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A note on the join of varieties of monoids with \mathbf{LI}

Nathan GROSSHANS*

Abstract

In this note, we give a characterisation in terms of identities of the join of \mathbf{V} with \mathbf{LI} for several well-known varieties of monoids \mathbf{V} by using classical algebraic-automata-theoretic techniques. To achieve this, we use the new notion of essentially- \mathbf{V} stamps defined by Grosshans, McKenzie and Segoufin and show that it actually coincides with the join of \mathbf{V} and \mathbf{LI} precisely when some natural condition on the variety of languages corresponding to \mathbf{V} is verified.

This work is a kind of rediscovery of the work of J. C. Costa 20 years ago from a rather different angle, since Costa's work relies on the use of advanced developments in profinite topology, whereas what is presented here essentially uses an algebraic, language-based approach.

1 Introduction

The general endeavour to understand the join of two varieties has attracted many research efforts (see [1, 9]) and a non-negligible part of those efforts were concentrated on the question of characterising the join of \mathbf{LI} and some variety of monoids in terms of identities (see [2]).

In this note, we give a characterisation in terms of identities of $\mathbf{V} \vee \mathbf{LI}$ for several well-known varieties of monoids \mathbf{V} by using classical algebraic-automata-theoretic techniques. To achieve this, we use the new notion of essentially- \mathbf{V} stamps defined by McKenzie, Segoufin and the author in [5] and show that it actually coincides with $\mathbf{V} \vee \mathbf{LI}$ precisely when some natural condition on $\mathcal{L}(\mathbf{V})$ is verified.

This work is a kind of rediscovery of the work of J. C. Costa [2] from a rather different angle, since Costa's work relies on the use of advanced developments in profinite topology, whereas what is presented here essentially uses an algebraic, language-based approach.

2 Preliminaries

For the basics and the classical results of automata theory, we refer the reader to the two classical references of the domain by Eilenberg [3, 4] and Pin [6]. For definitions and results specific to \mathcal{C} -varieties of stamps and associated profinite identities, see the article by Pin and Straubing [7].

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Note that if \mathbf{V} is a variety of monoids, then $\langle \mathbf{V} \rangle_{all}$ is the *all*-variety of stamps of all stamps $\varphi: \Sigma^* \rightarrow M$ such that $M \in \mathbf{V}$. Note that in that case, the variety of languages corresponding to \mathbf{V} is exactly the *all*-variety of languages corresponding to $\langle \mathbf{V} \rangle_{all}$.

Similarly, if \mathbf{V} is a variety of semigroups, then $\langle \mathbf{V} \rangle_{ne}$ is the *ne*-variety of stamps of all stamps $\varphi: \Sigma^* \rightarrow M$ such that $\varphi(\Sigma^+) \in \mathbf{V}$. In that case, we consider $\mathcal{L}(\mathbf{V})$ to be the *ne*-variety of languages corresponding to $\langle \mathbf{V} \rangle_{ne}$.

The variety \mathbf{LI} of finite locally trivial semigroups is well-known to be defined by the identity $x^\omega y x^\omega = x^\omega$ and such that $\mathcal{L}(\mathbf{LI})$ is the class of languages that are Boolean combinations of languages of the form $u\Sigma^*$ or Σ^*u for Σ any alphabet and $u \in \Sigma^+$, or equivalently the class of languages of the form $U\Sigma^*V \cup W$ with $U, V, W \subseteq \Sigma^*$ finite for Σ any alphabet (see [6, p. 38]).

Given some variety of monoids \mathbf{V} , the join of \mathbf{V} and \mathbf{LI} , denoted by $\mathbf{V} \vee \mathbf{LI}$, is the inclusion-wise smallest variety of semigroups containing both \mathbf{V} and \mathbf{LI} . In fact, a finite semigroup S belongs to $\mathbf{V} \vee \mathbf{LI}$ if and only if there exist $M \in \mathbf{V}$ and $T \in \mathbf{LI}$ such that S divides the semigroup $M \times T$. (See [4, Chapter V, Exercise 1.1].)

Proposition 2.1. *Let \mathbf{V} be a variety of monoids. Then $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ is the inclusion-wise smallest *ne*-variety of languages containing both $\mathcal{L}(\mathbf{V})$ and $\mathcal{L}(\mathbf{LI})$ and is equal to the Boolean closure of $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{LI})$.*

3 Essentially- \mathbf{V} stamps

In this section, we give a characterisation of essentially- \mathbf{V} stamps (defined in [5]), for \mathbf{V} a variety of monoids, in terms of identities. We first recall the definition.

Definition 3.1. Let \mathbf{V} be a variety of monoids. Let $\varphi: \Sigma^* \rightarrow M$ be a stamp from an alphabet Σ to a finite monoid M . Let s be the stability index of φ .

We say that φ is *essentially- \mathbf{V}* whenever there exists a stamp $\mu: \Sigma^* \rightarrow N$ with $N \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s).$$

We will denote by \mathbf{EV} the class of all essentially- \mathbf{V} stamps.

Now, we give an equivalent condition for a stamp to be essentially- \mathbf{V} , based on a specific congruence depending on that stamp.

Definition 3.2. Let $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid be a stamp and let s be its stability index. We define the equivalence relation \equiv_φ on Σ^* by $u \equiv_\varphi v$ for $u, v \in \Sigma^*$ whenever $\varphi(xuy) = \varphi(xvy)$ for all $x, y \in \Sigma^{\geq s}$.

Proposition 3.3. *Let $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid be a stamp. Then \equiv_φ is a congruence of finite index and for any variety of monoids \mathbf{V} , we have $\varphi \in \mathbf{EV}$ if and only if $\Sigma^*/\equiv_\varphi \in \mathbf{V}$.*

Proof. Let us denote by s the stability index of φ .

The equivalence relation \equiv_φ is a congruence because given $u, v \in \Sigma^*$ verifying $u \equiv_\varphi v$, for all $\alpha, \beta \in \Sigma^*$, we have $\alpha u \beta \equiv_\varphi \alpha v \beta$ since for any $x, y \in \Sigma^{\geq s}$, it holds that $\varphi(x\alpha u \beta y) = \varphi(x\alpha v \beta y)$ because $x\alpha, \beta y \in \Sigma^{\geq s}$. Furthermore, this

congruence is of finite index because for all $u, v \in \Sigma^*$, we have that $\varphi(u) = \varphi(v)$ implies $u \equiv_\varphi v$.

Let now \mathbf{V} be a variety of monoids. Assume first that $\Sigma^*/\equiv_\varphi \in \mathbf{V}$. It is quite direct to see that $\varphi \in \mathbf{EV}$, as the stamp $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$ witnesses this fact. Assume then that $\varphi \in \mathbf{EV}$. This means that there exists a stamp $\mu: \Sigma^* \rightarrow N$ with $N \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

Now consider $u, v \in \Sigma^*$ such that $\mu(u) = \mu(v)$. For any $x, y \in \Sigma^{\geq s}$, we have that $x = x_1x_2$ with $x_1 \in \Sigma^*$ and $x_2 \in \Sigma^s$ as well as $y = y_1y_2$ with $y_1 \in \Sigma^s$ and $y_2 \in \Sigma^*$, so that $\varphi(xuy) = \varphi(x_1)\varphi(x_2uy_1)\varphi(y_2) = \varphi(x_1\varphi(x_2vy_1))\varphi(y_2) = \varphi(xvy)$. Hence, $u \equiv_\varphi v$. Therefore, for all $u, v \in \Sigma^*$, we have that $\mu(u) = \mu(v)$ implies $u \equiv_\varphi v$, so we can define the application $\alpha: N \rightarrow \Sigma^*/\equiv_\varphi$ such that $\alpha(\mu(w)) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. It is easy to check that α is actually a surjective morphism. Thus, we can conclude that Σ^*/\equiv_φ , which divides N , belongs to \mathbf{V} . \square

Using this equivalent condition, we prove that given a set of ne -identities defining $\langle \mathbf{V} \rangle_{all}$ for some variety of monoids \mathbf{V} , we get a set of ne -identities defining \mathbf{EV} , which actually shows that \mathbf{EV} is an ne -variety of stamps along the way.

Proposition 3.4. *Let \mathbf{V} be a variety of monoids and let E be a set of identities such that $\langle \mathbf{V} \rangle_{all} = \llbracket E \rrbracket_{ne}$. Then*

$$\mathbf{EV} = \llbracket x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \text{alph}(u) \cup \text{alph}(v) \rrbracket_{ne} .$$

Proof. Let

$$F = \{x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \text{alph}(u) \cup \text{alph}(v)\} .$$

Central to the proof is the following claim.

Claim 3.5. *Let $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid be a stamp. Consider the stamp $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. It holds that for all $u, v \in \widehat{\Sigma}^*$,*

$$\widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow (\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+) .$$

Using this claim, we can prove that $\mathbf{EV} = \llbracket F \rrbracket_{ne}$.

Inclusion from left to right. Let $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid be a stamp in \mathbf{EV} . Consider the stamp $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. Since $\varphi \in \mathbf{EV}$, Proposition 3.3 tells us that $\Sigma^*/\equiv_\varphi \in \mathbf{V}$, hence $\mu \in \langle \mathbf{V} \rangle_{all}$.

Let us consider any identity $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$. It is thus defined on an alphabet B with $u = v \in E$ defined on an alphabet $A \subseteq B$ and $x, y, z, t \in B \setminus A$. Let $f: B^* \rightarrow \Sigma^*$ be an ne -morphism. Since $\mu \in \langle \mathbf{V} \rangle_{all}$, we have that μ satisfies the identity $u = v$, so that $\widehat{\mu}(f(u)) = \widehat{\mu}(f(v))$. Notice that we have that

$\widehat{f}(x^\omega) = f(x)^\omega$ as well as $\widehat{f}(t^\omega) = f(t)^\omega$ and that $f(x), f(y), f(z), f(t) \in \Sigma^+$ because f is non-erasing. Therefore, we have

$$\begin{aligned}\widehat{\varphi}(\widehat{f}(x^\omega yuzt^\omega)) &= \widehat{\varphi}(f(x)^\omega f(y)\widehat{f}(u)f(z)f(t)^\omega) \\ &= \widehat{\varphi}(f(x)^\omega f(y)\widehat{f}(v)f(z)f(t)^\omega) \\ &= \widehat{\varphi}(\widehat{f}(x^\omega yvzt^\omega))\end{aligned}$$

by Claim 3.5. As this holds for any ne -morphism $f: B^* \rightarrow \Sigma^*$, we can conclude that φ satisfies the identity $x^\omega yuzt^\omega = x^\omega yvzt^\omega$.

This is true for any identity in F , so $\varphi \in \llbracket F \rrbracket_{ne}$. In conclusion, $\mathbf{EV} \subseteq \llbracket F \rrbracket_{ne}$.

Inclusion from right to left. Let $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid be a stamp in $\llbracket F \rrbracket_{ne}$. Consider the stamp $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. We are now going to show that $\mu \in \langle \mathbf{V} \rangle_{all}$.

Take any identity $u = v \in E$ defined on an alphabet A . There is thus an identity $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$ defined on an alphabet B such that $A \subseteq B$ and $x, y, z, t \in B \setminus A$. Let $f: A^* \rightarrow \Sigma^*$ be an ne -morphism.

Take any $\alpha, \beta, \gamma, \delta \in \Sigma^+$. Let us define the ne -morphism $g: B^* \rightarrow \Sigma^*$ as the unique one such that for all $b \in B$,

$$g(b) = \begin{cases} f(b) & \text{if } b \in A \\ \alpha & \text{if } b = x \\ \beta & \text{if } b = y \\ \gamma & \text{if } b = z \\ \delta & \text{if } b = t \end{cases} .$$

Observe in particular that $\widehat{g}(w) = \widehat{f}(w)$ for any $w \in \widehat{A}^*$ and that $\widehat{g}(x^\omega) = g(x)^\omega = \alpha^\omega$ as well as $\widehat{g}(t^\omega) = \delta^\omega$. Now, as φ satisfies $x^\omega yuzt^\omega = x^\omega yvzt^\omega$, we have that

$$\begin{aligned}\widehat{\varphi}(\alpha^\omega \beta \widehat{f}(u) \gamma \delta^\omega) &= \widehat{\varphi}(\widehat{g}(x^\omega yuzt^\omega)) \\ &= \widehat{\varphi}(\widehat{g}(x^\omega yvzt^\omega)) \\ &= \widehat{\varphi}(\alpha^\omega \beta \widehat{f}(v) \gamma \delta^\omega) .\end{aligned}$$

Since this holds for any $\alpha, \beta, \gamma, \delta \in \Sigma^+$, by Claim 3.5, we have that $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$.

Therefore, $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$ for any ne -morphism $f: A^* \rightarrow \Sigma^*$, which means that μ satisfies $u = v$.

So, to conclude, since this is true for any $u = v \in E$, we have that $\mu \in \langle \mathbf{V} \rangle_{all}$, which implies that $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ and thus $\varphi \in \mathbf{EV}$ by Proposition 3.3. In conclusion, $\llbracket F \rrbracket_{ne} \subseteq \mathbf{EV}$.

The claim still needs to be proved.

Proof of Claim 3.5. Let $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid be a stamp of stability index s . Consider the stamp $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. We now want to show that for all $u, v \in \widehat{\Sigma}^*$,

$$\widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow (\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+) .$$

Let $u, v \in \widehat{\Sigma}^*$. There exist two Cauchy sequences $(u_n)_{n \geq 0}$ and $(v_n)_{n \geq 0}$ in Σ^* such that $u = \lim_{n \rightarrow \infty} u_n$ and $v = \lim_{n \rightarrow \infty} v_n$. This means that

$$\widehat{\mu}(u) = \lim_{n \rightarrow \infty} \mu(u_n) \quad \text{and} \quad \widehat{\varphi}(u) = \lim_{n \rightarrow \infty} \varphi(u_n)$$

as well as

$$\widehat{\mu}(v) = \lim_{n \rightarrow \infty} \mu(v_n) \quad \text{and} \quad \widehat{\varphi}(v) = \lim_{n \rightarrow \infty} \varphi(v_n).$$

But as $\Sigma^*/\equiv_{\varphi}$ and M are discrete, we have that all four Cauchy sequences $(\mu(u_n))_{n \geq 0}$, $(\varphi(u_n))_{n \geq 0}$, $(\mu(v_n))_{n \geq 0}$ and $(\varphi(v_n))_{n \geq 0}$ are ultimately constant. So there exists $k \in \mathbb{N}$ such that $\widehat{\mu}(u) = \mu(u_k)$, $\widehat{\varphi}(u) = \varphi(u_k)$, $\widehat{\mu}(v) = \mu(v_k)$ and $\widehat{\varphi}(v) = \varphi(v_k)$.

Assume first that $\widehat{\mu}(u) = \widehat{\mu}(v)$. Take any $\alpha, \beta, \gamma, \delta \in \Sigma^+$. We have that

$$\widehat{\varphi}(\alpha^{\omega}) = \widehat{\varphi}(\lim_{n \rightarrow \infty} \alpha^{n!}) = \lim_{n \rightarrow \infty} \varphi(\alpha^{n!})$$

and $\widehat{\varphi}(\delta^{\omega}) = \lim_{n \rightarrow \infty} \varphi(\delta^{n!})$. Since M is discrete, both Cauchy sequences $(\varphi(\alpha^{n!}))_{n \geq 0}$ and $(\varphi(\delta^{n!}))_{n \geq 0}$ are ultimately constant. So there exists $l \in \mathbb{N}$ such that for all $m \in \mathbb{N}, m \geq l$, we have $\lim_{n \rightarrow \infty} \varphi(\alpha^{n!}) = \varphi(\alpha^{m!})$ and $\lim_{n \rightarrow \infty} \varphi(\delta^{n!}) = \varphi(\delta^{m!})$. Hence, taking some $m \in \mathbb{N}, m \geq l$ such that $|\alpha^{m!}\beta| \geq s$ and $|\gamma\delta^{m!}| \geq s$, it follows that

$$\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \varphi(\alpha^{m!} \beta u_k \gamma \delta^{m!}) = \varphi(\alpha^{m!} \beta v_k \gamma \delta^{m!}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$$

because $[u_k]_{\equiv_{\varphi}} = \widehat{\mu}(u) = \widehat{\mu}(v) = [v_k]_{\equiv_{\varphi}}$. Thus, we have that

$$\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$$

for all $\alpha, \beta, \gamma, \delta \in \Sigma^+$.

Assume then that $\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$ for all $\alpha, \beta, \gamma, \delta \in \Sigma^+$. Take any $\alpha, \beta \in \Sigma^{\geq s}$. Since $\varphi(\Sigma^s)$ is a finite semigroup and verifies that $\varphi(\Sigma^s) = \varphi(\Sigma^s)^2$, by a classical result in finite semigroup theory (see e.g. [6, Chapter 1, Proposition 1.12]), we have that there exist $\alpha_1, e, f, \beta_2 \in \Sigma^s$ and $\alpha_2, \beta_1 \in \Sigma^{\geq s}$ such that $\varphi(\alpha_1 e \alpha_2) = \varphi(\alpha)$ and $\varphi(\beta_1 f \beta_2) = \varphi(\beta)$ with $\varphi(e)$ and $\varphi(f)$ idempotents. Now, since $\varphi(e)$ is idempotent, we have that

$$\widehat{\varphi}(e^{\omega}) = \widehat{\varphi}(\lim_{n \rightarrow \infty} e^{n!}) = \lim_{n \rightarrow \infty} \varphi(e^{n!}) = \lim_{n \rightarrow \infty} \varphi(e)^{n!} = \varphi(e)$$

and similarly, $\widehat{\varphi}(f^{\omega}) = \varphi(f)$. So it follows that

$$\begin{aligned} \varphi(\alpha u_k \beta) &= \varphi(\alpha_1 e \alpha_2 u_k \beta_1 f \beta_2) \\ &= \widehat{\varphi}(\alpha_1 e^{\omega} \alpha_2 u \beta_1 f^{\omega} \beta_2) \\ &= \widehat{\varphi}(\alpha_1 e^{\omega} \alpha_2 v \beta_1 f^{\omega} \beta_2) \\ &= \varphi(\alpha_1 e \alpha_2 v_k \beta_1 f \beta_2) \\ &= \varphi(\alpha v_k \beta). \end{aligned}$$

As this is true for any $\alpha, \beta \in \Sigma^{\geq s}$, by definition it holds that $u_k \equiv_{\varphi} v_k$, hence $\widehat{\mu}(u) = \mu(u_k) = \mu(v_k) = \widehat{\mu}(v)$. \square

This concludes the proof of the proposition. \square

4 Essentially- \mathbf{V} stamps and the join of \mathbf{V} and \mathbf{LI}

In this section, we establish the link between essentially- \mathbf{V} stamps and $\mathbf{V} \vee \mathbf{LI}$ by giving a criterion that characterises exactly when they do correspond.

A first result we can prove is that any stamp whose semigroup obtained as image of the set of non-empty words belongs to $\mathbf{V} \vee \mathbf{LI}$ is essentially- \mathbf{V} .

Lemma 4.1. *Let \mathbf{V} be a variety of monoids and $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid a stamp such that $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$. Then φ is essentially- \mathbf{V} .*

Proof. Let $S = \varphi(\Sigma^+)$ and let s be the stability index of φ . Since $S \in \mathbf{V} \vee \mathbf{LI}$, there exist $N \in \mathbf{V}$ and $T \in \mathbf{LI}$ such that S divides the semigroup $N \times T$. So there exist a subsemigroup S' of $N \times T$ and a surjective morphism $\alpha: S' \rightarrow S$. We define $\pi_1: N \times T \rightarrow N$ and $\pi_2: N \times T \rightarrow T$ to be the projection morphisms from $N \times T$ onto N and T , respectively.

Let $h: S \rightarrow S'$ be an arbitrary mapping from S to S' such that $\alpha(h(r)) = r$ for all $r \in S$. We can then uniquely define the morphism $\varphi': \Sigma^+ \rightarrow S'$ such that $\varphi'(a) = h(\varphi(a))$ for all $a \in \Sigma$. We can also define the unique stamp $\mu: \Sigma^* \rightarrow N'$ such that $\mu(a) = \pi_1(\varphi'(a))$ for all $a \in \Sigma$ and $N' \in \mathbf{V}$ is the submonoid of N generated by $\{\pi_1(\varphi'(a)) \mid a \in \Sigma\}$. Hence, for all $w \in \Sigma^+$, we have that $\mu(w) = \pi_1(\varphi'(w))$ and that $\alpha(\varphi'(w)) = \varphi(w)$.

Now, take $u, v \in \Sigma^*$ such that $\mu(u) = \mu(v)$ and take any $x, y \in \Sigma^s$. Since $\varphi(\Sigma^s)$ is a finite semigroup and verifies that $\varphi(\Sigma^s) = \varphi(\Sigma^s)^2$, by a classical result in finite semigroup theory (see e.g. [6, Chapter 1, Proposition 1.12]), we have that there exist $x_1, e, x_2, y_1, f, y_2 \in \Sigma^s$ such that $\varphi(x_1 e x_2) = \varphi(x)$ and $\varphi(y_1 f y_2) = \varphi(y)$ with $\varphi(e)$ and $\varphi(f)$ idempotents. If we denote by ω the idempotent power of T , since S' is a subsemigroup of $N \times T$ with $T \in \mathbf{LI}$, it follows that

$$\begin{aligned} \pi_2(\varphi'(e^\omega x_2 u y_1 f^\omega)) &= \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(x_2 u y_1)) \pi_2(\varphi'(f))^\omega \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(f))^\omega \\ &= \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(f))^\omega \\ &= \pi_2(\varphi'(e^\omega x_2 v y_1 f^\omega)) , \end{aligned}$$

so that

$$\pi_2(\varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2)) = \pi_2(\varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2)) .$$

Moreover, from the hypothesis that $\mu(u) = \mu(v)$, we can deduce that

$$\begin{aligned} \pi_1(\varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2)) &= \pi_1(\varphi'(x_1 e^\omega x_2)) \mu(u) \pi_1(\varphi'(y_1 f^\omega y_2)) \\ &= \pi_1(\varphi'(x_1 e^\omega x_2)) \mu(v) \pi_1(\varphi'(y_1 f^\omega y_2)) \\ &= \pi_1(\varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2)) . \end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned} \varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2) &= \varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2) \\ \varphi(x_1 e^\omega x_2 u y_1 f^\omega y_2) &= \varphi(x_1 e^\omega x_2 v y_1 f^\omega y_2) \\ \varphi(x_1 e x_2 u y_1 f y_2) &= \varphi(x_1 e x_2 v y_1 f y_2) \\ \varphi(x u y) &= \varphi(x v y) . \end{aligned}$$

Hence, for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s),$$

so that $\varphi \in \mathbf{EV}$. \square

When does the converse of this statement hold? Consider the following criterion for a variety of monoids \mathbf{V} .

Criterion (A). For any $L \in \mathcal{L}(\mathbf{V})$ over some alphabet Σ , we have $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ for all $x, y \in \Sigma^*$.

It is a kind of mild closure condition that appears naturally as a necessary condition for the converse of Lemma 4.1 to be true. (Basically because for any variety of monoids \mathbf{V} , we always have that $xLy \in \mathcal{L}(\mathbf{EV})$ for any $L \in \mathcal{L}(\mathbf{V})$ over some alphabet Σ and $x, y \in \Sigma^*$.)

Lemma 4.2. Let \mathbf{V} be a variety of monoids such that for any stamp $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid, we have $\varphi \in \mathbf{EV}$ if and only if $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$. Then \mathbf{V} verifies criterion (A).

Proof. Let $L \in \mathcal{L}(\mathbf{V})$ over some alphabet Σ . Let $\mu: \Sigma^* \rightarrow N$ be its syntactic morphism: this means that $N \in \mathbf{V}$ and that there exists $F \subseteq N$ such that $L = \mu^{-1}(F)$.

Now take $x, y \in \Sigma^*$. Let $\varphi: \Sigma^* \rightarrow M$ be the syntactic morphism of the language $xLy = x\Sigma^*y \cap \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|}$ and let s be its stability index. We then consider some $u, v \in \Sigma^*$ such that $\mu(u) = \mu(v)$. Take any $x', y' \in \Sigma^*$ such that $|x'| \geq |x|$ and $|y'| \geq |y|$. We clearly have that $x'uy' \in x\Sigma^*y$ if and only if $x'vy' \in x\Sigma^*y$. Moreover, $x' = x'_1x'_2$ for some $x'_1 \in \Sigma^{|x|}$ and $x'_2 \in \Sigma^*$ and $y' = y'_1y'_2$ for some $y'_1 \in \Sigma^*$ and $y'_2 \in \Sigma^{|y|}$, so that

$$\begin{aligned} x'uy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|} &\Leftrightarrow \mu(x'_2uy'_1) \in F \\ &\Leftrightarrow \mu(x'_2vy'_1) \in F \\ &\Leftrightarrow x'vy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|}. \end{aligned}$$

Hence, $x'uy' \in xLy$ if and only if $x'vy' \in xLy$ for all $x', y' \in \Sigma^*$ such that $|x'| \geq |x|$ and $|y'| \geq |y|$, so that, by definition of the stability index s of φ , we have $\varphi(x'uy') = \varphi(x'vy')$ for all $x', y' \in \Sigma^s$. Thus, it follows that $\varphi \in \mathbf{EV}$.

By hypothesis, this means that $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$, that is, the syntactic semigroup of xLy belongs to $\mathbf{V} \vee \mathbf{LI}$. Therefore, $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$.

As it is true for any $L \in \mathcal{L}(\mathbf{V})$ over some alphabet Σ and any $x, y \in \Sigma^*$, we can conclude that \mathbf{V} verifies criterion (A). \square

But criterion (A) is also a sufficient condition for the converse of Lemma 4.1 to be true, as we will now prove.

Lemma 4.3. Let \mathbf{V} be a variety of monoids that verifies criterion (A). Then for any stamp $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid, we have $\varphi \in \mathbf{EV}$ if and only if $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$.

Proof. Let $\varphi: \Sigma^* \rightarrow M$ for Σ an alphabet and M a finite monoid be a stamp.

Assume first that $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$. Then, by Lemma 4.1, we have that $\varphi \in \mathbf{EV}$.

Assume now that $\varphi \in \mathbf{EV}$. Given s its stability index, this means there exists a stamp $\mu: \Sigma^* \rightarrow N$ with $N \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

As \mathbf{V} verifies criterion (A), for any $m \in N$ and $x, y \in \Sigma^s$, the syntactic morphism $\eta_{m,x,y}: \Sigma^* \rightarrow M_{m,x,y}$ of the language $x\mu^{-1}(m)y$ verifies $\eta_{m,x,y}(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$. Let $E = N \times \Sigma^s \times \Sigma^s$ and let us now define the unique stamp $\varphi': \Sigma^* \rightarrow S'^1$ such that $\varphi'(a) = \prod_{(m,x,y) \in E} \eta_{m,x,y}(a)$ for all $a \in \Sigma$ and S' is the subsemigroup of $\prod_{(m,x,y) \in E} \eta_{m,x,y}(\Sigma^+)$ generated by $\{\prod_{(m,x,y) \in E} \eta_{m,x,y}(a) \mid a \in \Sigma\}$. It is obvious that S' belongs to $\mathbf{V} \vee \mathbf{LI}$.

Let $w, w' \in \Sigma^+$ such that $\varphi'(w) = \varphi'(w')$: this means that $\eta_{m,x,y}(w) = \eta_{m,x,y}(w')$ for all $m \in N$ and $x, y \in \Sigma^s$. We are going to show that $\varphi(w) = \varphi(w')$, by considering two different cases.

- Either it is the case that $|w| \geq 2s$ or $|w'| \geq 2s$. Assume without loss of generality that $|w| \geq 2s$. We thus have that $w = xuy$ for some $x, y \in \Sigma^s$ and $u \in \Sigma^*$. This means that $xuy \in x\mu^{-1}(\mu(u))y$, but since $\eta_{\mu(u),x,y}(w) = \eta_{\mu(u),x,y}(w')$, we also have that $w' \in x\mu^{-1}(\mu(u))y$. Hence, $w' = xvy$ with $v \in \Sigma^*$ verifying $\mu(v) = \mu(u)$. So, to conclude, $\varphi(w) = \varphi(xuy) = \varphi(xvy) = \varphi(w')$.
- Or it is the case that $|w| < 2s$ and $|w'| < 2s$. Assume without loss of generality that $|w'| \leq |w| < 2s$. Take an arbitrary $z \in \Sigma^{2s-|w|}$: we thus have that $wz = xy$ for some $x, y \in \Sigma^s$, so that $wz \in x\mu^{-1}(1)y$ where we denote by 1 the identity of N . But since $\eta_{1,x,y}(w) = \eta_{1,x,y}(w')$, it must be that $w'z \in x\mu^{-1}(1)y$. However, the sole member of $x\mu^{-1}(1)y$ of length at most $2s$ is xy , so we have that $w'z = wz$ and hence necessarily that $w = w'$. This entails that $\varphi(w) = \varphi(w')$.

Hence, for all $w, w' \in \Sigma^+$, we have that $\varphi'(w) = \varphi'(w') \Rightarrow \varphi(w) = \varphi(w')$.

Let $S = \varphi(\Sigma^+)$. Let $\rho: S' \rightarrow \Sigma^+$ be an arbitrary mapping from S' to Σ^+ such that $\varphi'(\rho(s')) = s'$ for all $s' \in S'$. Define the mapping $\alpha: S' \rightarrow S$ by $\alpha(s') = \varphi(\rho(s'))$ for all $s' \in S'$.

Let $s'_1, s'_2 \in S'$. We have that $\alpha(s'_1 s'_2) = \varphi(\rho(s'_1 s'_2))$. Now, $\varphi'(\rho(s'_1 s'_2)) = s'_1 s'_2 = \varphi'(\rho(s'_1))\varphi'(\rho(s'_2)) = \varphi'(\rho(s'_1)\rho(s'_2))$. Hence,

$$\varphi(\rho(s'_1 s'_2)) = \varphi(\rho(s'_1)\rho(s'_2)) = \varphi(\rho(s'_1))\varphi(\rho(s'_2)) = \alpha(s'_1)\alpha(s'_2) .$$

So α is a morphism. Moreover, it is surjective because for all $t \in S$, there exists $w \in \Sigma^+$ such that $\varphi(w) = t$ and thus $\alpha(\varphi'(w)) = \varphi(\rho(\varphi'(w))) = \varphi(w)$ since $\varphi'(\rho(\varphi'(w))) = \varphi'(w)$.

This allows us to conclude that S divides S' and, thus, that $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$.

This concludes the proof of the lemma. \square

The following proposition summarises the results of this section.

Proposition 4.4. *Let \mathbf{V} be a variety of monoids. Then $(\mathbf{V} \vee \mathbf{LI})_{\text{ne}} \subseteq \mathbf{EV}$ and there is equality if and only if \mathbf{V} verifies criterion (A).*

5 Applications

In this last section, we use the link between essentially- \mathbf{V} stamps and $\mathbf{V} \vee \mathbf{LI}$ to reprove some characterisations of joins between \mathbf{LI} and some well-known varieties of monoids in terms of identities.

We first give an equivalent formulation of criterion (A) that solely depends on the variety of monoids \mathbf{V} , without explicit reference to \mathbf{LI} .

Proposition 5.1. *Let \mathbf{V} be a variety of monoids. Then \mathbf{V} satisfies criterion (A) if and only if for any $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ for some alphabet Σ and $x, y \in \Sigma^*$, there exist $k, l \in \Sigma^*$ such that for all $u \in \Sigma^k, v \in \Sigma^l$, there exists some language $K \in \mathcal{L}(\mathbf{V})$ over alphabet Σ verifying $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$.*

Proof. Let us first observe that given any alphabet Σ , given any language K on that alphabet and given any two words $x, y \in \Sigma^*$, we have that $x(x^{-1}Ky^{-1})y = x\Sigma^*y \cap K$ and $x^{-1}(xKy)y^{-1} = K$.

Inclusion from right to left. Assume that for any $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ for some alphabet Σ and $x, y \in \Sigma^*$, there exist $k, l \in \Sigma^*$ such that for all $u \in \Sigma^k, v \in \Sigma^l$, there exists some language $K \in \mathcal{L}(\mathbf{V})$ over alphabet Σ verifying $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$.

This implies, by our observation at the beginning of the proof, that

$$x(u\Sigma^*v \cap L)y = xu(u^{-1}Lv^{-1})vy = xu((xu)^{-1}K(vy)^{-1})vy = xu\Sigma^*vy \cap K$$

for all $u \in \Sigma^k, v \in \Sigma^l$. Using Proposition 2.1, we thus have that $x(u\Sigma^*v \cap L)y \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ for all $u \in \Sigma^k, v \in \Sigma^l$. Moreover, since we have that the set of words of L of length at least $k+l$ is

$$\Sigma^{k+l} \cap L = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (u\Sigma^*v \cap L)$$

and since

$$L = (\Sigma^{\geq k+l} \cap L) \cup F$$

where F is a finite set of words on Σ of length less than $k+l$, we have that

$$xLy = x((\Sigma^{\geq k+l} \cap L) \cup F)y = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} x(u\Sigma^*v \cap L)y \cup xFy.$$

We can thus conclude that $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ by using Proposition 2.1 as well as the fact that $xFy \in \mathcal{L}(\mathbf{LI})(\Sigma)$ and because $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ is closed under unions.

Inclusion from left to right. Assume that \mathbf{V} satisfies criterion (A). Take some $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ for some alphabet Σ and take $x, y \in \Sigma^*$. By hypothesis, we know that $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$.

By Proposition 2.1, this means that xLy is a Boolean combination of languages in $\mathcal{L}(\mathbf{V})(\Sigma) \cup \mathcal{L}(\mathbf{LI})(\Sigma)$. Further, this implies that xLy can be written as the union of intersections of languages of $\mathcal{L}(\mathbf{V})(\Sigma)$ and $\mathcal{L}(\mathbf{LI})(\Sigma)$ or their complements, which in turn implies, by closure of $\mathcal{L}(\mathbf{V})(\Sigma)$ and $\mathcal{L}(\mathbf{LI})(\Sigma)$ under Boolean operations, that xLy can be written as a finite union of languages

of the form $K \cap (U\Sigma^*V \cup W)$ with $K \in \mathcal{L}(\mathbf{V})(\Sigma)$ and $U, V, W \subseteq \Sigma^*$ finite. Since any word in xLy must be of length at least $|xy|$ and have x as a prefix and y as a suffix, we can assume that any language $K \cap (U\Sigma^*V \cup W)$ appearing in a finite union as described above verifies that $U \subseteq x\Sigma^*$, that $V \subseteq \Sigma^*y$ and that $W \subseteq x\Sigma^*y$. Now, if we take $k, l \in \mathbb{N}$ big enough, we thus have that

$$xLy = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (K_{u,v} \cap xu\Sigma^*vy) \cup F$$

where $K_{u,v} \in \mathcal{L}(\mathbf{V})(\Sigma)$ for all $u \in \Sigma^k, v \in \Sigma^l$ and $F \subseteq \Sigma^{<|xy|+k+l}$. Hence, for all $u \in \Sigma^k, v \in \Sigma^l$, we have

$$\begin{aligned} u^{-1}Lv^{-1} &= u^{-1}(x^{-1}(xLy)y^{-1})v^{-1} \\ &= (xu)^{-1} \left(\bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (K_{u',v'} \cap xu'\Sigma^*v'y) \cup F \right) (vy)^{-1} \\ &= \bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (xu)^{-1} \left((xu')^{-1} K_{u',v'} (v'y)^{-1} v'y \right) (vy)^{-1} \cup \\ &\quad (xu)^{-1} F (vy)^{-1} \\ &= (xu)^{-1} K_{u,v} (vy)^{-1}, \end{aligned}$$

observing that $(xu)^{-1}K(vy)^{-1} = \emptyset$ for any $K \subseteq \Sigma^*$ such that $K \cap xu\Sigma^*vy = \emptyset$. \square

We now prove the announced characterisations of joins between **LI** and some well-known varieties of monoids in terms of identities.

Theorem 5.2. *We have the following.*

1. $\langle \mathbf{R} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{ER} = \llbracket x^\omega y(ab)^\omega azt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$.
2. $\langle \mathbf{L} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EL} = \llbracket x^\omega yb(ab)^\omega zt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$.
3. $\langle \mathbf{J} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EJ} = \llbracket x^\omega y(ab)^\omega zt^\omega = x^\omega y(ba)^\omega zt^\omega, a^{\omega+1} = a^\omega \rrbracket_{\text{ne}}$.
4. $\langle \mathbf{H} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EH}$ for any variety of groups **H**.

Proof. In each case, we prove that the variety of monoids under consideration satisfies criterion (A) using Proposition 5.1. We then use Proposition 4.4.

Proof of 1. It is well-known that given an alphabet Σ , the set $\mathcal{L}(\mathbf{R})(\Sigma)$ contains all languages that are disjoint unions of languages that are of the form $A_0^*a_1A_1^*\cdots a_kA_k^*$ where $k \in \mathbb{N}$, $a_1, \dots, a_k \in \Sigma$, $A_0, A_1, \dots, A_k \subseteq \Sigma^*$ and $a_i \notin A_{i-1}$ for all $i \in [k]$ (see [6, Chapter 4, Theorem 3.3]).

Let Σ be an alphabet and take some language $A_0^*a_1A_1^*\cdots a_kA_k^*$ where $k \in \mathbb{N}$, $a_1, \dots, a_k \in \Sigma$, $A_0, A_1, \dots, A_k \subseteq \Sigma^*$ and $a_i \notin A_{i-1}$ for all $i \in [k]$. Take $x, y \in \Sigma^*$. Observe that y can be uniquely written as $y = zt$ where $z, t \in \Sigma^*$ and either $|t| = 0$ and $z \in A_k^*$, or $|t| > 0$ and $t_1 \notin A_k$. We have

$$\begin{aligned} &A_0^*a_1A_1^*\cdots a_kA_k^* \\ &= x^{-1} \left(xA_0^*a_1A_1^*\cdots a_kA_k^*t \cap \bigcap_{v \in A_k^{<|z|}} (xA_0^*a_1A_1^*\cdots a_kA_k^*vt)^{\mathfrak{G}} \right) y^{-1} \end{aligned}$$

where $x A_0^* a_1 A_1^* \cdots a_k A_k^* t \cap \bigcap_{v \in A_k^{<|z|}} (x A_0^* a_1 A_1^* \cdots a_k A_k^* t)^{\mathbf{G}}$ does belong to the set $\mathcal{L}(\mathbf{R})(\Sigma)$ because the latter is closed under Boolean operations and by construction of z and t . We can conclude that for each $L \in \mathcal{L}(\mathbf{R})(\Sigma)$ and $x, y \in \Sigma^*$, there exists $K \in \mathcal{L}(\mathbf{R})(\Sigma)$ such that $L = x^{-1} K y^{-1}$ by using the characterisation of $\mathcal{L}(\mathbf{R})(\Sigma)$ and the fact that $x^{-1} K_1 y^{-1} \cup x^{-1} K_2 y^{-1} = x^{-1} (K_1 \cup K_2) y^{-1}$ for all $K_1, K_2 \in \Sigma^*$.

Proof of 2. It is also well-known that given an alphabet Σ , the set $\mathcal{L}(\mathbf{L})(\Sigma)$ contains all languages that are disjoint unions of languages that are of the form $A_0^* a_1 A_1^* \cdots a_k A_k^*$ where $k \in \mathbb{N}$, $a_1, \dots, a_k \in \Sigma$, $A_0, A_1, \dots, A_k \subseteq \Sigma^*$ and $a_i \notin A_i$ for all $i \in [k]$ (see [6, Chapter 4, Theorem 3.4]). The proof is then dual to the previous case.

Proof of 3. Given an alphabet Σ , for each $k \in \mathbb{N}$, we define the equivalence relation \sim_k on Σ^* by $u \sim_k v$ for $u, v \in \Sigma^*$ whenever u and v have the same set of subwords of length at most k . This relation is a congruence of finite index on Σ^* . Simon proved [8] that for all any alphabet Σ , a language belongs to $\mathcal{L}(\mathbf{J})(\Sigma)$ if and only if it is equal to a union of \sim_k -classes for some $k \in \mathbb{N}$.

Let Σ be an alphabet and take some $L \in \mathcal{L}(\mathbf{J})(\Sigma)$ as well as $x, y \in \Sigma^*$. Thus, there exists $k \in \mathbb{N}$ such that L is a union of \sim_k -classes. Define the language $K = \bigcup_{w \in L} [xwy]_{\sim_{|xy|+k}}$: it belongs to $\mathcal{L}(\mathbf{J})(\Sigma)$ by construction. We now show that $L = x^{-1} K y^{-1}$, which concludes the proof. Let $w \in L$: we have that $xwy \in [xwy]_{\sim_{|xy|+k}} \subseteq K$, so that $w \in x^{-1} K y^{-1}$. Let conversely $w \in x^{-1} K y^{-1}$. This means that $xwy \in K$, which implies that there exists $w' \in L$ such that $xwy \sim_{|xy|+k} xw'y$. Actually, it holds that any $u \in \Sigma^*$ of length at most k is a subword of w if and only if it is a subword of w' , because xuy is a subword of xwy if and only if it is a subword of $xw'y$. Hence, $w \sim_k w'$, which implies that $w \in L$.

Proof of 3. Consider some variety of groups \mathbf{H} . Take some language $L \in \mathcal{L}(\mathbf{H})(\Sigma)$ for an alphabet Σ and let $x, y \in \Sigma^*$. Consider the syntactic morphism $\eta: \Sigma^* \rightarrow M$ of L : we have that M is a group in \mathbf{H} . Define the language $K = \eta^{-1}(\eta(x)\eta(L)\eta(y))$. We now show that $L = x^{-1} K y^{-1}$, which concludes the proof. Let $w \in L$: we have that $\eta(xwy) \in \eta(x)\eta(L)\eta(y)$, so that $w \in x^{-1} K y^{-1}$. Conversely, let $w \in x^{-1} K y^{-1}$. We have that $xwy \in K$, which means that $\eta(xwy) = \eta(x)\eta(w')\eta(y)$ for some $w' \in L$, so that $\eta(w) = \eta(w') \in \eta(L)$, as any element in \mathbf{H} is invertible. Thus, $w \in L$. \square

6 Conclusion

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