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A NEW BOUNDING TECHNIQUE BASED ON INFINITE PRODUCT DECOMPOSITION

CHRISTOPHE CHESNEAU AND YOGESH J. BAGUL

ABSTRACT. In this note, a new strategy is proposed to obtain bounds for functions having product decompositions. Applications are given for trigonometric and hyperbolic functions, thus improving some existing inequalities in the literature. Some graphics illustrate the findings.

1. A NEW RESULT

As stated in [5], infinite products are rarely used to establish analytical inequalities. The main aim of this note is to propose a new technique for obtaining bounds of some functions by the excellent use of their infinite product representations. The next theorem is the main result of the paper.

Theorem 1. *Let us consider a differentiable function $f(x)$ on $(0, +\infty)$ such that we can express $f(x)$ as either:*

- *Case I: $f(x) = \prod_{k \in K} (1 + a_k x^c)$, or*
- *Case II: $f(x) = \prod_{k \in K} (1 - a_k x^c)$,*

for $x \in C$, where C is a subset of $(0, +\infty)$ and K is a subset of \mathbb{Z} , c is a positive real number and $(a_k)_{k \in K}$ is a sequence of positive real numbers. Then,

- *for $f(x)$ as defined in Case I and any x such that $2^{1/c}x \in C$, the following inequality holds:*

$$\frac{f'(x)}{f(x)} < \frac{c}{2x} \log \left[f(2^{1/c}x) \right],$$

- *for $f(x)$ as defined in Case II and any x such that $2^{1/c}x \in C$ and $x^c \sup_{k \in K} a_k < 1$, the following inequality holds:*

$$\frac{f'(x)}{f(x)} > \frac{c}{2x} \log \left[f(2^{1/c}x) \right].$$

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Proof. In order to prove the inequality of the first item, let us recall a result by [6] saying that, for any $y > 0$,

$$\log(1 + y) > \frac{y}{1 + y/2}. \quad (1.1)$$

Now, we have

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \{\log[f(x)]\}' = \left\{ \sum_{k \in K} \log(1 + a_k x^c) \right\}' = \sum_{k \in K} \frac{c a_k x^{c-1}}{1 + a_k x^c} \\ &= \frac{c}{2x} \sum_{k \in K} \frac{2 a_k x^c}{1 + a_k x^c} < \frac{c}{2x} \sum_{k \in K} \log(1 + 2 a_k x^c) \\ &= \frac{c}{2x} \sum_{k \in K} \log(1 + a_k (2^{1/c} x)^c) = \frac{c}{2x} \log[f(2^{1/c} x)]. \end{aligned} \quad (1.2)$$

This ends the proof of the first inequality. The proof of the inequality in the second item is similar; it is enough to use the reverse of the result of [6] which holds for any $y \in (0, 1)$, that is $\log(1 - y) < -y/(1 - y/2)$. By proceeding as in (1.2), we get the desired inequality. The proof of Theorem 1 is complete. \square

The rest of the study concerns several applications of Theorem 1, showing that it can produce sharp bounds for diverse trigonometric and hyperbolic functions of interest.

2. APPLICATIONS

The inequalities

$$\frac{\sin x}{x} < \exp\left(\log\left(\frac{\pi}{2}\right)(x \cot x - 1)\right), \quad x \in \left(0, \frac{\pi}{2}\right), \quad (2.1)$$

and

$$\exp\left(\frac{1}{2}(x \coth x - 1)\right) < \frac{\sinh x}{x}, \quad x > 0 \quad (2.2)$$

were established by [3] and [7], respectively. See also [1] for an alternative corrected proof of (2.2). We first refine inequality (2.1) for most of the values of $x \in (0, \pi/2)$. Moreover, the obtained upper bound is valid in a larger interval $(0, \pi)$.

Proposition 1. *For any $x \in (0, \pi/\sqrt{2})$, we have*

$$\cot x > \frac{1}{x} \left\{ 1 + \log \left[\frac{\sin(x\sqrt{2})}{x\sqrt{2}} \right] \right\}.$$

Equivalently, for any $x \in (0, \pi)$, we have

$$\frac{\sin x}{x} < \exp \left(\frac{x}{\sqrt{2}} \cot \left(\frac{x}{\sqrt{2}} \right) - 1 \right).$$

Proof. The proof follows from Theorem 1 applied with $f(x) = (\sin x)/x$. Since

$$\frac{\sin x}{x} = \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\pi^2 k^2} \right), \quad x \in (0, \pi),$$

we are in the Case II with $C = (0, \pi)$, $K = \mathbb{N}/\{0\}$, $a_k = 1/(\pi^2 k^2)$ and $c = 2$. The proof ends by remarking that $f'(x)/f(x) = \cot x - 1/x$. \square

The refinement of (2.1) in Proposition 1 is illustrated in Figure 1 where the curve of the difference function of the upper bounds is plotted, governed by the following equation:

$$d(x) = \exp \left(\log \left(\frac{\pi}{2} \right) (x \cot x - 1) \right) - \exp \left(\frac{x}{\sqrt{2}} \cot \left(\frac{x}{\sqrt{2}} \right) - 1 \right), \quad x \in \left(0, \frac{\pi}{2} \right).$$

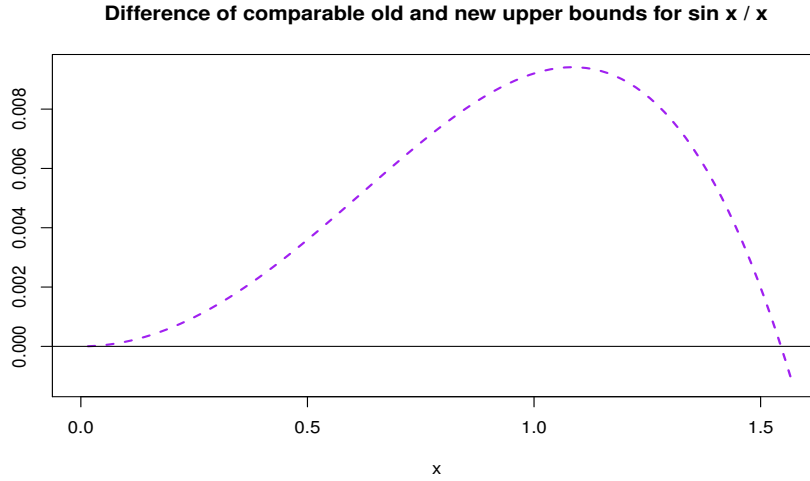


FIGURE 1. Curve for the function $d(x)$ for $x \in (0, \pi/2)$.

From Figure 1, it is clear that the new upper bound is sharper than the one established by [3], except for a small range of values included into $(\lambda, \pi/2)$ where $\lambda \approx 1.545$.

Also, we refine inequality (2.2) in the following proposition.

Proposition 2. *For any $x > 0$, we have*

$$\coth x < \frac{1}{x} \left\{ 1 + \log \left[\frac{\sinh(x\sqrt{2})}{x\sqrt{2}} \right] \right\}.$$

Equivalently, for any $x > 0$, we have

$$\exp \left(\frac{x}{\sqrt{2}} \coth \left(\frac{x}{\sqrt{2}} \right) - 1 \right) < \frac{\sinh x}{x}.$$

Proof. The proof follows from Theorem 1 applied with $f(x) = (\sinh x)/x$. Since

$$f(x) = \prod_{k=1}^{+\infty} \left(1 + \frac{x^2}{\pi^2 k^2} \right), \quad x > 0,$$

we are in the Case I with $C = (0, +\infty)$, $K = \mathbb{N}/\{0\}$, $a_k = 1/(\pi^2 k^2)$ and $c = 2$. The proof ends by remarking that $f'(x)/f(x) = \coth x - 1/x$. \square

The improvement of the new lower bound is illustrated in Figure 2, where the following difference function is plotted:

$$j(x) = \exp \left(\frac{1}{2}(x \coth x - 1) \right) - \exp \left(\frac{x}{\sqrt{2}} \coth \left(\frac{x}{\sqrt{2}} \right) - 1 \right), \quad x \in (0, 1.5).$$

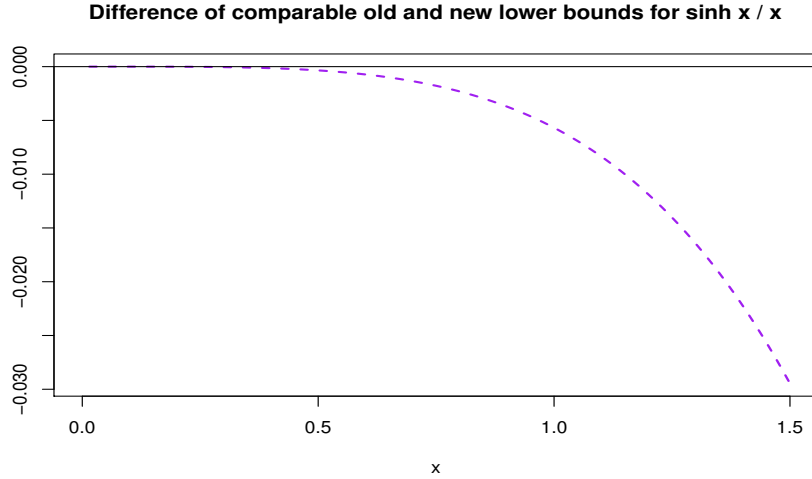


FIGURE 2. Curve for the function $j(x)$ for $x \in (0, 1.5)$.

From Figure 2, we see that the new lower bound is sharper than the one proposed by [7].

In [4] and [2], it is respectively proved that

$$\cos x < \exp\left(-\frac{x^2}{2}\right), \quad x \in (0, \pi/2) \quad (2.3)$$

and

$$\exp(\theta x^2) < \cosh x, \quad x \in (0, \alpha), \quad (2.4)$$

where $\alpha > 0$ and $\theta = \log(\cosh \alpha)/\alpha^2$.

Sharper exponential bounds for $\cos x$ and $\cosh x$ can be given by refining (2.3) and (2.4) as follows:

Proposition 3. *For any $x \in (0, \pi/(2\sqrt{2}))$, we have*

$$\tan x < -\frac{1}{x} \log \left[\cos(x\sqrt{2}) \right].$$

Equivalently, for any $x \in (0, \pi/2)$, we have

$$\cos x < \exp\left(-\frac{x}{\sqrt{2}} \tan\left(\frac{x}{\sqrt{2}}\right)\right).$$

Proof. The proof follows from Theorem 1 applied with $f(x) = \cos x$. Since

$$\cos x = \prod_{k=0}^{+\infty} \left(1 - \frac{4x^2}{\pi^2(2k+1)^2}\right), \quad x \in (0, \pi/2),$$

we are in the Case II with $C = (0, \pi/2)$, $K = \mathbb{N}$, $a_k = 4/[\pi^2(2k+1)^2]$ and $c = 2$. The proof ends by remarking that $f'(x)/f(x) = -\tan x$. \square

The upper bound in Proposition 3 refines (2.3) since $\tan x > x$ for $x \in (0, \pi/2)$.

Proposition 4. *For any $x > 0$, we have*

$$\tanh x < \frac{1}{x} \log \left[\cosh(x\sqrt{2}) \right].$$

Equivalently, for any $x > 0$, we have

$$\exp\left(\frac{x}{\sqrt{2}} \tanh\left(\frac{x}{\sqrt{2}}\right)\right) < \cosh x.$$

Proof. The proof follows from Theorem 1 applied with $f(x) = \cosh x$. Since

$$f(x) = \prod_{k=0}^{+\infty} \left(1 + \frac{4x^2}{\pi^2(2k+1)^2}\right), \quad x > 0,$$

we are in the Case I with $C = (0, +\infty)$, $K = \mathbb{N}$, $a_k = 4/[\pi^2(2k+1)^2]$ and $c = 2$. The proof ends by remarking that $f'(x)/f(x) = \tanh x$. \square

Applications of the previous result can be of various kinds, allowing the determination of simple bounds for various integral or series bounds having no closed-form. For instance, the integral $\int_0^{x_*} e^{y \tanh(y)} dy$ with $x_* > 0$ is complex to treat mathematically, but thanks to Proposition 4, we have

$$\int_0^{x_*} e^{y \tanh(y)} dy \leq \int_0^{x_*} \cosh(y\sqrt{2}) dy = \frac{1}{\sqrt{2}} \sinh(x_*\sqrt{2}).$$

We also established the following consequence of Theorem 1.

Proposition 5. *Let us adopt the setting and notations of Theorem 1. Let $f_1(x)$ and $f_2(x)$ be functions satisfying the Case I and Case II, respectively, with the same quantities involved. Then, for any x such that $2^{1/(2c)}x \in C$ and $x^c \sup_{k \in K} a_k < 1$, the following inequality holds:*

$$\frac{f'_1(x)}{f_1(x)} + \frac{f'_2(x)}{f_2(x)} > \frac{c}{x} \left\{ \log \left[f_1(2^{1/(2c)}x) \right] + \log \left[f_2(2^{1/(2c)}x) \right] \right\}.$$

Proof. The proof follows from Theorem 1 applied with $f(x) = f_1(x)f_2(x)$. Since

$$f_1(x)f_2(x) = \prod_{k \in K} (1 - a_k x^c) \prod_{k \in K} (1 + a_k x^c) = \prod_{k \in K} (1 - a_k^2 x^{2c}),$$

we are in the Case II with the sequence $(a_k^2)_{k \in K}$ and the constant $2c$. The proof ends by remarking that $f'(x)/f(x) = [f'_1(x)f_2(x) + f_1(x)f'_2(x)]/[f_1(x)f_2(x)] = f'_1(x)/f_1(x) + f'_2(x)/f_2(x)$. \square

As an example of application of Proposition 5, by considering $f_1(x) = \cosh x$ and $f_2(x) = \cos x$, then we can apply this result with $K = \mathbb{N}$, $a_k = 4/[\pi^2(2k+1)^2]$ and $c = 2$, for $x \in (0, \pi/2^{5/4})$, we get

$$\tanh x - \tan x > \frac{2}{x} \left\{ \log \left[\cosh(2^{1/4}x) \right] + \log \left[\cos(2^{1/4}x) \right] \right\}. \quad (2.5)$$

This inequality is shown in Figure 3 for $x \in (1, 1.3)$, illustrating its sharpness.

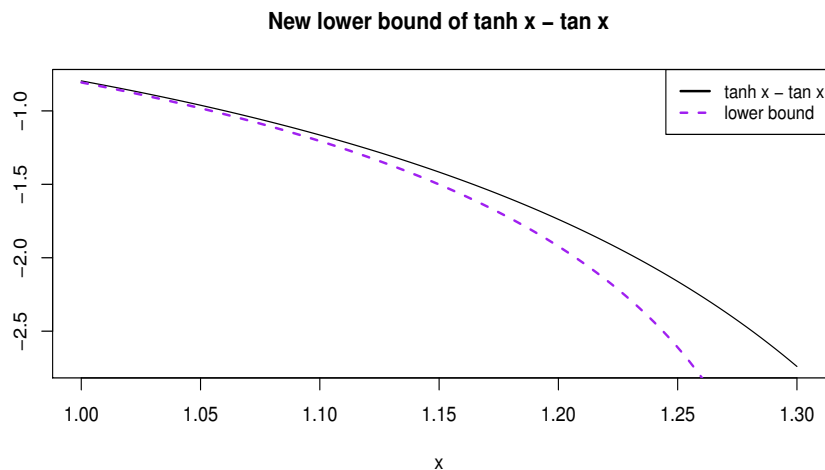


FIGURE 3. Illustration of the lower bound of $\tanh x - \tan x$ as formulated in (2.5).

REFERENCES

- [1] Bagul, Y. J. *On a result of Bhayo and Sándor*. Anal. Math., 47, 33–36 (2021). <https://doi.org/10.1007/s10476-020-0060-8>
- [2] Bagul, Y. J. *On exponential bounds of hyperbolic cosine*. Bull. Int. Math. Virtual Inst., 8(2), 365–367 (2018). Doi: 10.7251/BIMVI1802365B
- [3] Bhayo, B. A. and Sándor J. *On certain old and new trigonometric and hyperbolic inequalities*. Anal. Math., 41, 3–15 (2015). <https://doi.org/10.1007/s10476-015-0102-9>
- [4] Chesneau, C. and Bagul, Y. J. *A note on some new bounds for trigonometric functions using infinite products*. Malays. J. Math. Sci., 14(2), 273–283 (2020).
- [5] Kostić, M., Bagul, Y. J. and Chesneau C. *Generalized inequalities for ratio functions of trigonometric and hyperbolic functions*. Indian J. Math., 62(2), 183–190 (2020).
- [6] Love, E. R. *The Mathematical Gazette*, 64, 427, 55–57 (1980).
- [7] Sándor, J. *Two sharp inequalities for trigonometric and hyperbolic functions*. Math. Inequal. Appl., 15(2), 409–413 (2012). [dx.doi.org/10.7153/mia-15-35](https://doi.org/10.7153/mia-15-35)

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