matrices $\mathbf{POU}^s \in \mathbb{R}^{n^s \times n^s}$ whose only non-zero entries are $(\mathbf{POU}^s)_{ij} := 1 / (\mathbf{R}^s \mathbf{mult} \mathbf{R}^{s\top})_{ij}$ if $(\mathbf{R}^s \mathbf{mult} \mathbf{R}^{s\top})_{ij} \neq 0$.

Finally, $\mathbf{B}^s \in \mathbb{R}^{n^s \times n^s}$ is defined to be the Hadamard product (or entrywise product) of the partition of unity matrix by the local matrix:

$$(\mathbf{B}^s)_{ij} := (\mathbf{POU}^s)_{ij} (\mathbf{A}^s)_{ij}, \quad \forall i, j \in \llbracket 1, n \rrbracket. \quad (3)$$

By definition these matrices are symmetric and they satisfy (2). \square

2.2 A splitting of the splitting

In the proof of the result for the GenEO coarse space, the positive semi-definiteness of the matrices in the splitting of \mathbf{A} is crucial. We are unable to provide such a splitting algebraically. Instead, a matrix \mathbf{A}_+ is defined that is *as closely as possible* related to \mathbf{A} and with a known spsd splitting.

We proceed in the following way for each subdomain $s = 1, \dots, N$:

1. Diagonalize \mathbf{B}^s . Matrix \mathbf{B}^s is symmetric so there exist a diagonal matrix $\mathbf{\Lambda}^s$ and an orthogonal matrix \mathbf{V}^s such that $\mathbf{B}^s = \mathbf{V}^s \mathbf{\Lambda}^s \mathbf{V}^{s\top}$. Recall that the eigenvalues of \mathbf{B}^s are on the diagonal of $\mathbf{\Lambda}^s$. Assume (without restriction) that they are sorted in non-decreasing order. Also recall that the i -th column in \mathbf{V}^s is an eigenvector corresponding to the eigenvalue $(\mathbf{\Lambda}^s)_{ii}$.
2. Form two diagonal submatrices of $\mathbf{\Lambda}^s$ denoted by $\mathbf{\Lambda}_-^s$ and $\mathbf{\Lambda}_+^s$ whose entries are, respectively, the non-positive and positive eigenvalues of \mathbf{B}^s (once again sorted in non-decreasing order). Then define \mathbf{V}_-^s and \mathbf{V}_+^s to be the two submatrices of \mathbf{V}^s whose columns are eigenvectors corresponding, respectively, to the non-positive and positive eigenvalues of $\mathbf{\Lambda}^s$. The following holds:

$$\mathbf{\Lambda}^s = \begin{pmatrix} \mathbf{\Lambda}_-^s & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_+^s \end{pmatrix}, \quad \mathbf{V}^s = [\mathbf{V}_-^s | \mathbf{V}_+^s], \quad \mathbf{\Lambda}_+^s \text{ is spd, } -\mathbf{\Lambda}_-^s \text{ is spsd.}$$

3. Define the two following matrices in $\mathbb{R}^{n^s \times n^s}$

$$\mathbf{A}_+^s := \mathbf{V}_+^s \mathbf{A}_+^s \mathbf{V}_+^{s\top} \text{ and } \mathbf{A}_-^s := -\mathbf{V}_-^s \mathbf{A}_-^s \mathbf{V}_-^{s\top}.$$

Both \mathbf{A}_+^s and \mathbf{A}_-^s are spsd matrices and $\mathbf{B}^s = \mathbf{A}_+^s - \mathbf{A}_-^s$.

Finally two global matrices \mathbf{A}_+ and \mathbf{A}_- are defined as

$$\mathbf{A}_+ := \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s, \text{ and } \mathbf{A}_- := \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_-^s \mathbf{R}^s.$$

It is rather straightforward that $\mathbf{A} = (\mathbf{A}_+ - \mathbf{A}_-)$; \mathbf{A}_- is spsd; and \mathbf{A}_+ is spd, indeed

$$\langle \mathbf{x}, \mathbf{A}_+ \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{A}_- \mathbf{x} \rangle \geq \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle \geq 0 \text{ with equality only if } \mathbf{x} = \mathbf{0}.$$

2.3 Two-level preconditioner for \mathbf{A}_+ with a GenEO coarse space

Following [13], there are many possible choices for a two-level preconditioner for \mathbf{A}_+ with a GenEO coarse space. This is not the novelty here so only one is given with no further comment on other possibilities. For a threshold $\tau > 1$, let $\mathbf{H}_+(\tau)$ be the two level Additive Schwarz preconditioner defined by

$$\mathbf{H}_+(\tau) := \sum_{s=1}^N \mathbf{R}^{s\top} (\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top})^{-1} \mathbf{R}^s + \mathbf{R}^0(\tau)^\top (\mathbf{R}^0(\tau) \mathbf{A}_+ \mathbf{R}^0(\tau)^\top)^{-1} \mathbf{R}^0(\tau), \quad (4)$$

where the lines of $\mathbf{R}^0(\tau)$ span the GenEO coarse space $V^0(\tau)$. The coarse space is defined according to [13][Definition 5]:

$$V^0(\tau) := \sum_{s=1}^N \text{range} \left(\mathbf{R}^{s\top} \mathbf{Y}_L(\tau^{-1}, (\mathbf{D}^s)^{-1} \mathbf{N}^s (\mathbf{D}^s)^{-1}, \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}) \right),$$

where the matrices $\mathbf{D}^s := \mathbf{R}^s \left(\sum_{t=1}^N \mathbf{R}^{t\top} \mathbf{R}^t \right)^{-1} \mathbf{R}^{s\top}$ form a partition of unity in the usual sense and the columns of $\mathbf{Y}_L(\tau^{-1}, (\mathbf{D}^s)^{-1} \mathbf{N}^s (\mathbf{D}^s)^{-1}, \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top})$ are the $(\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top})$ -normalized eigenvectors \mathbf{y}^s corresponding to an eigenvalue $\lambda^s < \tau$ in the generalized eigenvalue problem:

$$(\mathbf{D}^s)^{-1} \mathbf{N}^s (\mathbf{D}^s)^{-1} \mathbf{y}^s = \lambda^s \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top} \mathbf{y}^s.$$

By [13][Remark 3, Corollary 4, Assumption 6], if $\tau > 1$ and \mathcal{N}_+ is the minimal number of colors that are needed to color each subdomain in such a way that two subdomains with the same color are \mathbf{A}_+ -orthogonal, then the eigenvalues of the preconditioned operator satisfy

$$\lambda(\mathbf{H}_+(\tau) \mathbf{A}_+) \in \left[((1 + 2\mathcal{N}_+) \tau)^{-1}, \mathcal{N}_+ + 1 \right], \quad (5)$$

2.4 New preconditioner for \mathbf{A}

The rank of \mathbf{A}_- , which we denote by n_- , satisfies $n_- \leq \sum_{s=1}^N n^s - n$. Indeed, it holds that

$$\text{rank}(\mathbf{A}_+) + \text{rank}(\mathbf{A}_-) \leq \sum_{s=1}^N n^s \text{ with } \text{rank}(\mathbf{A}_+) = n.$$

Consequently, the rank of \mathbf{A}_- is small compared to the rank n of \mathbf{A} ($n_- \ll n$) as long as there is little overlap between subdomains. Note that n_- can (and hopefully is) much smaller even than $\sum_{s=1}^N n^s - n$. Following this observation it is natural to write \mathbf{A} as a low rank modification of \mathbf{A}_+ . To this end, let $\mathbf{\Lambda}_- \in \mathbb{R}^{n_- \times n_-}$ and $\mathbf{V}_- \in \mathbb{R}^{n \times n_-}$ be the diagonal matrix and the orthogonal matrix that are obtained by removing the null part of \mathbf{A}_- from its diagonalization in such a way that

$$\mathbf{A}_- = \mathbf{V}_- \mathbf{\Lambda}_- \mathbf{V}_-^\top.$$

It now holds that $\mathbf{A} = \mathbf{A}_+ - \mathbf{V}_- \mathbf{\Lambda}_- \mathbf{V}_-^\top$ with \mathbf{A} , \mathbf{A}_+ , $\mathbf{\Lambda}_-$ spd matrices and \mathbf{V}_- a full rank matrix. The Woodbury matrix identity applied to computing the inverse of \mathbf{A} gives

$$\mathbf{A}^{-1} = \mathbf{A}_+^{-1} + \mathbf{A}_+^{-1} \mathbf{V}_- (\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}.$$

This identity leads, rather straightforwardly, to a new preconditioner for the original matrix \mathbf{A} that is defined in the following theorem.

Theorem 2 For $\tau > 1$, let the new preconditioner be defined as

$$\mathbf{H}(\tau) := \mathbf{H}_+(\tau) + \mathbf{A}_+^{-1} \mathbf{V}_- (\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}.$$

The eigenvalues of the preconditioned operator satisfy

$$\lambda(\mathbf{H}(\tau)\mathbf{A}) \in \left[((1 + 2\mathcal{N}_+)\tau)^{-1}, \mathcal{N}_+ + 1 \right], \quad (6)$$

where, once more \mathcal{N}_+ is the coloring constant with respect to the operator \mathbf{A}_+ .

Proof The estimate for the eigenvalues of $\mathbf{H}_+(\tau)\mathbf{A}_+$ in (5) is equivalent to

$$((1 + 2\mathcal{N}_+)\tau)^{-1} \langle \mathbf{x}, \mathbf{A}_+^{-1} \mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{H}_+(\tau) \mathbf{x} \rangle \leq (\mathcal{N}_+ + 1) \langle \mathbf{x}, \mathbf{A}_+^{-1} \mathbf{x} \rangle, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Adding, $\langle \mathbf{x}, \mathbf{A}_+^{-1} \mathbf{V}_- (\mathbf{D}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{x} \rangle$ to each term, it holds that

$$((1 + 2\mathcal{N}_+)\tau)^{-1} \langle \mathbf{x}, \mathbf{A}^{-1} \mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{H}(\tau) \mathbf{x} \rangle \leq (\mathcal{N}_+ + 1) \langle \mathbf{x}, \mathbf{A}^{-1} \mathbf{x} \rangle, \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

where the Woodbury matrix identity was applied as well as $\mathcal{N}_+ \geq 1$ and $\tau \geq 1$. This is equivalent to (6). \square

In order to apply the preconditioner, the matrix $\mathbf{A}_+^{-1} \mathbf{V}_-$ must be formed. This can be done by solving iteratively n_- linear systems preconditioned by $\mathbf{H}_+(\tau)$. It is likely that block Krylov methods would be advantageous.

3 Numerical Illustration

The results in this section are obtained using the software FreeFem++ [8], GNU Octave [4] and METIS [9]. The linear systems that are considered arise from discretizing with \mathbb{P}_1 finite elements some two-dimensional linear elasticity problems.



Fig. 1 Testcase 1 – partition into $N = 4$ subdomains and distribution of E (10^8 in white layers and 10^3 in dark layers)

The first test case is posed on the domain $\Omega = [4, 1]$ discretized by 112×28 elements. The problem size is $n = 6496$ degrees of freedom. The coefficients in the linear elasticity equation are $\nu = 0.3$ for Poisson's ratio and

$$E(x, y) = 10^8 \text{ if } y \in [1/7, 2/7] \cup [3/7, 4/7] \cup [5/7, 6/7]; \quad E(x, y) = 10^3 \text{ otherwise.}$$

The domain is partitioned into 4 subdomains with Metis. No overlap is added. Figure 1 shows both the partition into subdomains and the distribution of E . For this problem, the coloring constants with respect to \mathbf{A} and \mathbf{A}_+ are $\mathcal{N} = 2$, and $\mathcal{N}_+ = 3$. The problem is solved with the one-level Additive Schwarz, the two-level Additive Schwarz with the GenEO coarse space from [13][Section 5.2.2] and the new method. The value of the threshold τ for the last two methods is chosen to be $\tau = 10$. The theoretical bounds for GenEO and the new method is that the eigenvalues are in the interval $[1/70 \approx 0.014, 3]$ and $[1/70, 4]$, respectively. The \mathbf{A} -norm of the error at each iteration of the preconditioned conjugate gradient is represented in Figure 2. The quantities of interest are in Table 1. The one-level method is not efficient on this problem. This was to be expected. Both the GenEO solver and the new solver converge fast. With $\tau = 10$ in both methods, the coarse space for the new method is larger than with GenEO (58 *versus* 49 coarse vectors). For the new method there is an additional problem of size 49. Note that it is posed in a space that is also a subspace of the coarse space. The results show that the new preconditioner converges a little bit faster than GenEO. A study with more values of all the parameters is needed to compare GenEO and the new solver as the parameter τ does not play exactly the same role in the setup of both preconditioners. Since there is a lot more information injected into GenEO (through the matrices \mathbf{N}^s), it is expected that GenEO will be more efficient. However the new method has the very significant advantage of being algebraic, and being almost as efficient as GenEO would be an achievement.

It is very good news that the coarse space and the space V_- did not explode on the previous test case. The second test case is a rather easy problem posed on $\Omega = [1, 1]$ with a distribution of both coefficients that is homogeneous: $\nu = 0.3$ and $E = 10^8$. Two partitions are considered: one into $N = 16$ regular subdomains and the other

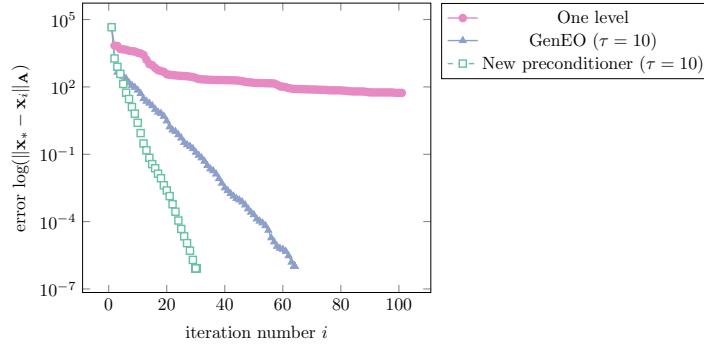


Fig. 2 Testcase 1 – Convergence history for the one-level method, the two-level GenEO method and the new method.

	λ_{\min}	λ_{\max}	κ	It	$\#V^0$	n_-
One-level Additive Schwarz	$2 \cdot 10^{-4}$	2.0	$1.0 \cdot 10^4$	>100	0	0
Two-level Additive Schwarz with GenEO	0.059	3.0	51	65	49	0
New method	0.24	2.93	12	30	58	49

Table 1 Testcase 1 – Extreme eigenvalues, condition number κ , Iteration count, size of coarse space, and n_- in the new method

into $N = 16$ subdomains with Metis. No overlap is added to the subdomains. The results are presented in Table 2. For the problem with regular subdomains, the new method selects a coarse space of size 44 (*versus* 40 for GenEO). This means, that even without the knowledge of the Neumann matrix, a coarse space is constructed that has almost the same number of vectors as the optimal coarse space for this problem which consists of $3 \times 12 = 36$ rigid body modes (there are 4 non-floating subdomains).

$N = 16$ regular subdomains						
	λ_{\min}	λ_{\max}	κ	It	$\#V^0$	n_-
One-level Additive Schwarz	$2 \cdot 10^{-3}$	4.0	1996	97	0	0
Two-level Additive Schwarz with GenEO	0.07	4.0	60	61	40	0
New method	0.19	4	21	39	44	24
$N = 16$ subdomains with Metis						
	λ_{\min}	λ_{\max}	κ	It	$\#V^0$	n_-
One-level Additive Schwarz	$1.7 \cdot 10^{-3}$	3.0	1817	>100	0	0
Two-level Additive Schwarz with GenEO	0.095	3.4	36	54	74	0
New method ($\tau = 10$)	0.26	3.0	11.3	31	117	94

Table 2 Testcase 2 – The coefficient distribution is uniform. – Extreme eigenvalues, condition number κ , Iteration count, size of coarse space, and n_- in the new method

4 Conclusion

A new preconditioner that is computed algebraically from \mathbf{A} was defined and bounds for the spectrum of the resulting preconditioned operator were proved. They are independent of the number of subdomains and any parameters in the problem. Future versions of the preconditioner will aim at improving some of its limitations: computation of $\mathbf{A}_+^{-1}\mathbf{V}_-$, rather exotic coarse solve, and cost of \mathbf{H}_+ .

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