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# A Note on the Join of Varieties of Monoids With LI

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## Abstract

In this note, we give a characterisation in terms of identities of the join of  $\mathbf{V}$  with the variety of finite locally trivial semigroups  $\mathbf{LI}$  for several well-known varieties of finite monoids  $\mathbf{V}$  by using classical algebraic-automata-theoretic techniques. To achieve this, we use the new notion of essentially- $\mathbf{V}$  stamps defined by Grosshans, McKenzie and Segoufin and show that it actually coincides with the join of  $\mathbf{V}$  and  $\mathbf{LI}$  precisely when some natural condition on the variety of languages corresponding to  $\mathbf{V}$  is verified.

This work is a kind of rediscovery of the work of J. C. Costa around 20 years ago from a rather different angle, since Costa's work relies on the use of advanced developments in profinite topology, whereas what is presented here essentially uses an algebraic, language-based approach.

**2012 ACM Subject Classification** Theory of computation  $\rightarrow$  Formal languages and automata theory; Theory of computation  $\rightarrow$  Algebraic language theory

**Keywords and phrases** Varieties of monoids, join, LI

## 1 Introduction

One of the most fundamental problems appearing in finite automata theory is the one of *characterisation*: given some subclass of the class of regular languages, find out whether there is a way to characterise those languages using some class of finite objects. This problem is often linked to and motivated by the problem of *decidability*: given some subclass of the class of regular languages, find out whether there exists an algorithm testing the membership of any regular language in that subclass. The obvious approach to try to find a characterisation of a class of regular languages would be to look for properties shared by all the minimal finite automata of those languages. If we find such characterising properties, we can then ask whether they can be checked by an algorithm to answer the problem of decidability for this class of languages. However, one of the most fruitful approaches in face of those two problems has been the *algebraic approach*, in which we basically replace automata with morphisms into monoids: a language  $L$  over some alphabet  $\Sigma$  is then said to be recognised by a morphism  $\varphi$  into a monoid  $M$  if and only if  $L$  is the inverse image by  $\varphi$  of a subset of  $M$ . Under this notion of recognition, each language has a minimal morphism recognising it, the *syntactic morphism* into the *syntactic monoid* of that language, that are minimal under some notion of division. The fundamental result on which this algebraic approach relies is that a language is regular if and only if its syntactic monoid is finite. One can thus try to find a characterisation of some class of regular languages by looking at the algebraic properties of the syntactic monoids of these languages.

And many such characterisations that are decidable were indeed successfully obtained since Schützenberger's seminal work in 1965 [16]. His famous result, that really started the field of *algebraic automata theory*, states that the star-free regular languages are exactly those whose syntactic monoids are finite and aperiodic. Another important early result in that vein is the one of Simon [17] characterising the piecewise testable languages as exactly those having a finite  $\mathfrak{J}$ -trivial syntactic monoid. Eilenberg [11] was the first to prove that such algebraic characterisations actually come as specific instances of a general bijective correspondence between varieties of finite monoids and varieties of languages — classes

46 of, respectively, finite monoids and regular languages closed under some respective natural  
 47 operations. Thus, a class of regular languages can indeed be characterised by the syntactic  
 48 monoids of these languages, as soon as it verifies some nice properties. Eilenberg's result was  
 49 later completed by Reiterman's theorem [15], that uses a notion of identities defined using  
 50 profinite topology and states that a class of finite monoids is a variety of finite monoids if and  
 51 only if it is defined by a set of profinite identities. Therefore, one can always characterise the  
 52 variety of finite monoids associated to a variety of languages by a set of profinite identities  
 53 and, additionally, this characterisation often leads to decidability, especially when this set is  
 54 finite. Several research works have been conducted to characterise varieties of finite monoids  
 55 or semigroups by profinite identities.

56 A kind of varieties of finite monoids or semigroups that has attracted many research  
 57 efforts aiming for characterisations through identities are the varieties defined as the join  
 58 of two other varieties. Given two varieties of finite monoids  $\mathbf{V}$  and  $\mathbf{W}$ , the *join of  $\mathbf{V}$*   
 59 *and  $\mathbf{W}$* , denoted by  $\mathbf{V} \vee \mathbf{W}$ , is the smallest variety of finite monoids containing both  $\mathbf{V}$   
 60 and  $\mathbf{W}$ . One of the main motivations to try to understand  $\mathbf{V} \vee \mathbf{W}$  is that the variety  
 61 of languages corresponding to it by the Eilenberg correspondence,  $\mathcal{L}(\mathbf{V} \vee \mathbf{W})$ , is the one  
 62 obtained by considering parallel computations of automata recognising languages from both  
 63  $\mathcal{L}(\mathbf{V})$  and  $\mathcal{L}(\mathbf{W})$ , the varieties of languages corresponding to, respectively,  $\mathbf{V}$  and  $\mathbf{W}$ . This  
 64 is a fundamental operation on automata, and while it is straightforward that  $\mathcal{L}(\mathbf{V} \vee \mathbf{W})$   
 65 is simply the smallest variety of languages containing both  $\mathcal{L}(\mathbf{V})$  and  $\mathcal{L}(\mathbf{W})$ , this does  
 66 not at all furnish a decidable characterisation of  $\mathcal{L}(\mathbf{V} \vee \mathbf{W})$ , let alone a set of identities  
 67 defining  $\mathbf{V} \vee \mathbf{W}$ . Generally speaking, the problem of finding a set of identities defining  
 68  $\mathbf{V} \vee \mathbf{W}$  is difficult (see [2, 21]), but it has been solved for many specific cases: have a look  
 69 at [1, 3, 5, 20, 19, 6, 8, 9] for some examples.

70 In this paper, we give a general method to find a set of identities defining the join of an  
 71 arbitrary variety of finite monoids  $\mathbf{V}$  and the *variety of finite locally trivial semigroups*  $\mathbf{LI}$ , as  
 72 soon as one has a set of identities defining  $\mathbf{V}$  and  $\mathbf{V}$  verifies some criterion. Joins of that sort  
 73 have been studied quite a lot in the literature we mentioned in the previous paragraph (e.g.  
 74 in [5, 19, 8, 9]), but while these works usually rely heavily on profinite topology with some  
 75 in-depth understanding of the structure of the elements of the so-called free pro- $\mathbf{V}$  monoids  
 76 and free pro- $\mathbf{LI}$  semigroups, we present a method that reduces the use of profinite topology  
 77 to the minimum and that relies mainly on algebraic and language-theoretic techniques. The  
 78 variety  $\mathbf{LI}$  is well-known to correspond to the class of languages for which membership only  
 79 depends on bounded-length prefixes and suffixes of words. In [12], McKenzie, Segoufin and  
 80 the author introduced the notion of *essentially- $\mathbf{V}$*  stamps (surjective morphisms  $\varphi: \Sigma^* \rightarrow M$   
 81 for  $\Sigma$  an alphabet and  $M$  a finite monoid) to characterise the built-in ability that programs  
 82 over monoids in  $\mathbf{V}$  have to treat separately some constant-length beginning and ending of a  
 83 word. Informally said, a stamp is *essentially- $\mathbf{V}$*  when it behaves like a stamp into a monoid  
 84 of  $\mathbf{V}$  as soon as a sufficiently long beginning and ending of the input word has been fixed.  
 85 Our method builds on two results, that we prove in this article.

- 86 1. The first result is a characterisation in terms of identities of the class  $\mathbf{EV}$  of essentially- $\mathbf{V}$   
 87 stamps given a set of identities  $E$  defining  $\mathbf{V}$ : a stamp is in  $\mathbf{EV}$  if and only if it satisfies  
 88 all identities  $x^\omega yuzt^\omega = x^\omega yvzt^\omega$  for  $u = v$  an identity in  $E$  and where  $x, y, z, t$  do appear  
 89 neither in  $u$  nor in  $v$ .
- 90 2. The second result says that  $\mathbf{EV}$  and  $\mathbf{V} \vee \mathbf{LI}$  do correspond if and only if  $\mathbf{V}$  verifies some  
 91 criterion, that can be formulated in terms of quotient-expressibility in  $\mathcal{L}(\mathbf{V})$ : any language  
 92  $L \in \mathcal{L}(\mathbf{V})$  must, for an arbitrary choice of  $x, y$ , be such that the quotient  $u^{-1}Lv^{-1}$  for  $u$   
 93 and  $v$  long enough can be expressed as the quotient  $(xu)^{-1}K(vy)^{-1}$  for some  $K \in \mathcal{L}(\mathbf{V})$ .

94 Using these results, we can find a set of identities defining the so-called *ne*-variety of stamps  
 95 corresponding to  $\mathbf{V} \vee \mathbf{LI}$  as soon as a set of identities defining  $\mathbf{V}$  is known by proving that  $\mathbf{V}$   
 96 verifies the criterion in point 2. Note that, for technical reasons, we work with the *ne*-variety  
 97 of stamps corresponding to  $\mathbf{V} \vee \mathbf{LI}$  rather than directly with the variety of finite semigroups  
 98  $\mathbf{V} \vee \mathbf{LI}$ , but this is not a problem since they correspond bijectively. We apply this method  
 99 to reprove characterisations of the join of  $\mathbf{LI}$  with each of the well-known varieties of finite  
 100 monoids  $\mathbf{R}$ ,  $\mathbf{L}$ ,  $\mathbf{J}$  and any variety of finite groups.

101 The author noticed after proving those results that his work actually forms a kind of  
 102 rediscovery of the work of J. C. Costa in [8]. He defines an operator  $U$  associating to each set  
 103 of identities  $E$  the exact same new set  $U(E)$  of identities as in point 1. Costa then defines a  
 104 property of cancellation for varieties of finite semigroups such that for any  $\mathbf{V}$  verifying it,  
 105  $U(E)$  defines  $\mathbf{V} \vee \mathbf{LI}$  for  $E$  defining  $\mathbf{V}$ . He finally uses this result to derive characterisations  
 106 of  $\mathbf{V} \vee \mathbf{LI}$  for all the cases we are treating in our paper and many more.

107 What is, then, the contribution of our article? In a nutshell, it does mainly use algebraic  
 108 and language-theoretic techniques while Costa's work relies heavily on profinite topology. In  
 109 our setting, once the stage is set, all proofs are quite straightforward without real difficulties  
 110 and rely on classical language-theoretic characterisations of the varieties under consideration.  
 111 This is to contrast with Costa's work, that for instance draws upon the difficult analysis of  
 112 the elements of free pro- $\mathbf{R}$  monoids by Almeida and Weil [4] to characterise  $\mathbf{R} \vee \mathbf{LI}$ .

113 **Organisation of the article.** Section 2 is dedicated to the necessary preliminaries. In  
 114 Section 3, we recall the definition of essentially- $\mathbf{V}$  stamps and prove the characterisation  
 115 by identities of point 1 above. Section 4 is then dedicated to the necessary and sufficient  
 116 criterion for  $\mathbf{EV}$  and  $\mathbf{V} \vee \mathbf{LI}$  to correspond presented in point 2 and finally those results are  
 117 applied to specific cases in Section 5. We finish with a short conclusion.

## 118 **2 Preliminaries**

119 We briefly introduce the mathematical material used in this paper. For the basics and the  
 120 classical results of automata theory, we refer the reader to the two classical references of the  
 121 domain by Eilenberg [10, 11] and Pin [13]. For definitions and results specific to varieties of  
 122 stamps and associated profinite identities, see the articles by Straubing [18] and by Pin and  
 123 Straubing [14]. We also assume some basic knowledge of topology.

124 **General notations.** Let  $i \in \mathbb{N}$  be a natural number. We shall denote by  $[i]$  the set of all  
 125 natural numbers  $n \in \mathbb{N}$  verifying  $1 \leq n \leq i$ .

126 **Words and languages.** Let  $\Sigma$  be a finite alphabet. We denote by  $\Sigma^*$  the set of all finite  
 127 words over  $\Sigma$ . We also denote by  $\Sigma^+$  the set of all finite non empty words over  $\Sigma$ , the empty  
 128 word being denoted by  $\varepsilon$ . All our alphabets and words will always be finite, without further  
 129 mention of this fact. Given some word  $w \in \Sigma^*$ , we denote its length by  $|w|$  and the set of  
 130 letters it contains by  $\text{alph}(w)$ . Given some  $n \in \mathbb{N}$ , we denote by  $\Sigma^{\geq n}$ ,  $\Sigma^n$  and  $\Sigma^{< n}$  the set of  
 131 words over  $\Sigma$  of length, respectively, at least  $n$ , exactly  $n$  and less than  $n$ .

132 A *language over  $\Sigma$*  is a subset of  $\Sigma^*$ . A language is *regular* if it is recognised by a  
 133 deterministic finite automaton. The *quotient of a language  $L$  over  $\Sigma$  relative to the words  $u$*   
 134 *and  $v$  over  $\Sigma$*  is the language, denoted by  $u^{-1}Lv^{-1}$ , of the words  $w$  such that  $uwv \in L$ .

135 **Monoids, semigroups and varieties.** A *semigroup* is a non-empty set equipped with an  
 136 associative law that we will write multiplicatively. A *monoid* is a semigroup with an identity.  
 137 An example of a semigroup is  $\Sigma^+$ , the free semigroup over  $\Sigma$ . Similarly  $\Sigma^*$  is the free monoid  
 138 over  $\Sigma$ . A *morphism*  $\varphi$  from a semigroup  $S$  to a semigroup  $T$  is a function from  $S$  to  $T$  such  
 139 that  $\varphi(xy) = \varphi(x)\varphi(y)$  for all  $x, y \in S$ . A morphism of monoids additionally requires that  
 140 the identity is preserved. A semigroup  $T$  is a *subsemigroup* of a semigroup  $S$  if  $T$  is a subset  
 141 of  $S$  and is equipped with the restricted law of  $S$ . Additionally the notion of submonoids  
 142 requires the presence of the identity. A semigroup  $T$  *divides* a semigroup  $S$  if  $T$  is the image  
 143 by a semigroup morphism of a subsemigroup of  $S$ . Division of monoids is defined in the same  
 144 way. The *Cartesian (or direct) product* of two semigroups is simply the semigroup given by  
 145 the Cartesian product of the two underlying sets equipped with the Cartesian product of  
 146 their laws. An element  $s$  of a semigroup is *idempotent* if  $ss = s$ .

147 A *variety of finite monoids* is a non-empty class of finite monoids closed under Cartesian  
 148 product and monoid division. A *variety of finite semigroups* is defined similarly. When  
 149 dealing with varieties, we consider only finite monoids and semigroups, so we will drop the  
 150 adjective finite when talking about varieties in the rest of this article.

151 **Varieties of stamps** Let  $f: \Sigma^* \rightarrow \Gamma^*$  be a morphism from the free monoid over an alphabet  
 152  $\Sigma$  to the free monoid over an alphabet  $\Gamma$ , that we might call an *all-morphism*. We say that  
 153  $f$  is an *ne-morphism* (non-erasing morphism) whenever  $f(\Sigma) \subseteq \Gamma^+$ .

154 We call *stamp* a surjective morphism  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite  
 155 monoid. We say that a stamp  $\varphi: \Sigma^* \rightarrow M$  *all-divides* (respectively *ne-divides*) a stamp  
 156  $\psi: \Gamma^* \rightarrow N$  whenever there exists an *all-morphism* (respectively *ne-morphism*)  $f: \Sigma^* \rightarrow \Gamma^*$   
 157 and a surjective morphism  $\alpha: \mathfrak{Im}(\psi \circ f) \rightarrow M$  such that  $\varphi = \alpha \circ \psi \circ f$ . The *direct product*  
 158 of two stamps  $\varphi: \Sigma^* \rightarrow M$  and  $\psi: \Gamma^* \rightarrow N$  is the stamp  $\varphi \times \psi: \Sigma^* \rightarrow K$  such that  $K$  is the  
 159 submonoid of  $M \times N$  generated by  $\{(\varphi(a), \psi(a)) \mid a \in \Sigma\}$  and  $\varphi \times \psi(a) = (\varphi(a), \psi(a))$  for  
 160 all  $a \in \Sigma$ .

161 An *all-variety of stamps* (respectively *ne-variety of stamps*) is a non-empty class of stamps  
 162 closed under direct product and *all-division* (respectively *ne-division*).

163 We will often use the following characteristic index of stamps, defined in [7]. Consider  
 164 some stamp  $\varphi: \Sigma^* \rightarrow M$ . As  $M$  is finite there is a  $k \in \mathbb{N}_{>0}$  such that  $\varphi(\Sigma^{2k}) = \varphi(\Sigma^k)$ : this  
 165 implies that  $\varphi(\Sigma^k)$  is a semigroup. The smallest such  $k$  is called the *stability index* of  $\varphi$ .

166 **Varieties of languages.** A language  $L$  over some alphabet  $\Sigma$  is *recognised by a monoid*  $M$   
 167 if there is a morphism  $\varphi: \Sigma^* \rightarrow M$  and  $F \subseteq M$  such that  $L = \varphi^{-1}(F)$ . We also say that  $\varphi$   
 168 *recognises*  $L$ . It is well known that a language is regular if and only if it is recognised by a  
 169 finite monoid. The *syntactic congruence* of  $L$ , denoted by  $\sim_L$ , is the equivalence relation  
 170 on  $\Sigma^*$  defined by  $u \sim_L v$  for  $u, v \in \Sigma^*$  whenever for all  $x, y \in \Sigma^*$ ,  $xuy \in L$  if and only if  
 171  $xvy \in L$ . The quotient  $\Sigma^*/\sim_L$  is a monoid, called *the syntactic monoid of  $L$* , that recognises  
 172  $L$  via the *syntactic morphism*  $\eta_L$  of  $L$  sending any word  $u$  to its equivalence class  $[u]_{\sim_L}$  for  
 173  $\sim_L$ . A stamp  $\varphi: \Sigma^* \rightarrow M$  recognises  $L$  if and only if there exists a surjective morphism  
 174  $\varphi: M \rightarrow \Sigma^*/\sim_L$  verifying  $\eta_L = \alpha \circ \varphi$ .

175 A *class of languages*  $\mathcal{C}$  is a correspondence that associates a set  $\mathcal{C}(\Sigma)$  to each alphabet  $\Sigma$ .  
 176 A (all-)variety of languages (respectively an *ne-variety of languages*)  $\mathcal{V}$  is a non-empty class  
 177 of regular languages closed under Boolean operations, quotients and inverses of *all-morphisms*  
 178 (respectively *ne-morphisms*). A classical result of Eilenberg [11, Chapter VII, Section 3]  
 179 says that there is a bijective correspondence between varieties of monoids and varieties of  
 180 languages: to each variety of monoids  $\mathbf{V}$  we can bijectively associate  $\mathcal{L}(\mathbf{V})$  the variety of

languages whose syntactic monoids belong to  $\mathbf{V}$ . This was generalised by Straubing [18] to the case of varieties of stamps: to each *all*-variety (respectively *ne*-variety) of stamps  $\mathbf{V}$  we can bijectively associate  $\mathcal{L}(\mathbf{V})$  the *all*-variety (respectively *ne*-variety) of languages whose syntactic morphisms belong to  $\mathbf{V}$ . Given two *all*-varieties (respectively *ne*-varieties) of stamps  $\mathbf{V}_1$  and  $\mathbf{V}_2$ , we have  $\mathbf{V}_1 \subseteq \mathbf{V}_2 \Leftrightarrow \mathcal{L}(\mathbf{V}_1) \subseteq \mathcal{L}(\mathbf{V}_2)$ .

For  $\mathbf{V}$  some variety of monoids, we define  $\langle \mathbf{V} \rangle_{all}$  the *all*-variety of all stamps  $\varphi: \Sigma^* \rightarrow M$  such that  $M \in \mathbf{V}$ . Of course, in that case  $\mathcal{L}(\mathbf{V}) = \mathcal{L}(\langle \mathbf{V} \rangle_{all})$ . Similarly, for  $\mathbf{V}$  some variety of semigroups, we define  $\langle \mathbf{V} \rangle_{ne}$  the *ne*-variety of all stamps  $\varphi: \Sigma^* \rightarrow M$  such that  $\varphi(\Sigma^+) \in \mathbf{V}$ . In that case, we consider  $\mathcal{L}(\mathbf{V})$  to be the *ne*-variety of languages corresponding to  $\langle \mathbf{V} \rangle_{ne}$ . The operations  $\langle \cdot \rangle_{all}$  and  $\langle \cdot \rangle_{ne}$  form bijective correspondences (see [18]).

**Identities.** Every finite monoid  $M$  is considered to be a complete metric space equipped with the discrete metric  $d$  defined by  $d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{otherwise} \end{cases}$  for all  $m, n \in M$ .

Let  $\Sigma$  be an alphabet. Given  $u, v \in \Sigma^*$ , we set

$$r(u, v) = \min\{|M| \mid \exists \varphi: \Sigma^* \rightarrow M \text{ stamp s.t. } \varphi(u) \neq \varphi(v)\}$$

and  $d(u, v) = 2^{-r(u, v)}$ , using the conventions that  $\min \emptyset = +\infty$  and  $2^{-\infty} = 0$ . Then  $d$  is a metric on  $\Sigma^*$ . The completion of the metric space  $(\Sigma^*, d)$ , denoted by  $(\widehat{\Sigma^*}, \widehat{d})$ , is a metric monoid called the *free profinite monoid on  $\Sigma^*$* . Its elements are all the formal limits  $\lim_{n \rightarrow \infty} x_n$  of Cauchy sequences  $(x_n)_{n \geq 0}$  in  $(\Sigma^*, d)$  and the metric  $d$  on  $\Sigma^*$  extends to a metric  $\widehat{d}$  on  $\widehat{\Sigma^*}$  defined by  $\widehat{d}(\lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n) = \lim_{n \rightarrow \infty} d(x_n, y_n)$  for Cauchy sequences  $(x_n)_{n \geq 0}$  and  $(y_n)_{n \geq 0}$  in  $(\Sigma^*, d)$ . Note that, when it is clear from the context, we usually do not make the metric explicit when talking about a metric space. One important example of elements of  $\widehat{\Sigma^*}$  is given by the elements  $x^\omega = \lim_{n \rightarrow \infty} x^{n!}$  for all  $x \in \Sigma^*$ .

Every stamp  $\varphi: \Sigma^* \rightarrow M$  extends uniquely to a uniformly continuous morphism  $\widehat{\varphi}: \widehat{\Sigma^*} \rightarrow M$  with  $\widehat{\varphi}(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} \varphi(x_n)$  for every Cauchy sequence  $(x_n)_{n \geq 0}$  in  $\Sigma^*$ . Similarly, every *all*-morphism  $f: \Sigma^* \rightarrow \Gamma^*$  extends uniquely to a uniformly continuous morphism  $\widehat{f}: \widehat{\Sigma^*} \rightarrow \widehat{\Gamma^*}$  with  $\widehat{f}(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} f(x_n)$  for every Cauchy sequence  $(x_n)_{n \geq 0}$  in  $\Sigma^*$ .

For  $u, v \in \widehat{A^*}$  with  $A$  an alphabet, we say that a stamp  $\Sigma^* \rightarrow M$  *all-satisfies* (respectively *ne-satisfies*) the identity  $u = v$  if for every *all*-morphism (respectively *ne*-morphism)  $f: A^* \rightarrow \Sigma^*$ , it holds that  $\widehat{\varphi} \circ \widehat{f}(u) = \widehat{\varphi} \circ \widehat{f}(v)$ . Given some set of identities  $E$ , we denote by  $\llbracket E \rrbracket_{all}$  (respectively  $\llbracket E \rrbracket_{ne}$ ) the class of stamps *all*-satisfying (respectively *ne*-satisfying) all the identities of  $E$ . When  $\llbracket E \rrbracket_{all}$  (respectively  $\llbracket E \rrbracket_{ne}$ ) is equal to some *all*-variety (respectively *ne*-variety) of stamps  $\mathbf{V}$ , we say that  $E$  *all-defines* (respectively *ne-defines*)  $\mathbf{V}$ .

► **Theorem 1.** *A class of stamps is an all-variety (respectively ne-variety) of stamps if and only if it can be all-defined (respectively ne-defined) by a set of identities.*

To give some examples, the classical varieties of monoids  $\mathbf{J}$ ,  $\mathbf{R}$  and  $\mathbf{L}$  can be characterised by identities in the following way:

$$\langle \mathbf{R} \rangle_{all} = \llbracket (ab)^\omega a = (ab)^\omega \rrbracket_{all} = \llbracket (ab)^\omega a = (ab)^\omega \rrbracket_{ne}$$

$$\langle \mathbf{L} \rangle_{all} = \llbracket b(ab)^\omega = (ab)^\omega \rrbracket_{all} = \llbracket b(ab)^\omega = (ab)^\omega \rrbracket_{ne}$$

$$\langle \mathbf{J} \rangle_{all} = \llbracket (ab)^\omega a = (ab)^\omega, b(ab)^\omega = (ab)^\omega \rrbracket_{all} = \llbracket (ab)^\omega a = (ab)^\omega, b(ab)^\omega = (ab)^\omega \rrbracket_{ne}.$$

**Finite locally trivial semigroups and the join operation.** The variety  $\mathbf{LI}$  of finite locally trivial semigroups is well-known to verify  $\langle \mathbf{LI} \rangle_{ne} = \llbracket x^\omega y x^\omega = x^\omega \rrbracket_{ne}$  and to be such that for

223 any alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{LI})(\Sigma)$  contains exactly all Boolean combinations of languages of  
 224 the form  $u\Sigma^*$  or  $\Sigma^*u$  for  $u \in \Sigma^*$ , or equivalently exactly all languages of the form  $U\Sigma^*V \cup W$   
 225 with  $U, V, W \subseteq \Sigma^*$  finite (see [13, p. 38]).

226 Given some variety of monoids  $\mathbf{V}$ , the join of  $\mathbf{V}$  and  $\mathbf{LI}$ , denoted by  $\mathbf{V} \vee \mathbf{LI}$ , is the  
 227 inclusion-wise smallest variety of semigroups containing both  $\mathbf{V}$  and  $\mathbf{LI}$ . In fact, a finite  
 228 semigroup  $S$  belongs to  $\mathbf{V} \vee \mathbf{LI}$  if and only if there exist  $M \in \mathbf{V}$  and  $T \in \mathbf{LI}$  such that  
 229  $S$  divides the semigroup  $M \times T$ . (See [11, Chapter V, Exercise 1.1].) We can prove the  
 230 following (see the appendix for the proof).

231 ► **Proposition 2.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{\text{ne}}$  is the inclusion-wise  
 232 smallest ne-variety of stamps containing both  $\langle \mathbf{V} \rangle_{\text{all}}$  and  $\langle \mathbf{LI} \rangle_{\text{ne}}$ . Moreover,  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  is the  
 233 inclusion-wise smallest ne-variety of languages containing both  $\mathcal{L}(\mathbf{V})$  and  $\mathcal{L}(\mathbf{LI})$  and verifies  
 234 that  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  is the Boolean closure of  $\mathcal{L}(\mathbf{V})(\Sigma) \cup \mathcal{L}(\mathbf{LI})(\Sigma)$  for each alphabet  $\Sigma$ .*

### 235 3 Essentially- $\mathbf{V}$ stamps

236 In this section, we give a characterisation of essentially- $\mathbf{V}$  stamps (defined in [12]), for  $\mathbf{V}$  a  
 237 variety of monoids, in terms of identities. We first recall the definition.

238 ► **Definition 3.** *Let  $\mathbf{V}$  be a variety of monoids. Let  $\varphi: \Sigma^* \rightarrow M$  be a stamp from an alphabet  
 239  $\Sigma$  to a finite monoid  $M$ . Let  $s$  be the stability index of  $\varphi$ .*

240 *We say that  $\varphi$  is essentially- $\mathbf{V}$  whenever there exists a stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$   
 241 such that for all  $u, v \in \Sigma^*$ , we have*

$$242 \quad \mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

243 *We will denote by  $\mathbf{EV}$  the class of all essentially- $\mathbf{V}$  stamps.*

244 Now, we give an equivalent condition for a stamp to be essentially- $\mathbf{V}$ , based on a specific  
 245 congruence depending on that stamp.

246 ► **Definition 4.** *Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp and let  
 247  $s$  be its stability index. We define the equivalence relation  $\equiv_\varphi$  on  $\Sigma^*$  by  $u \equiv_\varphi v$  for  $u, v \in \Sigma^*$   
 248 whenever  $\varphi(xuy) = \varphi(xvy)$  for all  $x, y \in \Sigma^{\geq s}$ .*

249 ► **Proposition 5.** *Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp.  
 250 Then  $\equiv_\varphi$  is a congruence of finite index and for any variety of monoids  $\mathbf{V}$ , we have  $\varphi \in \mathbf{EV}$   
 251 if and only if  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ .*

252 **Proof.** Let us denote by  $s$  the stability index of  $\varphi$ .

253 The equivalence relation  $\equiv_\varphi$  is a congruence because given  $u, v \in \Sigma^*$  verifying  $u \equiv_\varphi v$ ,  
 254 for all  $\alpha, \beta \in \Sigma^*$ , we have  $\alpha u \beta \equiv_\varphi \alpha v \beta$  since for any  $x, y \in \Sigma^{\geq s}$ , it holds that  $\varphi(x\alpha u \beta y) =$   
 255  $\varphi(x\alpha v \beta y)$  because  $x\alpha, \beta y \in \Sigma^{\geq s}$ . Furthermore, this congruence is of finite index because for  
 256 all  $u, v \in \Sigma^*$ , we have that  $\varphi(u) = \varphi(v)$  implies  $u \equiv_\varphi v$ .

257 Let now  $\mathbf{V}$  be a variety of monoids. Assume first that  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ . It is quite direct to  
 258 see that  $\varphi \in \mathbf{EV}$ , as the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$   
 259 witnesses this fact. Assume then that  $\varphi \in \mathbf{EV}$ . This means that there exists a stamp  
 260  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$261 \quad \mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

262 Now consider  $u, v \in \Sigma^*$  such that  $\mu(u) = \mu(v)$ . For any  $x, y \in \Sigma^{\geq s}$ , we have that  $x = x_1x_2$   
 263 with  $x_1 \in \Sigma^*$  and  $x_2 \in \Sigma^s$  as well as  $y = y_1y_2$  with  $y_1 \in \Sigma^s$  and  $y_2 \in \Sigma^*$ , so that



264  $\varphi(xuy) = \varphi(x_1)\varphi(x_2uy_1)\varphi(y_2) = \varphi(x_1)\varphi(x_2vy_1)\varphi(y_2) = \varphi(xvy)$ . Hence,  $u \equiv_\varphi v$ . Therefore,  
 265 for all  $u, v \in \Sigma^*$ , we have that  $\mu(u) = \mu(v)$  implies  $u \equiv_\varphi v$ , so we can define the mapping  
 266  $\alpha: N \rightarrow \Sigma^*/\equiv_\varphi$  such that  $\alpha(\mu(w)) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . It is easy to check that  $\alpha$   
 267 is actually a surjective morphism. Thus, we can conclude that  $\Sigma^*/\equiv_\varphi$ , which divides  $N$ ,  
 268 belongs to  $\mathbf{V}$ .  $\blacktriangleleft$

269 Using this equivalent condition, we prove that given a set of identities  $ne$ -defining  $\langle \mathbf{V} \rangle_{all}$   
 270 for some variety of monoids  $\mathbf{V}$ , we get a set of identities  $ne$ -defining  $\mathbf{EV}$ , which actually  
 271 shows that  $\mathbf{EV}$  is an  $ne$ -variety of stamps along the way.

272 **► Proposition 6.** *Let  $\mathbf{V}$  be a variety of monoids and let  $E$  be a set of identities such that*  
 273  $\langle \mathbf{V} \rangle_{all} = \llbracket E \rrbracket_{ne}$ . *Then*

$$274 \quad \mathbf{EV} = \llbracket x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \text{alph}(u) \cup \text{alph}(v) \rrbracket_{ne} .$$

275 **Proof.** Let

$$276 \quad F = \{x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \text{alph}(u) \cup \text{alph}(v)\} .$$

277 Central to the proof is the following claim.

278  $\triangleright$  **Claim 7.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp. Consider  
 279 the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . It holds that for all  
 280  $u, v \in \widehat{\Sigma}^*$ ,

$$281 \quad \widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow (\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+) .$$

282 Using this claim, we can prove that  $\mathbf{EV} = \llbracket F \rrbracket_{ne}$ .

283 **Inclusion from left to right.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid  
 284 be a stamp in  $\mathbf{EV}$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  
 285  $w \in \Sigma^*$ . Since  $\varphi \in \mathbf{EV}$ , Proposition 5 tells us that  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ , hence  $\mu \in \langle \mathbf{V} \rangle_{all}$ .

286 Let us consider any identity  $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$ . It is thus defined on an alphabet  
 287  $B$  with  $u = v \in E$  defined on an alphabet  $A \subseteq B$  and  $x, y, z, t \in B \setminus A$ . Let  $f: B^* \rightarrow \Sigma^*$  be  
 288 an  $ne$ -morphism. Since  $\mu \in \langle \mathbf{V} \rangle_{all}$ , we have that  $\mu$   $ne$ -satisfies the identity  $u = v$ , so that  
 289  $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$ . Notice that we have that  $\widehat{f}(x^\omega) = f(x)^\omega$  as well as  $\widehat{f}(t^\omega) = f(t)^\omega$  and  
 290 that  $f(x), f(y), f(z), f(t) \in \Sigma^+$  because  $f$  is non-erasing. Therefore, we have

$$291 \quad \begin{aligned} \widehat{\varphi}(\widehat{f}(x^\omega yuzt^\omega)) &= \widehat{\varphi}(f(x)^\omega f(y)\widehat{f}(u)f(z)f(t)^\omega) \\ 292 \quad &= \widehat{\varphi}(f(x)^\omega f(y)\widehat{f}(v)f(z)f(t)^\omega) \\ 293 \quad &= \widehat{\varphi}(\widehat{f}(x^\omega yvzt^\omega)) \end{aligned}$$

295 by Claim 7. As this holds for any  $ne$ -morphism  $f: B^* \rightarrow \Sigma^*$ , we can conclude that  $\varphi$   
 296  $ne$ -satisfies the identity  $x^\omega yuzt^\omega = x^\omega yvzt^\omega$ .

297 This is true for any identity in  $F$ , so  $\varphi \in \llbracket F \rrbracket_{ne}$ . In conclusion,  $\mathbf{EV} \subseteq \llbracket F \rrbracket_{ne}$ .

298 **Inclusion from right to left.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid  
 299 be a stamp in  $\llbracket F \rrbracket_{ne}$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  
 300  $w \in \Sigma^*$ . We are now going to show that  $\mu \in \langle \mathbf{V} \rangle_{all}$ .

301 Take any identity  $u = v \in E$  defined on an alphabet  $A$ . There is thus an identity  
 302  $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$  defined on an alphabet  $B$  such that  $A \subseteq B$  and  $x, y, z, t \in B \setminus A$ .  
 303 Let  $f: A^* \rightarrow \Sigma^*$  be an  $ne$ -morphism.



304 Take any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . Let us define the *ne*-morphism  $g: B^* \rightarrow \Sigma^*$  as the unique one  
305 such that for all  $b \in B$ ,

$$306 \quad g(b) = \begin{cases} f(b) & \text{if } b \in A \\ \alpha & \text{if } b = x \\ \beta & \text{if } b = y \\ \gamma & \text{if } b = z \\ \delta & \text{if } b = t \end{cases} .$$

307 Observe in particular that  $\widehat{g}(w) = \widehat{f}(w)$  for any  $w \in \widehat{A}^*$  and that  $\widehat{g}(x^\omega) = g(x)^\omega = \alpha^\omega$  as  
308 well as  $\widehat{g}(t^\omega) = \delta^\omega$ . Now, as  $\varphi$  *ne*-satisfies  $x^\omega y u z t^\omega = x^\omega y v z t^\omega$ , we have that

$$309 \quad \widehat{\varphi}(\alpha^\omega \beta \widehat{f}(u) \gamma \delta^\omega) = \widehat{\varphi}(\widehat{g}(x^\omega y u z t^\omega)) = \widehat{\varphi}(\widehat{g}(x^\omega y v z t^\omega)) = \widehat{\varphi}(\alpha^\omega \beta \widehat{f}(v) \gamma \delta^\omega) .$$

310 Since this holds for any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ , by Claim 7, we have that  $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$ .

311 Therefore,  $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$  for any *ne*-morphism  $f: A^* \rightarrow \Sigma^*$ , which means that  $\mu$   
312 *ne*-satisfies  $u = v$ .

313 So, to conclude, since this is true for any  $u = v \in E$ , we have that  $\mu \in \langle \mathbf{V} \rangle_{all}$ , which  
314 implies that  $\Sigma^* / \equiv_\varphi \in \mathbf{V}$  and thus  $\varphi \in \mathbf{EV}$  by Proposition 5. In conclusion,  $\llbracket F \rrbracket_{ne} \subseteq \mathbf{EV}$ .

315 The claim still needs to be proved.

316 Proof of Claim 7. Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp  
317 of stability index  $s$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^* / \equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  
318  $w \in \Sigma^*$ . We now want to show that for all  $u, v \in \widehat{\Sigma}^*$ ,

$$319 \quad \widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow (\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+) .$$

320 Let  $u, v \in \widehat{\Sigma}^*$ . There exist two Cauchy sequences  $(u_n)_{n \geq 0}$  and  $(v_n)_{n \geq 0}$  in  $\Sigma^*$  such that  
321  $u = \lim_{n \rightarrow \infty} u_n$  and  $v = \lim_{n \rightarrow \infty} v_n$ . As  $\Sigma^* / \equiv_\varphi$  and  $M$  are discrete, we have that all  
322 four Cauchy sequences  $(\mu(u_n))_{n \geq 0}$ ,  $(\varphi(u_n))_{n \geq 0}$ ,  $(\mu(v_n))_{n \geq 0}$  and  $(\varphi(v_n))_{n \geq 0}$  are ultimately  
323 constant. So there exists  $k \in \mathbb{N}$  such that  $\widehat{\mu}(u) = \mu(u_k)$ ,  $\widehat{\varphi}(u) = \varphi(u_k)$ ,  $\widehat{\mu}(v) = \mu(v_k)$  and  
324  $\widehat{\varphi}(v) = \varphi(v_k)$ .

325 Assume first that  $\widehat{\mu}(u) = \widehat{\mu}(v)$ . Take any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . Since  $M$  is discrete, both  
326 Cauchy sequences  $(\varphi(\alpha^{n!}))_{n \geq 0}$  and  $(\varphi(\delta^{n!}))_{n \geq 0}$  are ultimately constant. So there exists  
327  $l \in \mathbb{N}$  such that for all  $m \in \mathbb{N}, m \geq l$ , we have  $\widehat{\varphi}(\alpha^\omega) = \varphi(\alpha^{m!})$  and  $\widehat{\varphi}(\delta^\omega) = \varphi(\delta^{m!})$ . Hence,  
328 taking some  $m \in \mathbb{N}, m \geq l$  such that  $|\alpha^{m!} \beta| \geq s$  and  $|\gamma \delta^{m!}| \geq s$ , it follows that

$$329 \quad \widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \varphi(\alpha^{m!} \beta u_k \gamma \delta^{m!}) = \varphi(\alpha^{m!} \beta v_k \gamma \delta^{m!}) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega)$$

330 because  $[u_k]_{\equiv_\varphi} = \widehat{\mu}(u) = \widehat{\mu}(v) = [v_k]_{\equiv_\varphi}$ . Thus, we have that

$$331 \quad \widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega)$$

332 for all  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ .

333 Assume then that  $\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega)$  for all  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . Take any  $\alpha, \beta \in$   
334  $\Sigma^{\geq s}$ . Since  $\varphi(\Sigma^s)$  is a finite semigroup and verifies that  $\varphi(\Sigma^s) = \varphi(\Sigma^s)^2$ , by a classical result  
335 in finite semigroup theory (see e.g. [13, Chapter 1, Proposition 1.12]), we have that there  
336 exist  $\alpha_1, e, f, \beta_2 \in \Sigma^s$  and  $\alpha_2, \beta_1 \in \Sigma^{\geq s}$  such that  $\varphi(\alpha_1 e \alpha_2) = \varphi(\alpha)$  and  $\varphi(\beta_1 f \beta_2) = \varphi(\beta)$   
337 with  $\varphi(e)$  and  $\varphi(f)$  idempotents. Now, since  $\varphi(e)$  is idempotent, we have that

$$338 \quad \widehat{\varphi}(e^\omega) = \widehat{\varphi}(\lim_{n \rightarrow \infty} e^{n!}) = \lim_{n \rightarrow \infty} \varphi(e^{n!}) = \lim_{n \rightarrow \infty} \varphi(e)^{n!} = \varphi(e)$$

339 and similarly,  $\widehat{\varphi}(f^\omega) = \varphi(f)$ . So it follows that

$$\begin{aligned}
 340 \quad \varphi(\alpha u_k \beta) &= \varphi(\alpha_1 e \alpha_2 u_k \beta_1 f \beta_2) \\
 341 \quad &= \widehat{\varphi}(\alpha_1 e^\omega \alpha_2 u \beta_1 f^\omega \beta_2) \\
 342 \quad &= \widehat{\varphi}(\alpha_1 e^\omega \alpha_2 v \beta_1 f^\omega \beta_2) \\
 343 \quad &= \varphi(\alpha_1 e \alpha_2 v_k \beta_1 f \beta_2) \\
 344 \quad &= \varphi(\alpha v_k \beta) . \\
 345
 \end{aligned}$$

346 As this is true for any  $\alpha, \beta \in \Sigma^{\geq s}$ , by definition it holds that  $u_k \equiv_\varphi v_k$ , hence  $\widehat{\mu}(u) = \mu(u_k) =$   
 347  $\mu(v_k) = \widehat{\mu}(v)$ .  $\triangleleft$

348 This concludes the proof of the proposition.  $\blacktriangleleft$

#### 349 **4 Essentially-V stamps and the join of V and LI**

350 In this section, we establish the link between essentially-V stamps and  $\mathbf{V} \vee \mathbf{LI}$  and give a  
 351 criterion that characterises exactly when they do correspond.

352 More precisely, consider the following criterion for a variety of monoids  $\mathbf{V}$ .

353 **► Criterion (A).** For any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  with  $\Sigma$  some alphabet, we have  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$   
 354 for all  $x, y \in \Sigma^*$ .

355 It is a kind of mild closure condition that appears to be a sufficient and necessary condition  
 356 for  $\mathbf{EV}$  and  $\mathbf{V} \vee \mathbf{LI}$  to correspond.

357 **► Proposition 8.** Let  $\mathbf{V}$  be a variety of monoids. Then  $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{\text{ne}} \subseteq \mathbf{EV}$  and there is  
 358 equality if and only if  $\mathbf{V}$  verifies criterion (A).

359 Why this proposition is actually useful to give characterisations of  $\mathbf{V} \vee \mathbf{LI}$  in terms of  
 360 identities will become clear in the next section. For now, we focus on its proof, that entirely  
 361 relies on the following characterisation of the languages recognised by essentially-V stamps.

362 **► Proposition 9.** Let  $\mathbf{V}$  be a variety of monoids. For any alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{EV})(\Sigma)$   
 363 contains exactly all Boolean combinations of languages of the form  $xLy$  for  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$   
 364 and  $x, y \in \Sigma^*$ .

365 **Proof.** Let  $\mathcal{C}$  be the class of languages such that for any alphabet  $\Sigma$ , the set  $\mathcal{C}(\Sigma)$  contains  
 366 exactly all Boolean combinations of languages of the form  $xLy$  for  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  and  $x, y \in \Sigma^*$ .

367 Let  $\Sigma$  be an alphabet. We are now going to show that  $\mathcal{L}(\mathbf{EV})(\Sigma) = \mathcal{C}(\Sigma)$ , which proves  
 368 the proposition as it holds for any alphabet  $\Sigma$ .

369 **Inclusion from right to left.** Let  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  and  $x, y \in \Sigma^*$ . Let  $\mu: \Sigma^* \rightarrow N$  be the  
 370 syntactic morphism of  $L$ : this means that  $N \in \mathbf{V}$  and that there exists  $F \subseteq N$  such  
 371 that  $L = \mu^{-1}(F)$ . Let also  $\varphi: \Sigma^* \rightarrow M$  be the syntactic morphism of the language  
 372  $xLy = x\Sigma^*y \cap \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|}$  and let  $s$  be its stability index. We then consider some  
 373  $u, v \in \Sigma^*$  such that  $\mu(u) = \mu(v)$ . Take any  $x', y' \in \Sigma^*$  such that  $|x'| \geq |x|$  and  $|y'| \geq |y|$ . We  
 374 clearly have that  $x'uy' \in x\Sigma^*y$  if and only if  $x'vy' \in x\Sigma^*y$ . Moreover,  $x' = x'_1x'_2$  for some  
 375  $x'_1 \in \Sigma^{|x|}$  and  $x'_2 \in \Sigma^*$  and  $y' = y'_1y'_2$  for some  $y'_1 \in \Sigma^*$  and  $y'_2 \in \Sigma^{|y|}$ , so that

$$\begin{aligned}
 376 \quad x'uy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|} &\Leftrightarrow \mu(x'_2uy'_1) \in F \\
 377 \quad &\Leftrightarrow \mu(x'_2vy'_1) \in F \\
 378 \quad &\Leftrightarrow x'vy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|} .
 \end{aligned}$$

380 Hence,  $x'uy' \in xLy$  if and only if  $x'vy' \in xLy$  for all  $x', y' \in \Sigma^*$  such that  $|x'| \geq |x|$  and  
 381  $|y'| \geq |y|$ , so that, by definition of the stability index  $s$  of  $\varphi$  and as  $\varphi$  is the syntactic morphism  
 382 of  $xLy$ , we have  $\varphi(x'uy') = \varphi(x'vy')$  for all  $x', y' \in \Sigma^s$ . Thus, it follows that  $\varphi \in \mathbf{EV}$ .

383 This implies that  $xLy \in \mathcal{L}(\mathbf{EV})(\Sigma)$ . Therefore, since this is true for any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$   
 384 and  $x, y \in \Sigma^*$  and since  $\mathcal{L}(\mathbf{EV})(\Sigma)$  is closed under Boolean operations, we can conclude that  
 385  $\mathcal{C}(\Sigma) \subseteq \mathcal{L}(\mathbf{EV})(\Sigma)$ .

386 **Inclusion from left to right.** Let  $L \in \mathcal{L}(\mathbf{EV})(\Sigma)$  and let  $\varphi: \Sigma^* \rightarrow M$  be its syntactic  
 387 morphism: it is an essentially- $\mathbf{V}$  stamp. Given  $s$  its stability index, this means there exists a  
 388 stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$389 \quad \mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

390 For each  $m \in N$  and  $x, y \in \Sigma^s$  consider the language  $x\mu^{-1}(m)y$ : it verifies that for any  
 391 two words  $w, w' \in x\mu^{-1}(m)y$ , we have  $w = xuy$  and  $w' = xvy$  with  $\mu(u) = \mu(v) = m$ , so  
 392 that  $\varphi(w) = \varphi(w')$ . By definition of the syntactic morphism, this means that for all  $m \in N$   
 393 and  $x, y \in \Sigma^s$ , either  $x\mu^{-1}(m)y \subseteq L$  or  $x\mu^{-1}(m)y \cap L = \emptyset$ . Therefore, there exists a set  
 394  $E \subseteq N \times \Sigma^s \times \Sigma^s$  such that  $L \cap \Sigma^{\geq 2s} = \bigcup_{(m,x,y) \in E} x\mu^{-1}y$ , hence

$$395 \quad L = \bigcup_{(m,x,y) \in E} x\mu^{-1}(m)y \cup F$$

396 for some  $F \subseteq \Sigma^{< 2s}$ .

397 Take some  $w \in F$ . We have that  $\{w\} = w\Sigma^* \cap \bigcap_{a \in \Sigma} (wa\Sigma^*)^c$  with necessarily  $\Sigma^* \in$   
 398  $\mathcal{L}(\mathbf{V})(\Sigma)$ . Thus, the singleton language  $\{w\}$  belongs to  $\mathcal{C}(\Sigma)$  and since this is true for any  
 399  $w \in F$  and  $F$  is finite, we can deduce from this that  $F$  is in  $\mathcal{C}(\Sigma)$ , as the latter is trivially  
 400 closed under Boolean operations.

401 Now, for all  $m \in N$ , the language  $\mu^{-1}(m)$  belongs to  $\mathcal{L}(\mathbf{V})(\Sigma)$ , so we finally have  $L \in \mathcal{C}(\Sigma)$ .  
 402 This is true for any  $L \in \mathcal{L}(\mathbf{EV})(\Sigma)$ , so in conclusion,  $\mathcal{L}(\mathbf{EV})(\Sigma) \subseteq \mathcal{C}(\Sigma)$ .  $\blacktriangleleft$

403 Proposition 8 then follows from the two next lemmata, that are both easy consequences  
 404 of Proposition 9. For completeness, we give the proofs in the appendix.

405 **► Lemma 10.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{\text{ne}} \subseteq \mathbf{EV}$ .*

406 **► Lemma 11.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $\mathbf{EV} \subseteq \langle \mathbf{V} \vee \mathbf{LI} \rangle_{\text{ne}}$  if and only if  $\mathbf{V}$   
 407 verifies criterion (A).*

## 408 5 Applications

409 In this last section, we use the link between essentially- $\mathbf{V}$  stamps and  $\mathbf{V} \vee \mathbf{LI}$  to reprove some  
 410 characterisations of joins between  $\mathbf{LI}$  and some well-known varieties of monoids in terms of  
 411 identities.

412 One thing that seems at a first glance a bit problematic about proving that some variety  
 413 of monoids  $\mathbf{V}$  satisfies criterion (A) when one's goal is precisely to characterise  $\mathbf{V} \vee \mathbf{LI}$  is  
 414 that one needs to prove that some languages belong to  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ , a class about which one  
 415 a priori only knows what is given by Proposition 2. But understanding  $\mathcal{L}(\mathbf{V})$  enough is  
 416 actually sufficient: if it holds that given any language  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and  
 417 any  $x, y \in \Sigma^*$ , there exists some language  $K \in \mathcal{L}(\mathbf{V})(\Sigma)$  such that  $L$  is equal to the quotient  
 418  $x^{-1}Ky^{-1}$ , then  $\mathbf{V}$  verifies criterion (A). We don't know whether this quotient-expressibility  
 419 condition that solely depends on the variety  $\mathbf{V}$  (without explicit reference to  $\mathbf{LI}$ ) is actually

420 equivalent to it satisfying criterion (A), but we can prove such an equivalence for a weaker  
421 quotient-expressibility condition for  $\mathbf{V}$ . The proof is to be found in the appendix.

422 ► **Proposition 12.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $\mathbf{V}$  satisfies criterion (A) if and*  
423 *only if for any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and  $x, y \in \Sigma^*$ , there exist  $k, l \in \mathbb{N}$*   
424 *such that for all  $u \in \Sigma^k, v \in \Sigma^l$ , there exists some language  $K \in \mathcal{L}(\mathbf{V})(\Sigma)$  verifying*  
425  *$u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$ .*

426 This quotient-expressibility condition appears to be particularly useful to prove that a  
427 variety of monoids  $\mathbf{V}$  does not satisfy criterion (A) without needing to understand what  
428  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  is. We demonstrate this for the variety of finite commutative and idempotent  
429 monoids  $\mathbf{J}_1$ .

430 ► **Proposition 13.**  *$\mathbf{J}_1$  does not satisfy criterion (A).*

431 **Proof.** Given an alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{J}_1)(\Sigma)$  contains exactly all Boolean combinations of  
432 languages of the form  $\Sigma^*a\Sigma^*$  for  $a \in \Sigma$  (see [13, Chapter 2, Proposition 3.10]).

433 Let  $L = \{a, b\}^*b\{a, b\}^* \in \mathcal{L}(\mathbf{J}_1)(\{a, b\})$  and  $x = b, y = \varepsilon$ . Take any  $k, l \in \mathbb{N}$  and set  
434  $u = a^k$  and  $v = a^l$ . Consider some  $K \in \mathcal{L}(\mathbf{J}_1)(\{a, b\})$ . We have that  $xuavy \in K \Leftrightarrow$   
435  $xuabvy \in K$  so that  $a \in (xu)^{-1}K(vy)^{-1} \Leftrightarrow ab \in (xu)^{-1}K(vy)^{-1}$ . But  $a \notin u^{-1}Lv^{-1}$   
436 and  $ab \in u^{-1}Lv^{-1}$ , hence  $u^{-1}Lv^{-1} \neq (xu)^{-1}K(vy)^{-1}$  and this holds for any choice of  $K$ .  
437 So for any  $k, l \in \mathbb{N}$ , there exists  $u \in \Sigma^k, v \in \Sigma^l$  such that no  $K \in \mathcal{L}(\mathbf{J}_1)(\{a, b\})$  verifies  
438  $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$ .

439 In conclusion, by Proposition 12,  $\mathbf{J}_1$  does not satisfy criterion (A). ◀

440 We now prove the announced characterisations of joins between  $\mathbf{LI}$  and some well-known  
441 varieties of monoids in terms of identities.

442 ► **Theorem 14.** *We have the following.*

- 443 1.  $\langle \mathbf{R} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{ER} = \llbracket x^\omega y(ab)^\omega azt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$ .
- 444 2.  $\langle \mathbf{L} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EL} = \llbracket x^\omega yb(ab)^\omega zt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$ .
- 445 3.  $\langle \mathbf{J} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EJ} = \llbracket x^\omega y(ab)^\omega azt^\omega = x^\omega y(ab)^\omega zt^\omega, x^\omega yb(ab)^\omega zt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$ .
- 446 4.  $\langle \mathbf{H} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EH}$  for any variety of groups  $\mathbf{H}$ .

447 **Proof.** In each case, we prove that the variety of monoids under consideration satisfies  
448 criterion (A) using Proposition 12. We then use Proposition 8.

449 **Proof of 1.** It is well-known that given an alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{R})(\Sigma)$  contains all  
450 languages that are disjoint unions of languages that are of the form  $A_0^*a_1A_1^* \cdots a_kA_k^*$  where  
451  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma$  and  $a_i \notin A_{i-1}$  for all  $i \in [k]$  (see [13, Chapter 4,  
452 Theorem 3.3]).

453 Let  $\Sigma$  be an alphabet and take some language  $A_0^*a_1A_1^* \cdots a_kA_k^*$  where  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in$   
454  $\Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma$  and  $a_i \notin A_{i-1}$  for all  $i \in [k]$ . Take  $x, y \in \Sigma^*$ . Observe that  $y$  can be  
455 uniquely written as  $y = zt$  where  $z \in A_k^*$  and  $t \in \{\varepsilon\} \cup (\Sigma \setminus A_k)\Sigma^*$ . We have

$$456 A_0^*a_1A_1^* \cdots a_kA_k^* = x^{-1} \left( xA_0^*a_1A_1^* \cdots a_kA_k^*t \cap \bigcap_{v \in A_k^{<|z|}} (xA_0^*a_1A_1^* \cdots a_kvt) \right) y^{-1}$$

457 using the convention that  $xA_0^*a_1A_1^* \cdots a_kvt = xvt$  for all  $v \in A_k^{<|z|}$  when  $k = 0$ . The language  
458  $xA_0^*a_1A_1^* \cdots a_kA_k^*t \cap \bigcap_{v \in A_k^{<|z|}} (xA_0^*a_1A_1^* \cdots a_kvt)$  does belong to the set  $\mathcal{L}(\mathbf{R})(\Sigma)$  because  
459 the latter is closed under Boolean operations and by definition of  $z$  and  $t$ . Thus, we can  
460 conclude that for each  $L \in \mathcal{L}(\mathbf{R})(\Sigma)$  and  $x, y \in \Sigma^*$ , there exists  $K \in \mathcal{L}(\mathbf{R})(\Sigma)$  such that  
461  $L = x^{-1}Ky^{-1}$  by using the characterisation of  $\mathcal{L}(\mathbf{R})(\Sigma)$ , the fact that quotients commute  
462 with unions [13, p. 20] and closure of  $\mathcal{L}(\mathbf{R})(\Sigma)$  under unions.

463 **Proof of 2.** It is also well-known that given an alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{L})(\Sigma)$  contains all  
 464 languages that are disjoint unions of languages that are of the form  $A_0^*a_1A_1^*\cdots a_kA_k^*$  where  
 465  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma$  and  $a_i \notin A_i$  for all  $i \in [k]$  (see [13, Chapter 4,  
 466 Theorem 3.4]). The proof is then dual to the previous case.

467 **Proof of 3.** Given an alphabet  $\Sigma$ , for each  $k \in \mathbb{N}$ , we define the equivalence relation  $\sim_k$   
 468 on  $\Sigma^*$  by  $u \sim_k v$  for  $u, v \in \Sigma^*$  whenever  $u$  and  $v$  have the same set of subwords of length  
 469 at most  $k$ . This relation is a congruence of finite index on  $\Sigma^*$ . Simon proved [17] that a  
 470 language belongs to  $\mathcal{L}(\mathbf{J})(\Sigma)$  if and only if it is equal to a union of  $\sim_k$ -classes for some  $k \in \mathbb{N}$ .

471 Let  $\Sigma$  be an alphabet and take some  $L \in \mathcal{L}(\mathbf{J})(\Sigma)$  as well as  $x, y \in \Sigma^*$ . Thus, there exists  
 472  $k \in \mathbb{N}$  such that  $L$  is a union of  $\sim_k$ -classes. Define the language  $K = \bigcup_{w \in L} [xwy]_{\sim_{|xy|+k}}$ : it  
 473 belongs to  $\mathcal{L}(\mathbf{J})(\Sigma)$  by construction. We now show that  $L = x^{-1}Ky^{-1}$ , which concludes  
 474 the proof. Let  $w \in L$ : we have that  $xwy \in [xwy]_{\sim_{|xy|+k}} \subseteq K$ , so that  $w \in x^{-1}Ky^{-1}$ . Let  
 475 conversely  $w \in x^{-1}Ky^{-1}$ . This means that  $xwy \in K$ , which implies that there exists  $w' \in L$   
 476 such that  $xwy \sim_{|xy|+k} xw'y$ . Actually, it holds that any  $u \in \Sigma^*$  of length at most  $k$  is a  
 477 subword of  $w$  if and only if it is a subword of  $w'$ , because  $xuy$  is a subword of  $xwy$  if and  
 478 only if it is a subword of  $xw'y$ . Hence,  $w \sim_k w'$ , which implies that  $w \in L$ .

479 **Proof of 4.** Consider some variety of groups  $\mathbf{H}$ . Take some language  $L \in \mathcal{L}(\mathbf{H})(\Sigma)$  for  
 480 an alphabet  $\Sigma$  and let  $x, y \in \Sigma^*$ . Consider the syntactic morphism  $\eta: \Sigma^* \rightarrow M$  of  $L$ : we  
 481 have that  $M$  is a group in  $\mathbf{H}$ . Define the language  $K = \eta^{-1}(\eta(x)\eta(L)\eta(y))$ : it belongs to  
 482  $\mathcal{L}(\mathbf{H})(\Sigma)$ . We now show that  $L = x^{-1}Ky^{-1}$ , which concludes the proof. Let  $w \in L$ : we  
 483 have that  $\eta(xwy) \in \eta(x)\eta(L)\eta(y)$ , so that  $w \in x^{-1}Ky^{-1}$ . Conversely, let  $w \in x^{-1}Ky^{-1}$ . We  
 484 have that  $xwy \in K$ , which means that  $\eta(xwy) = \eta(x)\eta(w')\eta(y)$  for some  $w' \in L$ , so that  
 485  $\eta(w) = \eta(w') \in \eta(L)$ , as any element in  $\mathbf{H}$  is invertible. Thus,  $w \in L$ . ◀

## 486 **6 Conclusion**

487 The general method presented in this paper actually allows to reprove in a straightforward  
 488 language-theoretic way even more characterisations of the join of **LI** with some variety of  
 489 finite monoids. This can for instance be done for the variety of finite commutative monoids  
 490 **Com** or the variety of finite commutative aperiodic monoids **ACom**.

491 In fact, as already observed in some sense by Costa [8], many varieties of finite monoids  
 492 seem to verify criterion (A). The main question left open by this present work is to understand  
 493 better what exactly those varieties are. Another question left open is whether Proposition 12  
 494 can be refined by using the stronger quotient-expressibility condition alluded to before the  
 495 statement of the proposition. The answers to both questions are unclear to the author, but  
 496 making progress on them may also lead to a better understanding of joins of varieties of  
 497 finite monoids with **LI**.

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501 ——— **References** ———

- 502 1 Jorge Almeida. Some pseudovariety joins involving the pseudovariety of finite groups. In  
503 *Semigroup Forum*, volume 37, pages 53–57. Springer, 1988.
- 504 2 Jorge Almeida. *Finite semigroups and universal algebra*, volume 3. World Scientific, 1994.
- 505 3 Jorge Almeida and Assis Azevedo. The join of the pseudovarieties of R-trivial and L-trivial  
506 monoids. *Journal of Pure and Applied Algebra*, 60(2):129–137, 1989.
- 507 4 Jorge Almeida and Pascal Weil. Free profinite  $\mathit{mathscrR}$ -trivial monoids. *Int. J. Algebra*  
508 *Comput.*, 7(5):625–672, 1997. doi:10.1142/S0218196797000289.
- 509 5 Assis Azevedo. *The Join of the Pseudovariety J with Permutative Pseudovarieties*, pages 1–11.  
510 Springer US, Boston, MA, 1990. doi:10.1007/978-1-4899-2608-1\_1.
- 511 6 Assis Azevedo and Marc Zeitoun. Three examples of join computations. In *Semigroup Forum*,  
512 volume 57, pages 249–277. Springer, 1998.
- 513 7 Laura Chaubard, Jean-Eric Pin, and Howard Straubing. First order formulas with modular  
514 predicates. In *21th IEEE Symposium on Logic in Computer Science (LICS 2006), 12-15*  
515 *August 2006, Seattle, WA, USA, Proceedings*, pages 211–220. IEEE Computer Society, 2006.  
516 doi:10.1109/LICS.2006.24.
- 517 8 José Carlos Costa. Some pseudovariety joins involving locally trivial semigroups. In *Semigroup*  
518 *Forum*, volume 64, pages 12–28. Springer, 2001.
- 519 9 José Carlos Costa. Some pseudovariety joins involving groups and locally trivial semigroups.  
520 In *Semigroups, Algorithms, Automata And Languages*, pages 341–348. World Scientific, 2002.
- 521 10 Samuel Eilenberg. *Automata, Languages, and Machines*, volume A. Academic Press, New  
522 York, 1974.
- 523 11 Samuel Eilenberg. *Automata, Languages, and Machines*, volume B. Academic Press, New  
524 York, 1976.
- 525 12 Nathan Grosshans, Pierre McKenzie, and Luc Segoufin. Tameness and the power of programs  
526 over monoids in DA. *CoRR*, abs/2101.07495, 2021. URL: [https://arxiv.org/abs/2101.](https://arxiv.org/abs/2101.07495)  
527 [07495](https://arxiv.org/abs/2101.07495), arXiv:2101.07495.
- 528 13 Jean-Éric Pin. *Varieties of formal languages*. North Oxford, London and Plenum, New-York,  
529 1986. (Traduction de Variétés de langages formels).
- 530 14 Jean-Éric Pin and Howard Straubing. Some results on C-varieties. *RAIRO-Theor. Inf. Appl.*,  
531 39(1):239–262, 2005.
- 532 15 Jan Reiterman. The Birkhoff theorem for finite algebras. *algebra universalis*, 14(1):1–10, 1982.
- 533 16 Marcel Paul Schützenberger. On finite monoids having only trivial subgroups. *Inf. Control.*,  
534 8(2):190–194, 1965. doi:10.1016/S0019-9958(65)90108-7.
- 535 17 Imre Simon. Piecewise testable events. In *Automata Theory and Formal Languages, 2nd GI*  
536 *Conference*, volume 33 of *Lecture Notes in Computer Science*, pages 214–222. Springer, 1975.
- 537 18 Howard Straubing. On logical descriptions of regular languages. In Sergio Rajsbaum, editor,  
538 *LATIN 2002: Theoretical Informatics, 5th Latin American Symposium, Cancun, Mexico, April*  
539 *3-6, 2002, Proceedings*, volume 2286 of *Lecture Notes in Computer Science*, pages 528–538.  
540 Springer, 2002. doi:10.1007/3-540-45995-2\_46.
- 541 19 Marc Zeitoun. The join of the pseudovarieties of idempotent semigroups and locally trivial  
542 semigroups. In *Semigroup Forum*, volume 50, pages 367–381. Springer, 1995.
- 543 20 Marc Zeitoun. On the decidability of the membership problem of the pseudovariety JVB. *Int.*  
544 *J. Algebra Comput.*, 5(1):47–64, 1995. doi:10.1142/S0218196795000057.
- 545 21 Marc Zeitoun. On the join of two pseudovarieties. *Semigroups, Automata and Languages, eds.*  
546 *J. Almeida, GMS Gomes, and PV Silva. World Scientific*, pages 281–288, 1996.

547 **A** **Missing proofs**

548 **Proof of Proposition 2.** Let  $\mathbf{W}$  be some  $ne$ -variety of stamps such that  $\langle \mathbf{V} \rangle_{all} \cup \langle \mathbf{LI} \rangle_{ne} \subseteq \mathbf{W}$ .  
549 There exists some variety of semigroups  $\mathbf{W}'$  such that  $\langle \mathbf{W}' \rangle_{ne} = \mathbf{W}$ .



550 Let  $S \in \mathbf{V} \cup \mathbf{LI}$ . We denote by  $S^1$  the monoid  $S$  if  $S$  is already a monoid and the monoid  
 551  $S \cup \{1\}$  otherwise. Then the evaluation morphism  $\eta_S: S^* \rightarrow S^1$  such that  $\eta_S(s) = s$  for  
 552 all  $s \in S$  verifies  $\eta_S(S^+) = S$  and additionally  $S^1 = S$  when  $S \in \mathbf{V}$ . This implies that  
 553  $\eta_S \in \langle \mathbf{V} \rangle_{all} \cup \langle \mathbf{LI} \rangle_{ne} \subseteq \mathbf{W}$ . But by definition of  $\mathbf{W}'$ , it must be that  $S = \eta_S(S^+) \in \mathbf{W}'$ .

554 Therefore,  $\mathbf{W}'$  contains both  $\mathbf{V}$  and  $\mathbf{LI}$ , which implies that  $\mathbf{V} \vee \mathbf{LI} \subseteq \mathbf{W}'$  by inclusion-wise  
 555 minimality of  $\mathbf{V} \vee \mathbf{LI}$ . By definition, we can then conclude that  $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne} \subseteq \langle \mathbf{W}' \rangle_{ne} = \mathbf{W}$ .  
 556 So  $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}$  is the inclusion-wise smallest  $ne$ -variety of stamps containing both  $\langle \mathbf{V} \rangle_{all}$  and  
 557  $\langle \mathbf{LI} \rangle_{ne}$ .

558 Let now  $\mathcal{W}$  be some  $ne$ -variety of languages such that  $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{LI}) \subseteq \mathcal{W}$ . It holds that  
 559  $\mathcal{W} = \mathcal{L}(\mathbf{W})$  for some  $ne$ -variety of stamps  $\mathbf{W}$ . We have that  $\langle \mathbf{V} \rangle_{all}$ , which is in particular an  
 560  $ne$ -variety of stamps, is included in  $\mathbf{W}$  because  $\mathcal{L}(\langle \mathbf{V} \rangle_{all}) = \mathcal{L}(\mathbf{V}) \subseteq \mathcal{W} = \mathcal{L}(\mathbf{W})$ , but also  
 561 that  $\langle \mathbf{LI} \rangle_{ne}$  is included in  $\mathbf{W}$  because  $\mathcal{L}(\langle \mathbf{LI} \rangle_{ne}) = \mathcal{L}(\mathbf{LI}) \subseteq \mathcal{W} = \mathcal{L}(\mathbf{W})$ . By inclusion-wise  
 562 minimality of  $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}$ , it follows that  $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne} \subseteq \mathbf{W}$ . Hence, using again the above  
 563 fact on the Eilenberg correspondence, we can conclude that  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI}) = \mathcal{L}(\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}) \subseteq$   
 564  $\mathcal{L}(\mathbf{W}) = \mathcal{W}$ . So  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  is the inclusion-wise smallest  $ne$ -variety of languages containing  
 565 both  $\mathcal{L}(\mathbf{V})$  and  $\mathcal{L}(\mathbf{LI})$ .

566 Consider now the class of languages  $\mathcal{C}$  such that  $\mathcal{C}(\Sigma)$  is the Boolean closure of  $\mathcal{L}(\mathbf{V})(\Sigma) \cup$   
 567  $\mathcal{L}(\mathbf{LI})(\Sigma)$  for each alphabet  $\Sigma$ . By closure under Boolean operations of  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ , we  
 568 have that  $\mathcal{C} \subseteq \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ . Now, as Boolean operations commute with both quotients [13,  
 569 p. 20] and inverses of  $ne$ -morphisms [13, Proposition 0.4], by closure of  $\mathcal{L}(\mathbf{V})$  and  $\mathcal{L}(\mathbf{LI})$   
 570 under quotients and inverses of  $ne$ -morphisms, we actually have that  $\mathcal{C}$  is an  $ne$ -variety  
 571 of languages. Therefore, by inclusion-wise minimality of  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ , we can conclude that  
 572  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI}) = \mathcal{C}$ . ◀

573 **Proof of Lemma 10.** We actually have that  $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{LI}) \subseteq \mathcal{L}(\mathbf{EV})$ , which allows us to  
 574 conclude by inclusion-wise minimality of  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  (Proposition 2) and by the fact that  
 575  $\mathcal{L}(\mathbf{EV})$  is an  $ne$ -variety of languages (Proposition 6).

576 Let  $\Sigma$  be an alphabet. The fact that  $\mathcal{L}(\mathbf{V})(\Sigma) \subseteq \mathcal{L}(\mathbf{EV})(\Sigma)$  follows trivially from  
 577 Proposition 9. Moreover, for all  $u \in \Sigma^*$ , since necessarily  $\Sigma^* \in \mathcal{L}(\mathbf{V})(\Sigma)$ , we have that both  
 578  $u\Sigma^*$  and  $\Sigma^*u$  belong to  $\mathcal{L}(\mathbf{LI})(\Sigma)$ . Thus, as  $\mathcal{L}(\mathbf{EV})(\Sigma)$  is closed under Boolean operations,  
 579 it follows that  $\mathcal{L}(\mathbf{LI})(\Sigma) \subseteq \mathcal{L}(\mathbf{EV})(\Sigma)$ .

580 This concludes the proof, since it holds for any alphabet  $\Sigma$ . ◀

581 **Proof of Lemma 11.** Assume that  $\mathbf{EV} \subseteq \langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}$ . For any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for  $\Sigma$  some al-  
 582 phabet and any  $x, y \in \Sigma^*$ , by Proposition 9, we have that  $xLy \in \mathcal{L}(\mathbf{EV})(\Sigma) \subseteq \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ .  
 583 Hence,  $\mathbf{V}$  verifies criterion (A).

584 Conversely, assume that  $\mathbf{V}$  verifies criterion (A). For any alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$   
 585 contains all languages of the form  $xLy$  for  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  and  $x, y \in \Sigma^*$ , so it contains all  
 586 Boolean combinations of languages of that form, since it is closed under Boolean operations.  
 587 Therefore, by Proposition 9, we have  $\mathcal{L}(\mathbf{EV}) \subseteq \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ , so that  $\mathbf{EV} \subseteq \langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}$ . ◀

588 **Proof of Proposition 12.** Let us first observe that given any alphabet  $\Sigma$ , given any language  
 589  $K$  on that alphabet and given any two words  $x, y \in \Sigma^*$ , we have that  $x(x^{-1}Ky^{-1})y =$   
 590  $x\Sigma^*y \cap K$  and  $x^{-1}(xKy)y^{-1} = K$ .

591 **Implication from right to left.** Assume that for any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and  
 592  $x, y \in \Sigma^*$ , there exist  $k, l \in \mathbb{N}$  such that for all  $u \in \Sigma^k, v \in \Sigma^l$ , there exists some language  
 593  $K \in \mathcal{L}(\mathbf{V})(\Sigma)$  verifying  $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$ . Take some  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some



594 alphabet  $\Sigma$  and take  $x, y \in \Sigma^*$ . Consider also some  $k, l \in \mathbb{N}$  that are guaranteed to exist by  
 595 the assumption we just made.

596 For all  $u \in \Sigma^k, v \in \Sigma^l$ , there exists some language  $K \in \mathcal{L}(\mathbf{V})(\Sigma)$  verifying  $u^{-1}Lv^{-1} =$   
 597  $(xu)^{-1}K(vy)^{-1}$ , so that by our observation at the beginning of the proof, we have

$$598 \quad x(u\Sigma^*v \cap L)y = xu(u^{-1}Lv^{-1})vy = xu((xu)^{-1}K(vy)^{-1})vy = xu\Sigma^*vy \cap K .$$

599 Using Proposition 2, we thus have that  $x(u\Sigma^*v \cap L)y \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  for all  $u \in \Sigma^k, v \in \Sigma^l$ .  
 600 Moreover, since we have that the set of words of  $L$  of length at least  $k + l$  is

$$601 \quad \Sigma^{\geq k+l} \cap L = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (u\Sigma^*v \cap L)$$

602 and since

$$603 \quad L = (\Sigma^{\geq k+l} \cap L) \cup F$$

604 where  $F$  is a finite set of words on  $\Sigma$  of length less than  $k + l$ , we have that

$$605 \quad xLy = x((\Sigma^{\geq k+l} \cap L) \cup F)y = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} x(u\Sigma^*v \cap L)y \cup xFy .$$

606 We can thus conclude that  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  since  $xFy \in \mathcal{L}(\mathbf{LI})(\Sigma)$  and because  
 607  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  is closed under unions.

608 **Implication from left to right.** Assume that  $\mathbf{V}$  satisfies criterion (A). Take some  $L \in$   
 609  $\mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and take  $x, y \in \Sigma^*$ . By hypothesis, we know that  $xLy \in$   
 610  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ .

611 By Proposition 2, this means that  $xLy$  is a Boolean combination of languages in  $\mathcal{L}(\mathbf{V})(\Sigma) \cup$   
 612  $\mathcal{L}(\mathbf{LI})(\Sigma)$ . Further, this implies that  $xLy$  can be written as the union of intersections of  
 613 languages of  $\mathcal{L}(\mathbf{V})(\Sigma)$  and  $\mathcal{L}(\mathbf{LI})(\Sigma)$  or their complements, which in turn implies, by closure  
 614 of  $\mathcal{L}(\mathbf{V})(\Sigma)$  and  $\mathcal{L}(\mathbf{LI})(\Sigma)$  under Boolean operations, that  $xLy$  can be written as a finite  
 615 union of languages of the form  $K \cap (U\Sigma^*V \cup W)$  with  $K \in \mathcal{L}(\mathbf{V})(\Sigma)$  and  $U, V, W \subseteq \Sigma^*$  finite.  
 616 Since any word in  $xLy$  must be of length at least  $|xy|$  and have  $x$  as a prefix and  $y$  as a  
 617 suffix, we can assume that any language  $K \cap (U\Sigma^*V \cup W)$  appearing in a finite union as  
 618 described above verifies that  $U \subseteq x\Sigma^*$ , that  $V \subseteq \Sigma^*y$  and that  $W \subseteq x\Sigma^*y$ . Now, if we take  
 619  $k, l \in \mathbb{N}$  big enough, we thus have that

$$620 \quad xLy = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (K_{u,v} \cap xu\Sigma^*vy) \cup F$$

621 where  $K_{u,v} \in \mathcal{L}(\mathbf{V})(\Sigma)$  for all  $u \in \Sigma^k, v \in \Sigma^l$  and  $F \subseteq \Sigma^{<|xy|+k+l}$ . Hence, for all  $u \in \Sigma^k, v \in$   
 622  $\Sigma^l$ , we have

$$\begin{aligned} 623 \quad u^{-1}Lv^{-1} &= u^{-1}(x^{-1}(xLy)y^{-1})v^{-1} \\ 624 \quad &= (xu)^{-1} \left( \bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (K_{u',v'} \cap xu'\Sigma^*v'y) \cup F \right) (vy)^{-1} \\ 625 \quad &= \bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (xu)^{-1} \left( xu'((xu')^{-1}K_{u',v'}(v'y)^{-1})v'y \right) (vy)^{-1} \cup \\ &\quad (xu)^{-1}F(vy)^{-1} \\ 626 \quad &= (xu)^{-1}K_{u,v}(vy)^{-1} , \end{aligned}$$

628 using classical formulae for quotients [13, p. 20] and observing that  $(xu)^{-1}K(vy)^{-1} = \emptyset$  for  
 629 any  $K \subseteq \Sigma^*$  such that  $K \cap xu\Sigma^*vy = \emptyset$ .  $\blacktriangleleft$