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# Strong Incidence Colouring of Graphs

Brahim BENMEDJDOUB<sup>1,2</sup>      Éric SOPENA<sup>3</sup>

March 29, 2021

## Abstract

An incidence of a graph  $G$  is a pair  $(v, e)$  where  $v$  is a vertex of  $G$  and  $e$  is an edge of  $G$  incident with  $v$ . Two incidences  $(v, e)$  and  $(w, f)$  of  $G$  are adjacent whenever (i)  $v = w$ , or (ii)  $e = f$ , or (iii)  $vw = e$  or  $f$ . An incidence  $p$ -colouring of  $G$  is a mapping from the set of incidences of  $G$  to the set of colours  $\{1, \dots, p\}$  such that every two adjacent incidences receive distinct colours. Incidence colouring has been introduced by Brualdi and Quinn Massey in 1993 and, since then, studied by several authors.

In this paper, we introduce and study the strong version of incidence colouring, where incidences adjacent to a same incidence must also get distinct colours. We determine the exact value of – or upper bounds on – the strong incidence chromatic number of several classes of graphs, namely cycles, wheel graphs, trees, ladder graphs and subclasses of Halin graphs.

**Keywords:** Strong incidence colouring; Incidence colouring; Tree; Halin graph; Ladder graph; Necklace; Double star; Wheel graph.

**MSC 2010:** 05C15.

## 1 Introduction

All graphs considered in this paper are simple and loopless undirected graphs. We denote by  $V(G)$  and  $E(G)$  the set of vertices and the set of edges of a graph  $G$ , respectively, by  $\Delta(G)$  the maximum degree of  $G$ , by  $N(v)$  the set of vertices adjacent to the vertex  $v$  and by  $\text{dist}_G(u, v)$  the distance between vertices  $u$  and  $v$  in  $G$ .

An *incidence* of a graph  $G$  is a pair  $(v, e)$  where  $v$  is a vertex of  $G$  and  $e$  is an edge of  $G$  incident with  $v$ . Two incidences  $(v, e)$  and  $(w, f)$  of  $G$  are *adjacent* whenever (i)  $v = w$ , or (ii)  $e = f$ , or (iii)  $vw = e$  or  $f$ .

An *incidence  $p$ -colouring* of  $G$  is a mapping from the set of incidences of  $G$  to the set of colours  $\{1, \dots, p\}$  such that every two adjacent incidences receive distinct colours. The smallest  $p$  for which  $G$  admits an incidence  $p$ -colouring is the *incidence chromatic number* of  $G$ , denoted by  $\chi_i(G)$ . Incidence colourings were first introduced and studied by Brualdi and Quinn Massey [2]. Incidence colourings of various graph families have attracted much interest in recent years, see for instance [3, 4, 6, 7, 10, 11, 12].

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A *strong edge  $p$ -colouring* of  $G$  is a mapping from the set of edges of  $G$  to the set of colours  $\{1, \dots, p\}$  such that any two edges meeting at a common vertex, or being adjacent to a same edge of  $G$ , are assigned different colours. The smallest  $p$  for which  $G$  admits a strong edge  $p$ -colouring is the *strong chromatic index* of  $G$ , denoted by  $\chi'_s(G)$ .

The strong version of incidence colouring is defined in a similar way. A *strong incidence  $p$ -colouring* of a graph  $G$  is a mapping from the set of incidences of  $G$  to a finite set of colours  $\{1, \dots, p\}$  such that any two incidences that are adjacent or adjacent to the same incidence receive distinct colours. The smallest  $p$  for which  $G$  admits a strong incidence  $p$ -colouring is the *strong incidence chromatic number*, denoted by  $\chi_i^s(G)$ .

Our paper is organised as follows. We first give some preliminary results in Section 2. We then study the strong incidence chromatic number of simple graph classes (stars, complete graphs, cycles, wheel graphs and trees) in Section 3, of ladder graphs in Section 4 and of subclasses of Halin graphs in Section 5. We finally propose some directions for future research in Section 6.

## 2 Preliminary results

We list in this section some basic results on the strong incidence chromatic number of various graph classes.

The *square*  $G^2$  of a graph  $G$  is the graph defined by  $V(G^2) = V(G)$  and  $uv \in E(G^2)$  if and only if  $\text{dist}_G(u, v) \leq 2$ . A colouring of  $G^2$  is called a *2-distance colouring* of  $G$  and the *2-distance chromatic number* of  $G$  is denoted by  $\chi_2(G)$ .

For any graph  $G$ , the *incidence graph* of  $G$ , denoted by  $I_G$ , introduced in [1], is the graph whose vertices are the incidences of  $G$ , two incidences being joined by an edge whenever they are adjacent. Clearly, every incidence colouring of  $G$  is nothing but a proper vertex colouring of  $I_G$ , and every strong incidence colouring of  $G$  is nothing but a 2-distance colouring of  $G$ , so that,  $\chi_i(G) = \chi(I_G)$  and  $\chi_i^s(G) = \chi_2(I_G)$ . Moreover, since every strong incidence colouring is an incidence colouring, we have  $\chi_i(G) \leq \chi_i^s(G)$  for every graph  $G$ .

For every vertex  $v$  in a graph  $G$ , we denote by  $A^-(v)$  the set of incidences of the form  $(v, vu)$ , and by  $A^+(v)$  the set of incidences of the form  $(u, uv)$  (see Figure 1). We thus have  $|A^-(v)| = |A^+(v)| = \text{deg}(v)$  for every vertex  $v$ . Every edge  $uv$  of  $G$  has two incidences  $(u, uv)$  and  $(v, vu)$ . We will say that two incidences are *strongly adjacent* if they are either adjacent or adjacent to a same incidence. The following observation will be useful.

**Observation 1** *For every incidence  $(v, vu)$  in a graph  $G$  with maximum degree  $\Delta$ , the set of incidences that are strongly adjacent to  $(v, vu)$  is*

$$\bigcup_{w \in N(v) \setminus u} A^+(w) \cup \bigcup_{w \in N(v)} A^-(w) \cup \bigcup_{w \in N(u) \setminus v} A^-(w),$$

whose cardinality is at most  $3\Delta^2 - 2\Delta$ .

Indeed, the cardinality of the set of incidences that are strongly adjacent to  $(v, vu)$  (see Figure 2) is

$$\sum_{w \in N(v) \setminus u} \text{deg}(w) + \sum_{w \in N(v)} \text{deg}(w) + \sum_{w \in N(u) \setminus v} \text{deg}(w) \leq 2\Delta(\Delta - 1) + \Delta^2.$$

Therefore, if we colour  $G$  with  $3\Delta^2 - 2\Delta + 1$  colours, then there exists at least one free colour that can be assigned to each incidence  $(v, vu)$  of  $G$ . We thus have the following upper bound on the strong incidence chromatic number of every graph.

**Proposition 2** *For every graph  $G$  with maximum degree  $\Delta$ ,  $\chi_i^s(G) \leq 3\Delta^2 - 2\Delta + 1$ .*

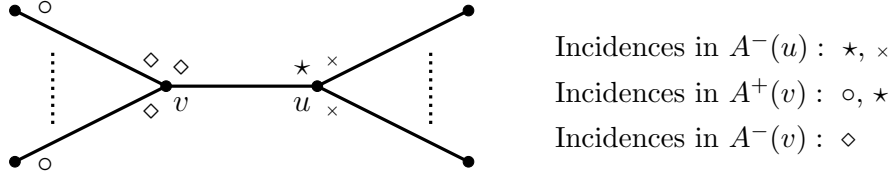


Figure 1: Adjacent incidences.

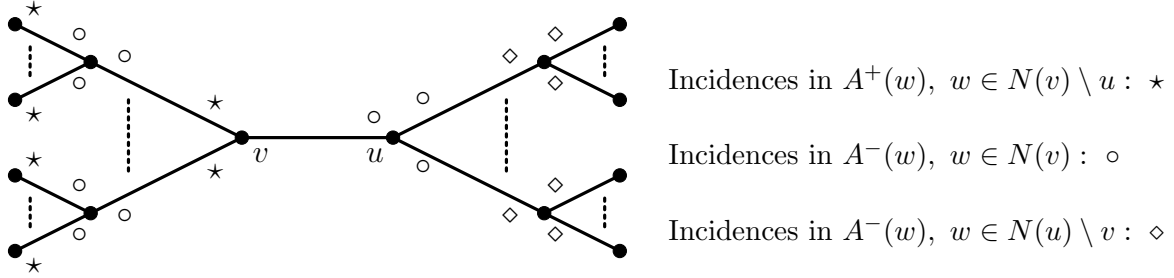


Figure 2: Strongly adjacent incidences.

For a given graph  $G$  with maximum degree  $\Delta$ , we let

$$\sigma(G) = \max_{uv \in E(G)} \{2 \deg_G(v) + \deg_G(u) - 1\}.$$

For every edge  $uv$  in  $E(G)$ , the incidences of the set  $A^-(v) \cup A^+(v) \cup A^-(u)$ , of cardinality  $2 \deg_G(v) + \deg_G(u) - 1$ , are pairwise strongly adjacent, which means that they must be assigned distinct colours. Therefore, using Proposition 2, we have the following inequalities.

**Proposition 3** *For every graph  $G$  with maximum degree  $\Delta$ ,  $\sigma(G) \leq \chi_i^s(G) \leq 3\Delta^2 - 2\Delta + 1$ .*

In the following proposition we give an upper bound on the strong incidence chromatic number of a graph  $G$  as a function of its strong chromatic index.

**Proposition 4** *For every graph  $G$ ,  $\chi_i^s(G) \leq 2\chi'_s(G)$ .*

**Proof.** Let  $\lambda$  be a strong edge  $p$ -colouring of  $G$ . From this colouring, a strong incidence  $2p$ -colouring  $\lambda'$  is obtained using the set of  $2p$  colours  $\{1, 1', \dots, p, p'\}$  as follows: for every edge  $uv \in E(G)$ , if  $\lambda(uv) = k$ ,  $k \in \{1, \dots, p\}$ , then  $\lambda'(u, uv) = k$  and  $\lambda'(v, vu) = k'$ . Indeed, if  $\lambda'(u, uv) = \lambda'(w, wx)$  for two incidences  $(u, uv)$  and  $(w, wx)$  of  $G$ , then  $\lambda(uv) = \lambda(wx)$ , which implies  $\text{dist}_G(uv, wx) \geq 3$ , and thus  $\text{dist}_{I_G}((u, uv), (w, wx)) \geq 3$ .  $\square$

### 3 Simple graph classes

In this section, we determine the strong incidence chromatic number of stars, complete graphs, cycles, trees and wheel graphs.

We denote by  $S_n$ ,  $n \geq 1$ , the star of order  $n + 1$ , by  $K_n$ ,  $n \geq 1$ , the complete graph of order  $n$  and by  $K_{m,n}$ ,  $m \geq n \geq 2$ , the complete bipartite graph with parts of size  $m$  and  $n$ . In [2], Brualdi and Massey showed that  $\chi_i(S_n) = n + 1$ ,  $\chi_i(K_n) = n$  and  $\chi_i(K_{m,n}) = m + 2$ , for all  $m \geq n \geq 2$ . Since all incidences of any graph in these classes of graphs are pairwise strongly adjacent, we have the following proposition.

**Proposition 5**

1. For every  $n \geq 1$ ,  $\chi_i^s(S_n) = 2n$ ,
2. for every  $n \geq 2$ ,  $\chi_i^s(K_n) = 2|E(G)|$ ,
3. for every  $m \geq n \geq 2$ ,  $\chi_i^s(K_{m,n}) = 2nm$ .

Let  $C_n$ ,  $n \geq 3$ , denote the cycle of order  $n$ . Observe that  $I_{C_n} = C_{2n}^2$ . Therefore, a strong incidence colouring of the cycle  $C_n$  is a 2-distance colouring of  $C_{2n}^2$ , that is nothing but a proper colouring of  $(C_{2n}^2)^2 = C_{2n}^4$ . We thus have the following result.

**Proposition 6** For every integer  $n \geq 3$ ,  $\chi_i^s(C_n) = \chi(C_{2n}^4)$ .

By setting  $a = 4$ , the following theorem gives the value of  $\chi(C_{2n}^4)$ .

**Theorem 7 (Prowse and Woodall [8])** Let  $n$  and  $a$  be positive integers such that  $n \geq 2a$  and  $n = q(a + 1) + r$ , with  $q > 0$  and  $0 \leq r \leq a$ . Then  $\chi(C_n^a) = a + 1 + \lceil r/q \rceil$ .

Using Proposition 6 and Theorem 7, we can infer the value of the strong incidence chromatic number of any cycle.

**Theorem 8** Let  $n$  be a positive integer such that  $n \geq 4$  and  $2n = 5q + r$ , with  $q > 0$  and  $0 \leq r \leq 4$ . Then  $\chi_i^s(C_n) = 5 + \lceil r/q \rceil$ .

We now determine the value of  $\chi_i^s(W_n)$ , where  $W_n$ ,  $n \geq 3$ , is the wheel graph of order  $n + 1$ , obtained from  $C_n$  by adding a universal vertex. It is easy to observe that  $\chi_i(W_n) = n + 1$  for every  $n \geq 3$ . Indeed, since the square of  $W_n$  is the complete graph  $K_{n+1}$ , we get

$$n + 1 = \Delta(W_n) + 1 \leq \chi_i(W_n) \leq \chi(W_n^2) = \chi(K_{n+1}) = n + 1.$$

Using Proposition 5 and Theorem 8, we can determine the strong incidence chromatic number of wheel graphs.

**Theorem 9** Let  $n$  be a positive integer such that  $n \geq 3$  and  $2n = 5q + r$ , with  $q > 0$  and  $0 \leq r \leq 4$ . Then  $\chi_i^s(W_n) = 5 + 2n + \lceil r/q \rceil$ .

**Proof.** Since  $W_3 = K_4$ , the result holds by Proposition 5 for  $n = 3$ . Suppose now  $n \geq 4$  and let  $T$  denote the spanning subgraph of  $W_n$  isomorphic to  $S_n$ . Since every incidence in  $T$  is strongly adjacent to every incidence not in  $T$ , we get  $\chi_i^s(W_n) = \chi_i^s(C_{W_n}) + \chi_i^s(S_n)$  and the result follows from Proposition 5 and Theorem 8.  $\square$

We finally determine the strong incidence chromatic number of trees.

**Theorem 10** If  $G$  is a tree then  $\chi_i^s(G) = \max_{uv \in E(G)} \{2 \deg_G(v) + \deg_G(u) - 1\} = \sigma(G)$ .

**Proof.** By Proposition 3, we have  $\chi_i^s(G) \geq \sigma(G)$ . The other direction is proved by induction on  $|V(G)|$ . If  $G$  is a star, then  $\sigma(G) = 2n$  and the result follows from Proposition 5. We can thus assume that  $G$  is not a star, so that  $|V(G)| \geq 4$ . Let  $u$  be a vertex of  $G$  of degree  $p + 1 \geq 3$  having only one neighbour, denoted by  $u'$ , which is not a leaf, and let  $A = \{v_1, \dots, v_p\}$  be the set of  $p$  leaves that are neighbours of  $u$ . Let  $\lambda$  be a strong incidence colouring of  $G \setminus A$ . Now, observe that each incidence of the form  $(u, uv_i)$ ,  $1 \leq i \leq p$ , has at most

$$2 \deg_{G \setminus A}(u') = 2 \deg_G(u') \leq \sigma(G) - \deg_G(u) + 1$$

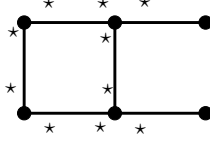


Figure 3: The graph  $L$ .

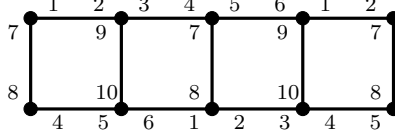


Figure 4: A strong incidence 10-colouring of  $L_5$ .

strongly adjacent incidences in  $G \setminus A$ , and each incidence of the form  $(v_i, v_i u)$ ,  $1 \leq i \leq p$ , has at most

$$\deg_{G \setminus A}(u') + 1 = \deg_G(u') + 1 \leq \sigma(G) - 2 \deg_G(u) + 1$$

strongly adjacent incidences in  $G \setminus A$ , so that  $\lambda$  can be extended to a strong incidence colouring of  $G$ , starting by colouring the incidences of the form  $(u, uv_i)$ ,  $1 \leq i \leq p$ , and then, the incidences of the form  $(v_i, v_i u)$ ,  $1 \leq i \leq p$ . This concludes the proof.  $\square$

## 4 Ladder graphs

The *ladder graph*, denoted by  $L_h$ , is obtained from two paths of order  $h$ ,  $h \geq 1$ ,  $P_h = v_1 \dots v_h$  and  $P'_h = v'_1 \dots v'_h$  by adding the edges  $v_i v'_i$ ,  $1 \leq i \leq h$ . In the following theorem, we give the value of  $\chi_i^s(L_h)$ .

**Theorem 11** *For every integer  $h \geq 3$ ,  $\chi_i^s(L_h) = 10$ .*

**Proof.** Note that when  $h \geq 3$ ,  $L_h$  contains a subgraph isomorphic to the graph  $L$  (see Figure 3) which contains 10 pairwise strongly adjacent incidences (marked with a star), implying  $\chi_i^s(L_h) \geq 10$  for every  $h \geq 3$ . To complete the proof, it suffices to give a strong incidence 10-colouring  $\lambda$  of  $L_h$ . Such a colouring can be obtained as follows (see Figure 4 for the case  $h = 5$ ).

1. We sequentially colour the incidences of the path  $P_h$  using the pattern 123456.
2. We sequentially colour the incidences of the path  $P'_h$  using the pattern 456123.
3. For every  $i$ ,  $1 \leq i \leq h$ , we set
  - $\lambda(v'_i, v'_i v_i) = 7$  and  $\lambda(v_i, v_i v'_i) = 8$ , if  $i$  is odd,
  - $\lambda(v'_i, v'_i v_i) = 9$  and  $\lambda(v_i, v_i v'_i) = 10$  if  $i$  is even.

The so-obtained colouring is clearly a strong incidence colouring of  $L_h$ . This concludes the proof.  $\square$

## 5 Subclasses of Halin graphs

Recall first that a *Halin graph*  $H$  is a planar graph obtained from a tree of order at least 4 with no vertex of degree 2, by adding a cycle connecting all its leaves [5]. We call this cycle the *outer cycle*

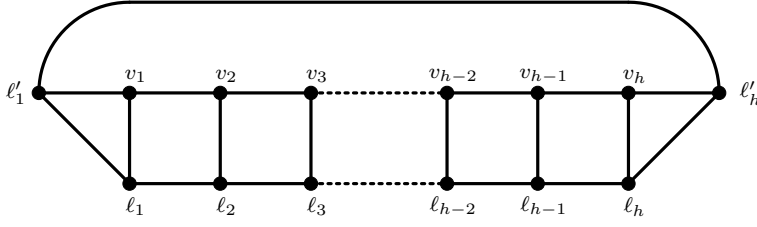


Figure 5: The graph  $N_h$ .

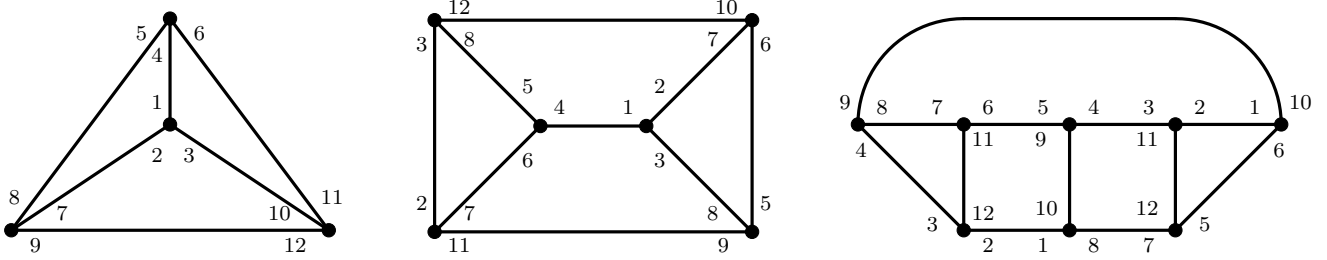


Figure 6: Strong incidence 12-colourings of  $N_1$ ,  $N_2$  and  $N_3$ .

of  $H$ . The subgraph  $T$  obtained by deleting all the edges of the outer cycle of  $H$  is thus a tree, called the *internal tree* of  $H$ .

In this section, we determine the exact value of – or upper bounds on – the strong incidence chromatic number of every Halin graph whose internal tree is either a comb or a double star.

## 5.1 Halin graphs whose internal tree is a comb

A tree is called a  $(3,1)$ -tree if the degree of each non-leaf vertex is 3. A caterpillar is a tree  $T$  such that, after deleting all its leaves, the remaining graph is a simple path called the *spine* of  $T$ . A *comb* is a caterpillar which is also a  $(3,1)$ -tree. It is easy to see that every Halin graph whose internal tree is a comb is a cubic Halin graph. In particular, if the spine has one vertex then this is the complete graph  $K_4$ .

For every integer  $h \geq 1$ , we construct a Halin graph  $H_h$  of order  $2h + 2$  whose internal tree  $T_h$  is a comb, using the construction given in [9]. Let  $P_h = v_1 v_2 \dots v_h$  be the spine of  $H_h$ . We denote by  $l_1$  and  $l'_1$  (resp.  $l_h$  and  $l'_h$ ) the two leaves of  $v_1$  (resp.  $v_h$ ), by  $l_i$  the unique leaf of  $v_i$ ,  $2 \leq i \leq h - 1$ , and by  $C_h$  the outer cycle of  $H_h$ .

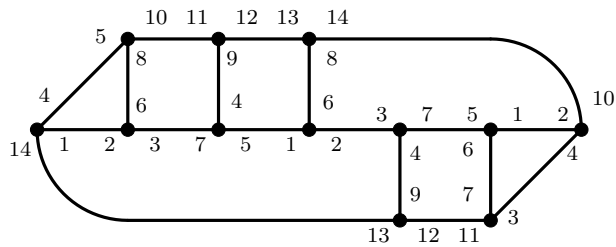
Let  $\mathcal{H}_h^c$  be the set of all Halin graphs whose internal tree is a comb of order  $2h + 2$ . A Halin graph  $H_h$  such that  $C_h = l_1 l_2 \dots l_h l_1$  is called a *necklace*. We denote by  $N_h$  the (unique) necklace of order  $2h + 2$ . Observe that  $\mathcal{H}_h^c = \{N_h\}$  for every  $h$ ,  $1 \leq h \leq 3$ .

It is easy to see that all incidences of  $N_1$  are pairwise strongly adjacent. Therefore,  $\chi_i^s(N_1) = 12$ . If  $G = N_2$ , then the incidences of the set

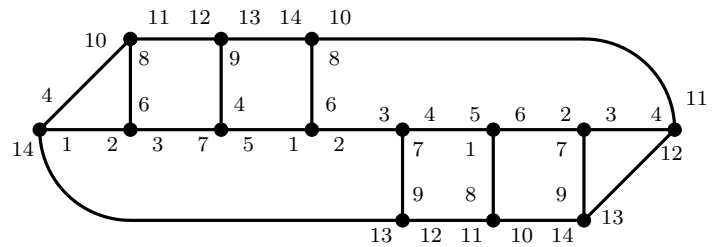
$$A^-(v_1) \cup A^+(v_1) \cup A^-(v_2) \cup \{(\ell'_1, \ell'_1 \ell'_2), (\ell'_2, \ell'_2 \ell'_1), (\ell_1, \ell_1 \ell_2), (\ell_2, \ell_2 \ell_1)\},$$

of cardinality 12, are pairwise strongly adjacent. Hence, we have  $\chi_i^s(N_2) \geq 12$ . Also, the cardinality

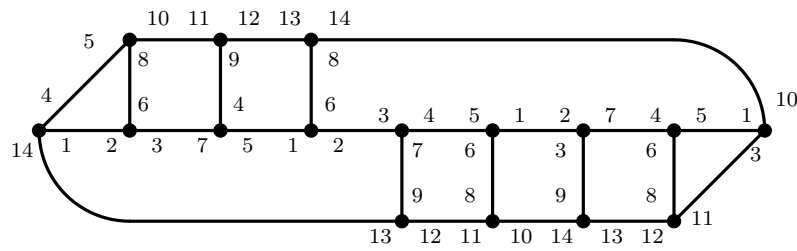




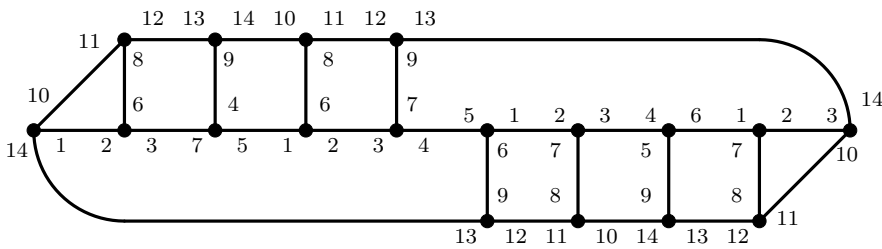
(a)



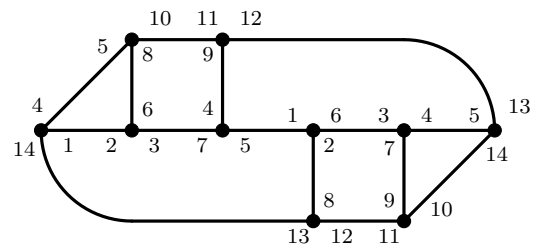
(b)



(c)



(d)



(e)

Figure 9: Strong incidence 14-colourings of graphs in  $\mathcal{H}_h^c$ ,  $4 \leq h \leq 8$ .

- We colour the incidences of the path  $\ell'_1, \ell'_h$  sequentially, from  $(\ell'_1, \ell'_1 v_1)$  to  $(\ell'_h, \ell'_h v_h)$  using the pattern 12345.
- For every integer  $i$ ,  $1 \leq i \leq h$ , we set  $\lambda(v_i, v_i \ell_i) = 6$  if  $i$  is odd and  $\lambda(v_i, v_i \ell_i) = 7$  otherwise.
- We colour circularly the incidences of the form  $(\ell_i, \ell_i v_i)$ ,  $1 \leq i \leq h$  according to their order in the outer cycle  $C_{2h+2}$  by alternating the colours 8 and 9.
- We exchange the colours of the two incidences  $(v_2, v_2 \ell_2)$  and  $(v_2, v_2 v_1)$ , and the colours of the two incidences  $(v_{h-1}, v_{h-1} v_h)$  and  $(v_{h-1}, v_{h-1} \ell_{h-1})$ .
- We now colour the incidences of the outer cycle  $C_{2h+2}$  according to the value of  $h \pmod 5$ :
  - $h = 5k$ ,  $k \geq 1$  (see Figure 9(a) for the case  $h = 5$ ).  
We first set  $\lambda(\ell'_1, \ell'_1 \ell_1) = \lambda(v_2, v_2 \ell_2)$ ,  $\lambda(\ell_1, \ell_1 \ell'_1) = \lambda(v_2, v_2 v_3)$ ,  $\lambda(\ell'_h, \ell'_h \ell_h) = \lambda(v_{h-1}, v_{h-1} \ell_{h-1})$  and  $\lambda(\ell_h, \ell_h \ell'_h) = \lambda(v_{h-1}, v_{h-1} v_h)$ . We then sequentially colour the uncoloured incidences of  $C_{2h+2}$ , starting from  $(\ell_1, \ell_1 \ell_2)$ , using the pattern 10.11.12.13.14.
  - $h = 5k + 1$ ,  $k \geq 1$  (see Figure 9(b) for the case  $h = 6$ ).  
We first set  $\lambda(\ell'_1, \ell'_1 \ell_1) = \lambda(v_2, v_2 \ell_2)$ . We then sequentially colour the uncoloured incidences of  $C_{2h+2}$ , starting from  $(\ell_1, \ell_1 \ell'_1)$ , using the pattern 10.11.12.13.14.
  - $h = 5k + 2$ ,  $k \geq 1$  (see Figure 9(c) for the case  $h = 7$ ).  
We first set  $\lambda(\ell'_1, \ell'_1 \ell_1) = \lambda(v_2, v_2 \ell_2)$ ,  $\lambda(\ell_1, \ell_1 \ell'_1) = \lambda(v_2, v_2 v_3)$  and  $\lambda(\ell'_h, \ell'_h \ell_h) = \lambda(v_{h-1}, v_{h-1} \ell_{h-1})$ . We then sequentially colour the uncoloured incidences of  $C_{2h+2}$ , starting from  $(\ell_1, \ell_1 \ell_2)$ , using the pattern 10.11.12.13.14.
  - $h = 5k + 3$ ,  $k \geq 1$  (see Figure 9(d) for the case  $h = 8$ ).  
We sequentially colour the incidences of  $C_{2h+2}$ , starting from  $(\ell'_1, \ell'_1 \ell_1)$ , using the pattern 10.11.12.13.14.
  - $h = 5k + 4$ ,  $k \geq 0$  (see Figure 9(e) for the case  $h = 4$ ).  
We first set  $\lambda(\ell'_1, \ell'_1 \ell_1) = \lambda(v_2, v_2 \ell_2)$  and  $\lambda(\ell_1, \ell_1 \ell'_1) = \lambda(v_2, v_2 v_3)$ . We then sequentially colour the uncoloured incidences of  $C_{2h+2}$ , starting from  $(\ell_1, \ell_1 \ell_2)$ , using the pattern 10.11.12.13.14.

In each case, the so-obtained colouring is clearly a strong incidence colouring of  $G \in \mathcal{H}_h^c$ . This completes the proof.  $\square$

We Now determine the value of the strong incidence chromatic number of necklaces.

**Theorem 14** *For every necklaces  $N_h$ ,  $h \geq 1$ , we have*

$$\chi_i^s(N_h) = \begin{cases} 12 & \text{if } h = 1, 2, 3, 5, \\ 11 & \text{otherwise.} \end{cases}$$

**Proof.** As we showed in the proof of Theorem 13,  $N_h$  contains a subgraph isomorphic to the graph  $F$  (see Figure 8), which implies  $\chi_i^s(N_h) \geq 11$  for every  $h \geq 1$ . The values of  $\chi_i^s(N_h)$ ,  $1 \leq h \leq 3$ , were given in the beginning of the subsection.

If  $h = 5$ , the cardinality of the set of the incidences of  $N_5$  is 36. Therefore, if we colour the graph with 11 colours, then at least three colours must be repeated at least four times, or one colour must be repeated at least five times and one colour must be repeated at least four times, or at least one colour must be repeated at least six times. We will prove that at most two colours can be repeated four times, which will imply  $\chi_i^s(N_5) \geq 12$ .

The set of incidences of  $N_5$  can be partitioned into four sets, the set of incidences marked with a star, the set of incidences marked with a diamond, the set of incidences marked with a circle

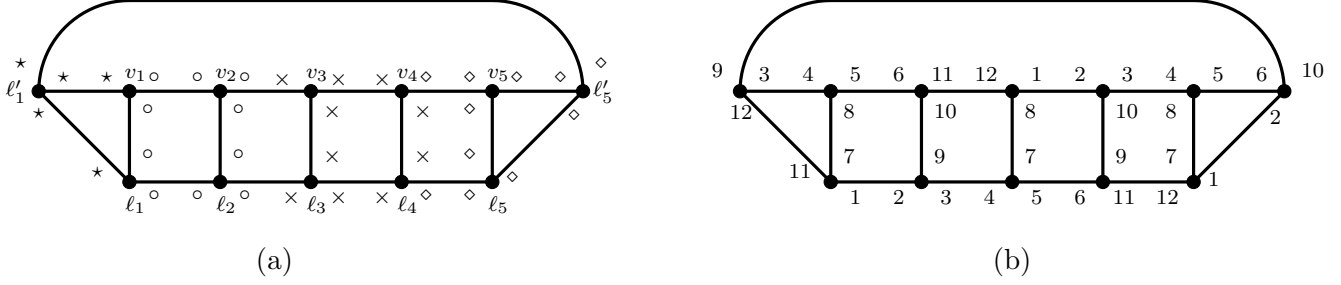


Figure 10: The necklace  $N_5$  (for the proof of Theorem 14).

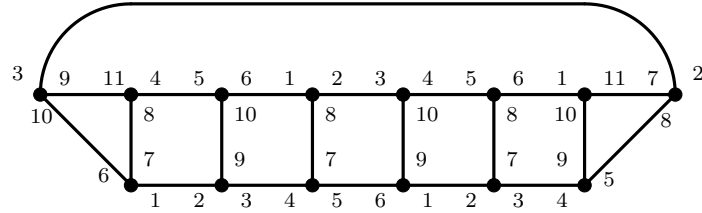
and the set of incidences marked with a cross (See Figure 10(a)). Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of any incidence of  $N_5$  can be repeated at most four times. Moreover, any colour repeated four times must be used exactly once in each of these sets. We now prove that among the colours of the incidences marked with a star in Figure 10(a), only two colours can be repeated four times.

- The incidence  $(\ell'_1, \ell'_1 \ell'_5)$ .  
The set of the incidences that are not strongly adjacent with this incidence can be partitioned into the two sets  $\{(v_2, v_2 l_2), (v_2, v_2 v_3), (l_2, l_2 v_2), (l_2, l_2 l_3), (v_3, v_3 v_2), (l_2, l_2 l_3)\}$  and  $\{(v_3, v_3 l_3), (v_3, v_3 v_4), (l_3, l_3 v_3), (l_3, l_3 l_4), (v_4, v_4 v_3), (v_4, v_4 l_4), (v_4, v_4 v_5), (l_4, l_4 l_3), (l_4, l_4 v_4), (l_4, l_4 l_5)\}$ . Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of the incidence  $(\ell'_1, \ell'_1 \ell'_5)$  can be repeated only two more times.
- The incidence  $(\ell'_1, \ell'_1 v_1)$ .  
The set of the incidences that are not strongly adjacent with this incidence can be partitioned into the two sets  $\{(l_2, l_2 v_2), (l_2, l_2 l_3), (v_3, v_3 v_2), (v_3, v_3 l_3), (v_3, v_3 v_4), (l_3, l_3 l_2), (l_3, l_3 v_3), (l_3, l_3 l_4)\}$  and  $\{(v_4, v_4 v_3), (v_4, v_4 l_4), (v_4, v_4 v_5), (l_4, l_4 l_3), (l_4, l_4 v_4), (l_4, l_4 l_5), (v_5, v_5 v_4), (v_5, v_5 l_5), (l_5, l_5 l_4), (l_5, l_5 v_5)\}$ . Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of the incidence  $(\ell'_1, \ell'_1 v_1)$  can be repeated only two more times.
- By symmetry, the case of the incidence  $(\ell'_1, \ell'_1 l_1)$  is similar to the case of the incidence  $(\ell'_1, \ell'_1 v_1)$ .

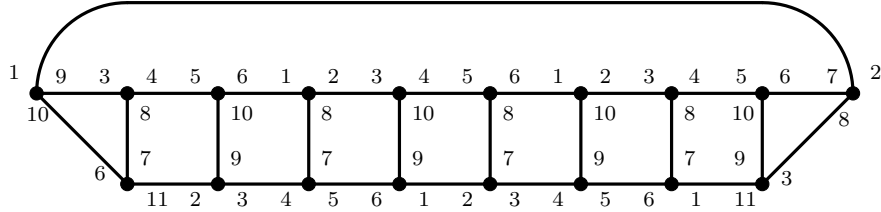
Based on the above, only the colours of the two incidences  $(l_1, l_1 \ell'_1)$  and  $(v_1, v_1 \ell'_1)$  can be repeated four times. Hence,  $\chi_i^s(N_5) \geq 12$ . A strong incidence 12-colouring of  $N_5$  is given in Figure 10(b), so that  $\chi_i^s(N_5) = 12$ .

Finally, if  $h = 4$  or  $h \geq 6$  then it suffices to construct a strong incidence 11-colouring of  $N_h$ . Such a colouring can be obtained as follows (see Figure 11). We consider two cases, depending on the parity of  $h$ .

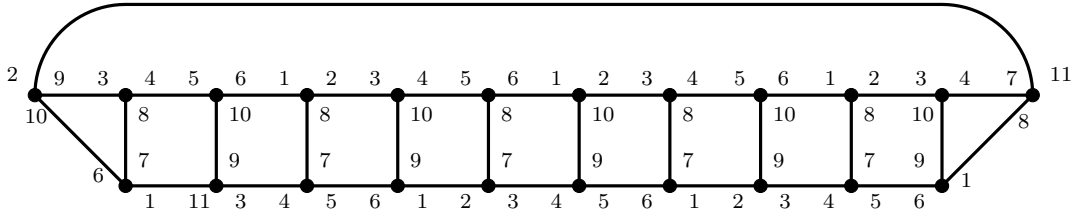
- $h$  is even.
  - We first colour the subgraph  $S_h$  induced by the set of vertices  $\{v_1, \dots, v_h, l_1, \dots, l_h\}$  (which is isomorphic to the ladder graph  $L_h$ ), as in the proof of Theorem 11.
  - We then modify the colouring of the subgraph  $S_h$  and we complete the colouring of  $N_h$  according to the value of  $h \pmod 6$ :
    - \*  $h = 6k, k \geq 1$  (see Figure 11(a) for the case  $h = 6$ ).  
We set  $\lambda(l_1, l_1 \ell'_1) = 6, \lambda(\ell'_1, \ell'_1 l_1) = 10, \lambda(\ell'_1, \ell'_1 v_1) = 9, \lambda(v_1, v_1 \ell'_1) = 11, \lambda(\ell'_1, \ell'_1 l'_h) = 3, \lambda(\ell'_h, \ell'_h l'_1) = 2, \lambda(\ell'_h, \ell'_h v_h) = 7, \lambda(\ell'_h, \ell'_h l_h) = 8, \lambda(v_h, v_h \ell'_h) = 11, \lambda(l_h, l_h l'_h) = 5$ .
    - \*  $h = 6k + 2, k \geq 1$  (see Figure 11(b) for the case  $h = 8$ ).  
We set  $\lambda(l_1, l_1 l_2) = 11, \lambda(l_1, l_1 \ell'_1) = 6, \lambda(\ell'_1, \ell'_1 l_1) = 10, \lambda(\ell'_1, \ell'_1 v_1) = 9, \lambda(v_1, v_1 \ell'_1) =$



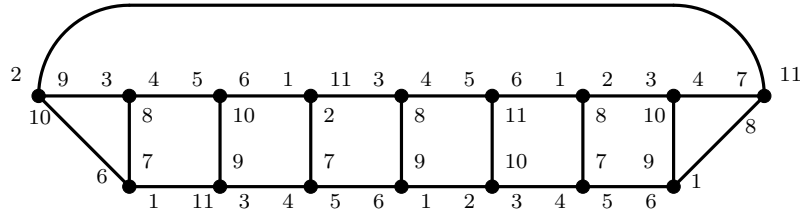
(a)



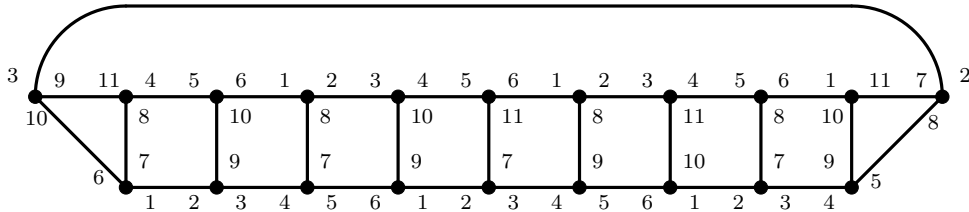
(b)



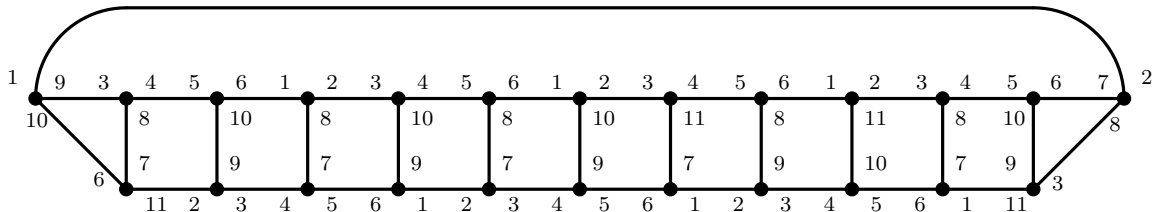
(c)



(d)



(e)



(f)

Figure 11: Strong incidence 11-colourings of  $N_h$ ,  $6 \leq h \leq 11$ .

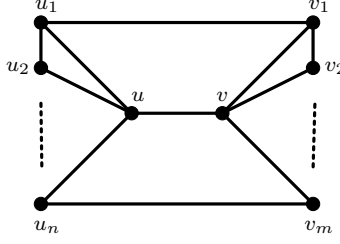


Figure 12: The Halin graph  $HD_{m,n}$ .

$$3, \lambda(\ell'_1, \ell'_1 \ell'_h) = 1, \lambda(\ell'_h, \ell'_h \ell'_1) = 2, \lambda(\ell'_h, \ell'_h v_h) = 7, \lambda(\ell'_h, \ell'_h \ell'_h) = 8, \lambda(v_h, v_h \ell'_h) = 6, \lambda(\ell_h, \ell_h \ell'_h) = 3.$$

\*  $h = 6k + 4, k \geq 0$  (see Figure 11(c) for the case  $h = 10$ ).

$$\text{We set } \lambda(\ell_2, \ell_2 \ell_1) = 11, \lambda(\ell_1, \ell_1 \ell'_1) = 6, \lambda(\ell'_1, \ell'_1 \ell_1) = 10, \lambda(\ell'_1, \ell'_1 v_1) = 9, \lambda(v_1, v_1 \ell'_1) = 3, \lambda(\ell'_1, \ell'_1 \ell'_h) = 2, \lambda(\ell'_h, \ell'_h \ell'_1) = 11, \lambda(\ell'_h, \ell'_h v_h) = 7, \lambda(\ell'_h, \ell'_h \ell'_h) = 8, \lambda(v_h, v_h \ell'_h) = 4, \lambda(\ell_h, \ell_h \ell'_h) = 1.$$

•  $h$  is odd.

– We first colour the subgraph  $S_h$  induced by the vertices  $\{v_1, \dots, v_h, \ell_1, \dots, \ell_h\}$  (which is isomorphic to the ladder graph  $L_h$ ), as in the proof of Theorem 11, and we make the following modifications.

$$\text{We set } \lambda(v_{h-4}, v_{h-4} \ell_{h-4}) = 11, \lambda(v_{h-3}, v_{h-3} \ell_{h-3}) = 8, \lambda(v_{h-2}, v_{h-2} \ell_{h-2}) = 11, \lambda(\ell_{h-2}, \ell_{h-2} v_{h-2}) = 10, \lambda(v_{h-1}, v_{h-1} \ell_{h-1}) = 8, \lambda(\ell_{h-1}, \ell_{h-1} v_{h-1}) = 7, \lambda(v_h, v_h \ell_h) = 10, \lambda(\ell_h, \ell_h v_h) = 9.$$

– We modify the colouring of the subgraph  $S_h$  and we complete the colouring of  $N_h$  according to the value of  $h \pmod 6$ :

\*  $h = 6k + 1, k \geq 1$  (see Figure 11(d) for the case  $h = 7$ ).

$$\text{We set } \lambda(\ell_2, \ell_2 \ell_1) = 11, \lambda(\ell_1, \ell_1 \ell'_1) = 6, \lambda(\ell'_1, \ell'_1 \ell_1) = 10, \lambda(\ell'_1, \ell'_1 v_1) = 9, \lambda(v_1, v_1 \ell'_1) = 3, \lambda(\ell'_1, \ell'_1 \ell'_h) = 2, \lambda(\ell'_h, \ell'_h \ell'_1) = 11, \lambda(\ell'_h, \ell'_h v_h) = 7, \lambda(\ell'_h, \ell'_h \ell'_h) = 8, \lambda(v_h, v_h \ell'_h) = 4, \lambda(\ell_h, \ell_h \ell'_h) = 1. \text{ if } h = 7 \text{ the we exchange the colours of the incidences } (v_3, v_3 v_4) \text{ and } (v_3, v_3 \ell_3).$$

\*  $h = 6k + 3, k \geq 1$  (see Figure 11(e) for the case  $h = 9$ ).

$$\text{We set } \lambda(\ell_1, \ell_1 \ell'_1) = 6, \lambda(\ell'_1, \ell'_1 \ell_1) = 10, \lambda(\ell'_1, \ell'_1 v_1) = 9, \lambda(v_1, v_1 \ell'_1) = 11, \lambda(\ell'_1, \ell'_1 \ell'_h) = 3, \lambda(\ell'_h, \ell'_h \ell'_1) = 2, \lambda(\ell'_h, \ell'_h v_h) = 7, \lambda(\ell'_h, \ell'_h \ell'_h) = 8, \lambda(v_h, v_h \ell'_h) = 11, \lambda(\ell_h, \ell_h \ell'_h) = 5.$$

\*  $h = 6k + 5, k \geq 1$  (see Figure 11(f) for the case  $h = 11$ ).

$$\text{We set } \lambda(\ell_1, \ell_1 \ell_2) = 11, \lambda(\ell_1, \ell_1 \ell'_1) = 6, \lambda(\ell'_1, \ell'_1 \ell_1) = 10, \lambda(\ell'_1, \ell'_1 v_1) = 9, \lambda(v_1, v_1 \ell'_1) = 3, \lambda(\ell'_1, \ell'_1 \ell'_h) = 1, \lambda(\ell'_h, \ell'_h \ell'_1) = 2, \lambda(\ell'_h, \ell'_h v_h) = 7, \lambda(\ell'_h, \ell'_h \ell'_h) = 8, \lambda(v_h, v_h \ell'_h) = 6, \lambda(\ell_h, \ell_h \ell'_h) = 3.$$

In each case, the so-obtained colouring is clearly a strong incidence colouring of  $N_h$ . This completes the proof.  $\square$

## 5.2 Halin graphs whose internal tree is a double star

The *double star*, denoted by  $S_{m,n}$ ,  $m \geq n \geq 2$ , is the graph obtained from the stars  $S_m$  and  $S_n$  by adding an edge joining the central vertex of  $S_m$  to the central vertex of  $S_n$ . The Halin graph  $HD_{m,n}$  (see Figure 12) is the Halin graph whose internal tree is the double star  $S_{m,n}$  and whose outer cycle is  $u_1 \dots u_n v_m \dots v_1 u_1$ . We denote by  $P$  the path  $v_1 \dots v_m$  and by  $P'$  the path  $u_1 \dots u_n$ .

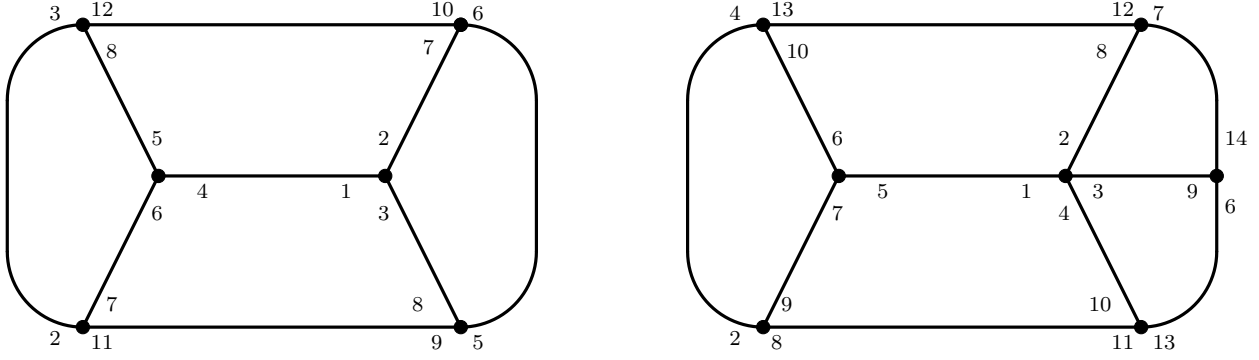


Figure 13: Strong incidence colourings of  $HD_{2,2}$  and  $HD_{3,2}$ .

It is easy to see that for every graph  $HD_{m,n}$ ,  $m \geq n \geq 2$ , the incidences of the set

$$A^-(v) \cup A^+(v) \cup A^-(u) \cup \{(v_1, v_1u_1)\},$$

of cardinality  $2 \deg(v) + \deg(u) = \sigma(HD_{m,n}) + 1$ , are pairwise strongly adjacent. Therefore, we have the following inequality.

**Proposition 15** For every two integers  $m$  and  $n$ ,  $m \geq n \geq 2$ ,  $\chi_i^s(HD_{m,n}) \geq 2m + n + 3 = \sigma(HD_{m,n}) + 1$ .

We first define a partial colouring  $\lambda$  of  $HD_{m,n}$ , for every  $m \geq n \geq 3$ , as follows.

- The incidences  $(v, vv)$ ,  $(v, vv_1)$ ,  $(v, vv_2), \dots, (v, vv_m)$  are coloured with the colours  $1, 2, 3, \dots, m + 1$ , respectively.
- The incidences  $(u, uv)$ ,  $(u, uu_1)$ ,  $(u, uu_2), \dots, (u, uu_n)$  are coloured with the colours  $m + 2, m + 3, m + 4, \dots, m + n + 2$ , respectively.
- The incidences  $(v_1, v_1v)$ ,  $(v_2, v_2v), \dots, (v_m, v_mv)$  are coloured with the colours  $m + n + 3, m + n + 4, \dots, 2m + n + 2$ , respectively.
- The incidence  $(v_1, v_1u_1)$  is coloured with the colour  $2m + n + 3$ .

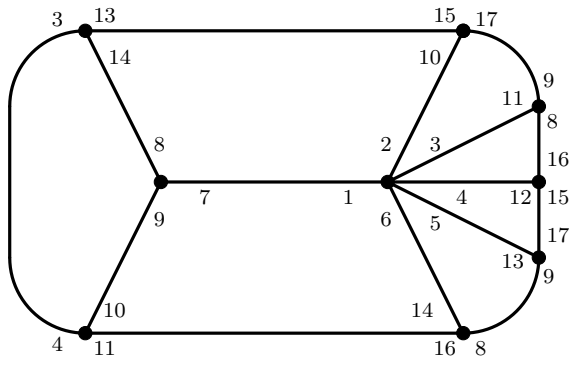
In the next lemmas, we will extend  $\lambda$  to a colouring of  $HD_{m,n}$ , according to the values of  $m$  and  $n$ .

**Lemma 16** For every integer  $m \geq 2$ ,

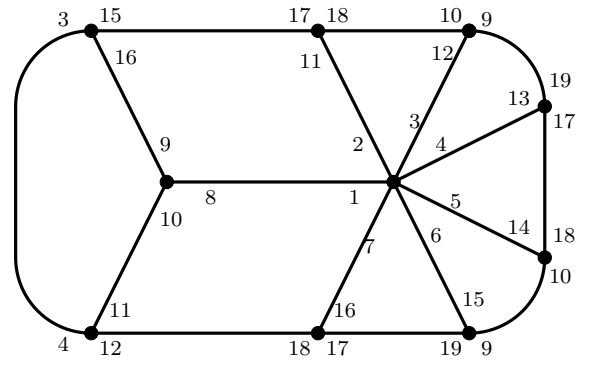
$$\chi_i^s(HD_{m,2}) = \begin{cases} \sigma(HD_{m,2}) + 4 & \text{if } m = 2, \\ \sigma(HD_{m,2}) + 3 & \text{otherwise.} \end{cases}$$

**Proof.** Strong incidence colourings of  $HD_{2,2}$  and  $HD_{3,2}$  are given in Figure 13. Suppose now  $m \geq 4$ . Observe that each colour of the set  $\{1, \dots, m + 1, m + 2, m + 5, \dots, 2m + 4\}$  of cardinality  $2m + 2$  is forbidden on the incidences of  $P$ , due to the partial colouring  $\lambda$ . Therefore,  $\chi_i^s(HD_{m,2}) \geq 2m + 7 = \sigma(HD_{m,2}) + 3$ . To colour the incidences of the path  $P$ , we use the three colours  $m + 3, m + 4$  and  $2m + 5$ , and two additional colours  $2m + 6$  and  $2m + 7$ .

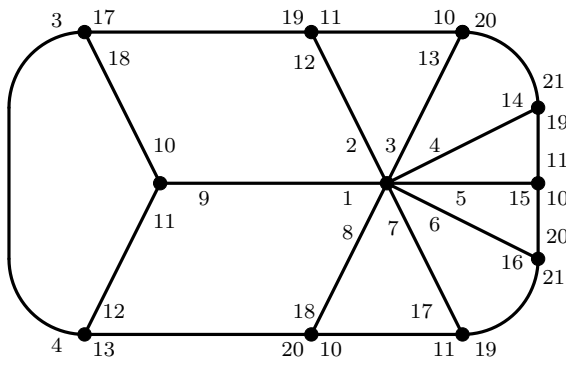
- We will sequentially colour the path  $P$ , according to the value of  $m \pmod{5}$ , as follows.



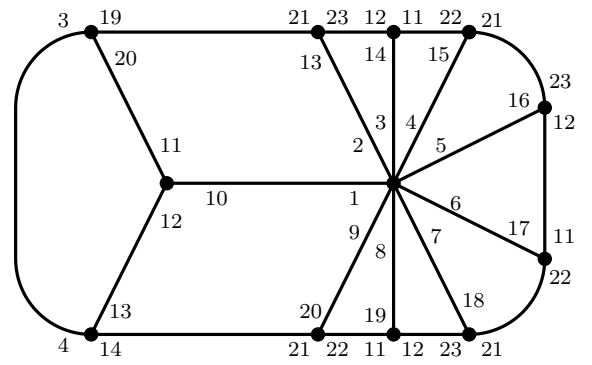
(a)



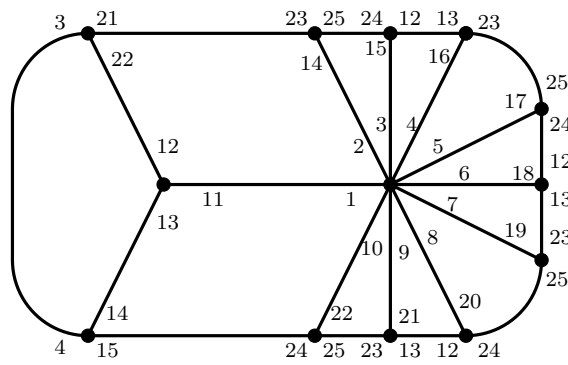
(b)



(c)



(d)



(e)

Figure 14: Strong incidence colourings of  $HD_{m,2}$ ,  $5 \leq m \leq 9$ .

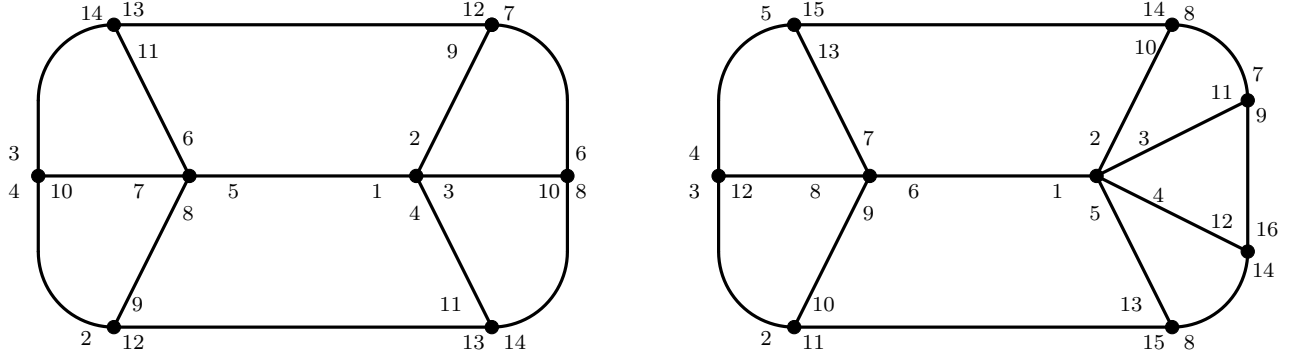


Figure 15: Strong incidence colourings of  $HD_{3,3}$  and  $HD_{4,3}$ .

- $m = 5k$ ,  $k \geq 1$  (see Figure 14(a) for the case  $m = 5$ ).  
We use the pattern  $(2m + 7)(m + 4)(m + 3)(2m + 6)(2m + 5)$ .
  - $m = 5k + 1$ ,  $k \geq 1$  (see Figure 14(b) for the case  $m = 6$ ).  
We use the pattern  $(2m + 6)(m + 4)(m + 3)(2m + 7)(2m + 5)$ .
  - $m = 5k + 2$ ,  $k \geq 1$  (see Figure 14(c) for the case  $m = 7$ ).  
We use the pattern  $(m + 4)(m + 3)(2m + 6)(2m + 7)(2m + 5)$ .
  - $m = 5k + 3$ ,  $k \geq 1$  (see Figure 14(d) for the case  $m = 8$ ).  
We use the pattern  $(2m + 7)(m + 4)(m + 3)(2m + 6)(2m + 5)$ .
  - $m = 5k + 4$ ,  $k \geq 1$  (see Figure 14(e) for the case  $m = 9$ ).  
We use the pattern  $(2m + 7)(2m + 6)(m + 3)(m + 4)(2m + 5)$ .
- We then set  $\lambda(v_m, v_m u_n) = 2m + 5$  if  $m = 5k + 3$ , and  $\lambda(v_m, v_m u_n) = 2m + 6$  otherwise.
  - We finally complete the colouring of  $HD_{m,2}$  by assigning the colours 3, 4,  $2m + 4$ ,  $m + 5$ ,  $2m + 3$  and  $m + 6$  to the incidences  $(u_1, u_1 u_2)$ ,  $(u_2, u_2 u_1)$ ,  $(u_1, u_1 u)$ ,  $(u_2, u_2 u)$ ,  $(u_1, u_1 v_1)$  and  $(u_2, u_2 v_m)$ , respectively.

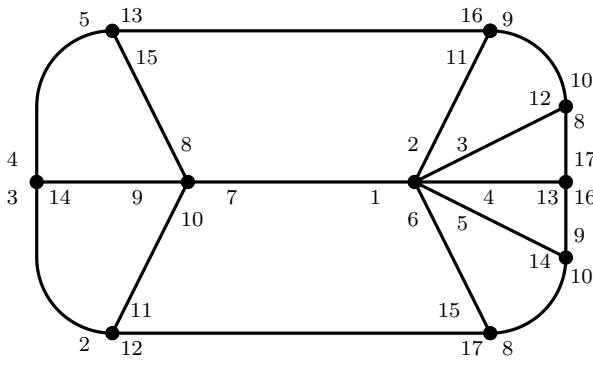
This concludes the proof. □

**Lemma 17** For every integer  $m \geq 3$ ,

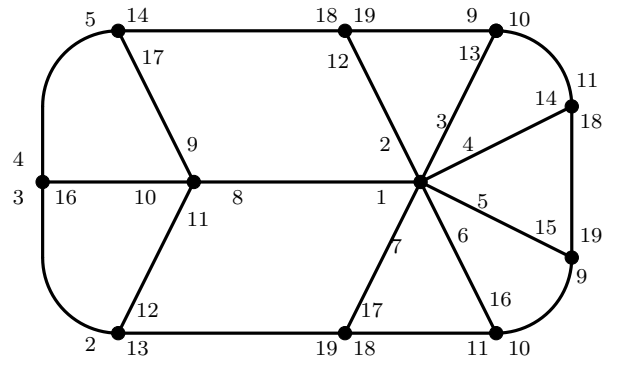
$$\chi_i^s(HD_{m,3}) = \begin{cases} \sigma(HD_{m,3}) + 3 & \text{if } m = 3 \text{ or } m = 4, \\ \sigma(HD_{m,3}) + 2 & \text{otherwise.} \end{cases}$$

**Proof.** Strong incidence colourings of  $HD_{3,3}$  and  $HD_{4,3}$  are given in Figure 15. Suppose now  $m \geq 5$ . Observe that each colour of the set  $\{1, \dots, m+1, m+2, m+6, \dots, 2m+5\}$  of cardinality  $2m+2$  is forbidden on the incidences of  $P$ , due to the partial colouring  $\lambda$ . Therefore,  $\chi_i^s(HD_{m,3}) \geq 2m+7 = \sigma(HD_{m,3}) + 2$ . To colour the incidences of the path  $P$ , we use the four colours  $m+3$ ,  $m+4$ ,  $m+5$  and  $2m+6$ , and an additional colour  $2m+7$ .

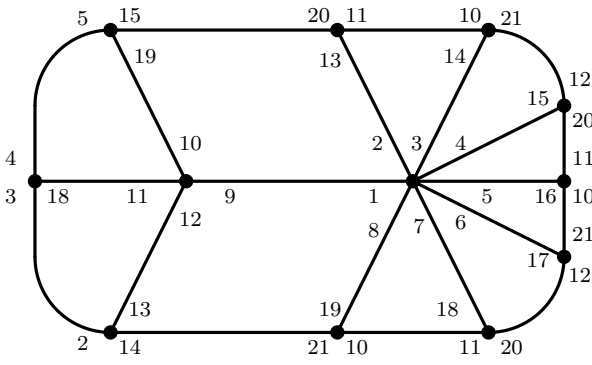
- We will sequentially colour the incidences of path  $P$ , starting from the incidence  $(v_1, v_1 v_2)$ , according to the value of  $m \pmod 5$ , as follows.
  - $m = 5k$ ,  $k \geq 1$  (see Figure 16(a) for the case  $m = 5$ ).  
We use the pattern  $(m + 4)(m + 5)(m + 3)(2m + 7)(2m + 6)$ .



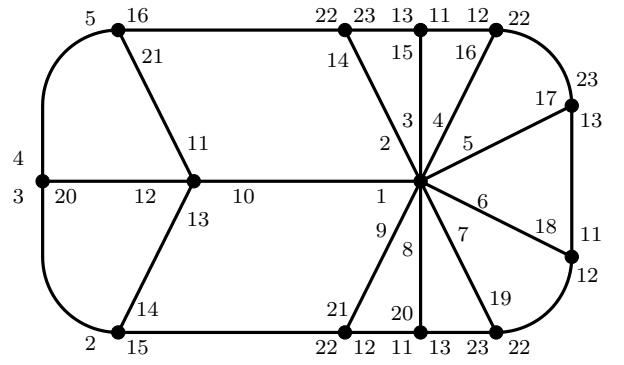
(a)



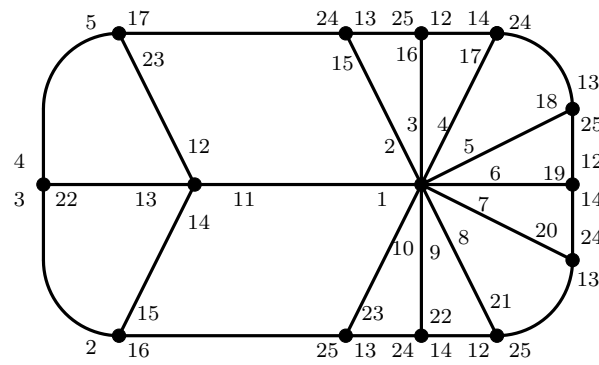
(b)



(c)



(d)



(e)

Figure 16: Strong incidence colourings of  $HD_{m,3}$ ,  $5 \leq m \leq 9$ .

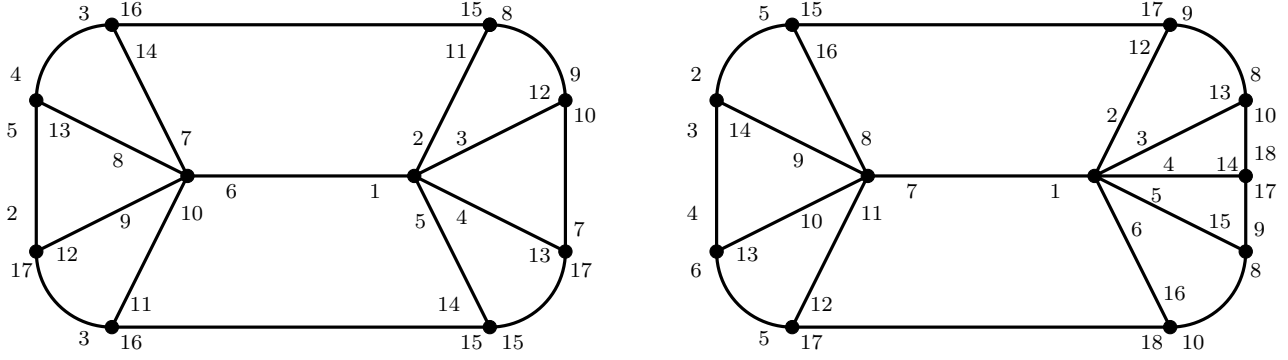


Figure 17: Strong incidence colourings of  $HD_{4,4}$  and  $HD_{5,4}$ .

- $m = 5k + 1$ ,  $k \geq 1$  (see Figure 16(b) for the case  $m = 6$ ).  
We use the pattern  $(2m + 7)(m + 3)(m + 4)(m + 5)(2m + 6)$ .
- $m = 5k + 2$ ,  $k \geq 1$  (see Figure 16(c) for the case  $m = 7$ ).  
We use the pattern  $(m + 4)(m + 3)(2m + 7)(m + 5)(2m + 6)$ .
- $m = 5k + 3$ ,  $k \geq 1$  (see Figure 16(d) for the case  $m = 8$ ).  
We use the pattern  $(2m + 7)(m + 5)(m + 3)(m + 4)(2m + 6)$ .
- $m = 5k + 4$ ,  $k \geq 1$  (see Figure 16(e) for the case  $m = 9$ ).  
We use the pattern  $(m + 4)(2m + 7)(m + 3)(m + 5)(2m + 6)$ .
- We then set  $\lambda(v_m, v_m u_n) = 2m + 6$  if  $m = 5k + 3$ , and  $\lambda(v_m, v_m u_n) = 2m + 7$  otherwise.
- We finally complete the colouring of  $HD_{m,3}$  by assigning the colours 5, 4, 3, 2,  $2m + 5$ ,  $2m + 4$ ,  $m + 6$ ,  $m + 8$  and  $m + 7$  to the incidences  $(u_1, u_1 u_2)$ ,  $(u_2, u_2 u_1)$ ,  $(u_2, u_2 u_3)$ ,  $(u_3, u_3 u_2)$ ,  $(u_1, u_1 u)$ ,  $(u_2, u_2 u)$ ,  $(u_3, u_3 u)$ ,  $(u_1, u_1 v_1)$  and  $(u_3, u_3 v_m)$ , respectively.

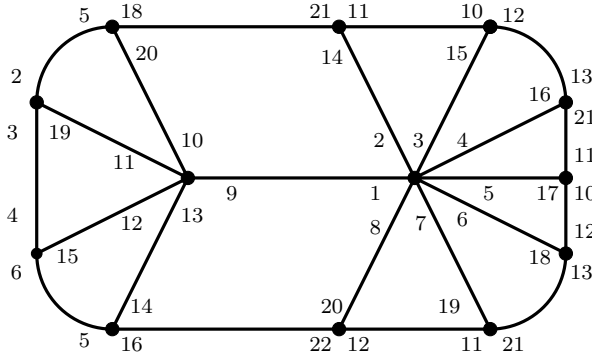
This concludes the proof.  $\square$

**Lemma 18** For every integer  $m \geq 4$ ,

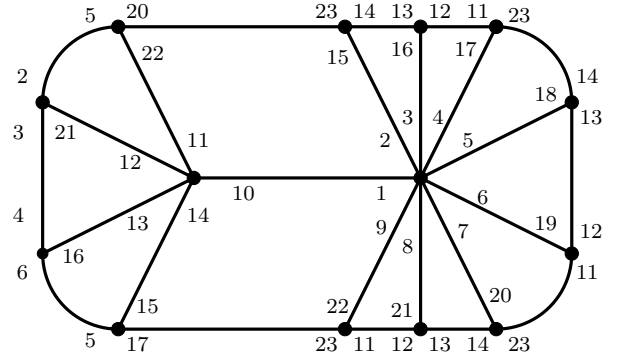
$$\chi_i^s(HD_{m,4}) = \begin{cases} \sigma(HD_{m,4}) + 3 & \text{if } m = 4, \\ \sigma(HD_{m,4}) + 1 & \text{if } m \equiv 3 \pmod{5}, \\ \sigma(HD_{m,4}) + 2 & \text{otherwise.} \end{cases}$$

**Proof.** Strong incidence colourings of  $HD_{4,4}$  and  $HD_{5,4}$  are given in Figure 17. Suppose now  $m \geq 5$ . Observe that each colour of the set  $\{1, \dots, m+1, m+2, m+7, \dots, 2m+6\}$  of cardinality  $2m+2$  is forbidden on the incidences of  $P$  and the incidence  $(v_m, v_m u_n)$ , due to the partial colouring  $\lambda$ . Therefore,  $\chi_i^s(HD_{m,4}) \geq 2m+7 = \sigma(HD_{m,4}) + 1$ . It should be noted that if  $\lambda(v_m, v_m u_n) = 2m+7$ , then we can colour the path  $P$  with the colours  $m+3$ ,  $m+4$ ,  $m+5$ ,  $m+6$  and  $2m+7$  if and only if  $m \equiv 3 \pmod{5}$ . We will sequentially colour the incidences of the path  $P$ , starting from the incidence  $(v_1, v_1 v_2)$ , as well as the incidence  $(v_m, v_m u_n)$ , according to the value of  $m \pmod{5}$ , as follows.

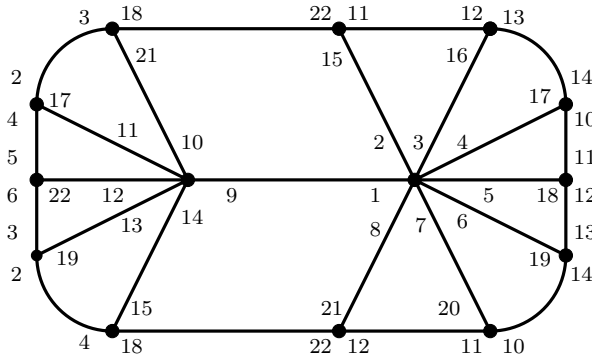
- $m \neq 5k + 3$  (see Figure 18(a) for the case  $m = 7$ ).  
We use the pattern  $(m+4)(m+3)(m+5)(m+6)(2m+7)$  for  $P$  and we set  $\lambda(v_m, v_m u_n) = 2m+8$ .



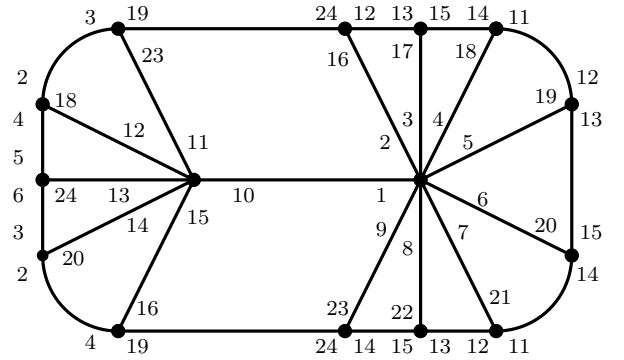
(a)



(b)

Figure 18: Strong incidence colourings of  $HD_{7,4}$  and  $HD_{8,4}$ .

(a)



(b)

Figure 19: Strong incidence colourings of  $HD_{7,5}$  and  $HD_{8,5}$ .

- $m = 5k + 3$ ,  $k \geq 1$  (see Figure 18(b) for the case  $m = 8$ ).

We use the pattern  $(m+6)(m+5)(m+4)(m+3)(2m+7)$  for  $P$  and we set  $\lambda(v_m, v_m u_n) = 2m+7$ .

We finally colour the remaining incidences of  $HD_{m,4}$  by assigning the colours 5, 2, 3, 4, 6, 5,  $2m+6$ ,  $2m+5$ ,  $m+8$ ,  $m+7$ ,  $2m+4$  and  $m+9$  to the incidences  $(u_1, u_1 u_2)$ ,  $(u_2, u_2 u_1)$ ,  $(u_2, u_2 u_3)$ ,  $(u_3, u_3 u_2)$ ,  $(u_3, u_3 u_4)$ ,  $(u_4, u_4 u_3)$ ,  $(u_1, u_1 u)$ ,  $(u_2, u_2 u)$ ,  $(u_3, u_3 u)$ ,  $(u_4, u_4 u)$ ,  $(u_1, u_1 v_1)$  and  $(u_4, u_4 v_m)$ , respectively. This concludes the proof.  $\square$

**Lemma 19** For every integer  $m \geq 5$ ,  $\chi_i^s(HD_{m,5}) = \sigma(HD_{m,5}) + 1$ .

**Proof.** By Proposition 15,  $\chi_i^s(HD_{m,5}) \geq 2m+8 = \sigma(HD_{m,5}) + 1$ . To complete the proof, we give a strong incidence  $(2m+8)$ -colouring by extending  $\lambda$  to a colouring of  $HD_{m,5}$  as follows.

- We first sequentially colour the incidences of the path  $P$ , starting from the incidence  $(v_1, v_1 v_2)$ , according to the value of  $m \pmod 5$  as follows.
  - $m \neq 5k + 3$ ,  $k \geq 1$  (see Figure 19(a) for the case  $m = 7$ ).  
We use the pattern  $(m+4)(m+5)(m+6)(m+7)(m+3)$ .

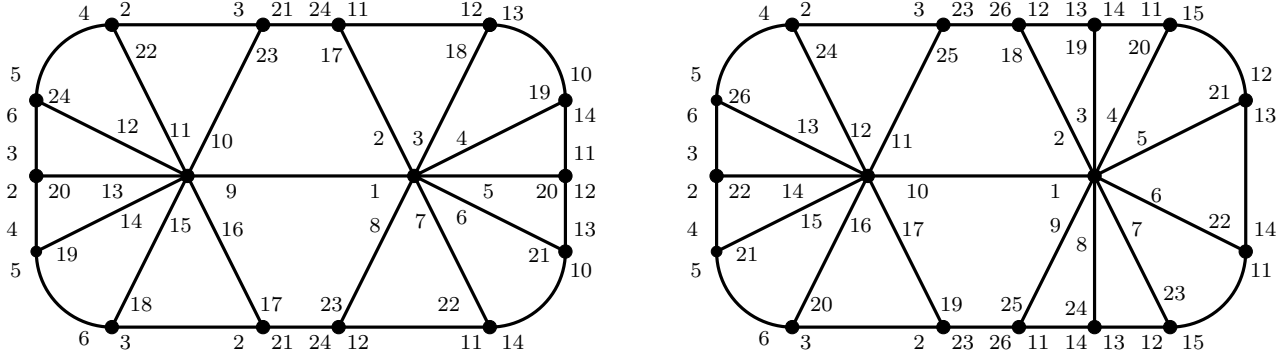


Figure 20: Strong incidence colourings of  $HD_{7,7}$  and  $HD_{8,7}$ .

–  $m = 5k + 3$ ,  $k \geq 1$  (see Figure 19(b) for the case  $m = 8$ ).

We use the pattern  $(m + 4)(m + 5)(m + 7)(m + 6)(m + 3)$ .

- We then sequentially colour the incidences of the path  $P'$ , starting from the incidence  $(u_1, u_1u_2)$ , using the pattern 32456.
- We finally colour the remaining incidences of  $HD_{m,5}$  by assigning the colours  $2m + 7$ ,  $m + 10$ ,  $2m + 8$ ,  $m + 12$ ,  $m + 8$ ,  $m + 11$ ,  $m + 11$  and  $2m + 8$  to the incidences  $(u_1, u_1u)$ ,  $(u_2, u_2u)$ ,  $(u_3, u_3u)$ ,  $(u_4, u_4u)$ ,  $(u_5, u_5u)$ ,  $(u_1, u_1v_1)$ ,  $(u_5, u_5v_m)$  and  $(v_m, v_mu_n)$ , respectively.

This concludes the proof.  $\square$

**Lemma 20** For every two integers  $m$  and  $n$ ,  $m \geq n \geq 6$ ,  $\chi_i^s(HD_{m,n}) = \sigma(HD_{m,n}) + 1$ .

**Proof.** By Proposition 15,  $\chi_i^s(HD_{m,n}) \geq 2m + n + 3 = \sigma(HD_{m,n}) + 1$ . To complete the proof, we give a strong incidence  $(2m + n + 3)$ -colouring by extending  $\lambda$  to a colouring of  $HD_{m,n}$  as follows (see Figure 20).

- We first sequentially colour the incidences of the path  $P$ , starting from the incidence  $(v_1, v_1v_2)$ , using the pattern  $(m + 4)(m + 5)(m + 6)(m + 3)(m + 7)$ .
- We then sequentially colour the incidences of the path  $P'$ , starting from the incidence  $(u_1, u_1u_2)$ , using the pattern 32456.
- We finally set  $\lambda(u_i, u_iu) = 2m + n + 3 - i$ , for every  $i$ ,  $i \in \{1, 2, 4, \dots, n\}$ ,  $\lambda(u_3, u_3u) = 2m + n + 3$ ,  $\lambda(u_n, u_nv_m) = \lambda(u_1, u_1v_1) = 2m + n$  and  $\lambda(v_m, v_mu_n) = 2m + n + 3$ .

This concludes the proof.  $\square$

Putting together Lemmas 16, 17, 18, 19 and 20, we finally get the following theorem.

**Theorem 21** For every two integers  $m$  and  $n$ ,  $m \geq n \geq 3$ ,

$$\chi_i^s(HD_{m,n}) = \begin{cases} \sigma(HD_{m,n}) + 4 & \text{if } n = 2 \text{ and } m = 2, \\ \sigma(HD_{m,n}) + 3 & \text{if } n = 2 \text{ and } m \neq 2, \\ & \text{or } (n, m) \in \{(3, 3), (3, 4), (4, 4)\}, \\ \sigma(HD_{m,n}) + 2 & \text{if } n = 3 \text{ and } m \notin \{3, 4\}, \\ & \text{or } n = 4 \text{ and } m \neq 4 \text{ and } m \not\equiv 3 \pmod{5}, \\ \sigma(HD_{m,n}) + 1 & \text{otherwise.} \end{cases}$$

## 6 Discussion

In this paper, we have introduced and studied the strong version of incidence colouring. We have determined the exact value of – or upper bounds on – the strong incidence chromatic number of several classes of graphs, namely cycles, wheel graphs, trees, ladder graphs and some subclasses of Halin graphs. We leave as open problems the following questions.

1. What is the best possible upper bound on the strong incidence chromatic number of graphs with bounded maximum degree? In particular, what about graphs with maximum degree 3?
2. What is the best possible upper bound on the strong incidence chromatic number of Halin graphs?
3. What is the best possible upper bound on the strong incidence chromatic number of  $d$ -degenerated graphs?

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