



HAL
open science

GenEO spectral coarse spaces in SPD domain decomposition

Nicole Spillane

► **To cite this version:**

| Nicole Spillane. GenEO spectral coarse spaces in SPD domain decomposition. 2025. hal-03186276v2

HAL Id: hal-03186276

<https://hal.science/hal-03186276v2>

Preprint submitted on 14 Feb 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GenEO spectral coarse spaces in SPD domain decomposition

NICOLE SPILLANE*

CNRS, CMAP, École polytechnique, 91128 Palaiseau Cedex, France

February 7, 2025

Abstract

Two-level domain decomposition methods are preconditioned Krylov solvers. What separates one and two-level domain decomposition method is the presence of a coarse space in the latter. The abstract Schwarz framework is a formalism that allows to define and study a large variety of two-level methods. The objective of this article is to define, in the abstract Schwarz framework, a family of coarse spaces called the GenEO coarse spaces (for Generalized Eigenvalues in the Overlaps). This is a generalization of methods that exist already for particular choices of domain decomposition methods. Bounds for the condition numbers of the preconditioned operators are proved that are independent of the parameters in the problem (*e.g.*, any coefficients in an underlying PDE or the number of subdomains). The coarse spaces are computed by finding low or high frequency spaces of some well chosen generalized eigenvalue problems in each subdomain. The abstract framework is illustrated by defining two-level Additive Schwarz, Neumann-Neumann and Inexact Schwarz preconditioners for a two dimensional linear elasticity problem. Explicit theoretical bounds as well as numerical results are provided for this example.

Keywords: linear solver, domain decomposition, coarse space, preconditioning, deflation, linear elasticity.

1 Introduction

The problem considered is the solution of linear systems of the form

$$Ax = \mathbf{b}, \text{ where } A \in \mathbb{R}^{n \times n} \text{ is symmetric positive and definite (spd).}$$

The applications to bare in mind are ones for which A is sparse and the number n of unknowns is very large. Hence, parallel solvers, and more specifically domain decomposition solvers, are studied. The purpose of the article is to provide unified definitions and theory for two-level domain decomposition methods with spectral coarse spaces. This is done in the *abstract Schwarz framework* by which it is referred to the formalism presented in Chapters 2 and 3 of the book by Toselli and Widlund [37]. This framework provides both a way of defining two-level domain decomposition preconditioners and to prove condition number bounds that involve them.

Having chosen a partition of the global computational domain into subdomains, one-level domain decomposition preconditioners are sums of inverses of some well-chosen local problems in each of the subdomains. Two-level methods have an extra ingredient that is the coarse space. Choosing the coarse space comes down to choosing an extra, low rank, problem that is shared

*Email: nicole.spillane@polytechnique.edu
This research received no external funding.

between all subdomains and solved at every iteration of the Krylov subspace solver. A good choice of coarse space can have a huge, positive, effect on the convergence of the method. It is with the introduction of coarse spaces that domain decomposition methods became scalable. Indeed, the first coarse spaces already ensured that, for some problems, the condition number of the two-level preconditioned operators did not depend on the number of subdomains and only weakly on the number of elements in each subdomain (see *e.g.*, [10, 24]).

A consensus seems to have occurred that it is worth enlarging, even quite significantly, the coarse space if this enlargement allows to achieve robustness and scalability. One popular way of doing this is to compute the coarse space by solving generalized eigenvalue problems in the subdomains. These generalized eigenvalue problems are chosen to seek out the vectors that make convergence slow. A first group of methods was tailored to the scalar elliptic problem with a varying conductivity in the Additive Schwarz framework. Among these are the two articles [11, 12] on one hand, and [28, 29, 7] on the other. The same two groups of authors contributed, with collaborators, to the set of articles [9] and [33, 34]. This time the methods apply to a much wider range of PDEs that include the linear elasticity equations. The method in [33, 34] is called GenEO for Generalized eigenvalues in the overlaps. A different version of the GenEO coarse space was proposed for FETI and BDD in [35]. The problems there are reduced to the interfaces between subdomains but the name GenEO was kept since these interfaces in some sense constitute an overlap between subdomains. The family of GenEO coarse spaces has grown since with *e.g.*, the contributions [15, 6] for Optimized Schwarz and [26] in the context of boundary element methods. In this article, the coarse spaces are referred to as GenEO coarse spaces since their construction generalizes the procedure for the two original GenEO coarse spaces: [34, 35].

The idea of solving generalized eigenvalue problems to design coarse spaces with guaranteed good convergence had in fact already been proposed, unknowingly to the authors previously mentioned. Indeed, the pioneering work [25] proposes such a technique for FETI-DP and BDDC. The authors make use of a ‘Local Indicator of the Condition Number Bound’ to fill a gap in what would be an otherwise complete proof of a condition number bound. The follow-up article [32] illustrates the efficiency of the method for BDDC in a multilevel framework and [22] (by different authors) makes the proof complete in two dimensions. It must also be noted that, as early as 1999, the authors of [3] proposed a multigrid smoothed aggregation algorithm with an enrichment technique that includes low frequency eigenmodes of the operator in the aggregate (which is like a subdomain). Thanks to this procedure, any convergence rate chosen *a priori* can be achieved. Spectral enrichment is also at the heart of the spectral algebraic multigrid method [5]. The field of coarse spaces based on generalized eigenproblems in subdomains has been so active that it is not realistic to list all contributions here. A very incomplete overview is [13, 17, 30, 21, 38]. A topic that is currently very active is the development of fully algebraic methods [14, 2].

In this article, the objective is to define coarse spaces for preconditioners in the abstract Schwarz framework. Compared to the framework in [37], quite a significant generalization is made by allowing the local solvers (in each subdomain) to be singular. It is fair to mention that the article [1] proposes a setting similar to the one here. One difference is that here all abstract results are proved from scratch to fit exactly the framework that is considered.

The outline of the article is the following. In Section 2, the Abstract Schwarz framework is presented with minimal assumptions and the main results are stated. In Section 3 the results are proved. This includes generalized versions of the technical lemmas from [37]. As an illustration, Section 4 considers a two-dimensional linear elasticity problem and presents exactly how the abstract framework applies to the Additive Schwarz, Neumann-Neumann, and inexact Schwarz preconditioners with numerical results.

Notation

- The abbreviations spd and spsd are used to mean symmetric positive definite and symmetric positive semi-definite.
- I is the identity matrix of the conforming size that is always clear in the context;
- $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y}$, and $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$;
- if M is an order m spd matrix, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$,

$$\langle \mathbf{x}, M\mathbf{y} \rangle = \mathbf{x}^\top M\mathbf{y}, \|\mathbf{x}\|_M = \langle \mathbf{x}, M\mathbf{x} \rangle^{1/2}, \text{ and } \mathbf{x} \perp^M \mathbf{y} \text{ if } \langle \mathbf{x}, M\mathbf{y} \rangle = 0;$$

- if M is an order n spsd matrix $|\mathbf{x}|_M = \langle \mathbf{x}, M\mathbf{x} \rangle^{1/2}$ for any $\mathbf{x} \in \mathbb{R}^n$;
- if M is a matrix, $\lambda(M)$ is any one of its eigenvalues;
- if M is a matrix, M^\dagger denotes its pseudo-inverse (also called its Moore-Penrose inverse) as defined *e.g.* in [19, Problem 7.3.P7].

2 Assumptions, GenEO coarse spaces and spectral results

The problem is, for a given $\mathbf{b} \in \mathbb{R}^n$, to find \mathbf{x} such that

$$A\mathbf{x} = \mathbf{b}, \text{ where } A \in \mathbb{R}^{n \times n} \text{ is spd} \quad (1)$$

2.1 Abstract Schwarz setting and preconditioners

We start with the components of the one-level abstract Schwarz preconditioner.

Assumption 1 (Local Setting). *Let $n \in \mathbb{N}$ denote the dimension of the problem matrix A . Let $N \in \mathbb{N}$ denote the chosen number of subdomains and $n^s \in \mathbb{N}$ ($s \in \llbracket 1, N \rrbracket$) denote the cardinality of each one. It is assumed that restriction operators $R^s \in \mathbb{R}^{n \times n^s}$ and local operators $\tilde{A}^s \in \mathbb{R}^{n^s \times n^s}$ have been defined for each $s \in \llbracket 1, N \rrbracket$ in such a way that:*

- $R^s R^{s\top} = I$ (i.e., the rows of R^s form an orthonormal basis of range $(R^{s\top})$),
- $\mathbb{R}^n = \sum_{s=1}^N \text{range}(R^{s\top})$ (i.e., the set of local subspaces forms a cover of the global space),
- each \tilde{A}^s is an spsd matrix,
- with $\tilde{A}^{s\dagger}$ denoting the pseudo-inverses of the matrices \tilde{A}^s , the one-level preconditioner $\left(\sum_{s=1}^N R^{s\top} \tilde{A}^{s\dagger} R^s \right)$ (later denoted by H – see Definition 6) is non-singular. Hence it is symmetric positive definite.

The coloring constant, whose definition is recalled next following [37][Section 2.5.1], plays an important role in the numerical performance and the theory of domain decomposition methods. For a given A , it depends only on the choice of local subspaces.

Definition 2 (Coloring constant). *Let $\mathcal{N} \in \mathbb{N}$ be such that there exists a set $\{\mathcal{C}_j; 1 \leq j \leq \mathcal{N}\}$ of subsets of $\llbracket 1, N \rrbracket$ satisfying*

$$\llbracket 1, N \rrbracket = \bigcup_{1 \leq j \leq \mathcal{N}} \mathcal{C}_j \text{ and } \forall j \in \llbracket 1, \mathcal{N} \rrbracket, \forall s, t \in \mathcal{C}_j, s \neq t \Rightarrow R^s A R^t \top = \mathbf{0}.$$

One can always choose $\mathcal{N} = N$ but in general there are values of \mathcal{N} that are significantly smaller than the number N of subdomains. The number \mathcal{N} is often referred to as the coloring constant since it can be viewed as the number of colors needed to color each subdomain in such a way that any two subdomains with the same color are A -orthogonal.

Assumption 3 (Splitting of A). *Assume that there exist a family of N spsd matrices $M^s \in \mathbb{R}^{n^s \times n^s}$ (for $s = 1, \dots, N$), and a real number $\mathcal{N}' > 0$ such that any $\mathbf{x} \in \mathbb{R}^n$ can be decomposed as*

$$\mathbf{x} = \sum_{s=1}^N R^s \mathbf{y}^s, \mathbf{y}^s \in \mathbb{R}^{n^s}, \text{ and } \sum_{s=1}^N \langle \mathbf{y}^s, M^s \mathbf{y}^s \rangle \leq \mathcal{N}' \langle \mathbf{x}, A \mathbf{x} \rangle, \text{ for every } \mathbf{x} \in \mathbb{R}^n. \quad (2)$$

The usual way of defining the GenEO coarse spaces is slightly less general and involves a family of matrices that form a partition of unity. Here, the partition of unity matrices are not assumed to be diagonal.

Assumption 4 (Simplification of Assumption 3). *Let $D^s \in \mathbb{R}^{n^s \times n^s}$ for $s = 1, \dots, N$ be a family of spd matrices that form a partition of unity in the sense that $I = \sum_{s=1}^N R^s \mathbf{D}^s R^s$. Assume that there exist a set of N spsd matrices $M^s \in \mathbb{R}^{n^s \times n^s}$ (for $s = 1, \dots, N$), and a real number $\mathcal{N}' > 0$ such that*

$$\sum_{s=1}^N \langle D^s R^s \mathbf{x}, M^s D^s R^s \mathbf{x} \rangle \leq \mathcal{N}' \langle \mathbf{x}, A \mathbf{x} \rangle, \text{ for every } \mathbf{x} \in \mathbb{R}^n. \quad (3)$$

Assumption 4 implies Assumption 3 by choosing the splitting $\mathbf{x} = \sum_{s=1}^N R^s \mathbf{y}^s$ with $\mathbf{y}^s = D^s R^s \mathbf{x}$.

To inject a second level into the preconditioners, a coarse space and a coarse solver must be chosen. The coarse space is the central topic of this article. It will be denoted by V^0 and the following assumption is made.

Assumption 5 (Coarse Setting). *Given a coarse space $V^0 \subset \mathbb{R}^n$, a basis for V^0 is stored in the rows of a matrix denoted R^0 :*

$$V^0 = \text{range}(R^{0\top}); \quad R^0 \in \mathbb{R}^{\dim(V^0) \times n}.$$

A solver must be chosen for the coarse space. In this article we will focus on the case where the coarse solver is the exact solver on the coarse space: $(R^0 A R^{0\top})^{-1}$.

Definition 6 (Abstract Schwarz preconditioners). *Under Assumption 1, the one-level abstract Schwarz preconditioner for linear system (1) is defined by:*

$$H := \sum_{s=1}^N R^s \mathbf{A}^s \mathbf{R}^s. \quad (4)$$

Under Assumptions 1 and 5, the two-level hybrid Schwarz preconditioner is defined by

$$H_{hyb} := \Pi H \Pi^\top + R^{0\top} (R^0 A R^{0\top})^{-1} R^0,$$

where

$$\Pi := I - R^{0\top} (R^0 A R^{0\top})^{-1} R^0 A$$

is the A -orthogonal projection satisfying $\text{Ker}(\Pi) = V^0$.

Under Assumptions 1 and 5, the two-level Additive Schwarz preconditioner is defined by

$$H_{ad} := H + R^{0\top} (R^0 A R^{0\top})^{-1} R^0.$$

2.2 Notation

Definition 7 ($\mathcal{Y}_L(\tau, M_A, M_B)$ and $\mathcal{Y}_H(\tau, M_A, M_B)$). Let $m \in \mathbb{N}^*$, let $M_A \in \mathbb{R}^{m \times m}$ be an spd matrix, let $M_B \in \mathbb{R}^{m \times m}$ be an spd matrix, and let $\tau > 0$ be a scalar. We define $\mathcal{Y}_L(\tau, M_A, M_B)$ and $\mathcal{Y}_H(\tau, M_A, M_B)$ to be the spaces of, respectively, low and high frequency eigenvectors of the generalized eigenvalue problem $M_A \mathbf{y} = \lambda M_B \mathbf{y}$:

$$\mathcal{Y}_L(\tau, M_A, M_B) := \text{span}\{\mathbf{y}; M_A \mathbf{y} = \lambda M_B \mathbf{y} \text{ with } \lambda < \tau\},$$

and

$$\mathcal{Y}_H(\tau, M_A, M_B) := \text{span}\{\mathbf{y}; M_A \mathbf{y} = \lambda M_B \mathbf{y} \text{ with } \lambda \geq \tau\}.$$

There are choices of τ for which \mathcal{Y}_L or \mathcal{Y}_H may be empty (the space spanned by an empty set of vectors is empty).

Definition 8 (Constant C_{\sharp}). Let $C_{\sharp} > 0$ be such that, for every $s = 1, \dots, N$ and every $\mathbf{x}^s \in \mathbb{R}^{n^s}$,

$$\|\tilde{\Pi}^s R^{s\top} \mathbf{x}^s\|_A^2 \leq C_{\sharp}^{-1} |\mathbf{x}^s|_{\tilde{A}^s}^2,$$

where $\tilde{\Pi}^s$ is the A -orthogonal projection characterized¹ by $\text{Ker}(\tilde{\Pi}^s) = R^{s\top} \text{Ker}(\tilde{A}^s)$. (If \tilde{A}^s is non-singular then $\tilde{\Pi}^s$ is the identity matrix.)

The existence of such a C_{\sharp} is clear if \tilde{A}^s is non-singular: C_{\sharp} is one of the constants in the equivalence of the norms induced by \tilde{A}^s and $R^s A R^{s\top}$. Otherwise, we notice that the terms on either side of the inequality are the semi-norms of \mathbf{x}^s induced by $R^s \tilde{\Pi}^{s\top} A \tilde{\Pi}^s R^{s\top}$ and \tilde{A}^s . The kernel of both these operators is $\text{Ker}(\tilde{A}^s)$ so they are both norms on $\text{range}(\tilde{A}^s)$ (by [19, Problem 5.1.P2] and C_{\sharp} is one of the constants in the equivalence of these norms.

2.3 GenEO coarse spaces and convergence results

Recall that for any matrix B , the notation $\lambda(B)$ refers to any of its eigenvalues. In the next two theorems, GenEO coarse spaces are defined and bounds for the eigenvalues of the preconditioned operators are provided. These do not depend on the number of subdomains or on any parameters in A . A well known result (see e.g., [37, Lemma C.10]) implies that the convergence of the (projected) preconditioned conjugate gradient does not depend on these quantities either.

Theorem 9 (Spectral results for the projected and hybrid preconditioners). *For all results in the theorem, it is assumed that Assumptions 1 and 5 hold. Let $\tau > 0$ be a user-chosen threshold. (Recall Definition 2 for the coloring constant \mathcal{N} and Definition 7 for the spaces \mathcal{Y}_L and \mathcal{Y}_R of low and high frequency eigenvectors of a certain generalized eigenvalue problem.)*

1. (Natural coarse space) With C_{\sharp} from Definition 8, if for any $s \in \llbracket 1, N \rrbracket$, $(R^{s\top} \text{Ker}(\tilde{A}^s)) \subset V^0$ then

$$\lambda(HA\Pi) \leq \mathcal{N} C_{\sharp}^{-1} \text{ and } \lambda(H_{hyb}A) \leq \max(1, \mathcal{N} C_{\sharp}^{-1}). \quad (5)$$

2. (GenEO for λ_{\max}) If for any $s \in \llbracket 1, N \rrbracket$, $(R^{s\top} \mathcal{Y}_L(\tau, \tilde{A}^s, R^s A R^{s\top})) \subset V^0$ then

$$\lambda(HA\Pi) \leq \mathcal{N} \tau^{-1} \text{ and } \lambda(H_{hyb}A) \leq \max(1, \mathcal{N} \tau^{-1}). \quad (6)$$

3. (GenEO for λ_{\min}) Under Assumption 3 (or 4), if for any $s \in \llbracket 1, N \rrbracket$,

¹If the columns in \tilde{Z}^s form a basis for $R^{s\top} \text{Ker}(\tilde{A}^s)$ then $\Pi^s := \tilde{Z}^s (\tilde{Z}^{s\top} A \tilde{Z}^s)^{-1} \tilde{Z}^{s\top} A$

- (a) either \tilde{A}^s is non-singular, and $\left(R^{s\top} \mathcal{Y}_L(\tau^{-1}, M^s, \tilde{A}^s)\right) \subset V^0$,
- (b) or M^s is non-singular and $\left(R^{s\top} \left[\text{Ker}(\tilde{A}^s) + \text{range}(\mathcal{Y}_H(\tau, \tilde{A}^s, M^s))\right]\right) \subset V^0$,
- (c) or $W^s \in \mathbb{R}^{n^s \times \text{rank}(M^s)}$ is a matrix whose columns form an ℓ_2 -orthonormal basis of $\text{range}(M^s)$ and $\left(R^{s\top} \left[\text{Ker}(\tilde{A}^s) + \text{Ker}(M^s) + \text{range}(W^s \mathcal{Y}_H(\tau, W^{s\top} \tilde{A}^s W^s, W^{s\top} M^s W^s))\right]\right) \subset V^0$

then

$$\left(\lambda(HA\Pi) = 0 \text{ or } (\tau\mathcal{N}')^{-1} \leq \lambda(HA\Pi)\right) \text{ and } \min(1, (\tau\mathcal{N}')^{-1}) \leq \lambda(H_{hyb}A), \quad (7)$$

(with \mathcal{N}' from Assumption 3 (or 4)).

Proof. By Lemma 12 (see below), (5) is a consequence of (6) with $\tau = C_{\sharp}$. We also notice that in the Assumptions for (7), Case 3b is already included in case 3c (setting $W^s = I$). It remains to prove (6) and (7) without the assumption labelled 3b. These proofs are core results of the article. Result (6) for $HA\Pi$ is rewritten and proved in Theorem 17. Result (7) for $HA\Pi$ is rewritten and proved in Theorem 18. The results for $H_{hyb}A$ are then deduced by applying Theorem 16. \square

The presence of W^s in the last case is to ensure that the matrix on the right hand side of the generalized eigenvalue problem is spd. Faced with the same difficulty, the authors in [6, Section 7] make use of projection operators.

Theorem 10 (Spectral results for the additive preconditioner). *For all results in the theorem, it is assumed that Assumptions 1 and 5 hold. It is also assumed that all local operators \tilde{A}^s are non-singular (for every $s \in \llbracket 1, N \rrbracket$). Finally, let $\tau, \nu > 0$ be two user-chosen thresholds. (Recall Definition 2 for the coloring constant \mathcal{N} and Definition 7 for the spaces \mathcal{Y}_L and \mathcal{Y}_R of low and high frequency eigenvectors of a certain generalized eigenvalue problem.)*

1. (Natural bound for λ_{\max}) With C_{\sharp} from Definition 8, it holds that

$$\lambda(H_{ad}A) \leq \mathcal{N}C_{\sharp}^{-1} + 1. \quad (8)$$

2. (GenEO for λ_{\min}) Under Assumption 3 (or 4), with C_{\sharp} from Definition 8, if for any $s \in \llbracket 1, N \rrbracket$,

- (a) either $\left(R^{s\top} \mathcal{Y}_L(\tau^{-1}, M^s, \tilde{A}^s)\right) \subset V^0$,
- (b) or M^s is non-singular, and $\left(R^{s\top} \text{range}(\mathcal{Y}_H(\tau, \tilde{A}^s, M^s))\right) \subset V^0$,
- (c) or $W^s \in \mathbb{R}^{n^s \times \text{rank}(M^s)}$ is a matrix whose columns form an ℓ_2 -orthonormal basis of $\text{range}(M^s)$ and $\left(R^{s\top} \left[\text{Ker}(M^s) + \text{range}(W^s \mathcal{Y}_H(\tau, W^{s\top} \tilde{A}^s W^s, W^{s\top} M^s W^s))\right]\right) \subset V^0$

then

$$\left[\max\left(2, 1 + 2\mathcal{N}C_{\sharp}^{-1}\right) \max(1, \mathcal{N}'\tau)\right]^{-1} \leq \lambda(H_{ad}A), \quad (9)$$

(with \mathcal{N}' from Assumption 3 (or 4)).

3. (Double GenEO for λ_{\min}) Under Assumption 3 (or 4), if for any $s \in \llbracket 1, N \rrbracket$, $R^{s\top} \mathcal{Y}_L(\nu, \tilde{A}^s, R^s A R^{s\top}) \subset V_0$ and

- (a) either $\left(R^{s\top} \mathcal{Y}_L(\tau^{-1}, M^s, \tilde{A}^s)\right) \subset V^0$,

- (b) or M^s is non-singular, and $\left(R^{s\top} \text{range}(\mathcal{Y}_H(\tau, \tilde{A}^s, M^s))\right) \subset V^0$,
- (c) or $W^s \in \mathbb{R}^{n^s \times \text{rank}(M^s)}$ is a matrix whose columns form an ℓ_2 -orthonormal basis of $\text{range}(M^s)$ and $\left(R^{s\top} \left[\text{Ker}(M^s) + \text{range}(W^s \mathcal{Y}_H(\tau, W^{s\top} \tilde{A}^s W^s, W^{s\top} M^s W^s)) \right]\right) \subset V^0$

then

$$\left[\max(2, 1 + 2\mathcal{N}v^{-1}) \max(1, \mathcal{N}'\tau) \right]^{-1} \leq \lambda(H_{ad}A), \quad (10)$$

(with \mathcal{N}' from Assumption 3 (or 4)).

Proof. It has been assumed that the \tilde{A}^s are non-singular so equation (5) in Theorem 9 applies to the one-level preconditioner to give $\lambda(HA) \leq \mathcal{N}C_{\#}^{-1}$. Moreover, $H_{ad}A - HA = R^{0\top} (R^0 A R^{0\top})^{-1} R^0 A$ which is a projection. Projections have eigenvalues in $\{0, 1\}$ so (8) holds with no restriction on the coarse space.

By Lemma 12 (see below), (9) is a consequence of (10) with $v = C_{\#}$ and non-singular \tilde{A}^s . We also notice that in the Assumptions for (10), Case 3b is already included in case 3c (setting $W^s = I$). It remains to prove (10) without the assumption labelled 3b. This follows from Theorem 19 with $\min(1, (\tau\mathcal{N}')^{-1}) \leq \lambda(H_{hyb}A)$ according to (7) in Theorem 9. \square

2.4 Introduction to the proofs

Subsets that are spanned by eigenvectors of a well-chosen generalized eigenvalue problem have very useful properties. Some of these are recalled for illustration and for further reference.

Lemma 11. *Let $m \in \mathbb{N}^*$, let $M_A \in \mathbb{R}^{m \times m}$ be an spsd matrix, let $M_B \in \mathbb{R}^{m \times m}$ be an spd matrix, and let $\tau > 0$. With the notation from Definition 7, the two following properties hold*

- *spectral estimates:*

$$\begin{cases} \|y\|_{M_A}^2 < \tau \|y\|_{M_B}^2 \text{ for any } \mathbf{y} \in \mathcal{Y}_L(\tau, M_A, M_B), \\ \|y\|_{M_A}^2 \geq \tau \|y\|_{M_B}^2 \text{ for any } \mathbf{y} \in \mathcal{Y}_H(\tau, M_A, M_B), \end{cases} \quad (11)$$

- *conjugacy :*

$$\begin{cases} (\mathcal{Y}_L(\tau, M_A, M_B))^{\perp \ell_2} = M_B \mathcal{Y}_H(\tau, M_A, M_B) \\ (\mathcal{Y}_H(\tau, M_A, M_B))^{\perp \ell_2} = M_B \mathcal{Y}_L(\tau, M_A, M_B). \end{cases} \quad (12)$$

Proof. By assumption M_B is spd, so the generalized eigenvalue problem $M_A \mathbf{y} = \lambda M_B \mathbf{y}$ from Definition 7 is equivalent to the classical eigenvalue problem

$$M_B^{-1/2} M_A M_B^{-1/2} \mathbf{z} = \lambda \mathbf{z}, \text{ with } \mathbf{z} = M_B^{1/2} \mathbf{y}.$$

By the spectral theorem [19, Theorem 4.1.5] applied to the spsd matrix $M_B^{-1/2} M_A M_B^{-1/2}$, this matrix is unitarily diagonalizable so there exists an orthonormal basis of \mathbb{R}^n formed of eigenvectors \mathbf{z}_k . Consequently, the vectors $\mathbf{y}_k = M_B^{-1/2} \mathbf{z}_k$ form an M_B -orthonormal basis of \mathbb{R}^n and everything else follows by analogy with the classical eigenvalue problem. \square

Lemma 12 (Natural coarse space as a spectral coarse space). *Let Assumptions 1 hold, then*

$$\mathcal{Y}_L(C_{\#}, \tilde{A}^s, R^s A R^{s\top}) = \text{Ker}(\tilde{A}^s), \text{ for any } s \in \llbracket 1, N \rrbracket$$

where $C_{\#} > 0$ comes from Definition 8.

Proof. Let $s \in \llbracket 1, N \rrbracket$. It is clear that $\text{Ker}(\tilde{A}^s) \subset \mathcal{Y}_L(C_{\sharp}^s, \tilde{A}^s, R^s A R^{s\top})$. It remains to prove that all non-zero eigenvalues λ in the eigenvalue problem that defines $\mathcal{Y}_L(C_{\sharp}^s, \tilde{A}^s, R^s A R^{s\top})$ are greater than C_{\sharp}^s . Let $\lambda \neq 0$ and $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ satisfy $\tilde{A}^s \mathbf{x} = \lambda R^s A R^{s\top} \mathbf{x}$.

Let $\tilde{\Pi}^s$ be as in Definition 8 (A -orthogonal projection with $\text{Ker}(\tilde{\Pi}^s) = R^{s\top} \text{Ker}(\tilde{A}^s)$). Next we prove that $\tilde{\Pi}^s R^{s\top} \mathbf{y} = R^{s\top} \mathbf{y}$ in the non-obvious case where \tilde{A}^s is singular:

$$\text{range}(\tilde{\Pi}^s) = \left(\text{Ker}(\tilde{\Pi}^s) \right)^{\perp A} = \left(R^{s\top} \text{Ker}(\tilde{A}^s) \right)^{\perp A} \quad \text{so } R^{s\top} \mathbf{y} \in \text{range}(\tilde{\Pi}^s) \Leftrightarrow \mathbf{y} \perp^{(R^s A R^{s\top})} \text{Ker}(\tilde{A}^s).$$

and this last assertion is true following the conjugacy property of eigenvectors ((12) in Lemma 11). We can now conclude since

$$\|\mathbf{y}\|_{R^s A R^{s\top}} = \|R^{s\top} \mathbf{y}\|_A^2 = \|\tilde{\Pi}^s R^{s\top} \mathbf{y}\|_A^2 \leq C_{\sharp}^{-1} |\mathbf{y}|_{\tilde{A}^s}^2 = C_{\sharp}^{-1} \lambda \|\mathbf{y}\|_{R^s A R^{s\top}},$$

where the definition of C_{\sharp}^s has been applied. Cancelling the common factor $\|\mathbf{y}\|_{R^s A R^{s\top}} \neq 0$ allows to conclude that $C_{\sharp}^{-1} \lambda \geq 1$. \square

3 Convergence proofs for the projected preconditioner

3.1 Spectral bounds in the abstract framework

The abstract Schwarz theory presented in [37][Chapters 2 and 3] provides theoretical results that greatly simplify the problem of finding eigenvalue bounds for the projected preconditioned operator. For the bound on the largest eigenvalue the results are [37][Assumption 2.3, Assumption 2.4, Lemma 2.6, Lemma 2.10 and Theorem 2.13]. For the bound on the smallest eigenvalue, the results for the projected operator can be found in [37][Theorem 2.13 under Assumption 2.12 (that weakens Assumption 2.2 by considering only elements in $\text{range}(\Pi)$)]. In this section, we state and prove very similar results with the generalization that \tilde{A}^s can be singular and with Π playing a more central role in the assumptions.

Remark 13. *The pseudo-inverse M^\dagger of any real matrix M is also called the Moore-Penrose inverse of M . It is defined e.g., in [19, Problem 7.3.P7] and has the following properties that we will refer back to in the proofs involving $\tilde{A}^{s\dagger}$:*

$$M^\dagger M M^\dagger = M^\dagger; \quad M M^\dagger M = M; \quad \text{and} \quad \text{range}(M^\dagger) = \text{range}(M^\top). \quad (13)$$

By symmetry, the last property implies that $\text{range}(\tilde{A}^{s\dagger}) = \text{range}(\tilde{A}^s)$.

Next, the abstract result used to bound λ_{\max} is given and proved. Note that a difference with [37][Assumption 2.4] is that the result must be proved for vectors in $\text{range}(\tilde{A}^{s\dagger} R^s \Pi^\top)$, instead of $\text{range}(\tilde{A}^{s\dagger} R^s)$. This subtlety is what will allow to choose the coarse space, and already appeared in [35][Lemma 2.8, Lemma 3.12] in the particular settings of BDD and FETI. Another difference, is the presence of a projection operator $\tilde{\Pi}^s$ in the assumption of the lemma. This weakens the assumption as long as the kernel of \tilde{A}^s (once extended to the global space) is in the coarse space.

Lemma 14 (Upper bound for λ_{\max}). *Assume that the kernels of the local solvers \tilde{A}^s contribute to the coarse space in the sense that $\sum_{s=1}^N R^{s\top} \text{Ker}(\tilde{A}^s) \subset V^0$. For each $s = 1, \dots, N$, let $\tilde{\Pi}^s$ be the A -orthogonal projection characterized by $\text{Ker}(\tilde{\Pi}^s) = R^{s\top} \text{Ker}(\tilde{A}^s)$. Assume that there exists $\omega > 0$ such that*

$$\|\tilde{\Pi}^s R^{s\top} \mathbf{x}^s\|_A^2 \leq \omega |\mathbf{x}^s|_{\tilde{A}^s}^2 \quad \text{for every } s = 1, \dots, N \text{ and every } \mathbf{x}^s \in \text{range}(\tilde{A}^{s\dagger} R^s \Pi^\top).$$

Then the largest eigenvalue λ_{\max} of $H A \Pi$ satisfies $\lambda_{\max} \leq \mathcal{N} \omega$, where \mathcal{N} is as in Definition 2.

Proof. Let $\mathbf{x} \in \text{range}(\Pi^\top)$. By assumption it holds that

$$\|\tilde{\Pi}^s R^{s\top} \tilde{A}^{s\top} R^s \mathbf{x}\|_A \leq \omega \langle \tilde{A}^{s\top} R^s \mathbf{x}, \tilde{A}^s \tilde{A}^{s\top} R^s \mathbf{x} \rangle, \text{ for any } s = 1, \dots, N.$$

With the notation $H^s := R^{s\top} \tilde{A}^{s\top} R^s$, this is equivalent to

$$\|\tilde{\Pi}^s H^s \mathbf{x}\|_A^2 \leq \omega |\mathbf{x}|_{H^s}^2. \quad (14)$$

We next prove the intermediary result $\|\Pi H \mathbf{x}\|_A^2 \leq \omega \mathcal{N} \|\mathbf{x}\|_H^2$ as follows

$$\begin{aligned} \|\Pi H \mathbf{x}\|_A^2 &= \left\| \sum_{s=1}^N \Pi H^s \mathbf{x} \right\|_A^2 = \left\| \sum_{j=1}^{\mathcal{N}} \sum_{s \in \mathcal{C}_j} \Pi H^s \mathbf{x} \right\|_A^2 \leq \left(\sum_{j=1}^{\mathcal{N}} \left\| \sum_{s \in \mathcal{C}_j} \Pi H^s \mathbf{x} \right\|_A \right)^2 \\ &\leq \mathcal{N} \sum_{j=1}^{\mathcal{N}} \left\| \sum_{s \in \mathcal{C}_j} \Pi H^s \mathbf{x} \right\|_A^2 \leq \mathcal{N} \sum_{j=1}^{\mathcal{N}} \left\| \sum_{s \in \mathcal{C}_j} \tilde{\Pi}^s H^s \mathbf{x} \right\|_A^2 = \mathcal{N} \sum_{j=1}^{\mathcal{N}} \sum_{s \in \mathcal{C}_j} \|\tilde{\Pi}^s H^s \mathbf{x}\|_A^2 \\ &\leq \mathcal{N} \sum_{j=1}^{\mathcal{N}} \sum_{s \in \mathcal{C}_j} \omega |\mathbf{x}|_{H^s}^2 = \mathcal{N} \sum_{s=1}^N \omega |\mathbf{x}|_{H^s}^2 = \omega \mathcal{N} \|\mathbf{x}\|_H^2, \end{aligned}$$

where in the first line the sets \mathcal{C}_j are as in Definition 2; in the second line the Cauchy-Schwarz estimate in the ℓ_2 -inner product, the definition of $\tilde{\Pi}^s$, as well as the definition of the sets \mathcal{C}_j are applied; and (14) is injected into the third line.

Next, we prove the bound for λ_{\max} starting with the definition of an eigenvalue:

$$\begin{aligned} \lambda_{\max} \text{ is an eigenvalue of } H A \Pi &\Leftrightarrow \lambda_{\max} \text{ is an eigenvalue of } \Pi^\top A H \\ &\Leftrightarrow \exists \mathbf{y} \in \mathbb{R}^n; \mathbf{y} \neq \mathbf{0} \text{ such that } \Pi^\top A H \mathbf{y} = \lambda_{\max} \mathbf{y}. \end{aligned} \quad (15)$$

Let \mathbf{y} be as in (15). It is obvious that $\mathbf{y} \in \text{range}(\Pi^\top)$. Taking the inner product of (15) by $H \mathbf{y}$, and injecting the intermediary result that was just proved gives

$$\lambda_{\max} \|\mathbf{y}\|_H^2 = \langle H \mathbf{y}, \Pi^\top A H \mathbf{y} \rangle = \|\Pi H \mathbf{y}\|_A^2 \leq \omega \mathcal{N} \|\mathbf{y}\|_H^2. \quad (16)$$

The common factor $\|\mathbf{y}\|_H^2$ can be cancelled because $\|\mathbf{y}\|_H^2 = 0$ would imply $\lambda_{\max} = 0$, and $\Pi = \mathbf{0}$. \square

The abstract Schwarz theory ([37][Theorem 2.13 under Assumption 2.12]) also provides a result for bounding the spectrum of the two-level operator from below. The result proved in the next Lemma is similar with the differences that are pointed out below the lemma.

Lemma 15 (Lower bound for λ_{\min}). *Assume that the kernels of the local solvers contribute to the coarse space in the sense that $\sum_{s=1}^N R^{s\top} \text{Ker}(\tilde{A}^s) \subset V^0$. If, for any $\mathbf{x} \in \text{range}(\Pi)$, there exist $\mathbf{z}^1, \dots, \mathbf{z}^n$ such that*

$$\mathbf{x} = \sum_{s=1}^N \Pi R^{s\top} \mathbf{z}^s \text{ and } \sum_{s=1}^N \langle \mathbf{z}^s, \tilde{A}^s \mathbf{z}^s \rangle \leq C_0^2 \langle \mathbf{x}, A \mathbf{x} \rangle \quad (\text{stable splitting of } \mathbf{x}),$$

then, the smallest eigenvalue λ_{\min} of $H A \Pi$, excluding zero, satisfies $\lambda_{\min} \geq C_0^{-2}$.

The differences compared to [37][Theorem 2.13] are the possible singularity of \tilde{A}^s , the extra presence of Π in the definition of a splitting, and the extra assumption on the minimal coarse space.

Proof. Let $\mathbf{x} \in \text{range}(\Pi)$ and $\{\mathbf{z}^s\}_{s=1,\dots,N}$ provide a stable splitting as defined in the lemma, then

$$\langle \mathbf{x}, A\mathbf{x} \rangle = \sum_{s=1}^N \langle \mathbf{x}, A\Pi R^s \Pi^\top \mathbf{z}^s \rangle = \sum_{s=1}^N \langle \tilde{A}^s \dagger R^s \Pi^\top A\mathbf{x}, \tilde{A}^s \mathbf{z}^s \rangle.$$

Indeed $\tilde{A}^s \tilde{A}^s \dagger R^s \Pi^\top = R^s \Pi^\top$ holds because of (13) and

$$\text{range}(R^s \Pi^\top) = \left(\text{Ker}(\Pi R^s \Pi^\top) \right)^{\perp \ell_2} \subset \left(\text{Ker}(\tilde{A}^s) \right)^{\perp \ell_2} = \text{range}(\tilde{A}^s),$$

recalling that $R^s \Pi^\top \text{Ker}(\tilde{A}^s) \subset V^0 = \text{Ker}(\Pi)$. Next, the generalized Cauchy-Schwarz inequality for the semi-norm induced by \tilde{A}^s , the first property in (13), the Cauchy-Schwarz inequality in the ℓ_2 -inner product, and the stable splitting assumption are applied in order to get

$$\begin{aligned} \langle \mathbf{x}, A\mathbf{x} \rangle &\leq \sum_{s=1}^N \langle \tilde{A}^s \dagger R^s \Pi^\top A\mathbf{x}, \tilde{A}^s \tilde{A}^s \dagger R^s \Pi^\top A\mathbf{x} \rangle^{1/2} \langle \mathbf{z}^s, \tilde{A}^s \mathbf{z}^s \rangle^{1/2} \\ &\leq \sum_{s=1}^N \langle \tilde{A}^s \dagger R^s \Pi^\top A\mathbf{x}, R^s \Pi^\top A\mathbf{x} \rangle^{1/2} \langle \mathbf{z}^s, \tilde{A}^s \mathbf{z}^s \rangle^{1/2} \\ &\leq \left[\sum_{s=1}^N \langle \tilde{A}^s \dagger R^s \Pi^\top A\mathbf{x}, R^s \Pi^\top A\mathbf{x} \rangle \right]^{1/2} \left[\sum_{s=1}^N \langle \mathbf{z}^s, \tilde{A}^s \mathbf{z}^s \rangle \right]^{1/2} \\ &\leq \langle \mathbf{x}, AH A\mathbf{x} \rangle^{1/2} C_0 \langle \mathbf{x}, A\mathbf{x} \rangle^{1/2}. \end{aligned}$$

Squaring and cancelling the common factor $\langle \mathbf{x}, A\mathbf{x} \rangle$ ($\neq 0$ if $\mathbf{x} \neq \mathbf{0}$) yields

$$\langle \mathbf{x}, AH A\mathbf{x} \rangle \geq C_0^{-2} \langle \mathbf{x}, A\mathbf{x} \rangle, \text{ for any } \mathbf{x} \in \text{range}(\Pi). \quad (17)$$

Finally, the bound for λ_{\min} is proved starting with the definition of an eigenvalue:

$$\lambda_{\min} \text{ is an eigenvalue of } HA\Pi \Leftrightarrow \exists \mathbf{x} \in \mathbb{R}^n; \mathbf{x} \neq \mathbf{0} \text{ such that } HA\Pi\mathbf{x} = \lambda_{\min}\mathbf{x}.$$

Let \mathbf{x} be such an eigenvector corresponding to eigenvalue λ_{\min} . By definition, $\lambda_{\min} \neq 0$ so $\Pi\mathbf{x} \neq \mathbf{0}$. Taking the inner product by $A\Pi\mathbf{x}$ gives

$$\langle \Pi\mathbf{x}, AH A\Pi\mathbf{x} \rangle = \lambda_{\min} \langle A\Pi\mathbf{x}, \mathbf{x} \rangle = \lambda_{\min} \langle A\Pi\mathbf{x}, \Pi\mathbf{x} \rangle \leq \lambda_{\min} C_0^2 \langle \Pi\mathbf{x}, AH A\Pi\mathbf{x} \rangle,$$

where the inequality comes from (17). Cancelling the common factor $\langle \Pi\mathbf{x}, AH A\Pi\mathbf{x} \rangle \neq 0$, leads to the conclusion that $\lambda_{\min} C_0^2 \geq 1$. \square

Theorem 16 (Spectral bounds for $HA\Pi \Rightarrow$ Spectral bounds for $H_{hyb}A$). *Let Assumptions 1 and 5 hold. If the eigenvalues of the projected and preconditioned operator satisfy*

$$\lambda(HA\Pi) \in \{0\} \cup [\lambda_{\min}, \lambda_{\max}],$$

then the eigenvalues of the operator preconditioned by H_{hyb} from Definition 6 satisfy

$$\lambda(H_{hyb}A) \in [\min(1, \lambda_{\min}), \max(1, \lambda_{\max})].$$

Proof. The connection between the spectra of the projected and hybrid/balanced preconditioned operators is well known (see e.g., [36, 23]) and easy to verify. Let $\mathbf{x} \in \mathbb{R}^n$, it holds that

$$\begin{aligned} \langle \mathbf{x}, AH_{hyb}A\mathbf{x} \rangle &= \langle \mathbf{x}, A\Pi H \Pi^\top A\mathbf{x} \rangle + \langle \mathbf{x}, AR^{0\top} (R^0 AR^{0\top})^{-1} R^0 A\mathbf{x} \rangle \\ &= \langle \Pi\mathbf{x}, AH A\Pi\mathbf{x} \rangle + \langle (I - \Pi)\mathbf{x}, A(I - \Pi)\mathbf{x} \rangle, \end{aligned}$$

so, with the result for the projected preconditioned operator:

$$\left\{ \begin{array}{l} \langle \mathbf{x}, AH_{hyb}A \rangle \geq \lambda_{\min} \langle \Pi \mathbf{x}, A \Pi \mathbf{x} \rangle + \langle (I - \Pi) \mathbf{x}, A(I - \Pi) \mathbf{x} \rangle \\ \text{and } \langle \mathbf{x}, AH_{hyb}A \rangle \leq \lambda_{\max} \langle \Pi \mathbf{x}, A \Pi \mathbf{x} \rangle + \langle (I - \Pi) \mathbf{x}, A(I - \Pi) \mathbf{x} \rangle \end{array} \right.$$

and the result follows by recalling that Π is an A -orthogonal projection so $\langle \mathbf{x}, A \mathbf{x} \rangle = \langle \Pi \mathbf{x}, A \Pi \mathbf{x} \rangle + \langle (I - \Pi) \mathbf{x}, A(I - \Pi) \mathbf{x} \rangle$. \square

3.2 Proof of (6) for $H A \Pi$

Theorem 17. *Let Assumptions 1 and 5 hold. Let $\tau > 0$. If for any $s \in \llbracket 1, N \rrbracket$, $(R^{s\top} \mathcal{Y}_L(\tau, \tilde{A}^s, R^s A R^{s\top})) \subset V^0$ then the largest eigenvalue λ_{\max} of $H A \Pi$ satisfies: $\lambda_{\max} \leq N \tau^{-1}$.*

Proof. It is assumed that $\tau > 0$, so for each $s = 1, \dots, N$, $\text{Ker}(\tilde{A}^s) \subset \mathcal{Y}_L(\tau, \tilde{A}^s, R^s A R^{s\top})$ and

$$R^{s\top} \text{Ker}(\tilde{A}^s) \subset V^0.$$

Then, according to the result in Lemma 14, a sufficient condition for the result in the theorem is that, for any $s = 1, \dots, N$,

$$\mathbf{x}^s \in \text{range}(\tilde{A}^s \dagger R^s \Pi^\top) \Rightarrow \|\tilde{\Pi}^s R^{s\top} \mathbf{x}^s\|_A^2 \leq \tau^{-1} \mathcal{N} |\mathbf{x}^s|_{\tilde{A}^s}^2. \quad (18)$$

Recall that $\tilde{\Pi}^s$ was defined in Lemma 14. Let $s = 1, \dots, N$ be fixed and, for the length of the proof let $\mathcal{Y}_L = \mathcal{Y}_L(\tau, \tilde{A}^s, R^s A R^{s\top})$ and $\mathcal{Y}_H = \mathcal{Y}_H(\tau, \tilde{A}^s, R^s A R^{s\top})$ in order to shorten notations.

We first characterize the space $\text{range}(\tilde{A}^s \dagger R^s \Pi^\top)$. The assumption is that $R^{s\top} \mathcal{Y}_L \subset V^0$, so $R^{s\top} \mathcal{Y}_L \subset \text{ker}(\Pi) = (\text{range}(\Pi^\top))^{\perp_{\ell_2}}$ which implies that $\mathcal{Y}_L \subset (\text{range}(R^s \Pi^\top))^{\perp_{\ell_2}}$ where the ℓ_2 -orthogonality is now in \mathbb{R}^{n^s} instead of \mathbb{R}^n . Taking the orthogonal again and applying (12) from Lemma 11 yields

$$\text{range}(R^s \Pi^\top) \subset (\mathcal{Y}_L)^{\perp_{\ell_2}} = R^s A R^{s\top} \mathcal{Y}_H = \tilde{A}^s \mathcal{Y}_H.$$

It then follows, by definition of $\tilde{A}^s \dagger$, that

$$\text{range}(\tilde{A}^s \dagger R^s \Pi^\top) \subset \tilde{A}^s \dagger \tilde{A}^s \mathcal{Y}_H \subset \mathcal{Y}_H + \text{Ker}(\tilde{A}^s),$$

Now, let $\mathbf{x}^s \in \text{range}(\tilde{A}^s \dagger R^s \Pi^\top)$. It has just been proved that there exist $\mathbf{y}^s \in \mathcal{Y}_H$ and $\mathbf{z}^s \in \text{Ker}(\tilde{A}^s)$ such that $\mathbf{x}^s = \mathbf{y}^s + \mathbf{z}^s$, so $\tilde{\Pi}^s R^{s\top} \mathbf{x}^s = \tilde{\Pi}^s R^{s\top} \mathbf{y}^s$. Moreover, $\tilde{\Pi}^s$ being an A -orthogonal projection, its range is the space

$$\text{range}(\tilde{\Pi}^s) = (\text{Ker}(\tilde{\Pi}^s))^{\perp_A} = (R^{s\top} \text{Ker}(\tilde{A}^s))^{\perp_A} \supset (R^{s\top} \mathcal{Y}_L)^{\perp_A}.$$

The last inclusion follows from $\text{Ker}(\tilde{A}^s) \subset \mathcal{Y}_L$. Another application of (12) from Lemma 11 guarantees that $\mathcal{Y}_H^{\perp_{\ell_2}} = R^s A R^{s\top} \mathcal{Y}_L$ so $R^{s\top} \mathcal{Y}_H \subset (R^{s\top} \mathcal{Y}_L)^{\perp_A} \subset \text{range}(\tilde{\Pi}^s)$. Consequently, $\tilde{\Pi}^s R^{s\top} \mathbf{x}^s = R^{s\top} \mathbf{y}^s$ and the desired estimate can finally be proved, using the second spectral estimate from (11) in Lemma 14 to get the inequality,

$$\|\tilde{\Pi}^s R^{s\top} \mathbf{x}^s\|_A^2 = \|R^{s\top} \mathbf{y}^s\|_A^2 = \|\mathbf{y}^s\|_{R^s A R^{s\top}}^2 \leq \tau^{-1} |\mathbf{y}^s|_{\tilde{A}^s}^2 = \tau^{-1} |\mathbf{x}^s|_{\tilde{A}^s}^2.$$

\square

3.3 Proof of (7) for $H\Pi$

Theorem 18. *Let Assumptions 1 and 5 hold. Let $\tau > 0$. Under Assumption 3 (or 4), if for any $s \in \llbracket 1, N \rrbracket$,*

1. *either \tilde{A}^s is non-singular, and $(R^{s\top} \mathcal{Y}_L(\tau^{-1}, M^s, \tilde{A}^s)) \subset V^0$,*
2. *or $W^s \in \mathbb{R}^{n^s \times \text{rank}(M^s)}$ is a matrix whose columns form an ℓ_2 -orthonormal basis of $\text{range}(M^s)$ and $(R^{s\top} [\text{Ker}(\tilde{A}^s) + \text{Ker}(M^s) + \text{range}(W^s \mathcal{Y}_H(\tau, W^{s\top} \tilde{A}^s W^s, W^{s\top} M^s W^s))]) \subset V^0$*

then the smallest non-zero eigenvalue λ_{\min} of $H\Pi$ satisfies: $(\tau \mathcal{N}')^{-1} < \lambda_{\min}$.

Proof. The proof consists in checking that the assumptions in Lemma 15 are satisfied. The fact that $\sum_{s=1}^N R^{s\top} \text{Ker}(\tilde{A}^s) \subset V^0$ is clear. It remains to prove that there exists a stable splitting of any $\mathbf{x} \in \text{range}(\Pi)$ with $C_0^2 = \tau \mathcal{N}'$.

Letting $\mathbf{x} \in \text{range}(\Pi)$, the idea is to start with the stable splitting from Assumption 3 or Assumption 4 which splits \mathbf{x} as

$$\mathbf{x} = \sum_{s=1}^N R^{s\top} \mathbf{y}^s \text{ satisfying } \sum_{s=1}^N \langle \mathbf{y}^s, M^s \mathbf{y}^s \rangle \leq \mathcal{N}' \langle \mathbf{x}, A\mathbf{x} \rangle.$$

If local components \mathbf{z}^s are found such that $\Pi R^{s\top} \mathbf{z}^s = \Pi R^{s\top} \mathbf{y}^s$ and $\langle \mathbf{z}^s, \tilde{A}^s \mathbf{z}^s \rangle \leq \tau \langle \mathbf{y}^s, M^s \mathbf{y}^s \rangle$, then it holds both that $\sum_{s=1}^N \Pi R^{s\top} \mathbf{z}^s = \Pi \sum_{s=1}^N R^{s\top} \mathbf{y}^s = \Pi \mathbf{x} = \mathbf{x}$ and $\sum_{s=1}^N \langle \mathbf{z}^s, \tilde{A}^s \mathbf{z}^s \rangle \leq \mathcal{N}' \tau \langle \mathbf{x}, A\mathbf{x} \rangle$. In other words finding such \mathbf{z}^s concludes the proof.

Let $s \in \llbracket 1, N \rrbracket$. We start with the case labelled 1 and set $\mathcal{Y}_L^s = \mathcal{Y}_L(\tau, W^{s\top} \tilde{A}^s W^s, W^{s\top} M^s W^s)$ for the length of the proof. Let

$$\mathbf{z}^s = W^s P^s W^{s\top} \mathbf{y}^s$$

where P^s is the $(W^{s\top} M^s W^s)$ -orthogonal projection onto \mathcal{Y}_L^s .

We first check that $\Pi R^{s\top} \mathbf{z}^s = \Pi R^{s\top} \mathbf{y}^s$, i.e., $\Pi R^{s\top} (I - W^s P^s W^{s\top}) \mathbf{y}^s = 0$. We proceed as follows

$$\text{range}(I - W^s P^s W^{s\top}) = \left(\text{Ker}(I - W^s P^s W^{s\top}) \right)^{\perp \ell_2} \subset (\text{range}(M^s W^s \mathcal{Y}_L^s))^{\perp \ell_2},$$

since $W^s P^s W^{s\top} (W^{s\top} M^s W^s) \mathcal{Y}_L^s = W^s (W^{s\top} M^s W^s) P^s \mathcal{Y}_L^s = M^s W^s \mathcal{Y}_L^s$ which implies that $\text{range}(M^s W^s \mathcal{Y}_L^s) \subset \text{Ker}(I - W^s P^s W^{s\top})$. It follows that

$$\begin{aligned} \text{range}(I - W^s P^s W^{s\top}) &= \text{Ker}(\mathcal{Y}_L^s W^{s\top} M^s) \\ &= \text{Ker}(M^s) + \text{range}(W^s \mathcal{Y}_H(\tau, W^{s\top} \tilde{A}^s W^s, W^{s\top} M^s W^s)) \\ &\subset \text{Ker}(\Pi R^{s\top}), \end{aligned}$$

where in the second step, one inclusion is easy to check with (12) from Lemma 11 and the dimensions of both spaces are equal. The stability property follows from

$$\begin{aligned} \langle \mathbf{z}^s, \tilde{A}^s \mathbf{z}^s \rangle &= \langle P^s W^{s\top} \mathbf{y}^s, (W^{s\top} \tilde{A}^s W^s) P^s W^{s\top} \mathbf{y}^s \rangle \\ &< \tau \langle P^s W^{s\top} \mathbf{y}^s, (W^{s\top} M^s W^s) P^s W^{s\top} \mathbf{y}^s \rangle \text{ (by (11) in Lemma 11)} \\ &\leq \tau \langle W^{s\top} \mathbf{y}^s, (W^{s\top} M^s W^s) W^{s\top} \mathbf{y}^s \rangle \text{ (} P^s \text{ is a } (W^{s\top} M^s W^s)\text{-orthogonal projection)} \\ &= \tau \langle \mathbf{y}^s, M^s \mathbf{y}^s \rangle \text{ since } M^s \mathbf{y}^s = M^s W^s W^{s\top} \mathbf{y}^s + \underbrace{M^s (I - W^s W^{s\top}) \mathbf{y}^s}_{\in \text{Ker}(M^s)}. \end{aligned}$$

We now address the case labelled 2. Let

$$\mathbf{z}^s = P^{s'} \mathbf{y}^s,$$

where $P^{s'}$ is the \tilde{A}^s -orthogonal projection onto $\mathcal{Y}_H(\tau^{-1}, M^s, \tilde{A}^s)$. We first check that $\Pi R^{s\top} \mathbf{z}^s = \Pi R^{s\top} \mathbf{y}^s$, i.e., $\Pi R^{s\top} (I - P^{s'}) \mathbf{y}^s = \mathbf{0}$. This is indeed the case since $\Pi R^{s\top} \mathcal{Y}_L(\tau^{-1}, M^s, \tilde{A}^s) = \mathbf{0}$ and

$$\text{range}(I - P^{s'}) = \text{Ker}(P^{s'}) = \left(\mathcal{Y}_H(\tau^{-1}, M^s, \tilde{A}^s) \right)^{\perp \tilde{A}^s} = \text{range}(\mathcal{Y}_L(\tau^{-1}, M^s, \tilde{A}^s)),$$

by (12) in Lemma 11.

The stability property follows from

$$\begin{aligned} \langle \mathbf{z}^s, \tilde{A}^s \mathbf{z}^s \rangle &= \langle P^{s'} \mathbf{y}^s, \tilde{A}^s P^{s'} \mathbf{y}^s \rangle \\ &\leq \tau \langle P^{s'} \mathbf{y}^s, M^s P^{s'} \mathbf{y}^s \rangle \quad (\text{by (11) in Lemma 11}) \\ &\leq \tau [\langle P^{s'} \mathbf{y}^s, M^s P^{s'} \mathbf{y}^s \rangle + \langle (I - P^{s'}) \mathbf{y}^s, M^s (I - P^{s'}) \mathbf{y}^s \rangle] \\ &\leq \tau \langle \mathbf{y}^s, M^s \mathbf{y}^s \rangle \quad (\text{by (12) in Lemma 11}). \end{aligned}$$

□

3.4 Proof of (10)

Theorem 19. *Let Assumptions 1, 5, and 3 (or 4) hold. Moreover, assume that the matrices \tilde{A}^s are non-singular. If for any $s \in \llbracket 1, N \rrbracket$, $\left(R^{s\top} \mathcal{Y}_L(v, \tilde{A}^s, R^s A R^{s\top}) \right) \subset V^0$ and $0 < \lambda_{\min, \text{hyb}} \leq \lambda(H_{\text{hyb}} A)$ is a lower bound for the eigenvalues of the hybrid preconditioned operator then*

$$\left[\max(2, 1 + 2\mathcal{N}v^{-1}) \lambda_{\min, \text{hyb}}^{-1} \right]^{-1} \leq \lambda(H_{\text{Ad}} A).$$

Proof. The proof comes down to applying Lemma 15 (stable splitting). Indeed, the two-level Additive preconditioner fits the abstract framework by considering that there are $N + 1$ subspaces ($\text{range}(R^{s\top})$ for $s = 0, \dots, N$) that play the same role. In other words, the coarse space V^0 is viewed just like any of the other subspaces with the local solver $\tilde{A}^0 = R^0 A R^{0\top}$ and the interpolation operator $R^{0\top}$. There is no coarse space that is treated by projection so the projection operator in Lemma 15 equals identity. Let $\mathbf{x} \in \mathbb{R}^n$, it suffices to prove that there exist $\mathbf{z}^s \in \mathbb{R}^{n^s}$ for any $s = 0, \dots, N$ such that

$$\mathbf{x} = \sum_{s=0}^N R^{s\top} \mathbf{z}^s \quad \text{and} \quad \sum_{s=1}^N \|\mathbf{z}^s\|_{\tilde{A}^s}^2 + \|\mathbf{z}^0\|_{\tilde{A}^0}^2 \leq C_0^2 \|\mathbf{x}\|_A^2; \quad C_0^2 = \max(2, 1 + 2\mathcal{N}v^{-1}) \lambda_{\min, \text{hyb}}^{-1}. \quad (19)$$

Inspired by [37][Lemma 2.5], the proof starts with $\langle \mathbf{x}, H_{\text{hyb}}^{-1} \mathbf{x} \rangle \leq \lambda_{\min, \text{hyb}}^{-1} \langle \mathbf{x}, A \mathbf{x} \rangle$ and follows with

$$\begin{aligned} \langle \mathbf{x}, H_{\text{hyb}}^{-1} \mathbf{x} \rangle &= \langle H_{\text{hyb}}^{-1} \mathbf{x}, H_{\text{hyb}} H_{\text{hyb}}^{-1} \mathbf{x} \rangle \\ &= \sum_{s=1}^N \langle H_{\text{hyb}}^{-1} \mathbf{x}, \Pi R^{s\top} \tilde{A}^s{}^{-1} R^s \Pi^\top H_{\text{hyb}}^{-1} \mathbf{x} \rangle + \langle H_{\text{hyb}}^{-1} \mathbf{x}, R^{0\top} \tilde{A}^0{}^{-1} R^0 H_{\text{hyb}}^{-1} \mathbf{x} \rangle \\ &= \sum_{s=1}^N \langle R^s \Pi^\top H_{\text{hyb}}^{-1} \mathbf{x}, \tilde{A}^s{}^{-1} \tilde{A}^s \tilde{A}^s{}^{-1} R^s \Pi^\top H_{\text{hyb}}^{-1} \mathbf{x} \rangle + \langle H_{\text{hyb}}^{-1} \mathbf{x}, R^{0\top} \tilde{A}^0{}^{-1} \tilde{A}^0 \tilde{A}^0{}^{-1} R^0 H_{\text{hyb}}^{-1} \mathbf{x} \rangle \\ &= \sum_{s=1}^N \|\mathbf{z}^s\|_{\tilde{A}^s}^2 + \|\mathbf{z}^0\|_{\tilde{A}^0}^2, \end{aligned}$$

with $\mathbf{z}^s := \tilde{A}^s{}^{-1}R^s\Pi^\top H_{hyb}^{-1}\mathbf{x}$ for $s = 1, \dots, N$, and $\mathbf{z}^{0'} := \tilde{A}^0{}^{-1}R^0H_{hyb}^{-1}\mathbf{x}$. This first splitting of \mathbf{x} satisfies $\mathbf{x} = \sum_{s=1}^N \Pi R^s{}^\top \mathbf{z}^s + R^0{}^\top \mathbf{z}^{0'}$ and

$$\sum_{s=1}^N \|\mathbf{z}^s\|_{\tilde{A}^s}^2 + \|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 \leq \lambda_{\min, hyb}^{-1}(\mathbf{x}, A\mathbf{x}). \quad (20)$$

The only problem is the presence of Π . Instead the splitting of \mathbf{x} is rewritten to suit the fully additive setting, i.e., so that it satisfies (19):

$$\mathbf{x} = \sum_{s=1}^N R^s{}^\top \mathbf{z}^s - (I - \Pi) \sum_{s=1}^N R^s{}^\top \mathbf{z}^s + R^0{}^\top \mathbf{z}^{0'} = \sum_{s=1}^N R^s{}^\top \mathbf{z}^s + R^0{}^\top \underbrace{\left[-(R^0 A R^0{}^\top)^{-1} R^0 A \sum_{s=1}^N R^s{}^\top \mathbf{z}^s + \mathbf{z}^{0'} \right]}_{:= \mathbf{z}^0 \in V^0}.$$

It remains to prove that the splitting satisfies (19). To this end we calculate

$$\begin{aligned} \|\mathbf{z}^0\|_{\tilde{A}^0}^2 &\leq 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\|(-R^0 A R^0{}^\top)^{-1} R^0 A \sum_{s=1}^N R^s{}^\top \mathbf{z}^s\|_{\tilde{A}^0}^2 \\ &= 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\left\langle \sum_{s=1}^N R^s{}^\top \mathbf{z}^s, A R^0{}^\top (R^0 A R^0{}^\top)^{-1} R^0 A \sum_{s=1}^N R^s{}^\top \mathbf{z}^s \right\rangle \\ &= 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\|(I - \Pi) \sum_{s=1}^N R^s{}^\top \mathbf{z}^s\|_A^2 \\ &= 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\|(I - \Pi) H \Pi^\top H_{hyb}^{-1} \mathbf{x}\|_A^2 \\ &\leq 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\|H \Pi^\top H_{hyb}^{-1} \mathbf{x}\|_A^2 \\ &\leq 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\mathcal{N} \sum_{s=1}^N \|R^s{}^\top \tilde{A}^s{}^{-1} R^s \Pi^\top H_{hyb}^{-1} \mathbf{x}\|_A^2 \quad (\text{Cauchy-Schwarz with Definition 2 of } \mathcal{N}) \\ &\leq 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\mathcal{N} \sum_{s=1}^N \|\tilde{A}^s{}^{-1} R^s \Pi^\top H_{hyb}^{-1} \mathbf{x}\|_{R^s A R^s{}^\top}^2 \\ &\leq 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\mathcal{N} v^{-1} \sum_{s=1}^N \|\tilde{A}^s{}^{-1} R^s \Pi^\top H_{hyb}^{-1} \mathbf{x}\|_{\tilde{A}^s}^2 \quad ((18) \text{ in the proof of Theorem 17 with } \tilde{\Pi}^s = I) \\ &= 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\mathcal{N} v^{-1} \sum_{s=1}^N \|\mathbf{z}^s\|_{\tilde{A}^s}^2 \end{aligned}$$

Finally, by putting this together with (20), it follows that

$$\begin{aligned} \sum_{s=1}^N \|\mathbf{z}^s\|_{\tilde{A}^s}^2 + \|\mathbf{z}^0\|_{\tilde{A}^0}^2 &\leq \sum_{s=1}^N \|\mathbf{z}^s\|_{\tilde{A}^s}^2 + 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 + 2\mathcal{N} v^{-1} \sum_{s=1}^N \|\mathbf{z}^s\|_{\tilde{A}^s}^2 \\ &= (1 + 2\mathcal{N} v^{-1}) \sum_{s=1}^N \|\mathbf{z}^s\|_{\tilde{A}^s}^2 + 2\|\mathbf{z}^{0'}\|_{\tilde{A}^0}^2 \\ &\leq \max(2, 1 + 2\mathcal{N} v^{-1}) \lambda_{\min}(H_{hyb} A)^{-1} \langle \mathbf{x}, A\mathbf{x} \rangle. \end{aligned}$$

□

4 Example: 2d linear elasticity with Additive Schwarz, Neumann-Neumann and Inexact Schwarz

In this Section, the abstract framework is made concrete. Its setup, analysis and numerical performance are described for a two-dimensional linear elasticity problem.

4.1 Geometry and PDE

Let $\Omega = [0, 2] \times [0, 1] \subset \mathbb{R}^2$ be the computational domain. Let $\partial\Omega_D$ be the left hand side boundary of Ω and let $\mathcal{V} = \{\mathbf{v} \in H^1(\Omega)^2; \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_D\}$. The linear elasticity equations posed in Ω with mixed boundary conditions are considered. A solution $\mathbf{u} \in \mathcal{V}$ is sought such that

$$\int_{\Omega} 2\mu\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) dx + \int_{\Omega} L \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v}) dx = \int_{\Omega} \mathbf{g} \cdot \mathbf{v} dx, \text{ for all } \mathbf{v} \in \mathcal{V}, \quad (21)$$

where, for $i, j = 1, 2$, $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$, $\mathbf{g} = (0, 1)^\top$ and the Lamé coefficients are functions of Young's modulus E and Poisson's ratio ν : $\mu = \frac{E}{2(1+\nu)}$, $L = \frac{E\nu}{(1+\nu)(1-2\nu)}$.

4.2 Discretization

The computational domain is discretized by a uniform mesh with element size $h = 1/42$ and the boundary value problem is solved numerically with standard piecewise linear (\mathbb{P}_1) Lagrange finite elements. Let \mathcal{V}_h be the space of \mathbb{P}_1 finite elements that satisfy the Dirichlet boundary condition. Let $\{\phi_k\}_{k=1}^n$ be a basis of \mathcal{V}_h . The linear system that is to be solved is

$$\text{Find } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{Ax} = \mathbf{b},$$

with $A_{ij} = \int_{\Omega} [2\mu\varepsilon(\phi_i) : \varepsilon(\phi_j) + L \operatorname{div}(\phi_i) \operatorname{div}(\phi_j)] dx$ and $\mathbf{b}_i = \int_{\Omega} \mathbf{g} \cdot \phi_i dx$. The dimension of the global problem is $n = 43 \times 84 \times 2 = 7224$ where it has been taken into account that there are two degrees of freedom at each grid point (the \mathbf{x} and \mathbf{y} displacements) and that there are no degrees of freedom where a Dirichlet boundary condition has been prescribed.

4.3 Domain Decomposition

The computational domain Ω is partitioned into $N = 8$ non-overlapping subdomains with Metis [20] (see Figure 1–left). The geometric subdomains are assumed to be mesh-conforming and they are denoted Ω^s for $s \in \llbracket 1, N \rrbracket$. Let $V^s = \{S_1^s, \dots, S_{n_s}^s\}$ be the set of mesh nodes that are in each Ω^s (for $s \in \llbracket 1, N \rrbracket$). These are also the local degrees of freedom. The restriction matrices $R^s \in \mathbb{R}^{n^s \times n}$ are defined by

$$(R^s)_{ij} = 1 \text{ if } j = S_i^s \text{ and } (R^s)_{ij} = 0 \text{ otherwise.}$$

Each R^s has exactly one 1 per row. These are in agreement with Assumption 1. By construction the degrees of freedom that are on the interfaces between subdomains are duplicated (and only these ones).

We may also assemble the matrices that correspond to the discretization of the problem (21) restricted to each subdomain $s \in \llbracket 1, N \rrbracket$:

$$A_{\Omega^s}^s \in \mathbb{R}^{n^s \times n^s} \text{ such that } (A_{\Omega^s}^s)_{ij} := \int_{\Omega^s} [2\mu\varepsilon(\phi_{S_i^s}) : \varepsilon(\phi_{S_j^s}) + L \operatorname{div}(\phi_{S_i^s}) \operatorname{div}(\phi_{S_j^s}) dx] \text{ for all } i, j.$$

These are frequently referred to as the local Neumann matrices as they arise from assembling the original problem over the subdomain Ω^s with natural boundary conditions. They can't be

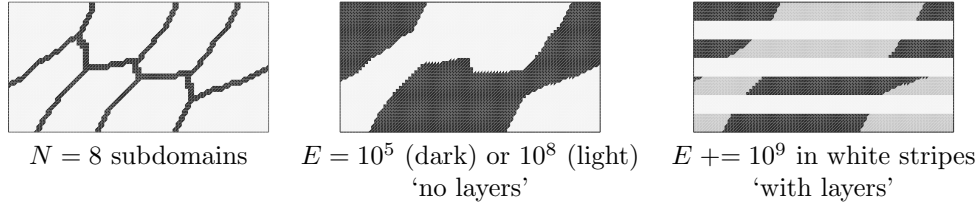


Figure 1: Partition into subdomains, distribution of E without and with harder layers.

computed from the global matrix A . Since we consider non-overlapping subdomains they satisfy the very useful property that

$$A = \sum_{s=1}^N R^s \top A_{|\Omega^s} R^s.$$

It is chosen to fulfill Assumption 3 through the use of a partition of unity as proposed in Assumption 4. Let the partition of unity be

- either the multiplicity or μ -scaling

$$D^s := R^s \left(\sum_{t=1}^N R^t \top R^t \right)^{-1} R^s \top \forall s \in \llbracket 1, N \rrbracket, \quad (22)$$

- or the k -scaling

$$D^s \in \mathbb{R}^{n^s \times n^s} \text{ diagonal with entries } (D_k^s)_{ii} := (A_{|\Omega^s}^s)_{ii} / (R^s A R^s \top)_{ii}, \forall i \in \llbracket 1, n^s \rrbracket, \quad (23)$$

Assumption 3 is fulfilled with $\mathcal{N}' = 1$ by setting

$$M^s := (D^s)^{-1} A_{|\Omega^s}^s (D^s)^{-1}; \text{ and } \mathbf{y}^s := D^s \mathbf{x}, \forall \mathbf{x} \in \mathbb{R}^{n^s}. \quad (24)$$

These matrices are typically singular and their kernel is the set of rigid body modes on the subdomain weighted by the partition of unity.

4.4 Choice of parameters in the PDE

We set $\nu = 0.4$ in all of the domain for all test cases. Two distributions of Young's modulus are considered. In the first E is constant per subdomain: $E = 10^5$ if s is odd and $E = 10^8$ if s is even. The second distribution of E is obtained by adding some rigid layers to the first one: E is augmented by 10^9 if $y \in [1/7, 2/7] \cup [3/7, 4/7] \cup [5/7, 6/7]$. These test cases are referred to as 'no layers' and 'with layers' and the coefficient distributions are plotted in Figure 1. It is well known (see, e.g., [31]) that the solution of (21) in a heterogeneous medium is challenging.

4.5 Domain Decomposition Preconditioners

4.5.1 Additive Schwarz preconditioner

Theorem 20 (Additive Schwarz preconditioner). *Let the Additive Schwarz preconditioner be defined by the choice $\hat{A}^s = R^s A R^s \top$ for every $s \in \llbracket 1, N \rrbracket$, which leads to the one-level preconditioner*

$$H := \sum_{s=1}^N R^s \top (R^s A R^s \top)^{-1} R^s. \quad (25)$$

With M^s from (24), and given any threshold $\tau > 1$, let the coarse space be defined

- either as $V^0 := \sum_{s=1}^N R^{s\top} \mathcal{Y}_L(\tau^{-1}, M^s, R^s A R^{s\top})$,
- or as $V^0 := \sum_{s=1}^N R^{s\top} \text{Ker}(M^s) + \sum_{s=1}^N R^{s\top} W^s \mathcal{Y}_H(\tau, W^{s\top} R^s A R^{s\top} W^s, W^{s\top} M^s W^s)$ where W^s contains an ℓ_2 -orthonormal basis of $\text{range}(M^s)$.

Under Assumption 5, the two level operators that result from applying Definition 6 satisfy

$$\lambda(HA) \leq \mathcal{N} \quad (26)$$

$$1/\tau \leq \lambda(HA\Pi) \leq \mathcal{N} \text{ if } \lambda(HA\Pi) \neq 0 \quad (27)$$

$$1/\tau \leq \lambda(H_{hyb}A) \leq \mathcal{N} \quad (28)$$

$$1/((1 + 2\mathcal{N})\tau) \leq \lambda(H_{ad}A) \leq \mathcal{N} + 1, \quad (29)$$

where \mathcal{N} is the coloring constant from Definition 2.

Proof. Apply Theorems 9 and 10 (with $C_{\sharp} = 1$, $\mathcal{N}' = 1$, $\mathcal{N} \geq 1$). The local matrices \tilde{A}^s in Theorem 20 are spd as a result of A being spd and the $R^{s\top}$ being full rank. \square

Remark 21 (Computation of V^0 in Theorem 20). *The coarse space V^0 is formed by contributions coming from each subdomain $s \in \llbracket 1, N \rrbracket$. They can be computed as follows. First, a Cholesky factorization with pivoting of the matrix M^s is performed. This gives both a factorization of M^s and an orthonormal basis Z^s for the kernel of M^s . Then an eigenvalue problem is solved partially to compute the highest or lowest frequency vectors. In both cases, the factorization of M^s is crucial since an application of $(M^s)^\dagger$ is necessary at each iteration. Although the first coarse space seems more simple than the second because its presentation is more compact, it does not vary much in actual computation work. The eigensolver SLEPc [18] provides an option for a deflation space. Setting it to $\text{Ker}(M^s)$ allows to solve the eigenvalue problem for the second coarse space (with W^s).*

Remark 22 (Choice of τ). *As τ decreases, the condition number of the preconditioned operators decreases. But the number of vectors in the coarse space becomes larger. It is not advised to choose $\tau < 1$ as this would lead to a very large coarse space. Indeed, by definition the matrices $R^s A R^{s\top}$ and M^s differ only at the interfaces between subdomains so eigenvalue 1 in the generalized eigenvalue problems is associated with a very large eigenspace. Indeed, it contains all the vectors that are 0 on the boundary of Ω^s .*

Remark 23. *The coarse vectors are $R^s A R^{s\top}$ -discrete harmonic inside the subdomains. This remark follows from the previous one: all eigenvectors that correspond to an eigenvalue other than 1 are $R^s A R^{s\top}$ -orthogonal to all vectors that are supported in the interior of a subdomain.*

4.5.2 Neumann-Neumann preconditioner

Theorem 24 (Neumann-Neumann preconditioner). *Let the Neumann-Neumann preconditioner be defined by the choice $\tilde{A}^s = M^s$ with M^s from (24) (for every $s \in \llbracket 1, N \rrbracket$), which leads to the one-level preconditioner*

$$H := \sum_{s=1}^N R^{s\top} D^s A_{|\Omega^s}^s \dagger D^s R^s. \quad (30)$$

Given any threshold $1 > \tau > 0$, let the coarse space be defined by

$$V^0 := \sum_{s=1}^N R^{s\top} \mathcal{Y}_L(\tau, M^s, R^s A R^{s\top}).$$

Under Assumption 5, the two level operators that result from applying Definition 6 satisfy

$$1 \leq \lambda(HA\Pi) \leq \mathcal{N}/\tau \text{ if } \lambda(HA\Pi) \neq 0 \quad (31)$$

$$1 \leq \lambda(H_{hyb}A) \leq \mathcal{N}/\tau. \quad (32)$$

where \mathcal{N} is the coloring constant from Definition 2.

Note that, V^0 is defined only for $\tau > 0$ so $\sum_{s=1}^N R^{s\top} \text{Ker}(M^s) \subset V^0$.

Proof. For the upper bounds, apply Theorem 9. For the lower bound of $HA\Pi$, apply Lemma 15 with $C_0^2 = 1$ (thanks to $\tilde{A}^s = M^s$ and Assumption 3 with $\mathcal{N}' = 1$). The lower bound for $H_{hyb}A$ follows from Theorem 16. \square

The choice of partition of unity enters into the one-level preconditioner as well as the coarse space. The local matrices \tilde{A}^s in Neumann-Neumann are singular unless there is a Dirichlet boundary condition for the subdomain numbered s . For the two-dimensional linear elasticity problem, the kernel of $A_{|\Omega^s}^s$ is the set of rigid body modes.

A remarkable feature is that the coarse space for Neumann-Neumann is the same as one of the coarse spaces for Additive Schwarz in Theorem 20: there is a set of coarse vectors that *fixes* both the Neumann-Neumann preconditioners and the Additive Schwarz preconditioners. This was already pointed out in [1].

The additive version of the Neumann-Neumann preconditioner is not considered as no results can be proved (and numerical performance is poor). There is no interesting result for the spectrum of the operator without a coarse space either. The closest thing to that is that $\lambda(HA\Pi) \geq 1$ with a coarse space consisting only of the kernels of the local solvers.

We refer the reader to the remarks in the previous paragraph which also apply here. In particular, the computation of V^0 is similar (or exactly the same), τ should be chosen to be less than 1, and the eigenvectors that enter into the coarse space are $R^s A R^{s\top}$ -harmonic in the interior of the subdomain.

4.5.3 Inexact Schwarz preconditioner

Theorem 25 (Inexact Schwarz preconditioner). *Let the Inexact Schwarz preconditioner be defined by the choice (for every $s \in \llbracket 1, N \rrbracket$) $\tilde{A}^s = L^s L^{s\top}$, where the triangular matrix L^s is the factor in the no-fill incomplete Cholesky factorization of A [4]. This leads to the one-level preconditioner*

$$H := \sum_{s=1}^N R^{s\top} (L^{s\top})^{-1} L^{s-1} R^s. \quad (33)$$

With M^s from (24), and given any two thresholds $\tau, v > 0$, let the coarse space be defined

- either as $V^0 := \sum_{s=1}^N R^{s\top} \left[\mathcal{Y}_L(v, L^s L^{s\top}, R^s A R^{s\top}) + \mathcal{Y}_L(\tau^{-1}, M^s, L^s L^{s\top}) \right]$
- or as $V^0 := \sum_{s=1}^N R^{s\top} \left[\text{Ker}(M^s) + \mathcal{Y}_L(v, L^s L^{s\top}, R^s A R^{s\top}) + W^s \mathcal{Y}_H(\tau, W^{s\top} L^s L^{s\top} W^s, W^{s\top} M^s W^s) \right]$, where W^s contains an ℓ_2 -orthonormal basis of $\text{range}(M^s)$.

Under Assumption 5, the two level operators that result from applying Definition 6 satisfy

$$1/\tau \leq \lambda(HA\Pi) \leq \mathcal{N}/v \text{ if } \lambda(HA\Pi) \neq 0 \quad (34)$$

$$1/\tau \leq \lambda(H_{hyb}A) \leq \mathcal{N}/v. \quad (35)$$

where \mathcal{N} is the coloring constant from Definition 2.

Proof. Apply Theorem 9 with $\mathcal{N}' = 1$. □

Remark 26 (Computation of V^0). *This time, the contributions to the coarse that come from each subdomain $s \in \llbracket 1, N \rrbracket$ require solving two generalized eigenvalue problems. For the first one $L^s L^{s\top} \mathbf{y}^s = \lambda R^s A R^{s\top} \mathbf{y}^s$, it is the low frequencies and their eigenvectors that must be computed. With an iterative solver this means solving linear systems with $L^s L^{s\top}$, a cost-effective task since L^s is triangular. The second eigenvalue problem resembles the one for Additive Schwarz (and Neumann-Neumann), in that the matrix that must be (pseudo-)inverted is the singular matrix M^s . We refer the reader to Remark 21.*

4.6 Numerical results

The results in this section were obtained with the software libraries FreeFem++ [16] and GNU Octave [8]. Let us recall that the size of the problem is $n = 7224$. The computational domain Ω is partitioned into $N = 8$ non-overlapping subdomains. Moreover, the value of the coloring constant from Definition 2 is $\mathcal{N} = 3$ and there are $n_\Gamma = 546$ degrees of freedom that belong to more than one subdomain.

Krylov Subspace Method The problem presented in Subsection 4.1 is solved by the preconditioned conjugate gradient method (PCG) with the Additive Schwarz, Neumann-Neumann, and Inexact Schwarz preconditioners. The problem is by no means a very large problem that requires state of the art parallel solvers. The purpose is to illustrate how the GenEO coarse spaces decrease the condition number and how many vectors per subdomain need to be added to the coarse space to achieve *fast* convergence. The stopping criterion for PCG is always that the error $\|\mathbf{x}_i - \mathbf{x}\|_A$ be less than $10^{-9} \|\mathbf{x}\|_A$ starting from a zero initial guess. This is not a practical stopping criterion but it has the advantage that it does not depend on the preconditioner. The numerical bounds for the spectrum of the preconditioned operators that are reported are the approximations given by the extreme Ritz values once the algorithm has converged (see *e.g.*, [27] for details on how to implement this procedure).

Results for the Additive Schwarz (AS) preconditioner from Theorem 20 In Figure 2, the condition numbers for the Additive and Hybrid preconditioners are plotted with respect to the threshold τ for both coefficient distributions and both choices of scaling. The theoretical upper bounds are also plotted and never exceeded. The theoretical bound is less sharp when τ becomes larger. As expected, preconditioning by the Hybrid preconditioner always leads to a lower condition number than preconditioning by the fully Additive variant.

Table 1 gives a lot more information. Only the test case with layers is considered. In all four configurations (hybrid/additive and μ -scaling/ k -scaling), the choice $\tau = 10$ seems to offer a good compromise between the condition number (or number of iterations) and the size of the coarse space. With the same value $\tau = 10$ the size of the coarse space with multiplicity scaling is 241 versus only 68 with k -scaling. This is due to the fact that the k -scaling already handles coefficient jumps that are across the subdomain interfaces (the ones that are already present in the ‘no layers’ test case). With multiplicity scaling, it is the coarse space that must handle also for these jumps. To better illustrate this behaviour, Figure 3 shows the eigenvalues of the generalized eigenvalue problem. It is clear that the even-numbered subdomains (in which E is three orders of magnitude larger) must contribute many more vectors. (Recall that it is the high-frequency vectors that get selected). This is not a failure of the GenEO method. In practice it is highly unlikely that an automatic graph partitioner would produce such a configuration. A human partitioner could produce such a configuration. If so, she would be aware of it and should choose the scaling accordingly.

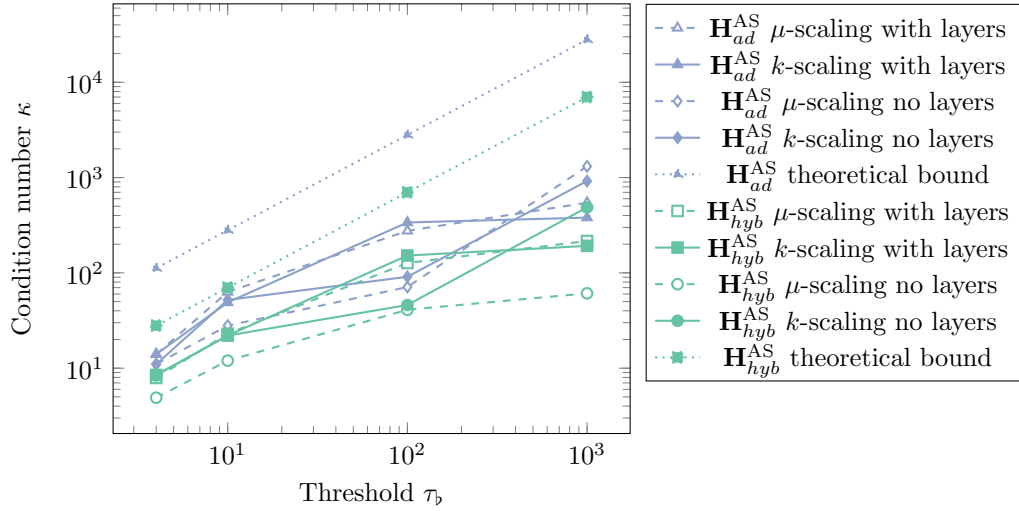


Figure 2: Condition numbers for Additive Schwarz preconditioners: additive and hybrid; μ -scaling and k -scaling; with and without layers; $\tau \in [4; 10; 100; 1000]$. All condition numbers are below the theoretical bound.

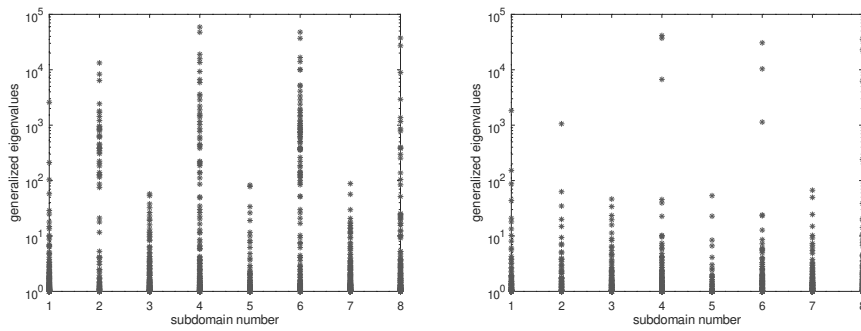


Figure 3: For each subdomain, solution of the generalized eigenvalue problem for computing V^0 in the case ‘with layers’. Left: μ -scaling, right: k -scaling.

Results also for the Neumann-Neumann (NN) and Inexact Schwarz (IS) preconditioners from Theorems 24 and 25 For lack of space these are presented in a lot less detail. Figure 4 (test cases with and without layers) shows the condition numbers with respect to the size of the coarse space for several values of the thresholds (that are not reported here) . In this plot, the best methods are the ones which have data points closest to the origin (small condition number with a small coarse space). It appears clearly that k -scaling gives better results. This was previously explained.

With k -scaling the methods from most to least efficient rank as follows: Neumann-Neumann, Additive Schwarz with hybrid preconditioner, Additive Schwarz with additive preconditioner, Inexact Schwarz. Again, this does not tell the whole story as the cost of one iteration depends on the choice of method. With inexact Schwarz, the local solves are cheapest. With additive Schwarz in the additive variant, the coarse solve can be done in parallel to the local solves. With Neumann-Neumann, the matrices that must be handled with most care numerically (the ones that are singular) are the local solvers whereas they only appear in the generalized eigenvalue problem for the other methods. A second word of caution about Neumann-Neumann with k -scaling is that the scaling matrices D_k^s can be very ill-conditioned and make the whole method less efficient again. This problem is well known and independent of GenEO. All these arguments lead only to one conclusion: this data does not tell us which method is most efficient overall. The answer would in any case be implementation and hardware dependant.

As a final remark on these results, let us comment on the *zig-zags* in the Inexact Schwarz data in Figure 4–Bottom. Since there are two parameters for the Inexact Schwarz coarse space, there are several choices of parameters that lead to coarse spaces that have the same size with possibly very different condition numbers. It is mostly only adding vectors to $Y_H(\tau, W^{s\top} L^s L^{s\top} W^s, W^{s\top} M^s W^s)$, that makes the method more efficient while adding vectors to $\mathcal{Y}_L(v, L^s L^{s\top}, R^s A R^{s\top})$ makes the coarse space grow very fast and the condition number decrease very little.

5 Conclusion

GenEO coarse spaces have been introduced for all domain decomposition methods in the abstract Schwarz framework that satisfy some clearly stated assumptions. By solving one or two generalized eigenvalue problems in each subdomain, it is possible to construct a method for which the eigenvalues of the preconditioned operator are bounded as desired. Proofs of these bounds were given for the projected preconditioned operators, the hybrid operators and, when possible, the additive operators. Finally, the method was applied to a linear elasticity problem discretized by \mathbb{P}_1 finite elements.

Additive Schwarz with μ-scaling (D^s from (22) in gevvp)						
		λ_{\min}	λ_{\max}	κ	It (final error)	$\#V^0$ (min $\#V_s^0$; max $\#V_s^0$)
one-level		$7.7 \cdot 10^{-4}$	3.0	3875	> 100 ($6 \cdot 10^{-3}$)	0 (0 ; 0)
$\tau = 10^{10}$ (only Ker(M^s))	hyb	0.003	3.0	959	> 100 ($1 \cdot 10^{-5}$)	18 (0 ; 3)
	ad	0.002	3.3	1517	> 100 ($3 \cdot 10^{-4}$)	18 (0 ; 3)
$\tau = 1000$	hyb	0.014	3.0	216	> 100 ($1 \cdot 10^{-9}$)	72 (1 ; 24)
	ad	0.007	4.0	545	> 100 ($1 \cdot 10^{-6}$)	72 (1 ; 24)
$\tau = 100$	hyb	0.024	3.0	127	92	159 (3 ; 60)
	ad	0.014	4.0	276	> 100 ($1 \cdot 10^{-7}$)	159 (3 ; 60)
$\tau = 10$	hyb	0.13	3.0	23	42	241 (10 ; 70)
	ad	0.06	4.0	63	64	241 (10 ; 70)
$\tau = 4$	hyb	0.37	3.0	7.9	23	303 (14 ; 77)
	ad	0.28	4.0	14	31	303 (14 ; 77)
Theory (τ)	hyb	$1/\tau$	$\mathcal{N} = 3$	3τ		
	ad	$1/(7\tau)$	$\mathcal{N} + 1 = 4$	28τ		
Additive Schwarz with k-scaling (D^s from (23) in gevvp)						
		λ_{\min}	λ_{\max}	κ	It (final error)	$\#V^0$ (min $\#V_s^0$; max $\#V_s^0$)
one-level		$7.7 \cdot 10^{-4}$	3.0	3875	> 100 ($6 \cdot 10^{-3}$)	0 (0 ; 0)
$\tau = 10^{10}$ (only Ker(M^s))	hyb	0.0030	3.0	1003	> 100 ($2 \cdot 10^{-5}$)	18 (0 ; 3)
	ad	0.0025	3.2	1271	> 100 ($2 \cdot 10^{-4}$)	18 (0 ; 3)
$\tau = 1000$	hyb	0.016	3.0	192	98	29 (1 ; 6)
	ad	0.0087	3.3	380	> 100 ($3 \cdot 10^{-7}$)	29 (1 ; 6)
$\tau = 100$	hyb	0.02	3.0	152	93	31 (1 ; 7)
	ad	0.098	3.3	338	> 100 ($2 \cdot 10^{-7}$)	31 (1 ; 7)
$\tau = 10$	hyb	0.13	3.0	22	43	68 (5 ; 13)
	ad	0.069	3.37	49	63	68 (5 ; 13)
$\tau = 4$	hyb	0.35	3.0	8.5	26	118 (8 ; 20)
	ad	0.25	3.4	14	34	118 (8 ; 20)
Theory (τ)	hyb	$1/\tau$	$\mathcal{N} = 3$	3τ		
	ad	$1/(7\tau)$	$\mathcal{N} + 1 = 4$	28τ		

Table 1: Test case with hard layers - All additive Schwarz methods - λ_{\min} and λ_{\max} : extreme eigenvalues, κ : condition number, It: iteration count (with relative error at iteration 100 in parenthesis if the method has not converged), $\#V^0$: dimension of the coarse space, min $\#V_s^0$: number of coarse vectors contributed by the subdomain that contributes the fewest vectors, max $\#V_s^0$: number of coarse vectors contributed by the subdomain that contributes the most eigenvectors, gevvp: generalized eigenvalue problem. The one-level method does not satisfy any theoretical bound for λ_{\min} .

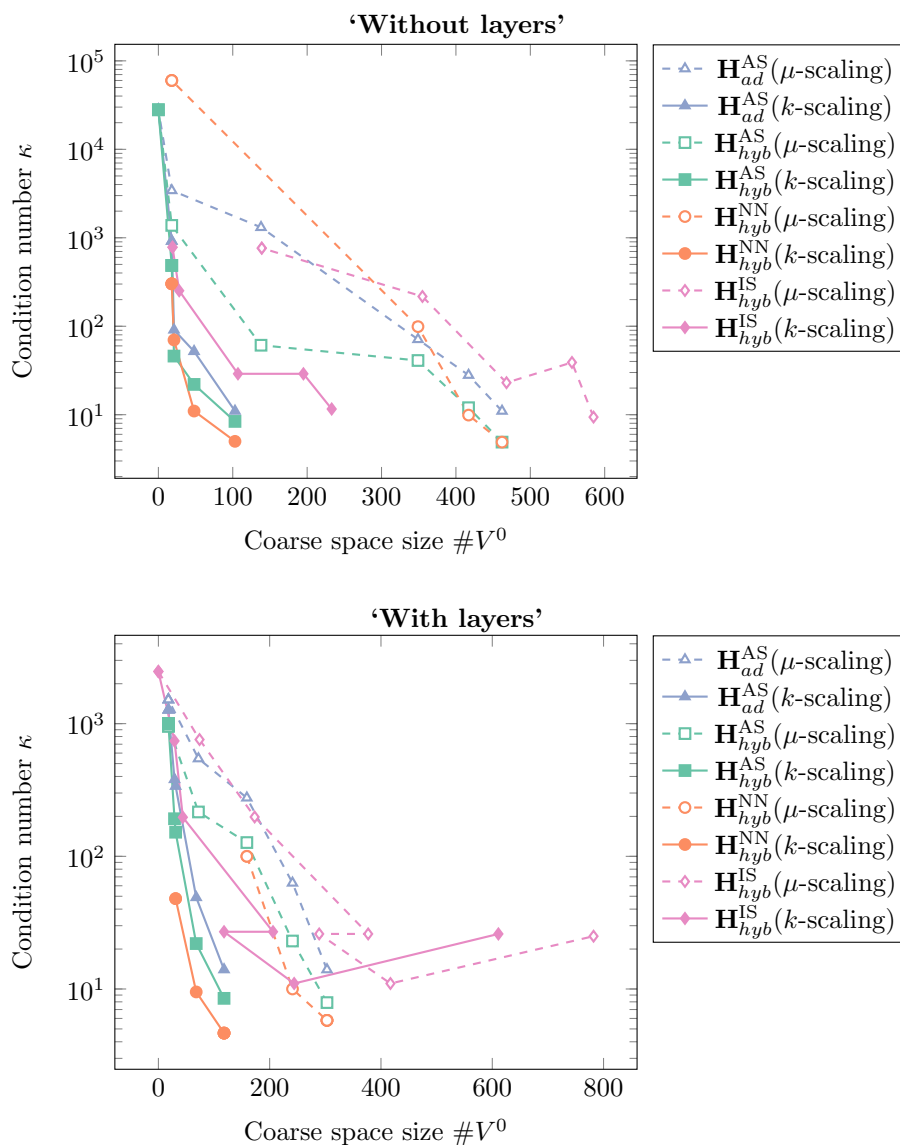


Figure 4: Efficiency of all methods (Condition number versus coarse space size). Top: the test case without layers in E . Bottom: test case with layers in E .

References

- [1] E. Agullo, L. Giraud, and L. Poirel. Robust preconditioners via generalized eigenproblems for hybrid sparse linear solvers. *SIAM Journal on Matrix Analysis and Applications*, 40(2):417–439, 2019.
- [2] H. Al Daas, P. Jolivet, and T. Rees. Efficient algebraic two-level Schwarz preconditioner for sparse matrices. *SIAM J. Sci. Comput.*, 45(3):a1199–a1213, 2023.
- [3] M. Brezina, C. Heberton, J. Mandel, and P. Vaněk. An iterative method with convergence rate chosen a priori. Technical Report 140, University of Colorado Denver, April 1999.
- [4] T. F. Chan and H. A. Van Der Vorst. Approximate and incomplete factorizations. In *Parallel numerical algorithms*, pages 167–202. Springer, 1997.
- [5] T. Chartier, R. D. Falgout, V. E. Henson, J. Jones, T. Manteuffel, S. McCormick, J. Ruge, and P. S. Vassilevski. Spectral AMGe (ρ AMGe). *SIAM J. Sci. Comput.*, 25(1):1–26, 2003.
- [6] V. Dolean, P. Jolivet, and F. Nataf. *An introduction to domain decomposition methods*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2015. Algorithms, theory, and parallel implementation.
- [7] V. Dolean, F. Nataf, R. Scheichl, and N. Spillane. Analysis of a two-level Schwarz method with coarse spaces based on local Dirichlet-to-Neumann maps. *Comput. Methods Appl. Math.*, 12(4):391–414, 2012.
- [8] J. W. Eaton, D. Bateman, S. Hauberg, and R. Wehbring. *GNU Octave version 5.2.0 manual: a high-level interactive language for numerical computations*, 2020.
- [9] Y. Efendiev, J. Galvis, R. Lazarov, and J. Willems. Robust domain decomposition preconditioners for abstract symmetric positive definite bilinear forms. *ESAIM Math. Model. Numer. Anal.*, 46(5):1175–1199, 2012.
- [10] C. Farhat and F.-X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Internat. J. Numer. Methods Engrg.*, 32:1205–1227, 1991.
- [11] J. Galvis and Y. Efendiev. Domain decomposition preconditioners for multiscale flows in high-contrast media. *Multiscale Model. Simul.*, 8(4):1461–1483, 2010.
- [12] J. Galvis and Y. Efendiev. Domain decomposition preconditioners for multiscale flows in high contrast media: reduced dimension coarse spaces. *Multiscale Model. Simul.*, 8(5):1621–1644, 2010.
- [13] M. J. Gander and A. Loneland. SHEM: An optimal coarse space for ras and its multiscale approximation. In *Domain decomposition methods in science and engineering XXIII*, pages 313–321. Springer, 2017.
- [14] L. Gouarin and N. Spillane. Fully algebraic domain decomposition preconditioners with adaptive spectral bounds. *ETNA, Electron. Trans. Numer. Anal.*, 60:169–196, 2024.
- [15] R. Haferssas, P. Jolivet, and F. Nataf. An additive Schwarz method type theory for Lions’s algorithm and a symmetrized optimized restricted additive Schwarz method. *SIAM Journal on Scientific Computing*, 39(4):A1345–A1365, 2017.
- [16] F. Hecht. New development in FreeFem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

- [17] A. Heinlein, A. Klawonn, J. Knepper, and O. Rheinbach. Adaptive GDSW coarse spaces for overlapping schwarz methods in three dimensions. *SIAM Journal on Scientific Computing*, 41(5):A3045–A3072, 2019.
- [18] V. Hernandez, J. E. Roman, and V. Vidal. SLEPc: A scalable and flexible toolkit for the solution of eigenvalue problems. *ACM Trans. Math. Software*, 31(3):351–362, 2005.
- [19] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, second edition, 2013.
- [20] G. Karypis and V. Kumar. A fast and high quality multilevel scheme for partitioning irregular graphs. *SIAM J. Sci. Comput.*, 20(1):359–392 (electronic), 1998.
- [21] A. Klawonn, M. Lanser, and O. Rheinbach. Toward extremely scalable nonlinear domain decomposition methods for elliptic partial differential equations. *SIAM Journal on Scientific Computing*, 37(6):C667–C696, 2015.
- [22] A. Klawonn, P. Radtke, and O. Rheinbach. A comparison of adaptive coarse spaces for iterative substructuring in two dimensions. *Electron. Trans. Numer. Anal.*, 45:75–106, 2016.
- [23] A. Klawonn and O. Rheinbach. Deflation, projector preconditioning, and balancing in iterative substructuring methods: connections and new results. *SIAM J. Sci. Comput.*, 34(1):A459–A484, 2012.
- [24] A. Klawonn and O. B. Widlund. FETI and Neumann-Neumann iterative substructuring methods: connections and new results. *Comm. Pure Appl. Math.*, 54(1):57–90, 2001.
- [25] J. Mandel and B. Sousedík. Adaptive selection of face coarse degrees of freedom in the BDDC and the FETI-DP iterative substructuring methods. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1389–1399, 2007.
- [26] P. Marchand, X. Claeys, P. Jolivet, F. Nataf, and P.-H. Tournier. Two-level preconditioning for h h-version boundary element approximation of hypersingular operator with GenEO. *Numerische Mathematik*, 146(3):597–628, 2020.
- [27] G. Meurant and P. Tichý. Approximating the extreme Ritz values and upper bounds for the A-norm of the error in CG. *Numerical Algorithms*, 82(3):937–968, 2019.
- [28] F. Nataf, H. Xiang, and V. Dolean. A two level domain decomposition preconditioner based on local Dirichlet-to-Neumann maps. *C. R. Math. Acad. Sci. Paris*, 348(21-22):1163–1167, 2010.
- [29] F. Nataf, H. Xiang, V. Dolean, and N. Spillane. A coarse space construction based on local Dirichlet-to-Neumann maps. *SIAM J. Sci. Comput.*, 33(4):1623–1642, 2011.
- [30] C. Pechstein and C. R. Dohrmann. A unified framework for adaptive BDDC. *Electron. Trans. Numer. Anal.*, 46(273-336):3, 2017.
- [31] C. Pechstein and R. Scheichl. Analysis of FETI methods for multiscale PDEs. Part II: interface variation. *Numer. Math.*, 118(3):485–529, 2011.
- [32] B. Sousedík, J. Šístek, and J. Mandel. Adaptive-Multilevel BDDC and its parallel implementation. *Computing*, 95(12):1087–1119, 2013.
- [33] N. Spillane, V. Dolean, P. Hauret, F. Nataf, C. Pechstein, and R. Scheichl. A robust two-level domain decomposition preconditioner for systems of PDEs. *C. R. Math. Acad. Sci. Paris*, 349(23-24):1255–1259, 2011.

- [34] N. Spillane, V. Dolean, P. Hauret, F. Nataf, C. Pechstein, and R. Scheichl. Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. Numer. Math., 126(4):741–770, 2014.
- [35] N. Spillane and D. J. Rixen. Automatic spectral coarse spaces for robust FETI and BDD algorithms. Int. J. Numer. Meth. Engng., 95(11):953–990, 2013.
- [36] J. M. Tang, R. Nabben, C. Vuik, and Y. A. Erlangga. Comparison of two-level preconditioners derived from deflation, domain decomposition and multigrid methods. J. Sci. Comput., 39(3):340–370, 2009.
- [37] A. Toselli and O. Widlund. Domain decomposition methods—algorithms and theory, volume 34 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 2005.
- [38] S. Zampini. PCBDDC: a class of robust dual-primal methods in PETSc. SIAM Journal on Scientific Computing, 38(5):S282–S306, 2016.