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# A note on the join of varieties of monoids with **LI**

Nathan GROSSHANS\*

## Abstract

In this note, we give a characterisation in terms of identities of the join of  $\mathbf{V}$  with **LI** for several well-known varieties of monoids  $\mathbf{V}$  by using classical algebraic-automata-theoretic techniques. To achieve this, we use the new notion of essentially- $\mathbf{V}$  stamps defined by Grosshans, McKenzie and Segoufin and show that it actually coincides with the join of  $\mathbf{V}$  and **LI** precisely when some natural condition on the variety of languages corresponding to  $\mathbf{V}$  is verified.

This work is a kind of rediscovery of the work of J. C. Costa 20 years ago from a rather different angle, since Costa's work relies on the use of advanced developments in profinite topology, whereas what is presented here essentially uses an algebraic, language-based approach.

## 1 Introduction

The general endeavour to understand the join of two varieties has attracted many research efforts (see [1, 9]) and a non-negligible part of those efforts were concentrated on the question of characterising the join of **LI** and some variety of monoids in terms of identities (see [2]).

In this note, we give a characterisation in terms of identities of  $\mathbf{V} \vee \mathbf{LI}$  for several well-known varieties of monoids  $\mathbf{V}$  by using classical algebraic-automata-theoretic techniques. To achieve this, we use the new notion of essentially- $\mathbf{V}$  stamps defined by McKenzie, Segoufin and the author in [5] and show that it actually coincides with  $\mathbf{V} \vee \mathbf{LI}$  precisely when some natural condition on  $\mathcal{L}(\mathbf{V})$  is verified.

This work is a kind of rediscovery of the work of J. C. Costa [2] from a rather different angle, since Costa's work relies on the use of advanced developments in profinite topology, whereas what is presented here essentially uses an algebraic, language-based approach.

## 2 Preliminaries

For the basics and the classical results of automata theory, we refer the reader to the two classical references of the domain by Eilenberg [3, 4] and Pin [6]. For definitions and results specific to  $\mathcal{C}$ -varieties of stamps and associated profinite identities, see the article by Pin and Straubing [7].

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Note that if  $\mathbf{V}$  is a variety of monoids, then  $\langle \mathbf{V} \rangle_{all}$  is the *all*-variety of stamps of all stamps  $\varphi: \Sigma^* \rightarrow M$  such that  $M \in \mathbf{V}$ . Note that in that case, the variety of languages corresponding to  $\mathbf{V}$  is exactly the *all*-variety of languages corresponding to  $\langle \mathbf{V} \rangle_{all}$ .

Similarly, if  $\mathbf{V}$  is a variety of semigroups, then  $\langle \mathbf{V} \rangle_{ne}$  is the *ne*-variety of stamps of all stamps  $\varphi: \Sigma^* \rightarrow M$  such that  $\varphi(\Sigma^+) \in \mathbf{V}$ . In that case, we consider  $\mathcal{L}(\mathbf{V})$  to be the *ne*-variety of languages corresponding to  $\langle \mathbf{V} \rangle_{ne}$ .

The variety  $\mathbf{LI}$  of finite locally trivial semigroups is well-known to be defined by the identity  $x^\omega y x^\omega = x^\omega$  and such that  $\mathcal{L}(\mathbf{LI})$  is the class of languages that are Boolean combinations of languages of the form  $u\Sigma^*$  or  $\Sigma^*u$  for  $\Sigma$  any alphabet and  $u \in \Sigma^+$ , or equivalently the class of languages of the form  $U\Sigma^*V \cup W$  with  $U, V, W \subseteq \Sigma^*$  finite for  $\Sigma$  any alphabet (see [6, p. 38]).

Given some variety of monoids  $\mathbf{V}$ , the join of  $\mathbf{V}$  and  $\mathbf{LI}$ , denoted by  $\mathbf{V} \vee \mathbf{LI}$ , is the inclusion-wise smallest variety of semigroups containing both  $\mathbf{V}$  and  $\mathbf{LI}$ . In fact, a finite semigroup  $S$  belongs to  $\mathbf{V} \vee \mathbf{LI}$  if and only if there exist  $M \in \mathbf{V}$  and  $T \in \mathbf{LI}$  such that  $S$  divides the semigroup  $M \times T$ . (See [4, Chapter V, Exercise 1.1].)

**Proposition 2.1.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  is the inclusion-wise smallest *ne*-variety of languages containing both  $\mathcal{L}(\mathbf{V})$  and  $\mathcal{L}(\mathbf{LI})$  and is equal to the Boolean closure of  $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{LI})$ .*

### 3 Essentially- $\mathbf{V}$ stamps

In this section, we give a characterisation of essentially- $\mathbf{V}$  stamps (defined in [5]), for  $\mathbf{V}$  a variety of monoids, in terms of identities. We first recall the definition.

**Definition 3.1.** Let  $\mathbf{V}$  be a variety of monoids. Let  $\varphi: \Sigma^* \rightarrow M$  be a stamp from an alphabet  $\Sigma$  to a finite monoid  $M$ . Let  $s$  be the stability index of  $\varphi$ .

We say that  $\varphi$  is *essentially- $\mathbf{V}$*  whenever there exists a stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s).$$

We will denote by  $\mathbf{EV}$  the class of all essentially- $\mathbf{V}$  stamps.

Now, we give an equivalent condition for a stamp to be essentially- $\mathbf{V}$ , based on a specific congruence depending on that stamp.

**Definition 3.2.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp and let  $s$  be its stability index. We define the equivalence relation  $\equiv_\varphi$  on  $\Sigma^*$  by  $u \equiv_\varphi v$  for  $u, v \in \Sigma^*$  whenever  $\varphi(xuy) = \varphi(xvy)$  for all  $x, y \in \Sigma^{\geq s}$ .

**Proposition 3.3.** *Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp. Then  $\equiv_\varphi$  is a congruence of finite index and for any variety of monoids  $\mathbf{V}$ , we have  $\varphi \in \mathbf{EV}$  if and only if  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ .*

*Proof.* Let us denote by  $s$  the stability index of  $\varphi$ .

The equivalence relation  $\equiv_\varphi$  is a congruence because given  $u, v \in \Sigma^*$  verifying  $u \equiv_\varphi v$ , for all  $\alpha, \beta \in \Sigma^*$ , we have  $\alpha u \beta \equiv_\varphi \alpha v \beta$  since for any  $x, y \in \Sigma^{\geq s}$ , it holds that  $\varphi(x\alpha u \beta y) = \varphi(x\alpha v \beta y)$  because  $x\alpha, \beta y \in \Sigma^{\geq s}$ . Furthermore, this

congruence is of finite index because for all  $u, v \in \Sigma^*$ , we have that  $\varphi(u) = \varphi(v)$  implies  $u \equiv_\varphi v$ .

Let now  $\mathbf{V}$  be a variety of monoids. Assume first that  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ . It is quite direct to see that  $\varphi \in \mathbf{EV}$ , as the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$  witnesses this fact. Assume then that  $\varphi \in \mathbf{EV}$ . This means that there exists a stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

Now consider  $u, v \in \Sigma^*$  such that  $\mu(u) = \mu(v)$ . For any  $x, y \in \Sigma^{\geq s}$ , we have that  $x = x_1x_2$  with  $x_1 \in \Sigma^*$  and  $x_2 \in \Sigma^s$  as well as  $y = y_1y_2$  with  $y_1 \in \Sigma^s$  and  $y_2 \in \Sigma^*$ , so that  $\varphi(xuy) = \varphi(x_1)\varphi(x_2uy_1)\varphi(y_2) = \varphi(x_1\varphi(x_2vy_1))\varphi(y_2) = \varphi(xvy)$ . Hence,  $u \equiv_\varphi v$ . Therefore, for all  $u, v \in \Sigma^*$ , we have that  $\mu(u) = \mu(v)$  implies  $u \equiv_\varphi v$ , so we can define the application  $\alpha: N \rightarrow \Sigma^*/\equiv_\varphi$  such that  $\alpha(\mu(w)) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . It is easy to check that  $\alpha$  is actually a surjective morphism. Thus, we can conclude that  $\Sigma^*/\equiv_\varphi$ , which divides  $N$ , belongs to  $\mathbf{V}$ .  $\square$

Using this equivalent condition, we prove that given a set of  $ne$ -identities defining  $\langle \mathbf{V} \rangle_{all}$  for some variety of monoids  $\mathbf{V}$ , we get a set of  $ne$ -identities defining  $\mathbf{EV}$ , which actually shows that  $\mathbf{EV}$  is an  $ne$ -variety of stamps along the way.

**Proposition 3.4.** *Let  $\mathbf{V}$  be a variety of monoids and let  $E$  be a set of identities such that  $\langle \mathbf{V} \rangle_{all} = \llbracket E \rrbracket_{ne}$ . Then*

$$\mathbf{EV} = \llbracket x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \text{alph}(u) \cup \text{alph}(v) \rrbracket_{ne} .$$

*Proof.* Let

$$F = \{x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \text{alph}(u) \cup \text{alph}(v)\} .$$

Central to the proof is the following claim.

**Claim 3.5.** *Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp. Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . It holds that for all  $u, v \in \widehat{\Sigma}^*$ ,*

$$\widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow (\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+) .$$

Using this claim, we can prove that  $\mathbf{EV} = \llbracket F \rrbracket_{ne}$ .

**Inclusion from left to right.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp in  $\mathbf{EV}$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . Since  $\varphi \in \mathbf{EV}$ , Proposition 3.3 tells us that  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ , hence  $\mu \in \langle \mathbf{V} \rangle_{all}$ .

Let us consider any identity  $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$ . It is thus defined on an alphabet  $B$  with  $u = v \in E$  defined on an alphabet  $A \subseteq B$  and  $x, y, z, t \in B \setminus A$ . Let  $f: B^* \rightarrow \Sigma^*$  be an  $ne$ -morphism. Since  $\mu \in \langle \mathbf{V} \rangle_{all}$ , we have that  $\mu$  satisfies the identity  $u = v$ , so that  $\widehat{\mu}(f(u)) = \widehat{\mu}(f(v))$ . Notice that we have that

$\widehat{f}(x^\omega) = f(x)^\omega$  as well as  $\widehat{f}(t^\omega) = f(t)^\omega$  and that  $f(x), f(y), f(z), f(t) \in \Sigma^+$  because  $f$  is non-erasing. Therefore, we have

$$\begin{aligned}\widehat{\varphi}(\widehat{f}(x^\omega yuzt^\omega)) &= \widehat{\varphi}(f(x)^\omega f(y)\widehat{f}(u)f(z)f(t)^\omega) \\ &= \widehat{\varphi}(f(x)^\omega f(y)\widehat{f}(v)f(z)f(t)^\omega) \\ &= \widehat{\varphi}(\widehat{f}(x^\omega yvzt^\omega))\end{aligned}$$

by Claim 3.5. As this holds for any  $ne$ -morphism  $f: B^* \rightarrow \Sigma^*$ , we can conclude that  $\varphi$  satisfies the identity  $x^\omega yuzt^\omega = x^\omega yvzt^\omega$ .

This is true for any identity in  $F$ , so  $\varphi \in \llbracket F \rrbracket_{ne}$ . In conclusion,  $\mathbf{EV} \subseteq \llbracket F \rrbracket_{ne}$ .

**Inclusion from right to left.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp in  $\llbracket F \rrbracket_{ne}$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . We are now going to show that  $\mu \in \langle \mathbf{V} \rangle_{all}$ .

Take any identity  $u = v \in E$  defined on an alphabet  $A$ . There is thus an identity  $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$  defined on an alphabet  $B$  such that  $A \subseteq B$  and  $x, y, z, t \in B \setminus A$ . Let  $f: A^* \rightarrow \Sigma^*$  be an  $ne$ -morphism.

Take any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . Let us define the  $ne$ -morphism  $g: B^* \rightarrow \Sigma^*$  as the unique one such that for all  $b \in B$ ,

$$g(b) = \begin{cases} f(b) & \text{if } b \in A \\ \alpha & \text{if } b = x \\ \beta & \text{if } b = y \\ \gamma & \text{if } b = z \\ \delta & \text{if } b = t \end{cases} .$$

Observe in particular that  $\widehat{g}(w) = \widehat{f}(w)$  for any  $w \in \widehat{A}^*$  and that  $\widehat{g}(x^\omega) = g(x)^\omega = \alpha^\omega$  as well as  $\widehat{g}(t^\omega) = \delta^\omega$ . Now, as  $\varphi$  satisfies  $x^\omega yuzt^\omega = x^\omega yvzt^\omega$ , we have that

$$\begin{aligned}\widehat{\varphi}(\alpha^\omega \beta \widehat{f}(u) \gamma \delta^\omega) &= \widehat{\varphi}(\widehat{g}(x^\omega yuzt^\omega)) \\ &= \widehat{\varphi}(\widehat{g}(x^\omega yvzt^\omega)) \\ &= \widehat{\varphi}(\alpha^\omega \beta \widehat{f}(v) \gamma \delta^\omega) .\end{aligned}$$

Since this holds for any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ , by Claim 3.5, we have that  $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$ .

Therefore,  $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$  for any  $ne$ -morphism  $f: A^* \rightarrow \Sigma^*$ , which means that  $\mu$  satisfies  $u = v$ .

So, to conclude, since this is true for any  $u = v \in E$ , we have that  $\mu \in \langle \mathbf{V} \rangle_{all}$ , which implies that  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$  and thus  $\varphi \in \mathbf{EV}$  by Proposition 3.3. In conclusion,  $\llbracket F \rrbracket_{ne} \subseteq \mathbf{EV}$ .

The claim still needs to be proved.

*Proof of Claim 3.5.* Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp of stability index  $s$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . We now want to show that for all  $u, v \in \widehat{\Sigma}^*$ ,

$$\widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow (\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+) .$$

Let  $u, v \in \widehat{\Sigma}^*$ . There exist two Cauchy sequences  $(u_n)_{n \geq 0}$  and  $(v_n)_{n \geq 0}$  in  $\Sigma^*$  such that  $u = \lim_{n \rightarrow \infty} u_n$  and  $v = \lim_{n \rightarrow \infty} v_n$ . This means that

$$\widehat{\mu}(u) = \lim_{n \rightarrow \infty} \mu(u_n) \quad \text{and} \quad \widehat{\varphi}(u) = \lim_{n \rightarrow \infty} \varphi(u_n)$$

as well as

$$\widehat{\mu}(v) = \lim_{n \rightarrow \infty} \mu(v_n) \quad \text{and} \quad \widehat{\varphi}(v) = \lim_{n \rightarrow \infty} \varphi(v_n).$$

But as  $\Sigma^*/\equiv_{\varphi}$  and  $M$  are discrete, we have that all four Cauchy sequences  $(\mu(u_n))_{n \geq 0}$ ,  $(\varphi(u_n))_{n \geq 0}$ ,  $(\mu(v_n))_{n \geq 0}$  and  $(\varphi(v_n))_{n \geq 0}$  are ultimately constant. So there exists  $k \in \mathbb{N}$  such that  $\widehat{\mu}(u) = \mu(u_k)$ ,  $\widehat{\varphi}(u) = \varphi(u_k)$ ,  $\widehat{\mu}(v) = \mu(v_k)$  and  $\widehat{\varphi}(v) = \varphi(v_k)$ .

Assume first that  $\widehat{\mu}(u) = \widehat{\mu}(v)$ . Take any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . We have that

$$\widehat{\varphi}(\alpha^{\omega}) = \widehat{\varphi}(\lim_{n \rightarrow \infty} \alpha^{n!}) = \lim_{n \rightarrow \infty} \varphi(\alpha^{n!})$$

and  $\widehat{\varphi}(\delta^{\omega}) = \lim_{n \rightarrow \infty} \varphi(\delta^{n!})$ . Since  $M$  is discrete, both Cauchy sequences  $(\varphi(\alpha^{n!}))_{n \geq 0}$  and  $(\varphi(\delta^{n!}))_{n \geq 0}$  are ultimately constant. So there exists  $l \in \mathbb{N}$  such that for all  $m \in \mathbb{N}, m \geq l$ , we have  $\lim_{n \rightarrow \infty} \varphi(\alpha^{n!}) = \varphi(\alpha^{m!})$  and  $\lim_{n \rightarrow \infty} \varphi(\delta^{n!}) = \varphi(\delta^{m!})$ . Hence, taking some  $m \in \mathbb{N}, m \geq l$  such that  $|\alpha^{m!}\beta| \geq s$  and  $|\gamma\delta^{m!}| \geq s$ , it follows that

$$\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \varphi(\alpha^{m!} \beta u_k \gamma \delta^{m!}) = \varphi(\alpha^{m!} \beta v_k \gamma \delta^{m!}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$$

because  $[u_k]_{\equiv_{\varphi}} = \widehat{\mu}(u) = \widehat{\mu}(v) = [v_k]_{\equiv_{\varphi}}$ . Thus, we have that

$$\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$$

for all  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ .

Assume then that  $\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$  for all  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . Take any  $\alpha, \beta \in \Sigma^{\geq s}$ . Since  $\varphi(\Sigma^s)$  is a finite semigroup and verifies that  $\varphi(\Sigma^s) = \varphi(\Sigma^s)^2$ , by a classical result in finite semigroup theory (see e.g. [6, Chapter 1, Proposition 1.12]), we have that there exist  $\alpha_1, e, f, \beta_2 \in \Sigma^s$  and  $\alpha_2, \beta_1 \in \Sigma^{\geq s}$  such that  $\varphi(\alpha_1 e \alpha_2) = \varphi(\alpha)$  and  $\varphi(\beta_1 f \beta_2) = \varphi(\beta)$  with  $\varphi(e)$  and  $\varphi(f)$  idempotents. Now, since  $\varphi(e)$  is idempotent, we have that

$$\widehat{\varphi}(e^{\omega}) = \widehat{\varphi}(\lim_{n \rightarrow \infty} e^{n!}) = \lim_{n \rightarrow \infty} \varphi(e^{n!}) = \lim_{n \rightarrow \infty} \varphi(e)^{n!} = \varphi(e)$$

and similarly,  $\widehat{\varphi}(f^{\omega}) = \varphi(f)$ . So it follows that

$$\begin{aligned} \varphi(\alpha u_k \beta) &= \varphi(\alpha_1 e \alpha_2 u_k \beta_1 f \beta_2) \\ &= \widehat{\varphi}(\alpha_1 e^{\omega} \alpha_2 u \beta_1 f^{\omega} \beta_2) \\ &= \widehat{\varphi}(\alpha_1 e^{\omega} \alpha_2 v \beta_1 f^{\omega} \beta_2) \\ &= \varphi(\alpha_1 e \alpha_2 v_k \beta_1 f \beta_2) \\ &= \varphi(\alpha v_k \beta). \end{aligned}$$

As this is true for any  $\alpha, \beta \in \Sigma^{\geq s}$ , by definition it holds that  $u_k \equiv_{\varphi} v_k$ , hence  $\widehat{\mu}(u) = \mu(u_k) = \mu(v_k) = \widehat{\mu}(v)$ .  $\square$

This concludes the proof of the proposition.  $\square$

## 4 Essentially- $\mathbf{V}$ stamps and the join of $\mathbf{V}$ and $\mathbf{LI}$

In this section, we establish the link between essentially- $\mathbf{V}$  stamps and  $\mathbf{V} \vee \mathbf{LI}$  by giving a criterion that characterises exactly when they do correspond.

A first result we can prove is that any stamp whose semigroup obtained as image of the set of non-empty words belongs to  $\mathbf{V} \vee \mathbf{LI}$  is essentially- $\mathbf{V}$ .

**Lemma 4.1.** *Let  $\mathbf{V}$  be a variety of monoids and  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid a stamp such that  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ . Then  $\varphi$  is essentially- $\mathbf{V}$ .*

*Proof.* Let  $S = \varphi(\Sigma^+)$  and let  $s$  be the stability index of  $\varphi$ . Since  $S \in \mathbf{V} \vee \mathbf{LI}$ , there exist  $N \in \mathbf{V}$  and  $T \in \mathbf{LI}$  such that  $S$  divides the semigroup  $N \times T$ . So there exist a subsemigroup  $S'$  of  $N \times T$  and a surjective morphism  $\alpha: S' \rightarrow S$ . We define  $\pi_1: N \times T \rightarrow N$  and  $\pi_2: N \times T \rightarrow T$  to be the projection morphisms from  $N \times T$  onto  $N$  and  $T$ , respectively.

Let  $h: S \rightarrow S'$  be an arbitrary mapping from  $S$  to  $S'$  such that  $\alpha(h(r)) = r$  for all  $r \in S$ . We can then uniquely define the morphism  $\varphi': \Sigma^+ \rightarrow S'$  such that  $\varphi'(a) = h(\varphi(a))$  for all  $a \in \Sigma$ . We can also define the unique stamp  $\mu: \Sigma^* \rightarrow N'$  such that  $\mu(a) = \pi_1(\varphi'(a))$  for all  $a \in \Sigma$  and  $N' \in \mathbf{V}$  is the submonoid of  $N$  generated by  $\{\pi_1(\varphi'(a)) \mid a \in \Sigma\}$ . Hence, for all  $w \in \Sigma^+$ , we have that  $\mu(w) = \pi_1(\varphi'(w))$  and that  $\alpha(\varphi'(w)) = \varphi(w)$ .

Now, take  $u, v \in \Sigma^*$  such that  $\mu(u) = \mu(v)$  and take any  $x, y \in \Sigma^s$ . Since  $\varphi(\Sigma^s)$  is a finite semigroup and verifies that  $\varphi(\Sigma^s) = \varphi(\Sigma^s)^2$ , by a classical result in finite semigroup theory (see e.g. [6, Chapter 1, Proposition 1.12]), we have that there exist  $x_1, e, x_2, y_1, f, y_2 \in \Sigma^s$  such that  $\varphi(x_1 e x_2) = \varphi(x)$  and  $\varphi(y_1 f y_2) = \varphi(y)$  with  $\varphi(e)$  and  $\varphi(f)$  idempotents. If we denote by  $\omega$  the idempotent power of  $T$ , since  $S'$  is a subsemigroup of  $N \times T$  with  $T \in \mathbf{LI}$ , it follows that

$$\begin{aligned} \pi_2(\varphi'(e^\omega x_2 u y_1 f^\omega)) &= \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(x_2 u y_1)) \pi_2(\varphi'(f))^\omega \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(f))^\omega \\ &= \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(f))^\omega \\ &= \pi_2(\varphi'(e^\omega x_2 v y_1 f^\omega)) , \end{aligned}$$

so that

$$\pi_2(\varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2)) = \pi_2(\varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2)) .$$

Moreover, from the hypothesis that  $\mu(u) = \mu(v)$ , we can deduce that

$$\begin{aligned} \pi_1(\varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2)) &= \pi_1(\varphi'(x_1 e^\omega x_2)) \mu(u) \pi_1(\varphi'(y_1 f^\omega y_2)) \\ &= \pi_1(\varphi'(x_1 e^\omega x_2)) \mu(v) \pi_1(\varphi'(y_1 f^\omega y_2)) \\ &= \pi_1(\varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2)) . \end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned} \varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2) &= \varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2) \\ \varphi(x_1 e^\omega x_2 u y_1 f^\omega y_2) &= \varphi(x_1 e^\omega x_2 v y_1 f^\omega y_2) \\ \varphi(x_1 e x_2 u y_1 f y_2) &= \varphi(x_1 e x_2 v y_1 f y_2) \\ \varphi(x u y) &= \varphi(x v y) . \end{aligned}$$

Hence, for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s),$$

so that  $\varphi \in \mathbf{EV}$ .  $\square$

When does the converse of this statement hold? Consider the following criterion for a variety of monoids  $\mathbf{V}$ .

**Criterion (A).** For any  $L \in \mathcal{L}(\mathbf{V})$  over some alphabet  $\Sigma$ , we have  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  for all  $x, y \in \Sigma^*$ .

It is a kind of mild closure condition that appears naturally as a necessary condition for the converse of Lemma 4.1 to be true. (Basically because for any variety of monoids  $\mathbf{V}$ , we always have that  $xLy \in \mathcal{L}(\mathbf{EV})$  for any  $L \in \mathcal{L}(\mathbf{V})$  over some alphabet  $\Sigma$  and  $x, y \in \Sigma^*$ .)

**Lemma 4.2.** Let  $\mathbf{V}$  be a variety of monoids such that for any stamp  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid, we have  $\varphi \in \mathbf{EV}$  if and only if  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ . Then  $\mathbf{V}$  verifies criterion (A).

*Proof.* Let  $L \in \mathcal{L}(\mathbf{V})$  over some alphabet  $\Sigma$ . Let  $\mu: \Sigma^* \rightarrow N$  be its syntactic morphism: this means that  $N \in \mathbf{V}$  and that there exists  $F \subseteq N$  such that  $L = \mu^{-1}(F)$ .

Now take  $x, y \in \Sigma^*$ . Let  $\varphi: \Sigma^* \rightarrow M$  be the syntactic morphism of the language  $xLy = x\Sigma^*y \cap \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|}$  and let  $s$  be its stability index. We then consider some  $u, v \in \Sigma^*$  such that  $\mu(u) = \mu(v)$ . Take any  $x', y' \in \Sigma^*$  such that  $|x'| \geq |x|$  and  $|y'| \geq |y|$ . We clearly have that  $x'uy' \in x\Sigma^*y$  if and only if  $x'vy' \in x\Sigma^*y$ . Moreover,  $x' = x'_1x'_2$  for some  $x'_1 \in \Sigma^{|x|}$  and  $x'_2 \in \Sigma^*$  and  $y' = y'_1y'_2$  for some  $y'_1 \in \Sigma^*$  and  $y'_2 \in \Sigma^{|y|}$ , so that

$$\begin{aligned} x'uy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|} &\Leftrightarrow \mu(x'_2uy'_1) \in F \\ &\Leftrightarrow \mu(x'_2vy'_1) \in F \\ &\Leftrightarrow x'vy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|}. \end{aligned}$$

Hence,  $x'uy' \in xLy$  if and only if  $x'vy' \in xLy$  for all  $x', y' \in \Sigma^*$  such that  $|x'| \geq |x|$  and  $|y'| \geq |y|$ , so that, by definition of the stability index  $s$  of  $\varphi$ , we have  $\varphi(x'uy') = \varphi(x'vy')$  for all  $x', y' \in \Sigma^s$ . Thus, it follows that  $\varphi \in \mathbf{EV}$ .

By hypothesis, this means that  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ , that is, the syntactic semigroup of  $xLy$  belongs to  $\mathbf{V} \vee \mathbf{LI}$ . Therefore,  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ .

As it is true for any  $L \in \mathcal{L}(\mathbf{V})$  over some alphabet  $\Sigma$  and any  $x, y \in \Sigma^*$ , we can conclude that  $\mathbf{V}$  verifies criterion (A).  $\square$

But criterion (A) is also a sufficient condition for the converse of Lemma 4.1 to be true, as we will now prove.

**Lemma 4.3.** Let  $\mathbf{V}$  be a variety of monoids that verifies criterion (A). Then for any stamp  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid, we have  $\varphi \in \mathbf{EV}$  if and only if  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ .

*Proof.* Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp.

Assume first that  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ . Then, by Lemma 4.1, we have that  $\varphi \in \mathbf{EV}$ .



Assume now that  $\varphi \in \mathbf{EV}$ . Given  $s$  its stability index, this means there exists a stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

As  $\mathbf{V}$  verifies criterion (A), for any  $m \in N$  and  $x, y \in \Sigma^s$ , the syntactic morphism  $\eta_{m,x,y}: \Sigma^* \rightarrow M_{m,x,y}$  of the language  $x\mu^{-1}(m)y$  verifies  $\eta_{m,x,y}(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ . Let  $E = N \times \Sigma^s \times \Sigma^s$  and let us now define the unique stamp  $\varphi': \Sigma^* \rightarrow S'^1$  such that  $\varphi'(a) = \prod_{(m,x,y) \in E} \eta_{m,x,y}(a)$  for all  $a \in \Sigma$  and  $S'$  is the subsemigroup of  $\prod_{(m,x,y) \in E} \eta_{m,x,y}(\Sigma^+)$  generated by  $\{\prod_{(m,x,y) \in E} \eta_{m,x,y}(a) \mid a \in \Sigma\}$ . It is obvious that  $S'$  belongs to  $\mathbf{V} \vee \mathbf{LI}$ .

Let  $w, w' \in \Sigma^+$  such that  $\varphi'(w) = \varphi'(w')$ : this means that  $\eta_{m,x,y}(w) = \eta_{m,x,y}(w')$  for all  $m \in N$  and  $x, y \in \Sigma^s$ . We are going to show that  $\varphi(w) = \varphi(w')$ , by considering two different cases.

- Either it is the case that  $|w| \geq 2s$  or  $|w'| \geq 2s$ . Assume without loss of generality that  $|w| \geq 2s$ . We thus have that  $w = xuy$  for some  $x, y \in \Sigma^s$  and  $u \in \Sigma^*$ . This means that  $xuy \in x\mu^{-1}(\mu(u))y$ , but since  $\eta_{\mu(u),x,y}(w) = \eta_{\mu(u),x,y}(w')$ , we also have that  $w' \in x\mu^{-1}(\mu(u))y$ . Hence,  $w' = xvy$  with  $v \in \Sigma^*$  verifying  $\mu(v) = \mu(u)$ . So, to conclude,  $\varphi(w) = \varphi(xuy) = \varphi(xvy) = \varphi(w')$ .
- Or it is the case that  $|w| < 2s$  and  $|w'| < 2s$ . Assume without loss of generality that  $|w'| \leq |w| < 2s$ . Take an arbitrary  $z \in \Sigma^{2s-|w|}$ : we thus have that  $wz = xy$  for some  $x, y \in \Sigma^s$ , so that  $wz \in x\mu^{-1}(1)y$  where we denote by 1 the identity of  $N$ . But since  $\eta_{1,x,y}(w) = \eta_{1,x,y}(w')$ , it must be that  $w'z \in x\mu^{-1}(1)y$ . However, the sole member of  $x\mu^{-1}(1)y$  of length at most  $2s$  is  $xy$ , so we have that  $w'z = wz$  and hence necessarily that  $w = w'$ . This entails that  $\varphi(w) = \varphi(w')$ .

Hence, for all  $w, w' \in \Sigma^+$ , we have that  $\varphi'(w) = \varphi'(w') \Rightarrow \varphi(w) = \varphi(w')$ .

Let  $S = \varphi(\Sigma^+)$ . Let  $\rho: S' \rightarrow \Sigma^+$  be an arbitrary mapping from  $S'$  to  $\Sigma^+$  such that  $\varphi'(\rho(s')) = s'$  for all  $s' \in S'$ . Define the mapping  $\alpha: S' \rightarrow S$  by  $\alpha(s') = \varphi(\rho(s'))$  for all  $s' \in S'$ .

Let  $s'_1, s'_2 \in S'$ . We have that  $\alpha(s'_1 s'_2) = \varphi(\rho(s'_1 s'_2))$ . Now,  $\varphi'(\rho(s'_1 s'_2)) = s'_1 s'_2 = \varphi'(\rho(s'_1))\varphi'(\rho(s'_2)) = \varphi'(\rho(s'_1)\rho(s'_2))$ . Hence,

$$\varphi(\rho(s'_1 s'_2)) = \varphi(\rho(s'_1)\rho(s'_2)) = \varphi(\rho(s'_1))\varphi(\rho(s'_2)) = \alpha(s'_1)\alpha(s'_2) .$$

So  $\alpha$  is a morphism. Moreover, it is surjective because for all  $t \in S$ , there exists  $w \in \Sigma^+$  such that  $\varphi(w) = t$  and thus  $\alpha(\varphi'(w)) = \varphi(\rho(\varphi'(w))) = \varphi(w)$  since  $\varphi'(\rho(\varphi'(w))) = \varphi'(w)$ .

This allows us to conclude that  $S$  divides  $S'$  and, thus, that  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ .

This concludes the proof of the lemma.  $\square$

The following proposition summarises the results of this section.

**Proposition 4.4.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $(\mathbf{V} \vee \mathbf{LI})_{\text{ne}} \subseteq \mathbf{EV}$  and there is equality if and only if  $\mathbf{V}$  verifies criterion (A).*

## 5 Applications

In this last section, we use the link between essentially- $\mathbf{V}$  stamps and  $\mathbf{V} \vee \mathbf{LI}$  to reprove some characterisations of joins between  $\mathbf{LI}$  and some well-known varieties of monoids in terms of identities.

We first give an equivalent formulation of criterion (A) that solely depends on the variety of monoids  $\mathbf{V}$ , without explicit reference to  $\mathbf{LI}$ .

**Proposition 5.1.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $\mathbf{V}$  satisfies criterion (A) if and only if for any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and  $x, y \in \Sigma^*$ , there exist  $k, l \in \Sigma^*$  such that for all  $u \in \Sigma^k, v \in \Sigma^l$ , there exists some language  $K \in \mathcal{L}(\mathbf{V})$  over alphabet  $\Sigma$  verifying  $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$ .*

*Proof.* Let us first observe that given any alphabet  $\Sigma$ , given any language  $K$  on that alphabet and given any two words  $x, y \in \Sigma^*$ , we have that  $x(x^{-1}Ky^{-1})y = x\Sigma^*y \cap K$  and  $x^{-1}(xKy)y^{-1} = K$ .

**Inclusion from right to left.** Assume that for any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and  $x, y \in \Sigma^*$ , there exist  $k, l \in \Sigma^*$  such that for all  $u \in \Sigma^k, v \in \Sigma^l$ , there exists some language  $K \in \mathcal{L}(\mathbf{V})$  over alphabet  $\Sigma$  verifying  $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$ .

This implies, by our observation at the beginning of the proof, that

$$x(u\Sigma^*v \cap L)y = xu(u^{-1}Lv^{-1})vy = xu((xu)^{-1}K(vy)^{-1})vy = xu\Sigma^*vy \cap K$$

for all  $u \in \Sigma^k, v \in \Sigma^l$ . Using Proposition 2.1, we thus have that  $x(u\Sigma^*v \cap L)y \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  for all  $u \in \Sigma^k, v \in \Sigma^l$ . Moreover, since we have that the set of words of  $L$  of length at least  $k+l$  is

$$\Sigma^{k+l} \cap L = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (u\Sigma^*v \cap L)$$

and since

$$L = (\Sigma^{\geq k+l} \cap L) \cup F$$

where  $F$  is a finite set of words on  $\Sigma$  of length less than  $k+l$ , we have that

$$xLy = x((\Sigma^{\geq k+l} \cap L) \cup F)y = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} x(u\Sigma^*v \cap L)y \cup xFy.$$

We can thus conclude that  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  by using Proposition 2.1 as well as the fact that  $xFy \in \mathcal{L}(\mathbf{LI})(\Sigma)$  and because  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  is closed under unions.

**Inclusion from left to right.** Assume that  $\mathbf{V}$  satisfies criterion (A). Take some  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and take  $x, y \in \Sigma^*$ . By hypothesis, we know that  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ .

By Proposition 2.1, this means that  $xLy$  is a Boolean combination of languages in  $\mathcal{L}(\mathbf{V})(\Sigma) \cup \mathcal{L}(\mathbf{LI})(\Sigma)$ . Further, this implies that  $xLy$  can be written as the union of intersections of languages of  $\mathcal{L}(\mathbf{V})(\Sigma)$  and  $\mathcal{L}(\mathbf{LI})(\Sigma)$  or their complements, which in turn implies, by closure of  $\mathcal{L}(\mathbf{V})(\Sigma)$  and  $\mathcal{L}(\mathbf{LI})(\Sigma)$  under Boolean operations, that  $xLy$  can be written as a finite union of languages

of the form  $K \cap (U\Sigma^*V \cup W)$  with  $K \in \mathcal{L}(\mathbf{V})(\Sigma)$  and  $U, V, W \subseteq \Sigma^*$  finite. Since any word in  $xLy$  must be of length at least  $|xy|$  and have  $x$  as a prefix and  $y$  as a suffix, we can assume that any language  $K \cap (U\Sigma^*V \cup W)$  appearing in a finite union as described above verifies that  $U \subseteq x\Sigma^*$ , that  $V \subseteq \Sigma^*y$  and that  $W \subseteq x\Sigma^*y$ . Now, if we take  $k, l \in \mathbb{N}$  big enough, we thus have that

$$xLy = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (K_{u,v} \cap xu\Sigma^*vy) \cup F$$

where  $K_{u,v} \in \mathcal{L}(\mathbf{V})(\Sigma)$  for all  $u \in \Sigma^k, v \in \Sigma^l$  and  $F \subseteq \Sigma^{<|xy|+k+l}$ . Hence, for all  $u \in \Sigma^k, v \in \Sigma^l$ , we have

$$\begin{aligned} u^{-1}Lv^{-1} &= u^{-1}(x^{-1}(xLy)y^{-1})v^{-1} \\ &= (xu)^{-1} \left( \bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (K_{u',v'} \cap xu'\Sigma^*v'y) \cup F \right) (vy)^{-1} \\ &= \bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (xu)^{-1} \left( xu'((xu')^{-1}K_{u',v'}(v'y)^{-1})v'y \right) (vy)^{-1} \cup \\ &\quad (xu)^{-1}F(vy)^{-1} \\ &= (xu)^{-1}K_{u,v}(vy)^{-1}, \end{aligned}$$

observing that  $(xu)^{-1}K(vy)^{-1} = \emptyset$  for any  $K \subseteq \Sigma^*$  such that  $K \cap xu\Sigma^*vy = \emptyset$ .  $\square$

We now prove the announced characterisations of joins between **LI** and some well-known varieties of monoids in terms of identities.

**Theorem 5.2.** *We have the following.*

1.  $\langle \mathbf{R} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{ER} = \llbracket x^\omega y(ab)^\omega azt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$ .
2.  $\langle \mathbf{L} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EL} = \llbracket x^\omega yb(ab)^\omega zt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$ .
3.  $\langle \mathbf{J} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EJ} = \llbracket x^\omega y(ab)^\omega zt^\omega = x^\omega y(ba)^\omega zt^\omega, a^{\omega+1} = a^\omega \rrbracket_{\text{ne}}$ .
4.  $\langle \mathbf{H} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EH}$  for any variety of groups **H**.

*Proof.* In each case, we prove that the variety of monoids under consideration satisfies criterion (A) using Proposition 5.1. We then use Proposition 4.4.

**Proof of 1.** It is well-known that given an alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{R})(\Sigma)$  contains all languages that are disjoint unions of languages that are of the form  $A_0^*a_1A_1^*\cdots a_kA_k^*$  where  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma^*$  and  $a_i \notin A_{i-1}$  for all  $i \in [k]$  (see [6, Chapter 4, Theorem 3.3]).

Let  $\Sigma$  be an alphabet and take some language  $A_0^*a_1A_1^*\cdots a_kA_k^*$  where  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma^*$  and  $a_i \notin A_{i-1}$  for all  $i \in [k]$ . Take  $x, y \in \Sigma^*$ . Observe that  $y$  can be uniquely written as  $y = zt$  where  $z, t \in \Sigma^*$  and either  $|t| = 0$  and  $z \in A_k^*$ , or  $|t| > 0$  and  $t_1 \notin A_k$ . We have

$$\begin{aligned} &A_0^*a_1A_1^*\cdots a_kA_k^* \\ &= x^{-1} \left( xA_0^*a_1A_1^*\cdots a_kA_k^*t \cap \bigcap_{v \in A_k^{<|z|}} (xA_0^*a_1A_1^*\cdots a_kA_k^*vt)^{\mathbb{G}} \right) y^{-1} \end{aligned}$$

where  $x A_0^* a_1 A_1^* \cdots a_k A_k^* t \cap \bigcap_{v \in A_k^{\leq |z|}} (x A_0^* a_1 A_1^* \cdots a_k A_k^* t)^{\mathbf{G}}$  does belong to the set  $\mathcal{L}(\mathbf{R})(\Sigma)$  because the latter is closed under Boolean operations and by construction of  $z$  and  $t$ . We can conclude that for each  $L \in \mathcal{L}(\mathbf{R})(\Sigma)$  and  $x, y \in \Sigma^*$ , there exists  $K \in \mathcal{L}(\mathbf{R})(\Sigma)$  such that  $L = x^{-1} K y^{-1}$  by using the characterisation of  $\mathcal{L}(\mathbf{R})(\Sigma)$  and the fact that  $x^{-1} K_1 y^{-1} \cup x^{-1} K_2 y^{-1} = x^{-1} (K_1 \cup K_2) y^{-1}$  for all  $K_1, K_2 \in \Sigma^*$ .

**Proof of 2.** It is also well-known that given an alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{L})(\Sigma)$  contains all languages that are disjoint unions of languages that are of the form  $A_0^* a_1 A_1^* \cdots a_k A_k^*$  where  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma^*$  and  $a_i \notin A_i$  for all  $i \in [k]$  (see [6, Chapter 4, Theorem 3.4]). The proof is then dual to the previous case.

**Proof of 3.** Given an alphabet  $\Sigma$ , for each  $k \in \mathbb{N}$ , we define the equivalence relation  $\sim_k$  on  $\Sigma^*$  by  $u \sim_k v$  for  $u, v \in \Sigma^*$  whenever  $u$  and  $v$  have the same set of subwords of length at most  $k$ . This relation is a congruence of finite index on  $\Sigma^*$ . Simon proved [8] that for all any alphabet  $\Sigma$ , a language belongs to  $\mathcal{L}(\mathbf{J})(\Sigma)$  if and only if it is equal to a union of  $\sim_k$ -classes for some  $k \in \mathbb{N}$ .

Let  $\Sigma$  be an alphabet and take some  $L \in \mathcal{L}(\mathbf{J})(\Sigma)$  as well as  $x, y \in \Sigma^*$ . Thus, there exists  $k \in \mathbb{N}$  such that  $L$  is a union of  $\sim_k$ -classes. Define the language  $K = \bigcup_{w \in L} [xwy]_{\sim_{|xy|+k}}$ : it belongs to  $\mathcal{L}(\mathbf{J})(\Sigma)$  by construction. We now show that  $L = x^{-1} K y^{-1}$ , which concludes the proof. Let  $w \in L$ : we have that  $xwy \in [xwy]_{\sim_{|xy|+k}} \subseteq K$ , so that  $w \in x^{-1} K y^{-1}$ . Let conversely  $w \in x^{-1} K y^{-1}$ . This means that  $xwy \in K$ , which implies that there exists  $w' \in L$  such that  $xwy \sim_{|xy|+k} xw'y$ . Actually, it holds that any  $u \in \Sigma^*$  of length at most  $k$  is a subword of  $w$  if and only if it is a subword of  $w'$ , because  $xuy$  is a subword of  $xwy$  if and only if it is a subword of  $xw'y$ . Hence,  $w \sim_k w'$ , which implies that  $w \in L$ .

**Proof of 3.** Consider some variety of groups  $\mathbf{H}$ . Take some language  $L \in \mathcal{L}(\mathbf{H})(\Sigma)$  for an alphabet  $\Sigma$  and let  $x, y \in \Sigma^*$ . Consider the syntactic morphism  $\eta: \Sigma^* \rightarrow M$  of  $L$ : we have that  $M$  is a group in  $\mathbf{H}$ . Define the language  $K = \eta^{-1}(\eta(x)\eta(L)\eta(y))$ . We now show that  $L = x^{-1} K y^{-1}$ , which concludes the proof. Let  $w \in L$ : we have that  $\eta(xwy) \in \eta(x)\eta(L)\eta(y)$ , so that  $w \in x^{-1} K y^{-1}$ . Conversely, let  $w \in x^{-1} K y^{-1}$ . We have that  $xwy \in K$ , which means that  $\eta(xwy) = \eta(x)\eta(w')\eta(y)$  for some  $w' \in L$ , so that  $\eta(w) = \eta(w') \in \eta(L)$ , as any element in  $\mathbf{H}$  is invertible. Thus,  $w \in L$ .  $\square$

## 6 Conclusion

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## References

- [1] Jorge Almeida. *Finite semigroups and universal algebra*, volume 3. World Scientific, 1994.
- [2] José Carlos Costa. Some pseudovariety joins involving locally trivial semigroups. In *Semigroup Forum*, volume 64, pages 12–28. Springer, 2001.
- [3] Samuel Eilenberg. *Automata, Languages, and Machines*, volume A. Academic Press, New York, 1974.
- [4] Samuel Eilenberg. *Automata, Languages, and Machines*, volume B. Academic Press, New York, 1976.
- [5] Nathan Grosshans, Pierre McKenzie, and Luc Segoufin. Tameness and the power of programs over monoids in DA. *CoRR*, abs/2101.07495, 2021. URL: <https://arxiv.org/abs/2101.07495>, arXiv:2101.07495.
- [6] Jean-Éric Pin. *Varieties of formal languages*. North Oxford, London and Plenum, New-York, 1986. (Traduction de Variétés de langages formels).
- [7] Jean-Éric Pin and Howard Straubing. Some results on  $\mathcal{C}$ -varieties. *RAIRO-Theor. Inf. Appl.*, 39(1):239–262, 2005.
- [8] Imre Simon. Piecewise testable events. In *Automata Theory and Formal Languages, 2nd GI Conference*, volume 33 of *Lecture Notes in Computer Science*, pages 214–222. Springer, 1975.
- [9] Marc Zeitoun. On the join of two pseudovarieties. *Semigroups, Automata and Languages*, eds. J. Almeida, GMS Gomes, and PV Silva. World Scientific, pages 281–288, 1996.