



**HAL**  
open science

## A note on the join of varieties of monoids with LI

Nathan Grosshans

► **To cite this version:**

| Nathan Grosshans. A note on the join of varieties of monoids with LI. 2021. hal-03183838v1

**HAL Id: hal-03183838**

**<https://hal.science/hal-03183838v1>**

Preprint submitted on 29 Mar 2021 (v1), last revised 27 Aug 2021 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A note on the join of varieties of monoids with **LI**

Nathan GROSSHANS\*

## Abstract

In this note, we give a characterisation in terms of identities of the join of  $\mathbf{V}$  with **LI** for several well-known varieties of monoids  $\mathbf{V}$  by using classical algebraic-automata-theoretic techniques. To achieve this, we use the new notion of essentially- $\mathbf{V}$  stamps defined by Grosshans, McKenzie and Segoufin and show that it actually coincides with the join of  $\mathbf{V}$  and **LI** precisely when some natural condition on the variety of languages corresponding to  $\mathbf{V}$  is verified.

This work is a kind of rediscovery of the work of J. C. Costa 20 years ago from a rather different angle, since Costa's work relies on the use of advanced developments in profinite topology, whereas what is presented here essentially uses an algebraic, language-based approach.

## 1 Introduction

The general endeavour to understand the join of two varieties has attracted many research efforts (see [1, 9]) and a non-negligible part of those efforts were concentrated on the question of characterising the join of **LI** and some variety of monoids in terms of identities (see [2]).

In this note, we give a characterisation in terms of identities of  $\mathbf{V} \vee \mathbf{LI}$  for several well-known varieties of monoids  $\mathbf{V}$  by using classical algebraic-automata-theoretic techniques. To achieve this, we use the new notion of essentially- $\mathbf{V}$  stamps defined by McKenzie, Segoufin and the author in [5] and show that it actually coincides with  $\mathbf{V} \vee \mathbf{LI}$  precisely when some natural condition on  $\mathcal{L}(\mathbf{V})$  is verified.

This work is a kind of rediscovery of the work of J. C. Costa [2] from a rather different angle, since Costa's work relies on the use of advanced developments in profinite topology, whereas what is presented here essentially uses an algebraic, language-based approach.

## 2 Preliminaries

For the basics and the classical results of automata theory, we refer the reader to the two classical references of the domain by Eilenberg [3, 4] and Pin [6]. For definitions and results specific to  $\mathcal{C}$ -varieties of stamps and associated profinite identities, see the article by Pin and Straubing [7].

---

\*Universität Kassel, Fachgebiet Elektrotechnik / Informatik, Kassel, Germany, [nathan.grosshans@polytechnique.edu](mailto:nathan.grosshans@polytechnique.edu), <https://nathan.grosshans.me>.

Note that if  $\mathbf{V}$  is a variety of monoids, then  $\langle \mathbf{V} \rangle_{all}$  is the *all*-variety of stamps of all stamps  $\varphi: \Sigma^* \rightarrow M$  such that  $M \in \mathbf{V}$ . Note that in that case, the variety of languages corresponding to  $\mathbf{V}$  is exactly the *all*-variety of languages corresponding to  $\langle \mathbf{V} \rangle_{all}$ .

Similarly, if  $\mathbf{V}$  is a variety of semigroups, then  $\langle \mathbf{V} \rangle_{ne}$  is the *ne*-variety of stamps of all stamps  $\varphi: \Sigma^* \rightarrow M$  such that  $\varphi(\Sigma^+) \in \mathbf{V}$ . In that case, we consider  $\mathcal{L}(\mathbf{V})$  to be the *ne*-variety of languages corresponding to  $\langle \mathbf{V} \rangle_{ne}$ .

The variety  $\mathbf{LI}$  of finite locally trivial semigroups is well-known to be defined by the identity  $x^\omega y x^\omega = x^\omega$  and such that  $\mathcal{L}(\mathbf{LI})$  is the class of languages that are Boolean combinations of languages of the form  $u\Sigma^*$  or  $\Sigma^*u$  for  $\Sigma$  any alphabet and  $u \in \Sigma^+$ , or equivalently the class of languages of the form  $U\Sigma^*V \cup W$  with  $U, V, W \subseteq \Sigma^*$  finite for  $\Sigma$  any alphabet (see [6, p. 38]).

Given some variety of monoids  $\mathbf{V}$ , the join of  $\mathbf{V}$  and  $\mathbf{LI}$ , denoted by  $\mathbf{V} \vee \mathbf{LI}$ , is the inclusion-wise smallest variety of semigroups containing both  $\mathbf{V}$  and  $\mathbf{LI}$ . In fact, a finite semigroup  $S$  belongs to  $\mathbf{V} \vee \mathbf{LI}$  if and only if there exist  $M \in \mathbf{V}$  and  $T \in \mathbf{LI}$  such that  $S$  divides the semigroup  $M \times T$ . (See [4, Chapter V, Exercise 1.1].)

**Proposition 2.1.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  is the inclusion-wise smallest *ne*-variety of languages containing both  $\mathcal{L}(\mathbf{V})$  and  $\mathcal{L}(\mathbf{LI})$  and is equal to the Boolean closure of  $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{LI})$ .*

### 3 Essentially- $\mathbf{V}$ stamps

In this section, we give a characterisation of essentially- $\mathbf{V}$  stamps (defined in [5]), for  $\mathbf{V}$  a variety of monoids, in terms of identities. We first recall the definition.

**Definition 3.1.** Let  $\mathbf{V}$  be a variety of monoids. Let  $\varphi: \Sigma^* \rightarrow M$  be a stamp from an alphabet  $\Sigma$  to a finite monoid  $M$ . Let  $s$  be the stability index of  $\varphi$ .

We say that  $\varphi$  is *essentially- $\mathbf{V}$*  whenever there exists a stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s).$$

We will denote by  $\mathbf{EV}$  the class of all essentially- $\mathbf{V}$  stamps.

Now, we give an equivalent condition for a stamp to be essentially- $\mathbf{V}$ , based on a specific congruence depending on that stamp.

**Definition 3.2.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp and let  $s$  be its stability index. We define the equivalence relation  $\equiv_\varphi$  on  $\Sigma^*$  by  $u \equiv_\varphi v$  for  $u, v \in \Sigma^*$  whenever  $\varphi(xuy) = \varphi(xvy)$  for all  $x, y \in \Sigma^{\geq s}$ .

**Proposition 3.3.** *Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp. Then  $\equiv_\varphi$  is a congruence of finite index and for any variety of monoids  $\mathbf{V}$ , we have  $\varphi \in \mathbf{EV}$  if and only if  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ .*

*Proof.* Let us denote by  $s$  the stability index of  $\varphi$ .

The equivalence relation  $\equiv_\varphi$  is a congruence because given  $u, v \in \Sigma^*$  verifying  $u \equiv_\varphi v$ , for all  $\alpha, \beta \in \Sigma^*$ , we have  $\alpha u \beta \equiv_\varphi \alpha v \beta$  since for any  $x, y \in \Sigma^{\geq s}$ , it holds that  $\varphi(x\alpha u \beta y) = \varphi(x\alpha v \beta y)$  because  $x\alpha, \beta y \in \Sigma^{\geq s}$ . Furthermore, this

congruence is of finite index because for all  $u, v \in \Sigma^*$ , we have that  $\varphi(u) = \varphi(v)$  implies  $u \equiv_\varphi v$ .

Let now  $\mathbf{V}$  be a variety of monoids. Assume first that  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ . It is quite direct to see that  $\varphi \in \mathbf{EV}$ , as the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$  witnesses this fact. Assume then that  $\varphi \in \mathbf{EV}$ . This means that there exists a stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

Now consider  $u, v \in \Sigma^*$  such that  $\mu(u) = \mu(v)$ . For any  $x, y \in \Sigma^{\geq s}$ , we have that  $x = x_1x_2$  with  $x_1 \in \Sigma^*$  and  $x_2 \in \Sigma^s$  as well as  $y = y_1y_2$  with  $y_1 \in \Sigma^s$  and  $y_2 \in \Sigma^*$ , so that  $\varphi(xuy) = \varphi(x_1)\varphi(x_2uy_1)\varphi(y_2) = \varphi(x_1\varphi(x_2vy_1))\varphi(y_2) = \varphi(xvy)$ . Hence,  $u \equiv_\varphi v$ . Therefore, for all  $u, v \in \Sigma^*$ , we have that  $\mu(u) = \mu(v)$  implies  $u \equiv_\varphi v$ , so we can define the application  $\alpha: N \rightarrow \Sigma^*/\equiv_\varphi$  such that  $\alpha(\mu(w)) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . It is easy to check that  $\alpha$  is actually a surjective morphism. Thus, we can conclude that  $\Sigma^*/\equiv_\varphi$ , which divides  $N$ , belongs to  $\mathbf{V}$ .  $\square$

Using this equivalent condition, we prove that given a set of  $ne$ -identities defining  $\langle \mathbf{V} \rangle_{all}$  for some variety of monoids  $\mathbf{V}$ , we get a set of  $ne$ -identities defining  $\mathbf{EV}$ , which actually shows that  $\mathbf{EV}$  is an  $ne$ -variety of stamps along the way.

**Proposition 3.4.** *Let  $\mathbf{V}$  be a variety of monoids and let  $E$  be a set of identities such that  $\langle \mathbf{V} \rangle_{all} = \llbracket E \rrbracket_{ne}$ . Then*

$$\mathbf{EV} = \llbracket x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \text{alph}(u) \cup \text{alph}(v) \rrbracket_{ne} .$$

*Proof.* Let

$$F = \{x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \text{alph}(u) \cup \text{alph}(v)\} .$$

Central to the proof is the following claim.

**Claim 3.5.** *Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp. Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . It holds that for all  $u, v \in \widehat{\Sigma}^*$ ,*

$$\widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow (\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+) .$$

Using this claim, we can prove that  $\mathbf{EV} = \llbracket F \rrbracket_{ne}$ .

**Inclusion from left to right.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp in  $\mathbf{EV}$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . Since  $\varphi \in \mathbf{EV}$ , Proposition 3.3 tells us that  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$ , hence  $\mu \in \langle \mathbf{V} \rangle_{all}$ .

Let us consider any identity  $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$ . It is thus defined on an alphabet  $B$  with  $u = v \in E$  defined on an alphabet  $A \subseteq B$  and  $x, y, z, t \in B \setminus A$ . Let  $f: B^* \rightarrow \Sigma^*$  be an  $ne$ -morphism. Since  $\mu \in \langle \mathbf{V} \rangle_{all}$ , we have that  $\mu$  satisfies the identity  $u = v$ , so that  $\widehat{\mu}(f(u)) = \widehat{\mu}(f(v))$ . Notice that we have that

$\widehat{f}(x^\omega) = f(x)^\omega$  as well as  $\widehat{f}(t^\omega) = f(t)^\omega$  and that  $f(x), f(y), f(z), f(t) \in \Sigma^+$  because  $f$  is non-erasing. Therefore, we have

$$\begin{aligned}\widehat{\varphi}(\widehat{f}(x^\omega yuzt^\omega)) &= \widehat{\varphi}(f(x)^\omega f(y)\widehat{f}(u)f(z)f(t)^\omega) \\ &= \widehat{\varphi}(f(x)^\omega f(y)\widehat{f}(v)f(z)f(t)^\omega) \\ &= \widehat{\varphi}(\widehat{f}(x^\omega yvzt^\omega))\end{aligned}$$

by Claim 3.5. As this holds for any  $ne$ -morphism  $f: B^* \rightarrow \Sigma^*$ , we can conclude that  $\varphi$  satisfies the identity  $x^\omega yuzt^\omega = x^\omega yvzt^\omega$ .

This is true for any identity in  $F$ , so  $\varphi \in \llbracket F \rrbracket_{ne}$ . In conclusion,  $\mathbf{EV} \subseteq \llbracket F \rrbracket_{ne}$ .

**Inclusion from right to left.** Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp in  $\llbracket F \rrbracket_{ne}$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . We are now going to show that  $\mu \in \langle \mathbf{V} \rangle_{all}$ .

Take any identity  $u = v \in E$  defined on an alphabet  $A$ . There is thus an identity  $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$  defined on an alphabet  $B$  such that  $A \subseteq B$  and  $x, y, z, t \in B \setminus A$ . Let  $f: A^* \rightarrow \Sigma^*$  be an  $ne$ -morphism.

Take any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . Let us define the  $ne$ -morphism  $g: B^* \rightarrow \Sigma^*$  as the unique one such that for all  $b \in B$ ,

$$g(b) = \begin{cases} f(b) & \text{if } b \in A \\ \alpha & \text{if } b = x \\ \beta & \text{if } b = y \\ \gamma & \text{if } b = z \\ \delta & \text{if } b = t \end{cases} .$$

Observe in particular that  $\widehat{g}(w) = \widehat{f}(w)$  for any  $w \in \widehat{A}^*$  and that  $\widehat{g}(x^\omega) = g(x)^\omega = \alpha^\omega$  as well as  $\widehat{g}(t^\omega) = \delta^\omega$ . Now, as  $\varphi$  satisfies  $x^\omega yuzt^\omega = x^\omega yvzt^\omega$ , we have that

$$\begin{aligned}\widehat{\varphi}(\alpha^\omega \beta \widehat{f}(u) \gamma \delta^\omega) &= \widehat{\varphi}(\widehat{g}(x^\omega yuzt^\omega)) \\ &= \widehat{\varphi}(\widehat{g}(x^\omega yvzt^\omega)) \\ &= \widehat{\varphi}(\alpha^\omega \beta \widehat{f}(v) \gamma \delta^\omega) .\end{aligned}$$

Since this holds for any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ , by Claim 3.5, we have that  $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$ .

Therefore,  $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$  for any  $ne$ -morphism  $f: A^* \rightarrow \Sigma^*$ , which means that  $\mu$  satisfies  $u = v$ .

So, to conclude, since this is true for any  $u = v \in E$ , we have that  $\mu \in \langle \mathbf{V} \rangle_{all}$ , which implies that  $\Sigma^*/\equiv_\varphi \in \mathbf{V}$  and thus  $\varphi \in \mathbf{EV}$  by Proposition 3.3. In conclusion,  $\llbracket F \rrbracket_{ne} \subseteq \mathbf{EV}$ .

The claim still needs to be proved.

*Proof of Claim 3.5.* Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp of stability index  $s$ . Consider the stamp  $\mu: \Sigma^* \rightarrow \Sigma^*/\equiv_\varphi$  defined by  $\mu(w) = [w]_{\equiv_\varphi}$  for all  $w \in \Sigma^*$ . We now want to show that for all  $u, v \in \widehat{\Sigma}^*$ ,

$$\widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow (\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+) .$$

Let  $u, v \in \widehat{\Sigma}^*$ . There exist two Cauchy sequences  $(u_n)_{n \geq 0}$  and  $(v_n)_{n \geq 0}$  in  $\Sigma^*$  such that  $u = \lim_{n \rightarrow \infty} u_n$  and  $v = \lim_{n \rightarrow \infty} v_n$ . This means that

$$\widehat{\mu}(u) = \lim_{n \rightarrow \infty} \mu(u_n) \quad \text{and} \quad \widehat{\varphi}(u) = \lim_{n \rightarrow \infty} \varphi(u_n)$$

as well as

$$\widehat{\mu}(v) = \lim_{n \rightarrow \infty} \mu(v_n) \quad \text{and} \quad \widehat{\varphi}(v) = \lim_{n \rightarrow \infty} \varphi(v_n).$$

But as  $\Sigma^*/\equiv_{\varphi}$  and  $M$  are discrete, we have that all four Cauchy sequences  $(\mu(u_n))_{n \geq 0}$ ,  $(\varphi(u_n))_{n \geq 0}$ ,  $(\mu(v_n))_{n \geq 0}$  and  $(\varphi(v_n))_{n \geq 0}$  are ultimately constant. So there exists  $k \in \mathbb{N}$  such that  $\widehat{\mu}(u) = \mu(u_k)$ ,  $\widehat{\varphi}(u) = \varphi(u_k)$ ,  $\widehat{\mu}(v) = \mu(v_k)$  and  $\widehat{\varphi}(v) = \varphi(v_k)$ .

Assume first that  $\widehat{\mu}(u) = \widehat{\mu}(v)$ . Take any  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . We have that

$$\widehat{\varphi}(\alpha^{\omega}) = \widehat{\varphi}(\lim_{n \rightarrow \infty} \alpha^{n!}) = \lim_{n \rightarrow \infty} \varphi(\alpha^{n!})$$

and  $\widehat{\varphi}(\delta^{\omega}) = \lim_{n \rightarrow \infty} \varphi(\delta^{n!})$ . Since  $M$  is discrete, both Cauchy sequences  $(\varphi(\alpha^{n!}))_{n \geq 0}$  and  $(\varphi(\delta^{n!}))_{n \geq 0}$  are ultimately constant. So there exists  $l \in \mathbb{N}$  such that for all  $m \in \mathbb{N}, m \geq l$ , we have  $\lim_{n \rightarrow \infty} \varphi(\alpha^{n!}) = \varphi(\alpha^{m!})$  and  $\lim_{n \rightarrow \infty} \varphi(\delta^{n!}) = \varphi(\delta^{m!})$ . Hence, taking some  $m \in \mathbb{N}, m \geq l$  such that  $|\alpha^{m!}\beta| \geq s$  and  $|\gamma\delta^{m!}| \geq s$ , it follows that

$$\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \varphi(\alpha^{m!} \beta u_k \gamma \delta^{m!}) = \varphi(\alpha^{m!} \beta v_k \gamma \delta^{m!}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$$

because  $[u_k]_{\equiv_{\varphi}} = \widehat{\mu}(u) = \widehat{\mu}(v) = [v_k]_{\equiv_{\varphi}}$ . Thus, we have that

$$\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$$

for all  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ .

Assume then that  $\widehat{\varphi}(\alpha^{\omega} \beta u \gamma \delta^{\omega}) = \widehat{\varphi}(\alpha^{\omega} \beta v \gamma \delta^{\omega})$  for all  $\alpha, \beta, \gamma, \delta \in \Sigma^+$ . Take any  $\alpha, \beta \in \Sigma^{\geq s}$ . Since  $\varphi(\Sigma^s)$  is a finite semigroup and verifies that  $\varphi(\Sigma^s) = \varphi(\Sigma^s)^2$ , by a classical result in finite semigroup theory (see e.g. [6, Chapter 1, Proposition 1.12]), we have that there exist  $\alpha_1, e, f, \beta_2 \in \Sigma^s$  and  $\alpha_2, \beta_1 \in \Sigma^{\geq s}$  such that  $\varphi(\alpha_1 e \alpha_2) = \varphi(\alpha)$  and  $\varphi(\beta_1 f \beta_2) = \varphi(\beta)$  with  $\varphi(e)$  and  $\varphi(f)$  idempotents. Now, since  $\varphi(e)$  is idempotent, we have that

$$\widehat{\varphi}(e^{\omega}) = \widehat{\varphi}(\lim_{n \rightarrow \infty} e^{n!}) = \lim_{n \rightarrow \infty} \varphi(e^{n!}) = \lim_{n \rightarrow \infty} \varphi(e)^{n!} = \varphi(e)$$

and similarly,  $\widehat{\varphi}(f^{\omega}) = \varphi(f)$ . So it follows that

$$\begin{aligned} \varphi(\alpha u_k \beta) &= \varphi(\alpha_1 e \alpha_2 u_k \beta_1 f \beta_2) \\ &= \widehat{\varphi}(\alpha_1 e^{\omega} \alpha_2 u \beta_1 f^{\omega} \beta_2) \\ &= \widehat{\varphi}(\alpha_1 e^{\omega} \alpha_2 v \beta_1 f^{\omega} \beta_2) \\ &= \varphi(\alpha_1 e \alpha_2 v_k \beta_1 f \beta_2) \\ &= \varphi(\alpha v_k \beta). \end{aligned}$$

As this is true for any  $\alpha, \beta \in \Sigma^{\geq s}$ , by definition it holds that  $u_k \equiv_{\varphi} v_k$ , hence  $\widehat{\mu}(u) = \mu(u_k) = \mu(v_k) = \widehat{\mu}(v)$ .  $\square$

This concludes the proof of the proposition.  $\square$

## 4 Essentially- $\mathbf{V}$ stamps and the join of $\mathbf{V}$ and $\mathbf{LI}$

In this section, we establish the link between essentially- $\mathbf{V}$  stamps and  $\mathbf{V} \vee \mathbf{LI}$  by giving a criterion that characterises exactly when they do correspond.

A first result we can prove is that any stamp whose semigroup obtained as image of the set of non-empty words belongs to  $\mathbf{V} \vee \mathbf{LI}$  is essentially- $\mathbf{V}$ .

**Lemma 4.1.** *Let  $\mathbf{V}$  be a variety of monoids and  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid a stamp such that  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ . Then  $\varphi$  is essentially- $\mathbf{V}$ .*

*Proof.* Let  $S = \varphi(\Sigma^+)$  and let  $s$  be the stability index of  $\varphi$ . Since  $S \in \mathbf{V} \vee \mathbf{LI}$ , there exist  $N \in \mathbf{V}$  and  $T \in \mathbf{LI}$  such that  $S$  divides the semigroup  $N \times T$ . So there exist a subsemigroup  $S'$  of  $N \times T$  and a surjective morphism  $\alpha: S' \rightarrow S$ . We define  $\pi_1: N \times T \rightarrow N$  and  $\pi_2: N \times T \rightarrow T$  to be the projection morphisms from  $N \times T$  onto  $N$  and  $T$ , respectively.

Let  $h: S \rightarrow S'$  be an arbitrary mapping from  $S$  to  $S'$  such that  $\alpha(h(r)) = r$  for all  $r \in S$ . We can then uniquely define the morphism  $\varphi': \Sigma^+ \rightarrow S'$  such that  $\varphi'(a) = h(\varphi(a))$  for all  $a \in \Sigma$ . We can also define the unique stamp  $\mu: \Sigma^* \rightarrow N'$  such that  $\mu(a) = \pi_1(\varphi'(a))$  for all  $a \in \Sigma$  and  $N' \in \mathbf{V}$  is the submonoid of  $N$  generated by  $\{\pi_1(\varphi'(a)) \mid a \in \Sigma\}$ . Hence, for all  $w \in \Sigma^+$ , we have that  $\mu(w) = \pi_1(\varphi'(w))$  and that  $\alpha(\varphi'(w)) = \varphi(w)$ .

Now, take  $u, v \in \Sigma^*$  such that  $\mu(u) = \mu(v)$  and take any  $x, y \in \Sigma^s$ . Since  $\varphi(\Sigma^s)$  is a finite semigroup and verifies that  $\varphi(\Sigma^s) = \varphi(\Sigma^s)^2$ , by a classical result in finite semigroup theory (see e.g. [6, Chapter 1, Proposition 1.12]), we have that there exist  $x_1, e, x_2, y_1, f, y_2 \in \Sigma^s$  such that  $\varphi(x_1 e x_2) = \varphi(x)$  and  $\varphi(y_1 f y_2) = \varphi(y)$  with  $\varphi(e)$  and  $\varphi(f)$  idempotents. If we denote by  $\omega$  the idempotent power of  $T$ , since  $S'$  is a subsemigroup of  $N \times T$  with  $T \in \mathbf{LI}$ , it follows that

$$\begin{aligned} \pi_2(\varphi'(e^\omega x_2 u y_1 f^\omega)) &= \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(x_2 u y_1)) \pi_2(\varphi'(f))^\omega \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(f))^\omega \\ &= \pi_2(\varphi'(e))^\omega \pi_2(\varphi'(f))^\omega \\ &= \pi_2(\varphi'(e^\omega x_2 v y_1 f^\omega)) , \end{aligned}$$

so that

$$\pi_2(\varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2)) = \pi_2(\varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2)) .$$

Moreover, from the hypothesis that  $\mu(u) = \mu(v)$ , we can deduce that

$$\begin{aligned} \pi_1(\varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2)) &= \pi_1(\varphi'(x_1 e^\omega x_2)) \mu(u) \pi_1(\varphi'(y_1 f^\omega y_2)) \\ &= \pi_1(\varphi'(x_1 e^\omega x_2)) \mu(v) \pi_1(\varphi'(y_1 f^\omega y_2)) \\ &= \pi_1(\varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2)) . \end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned} \varphi'(x_1 e^\omega x_2 u y_1 f^\omega y_2) &= \varphi'(x_1 e^\omega x_2 v y_1 f^\omega y_2) \\ \varphi(x_1 e^\omega x_2 u y_1 f^\omega y_2) &= \varphi(x_1 e^\omega x_2 v y_1 f^\omega y_2) \\ \varphi(x_1 e x_2 u y_1 f y_2) &= \varphi(x_1 e x_2 v y_1 f y_2) \\ \varphi(x u y) &= \varphi(x v y) . \end{aligned}$$

Hence, for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s),$$

so that  $\varphi \in \mathbf{EV}$ .  $\square$

When does the converse of this statement hold? Consider the following criterion for a variety of monoids  $\mathbf{V}$ .

**Criterion (A).** For any  $L \in \mathcal{L}(\mathbf{V})$  over some alphabet  $\Sigma$ , we have  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  for all  $x, y \in \Sigma^*$ .

It is a kind of mild closure condition that appears naturally as a necessary condition for the converse of Lemma 4.1 to be true. (Basically because for any variety of monoids  $\mathbf{V}$ , we always have that  $xLy \in \mathcal{L}(\mathbf{EV})$  for any  $L \in \mathcal{L}(\mathbf{V})$  over some alphabet  $\Sigma$  and  $x, y \in \Sigma^*$ .)

**Lemma 4.2.** Let  $\mathbf{V}$  be a variety of monoids such that for any stamp  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid, we have  $\varphi \in \mathbf{EV}$  if and only if  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ . Then  $\mathbf{V}$  verifies criterion (A).

*Proof.* Let  $L \in \mathcal{L}(\mathbf{V})$  over some alphabet  $\Sigma$ . Let  $\mu: \Sigma^* \rightarrow N$  be its syntactic morphism: this means that  $N \in \mathbf{V}$  and that there exists  $F \subseteq N$  such that  $L = \mu^{-1}(F)$ .

Now take  $x, y \in \Sigma^*$ . Let  $\varphi: \Sigma^* \rightarrow M$  be the syntactic morphism of the language  $xLy = x\Sigma^*y \cap \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|}$  and let  $s$  be its stability index. We then consider some  $u, v \in \Sigma^*$  such that  $\mu(u) = \mu(v)$ . Take any  $x', y' \in \Sigma^*$  such that  $|x'| \geq |x|$  and  $|y'| \geq |y|$ . We clearly have that  $x'uy' \in x\Sigma^*y$  if and only if  $x'vy' \in x\Sigma^*y$ . Moreover,  $x' = x'_1x'_2$  for some  $x'_1 \in \Sigma^{|x|}$  and  $x'_2 \in \Sigma^*$  and  $y' = y'_1y'_2$  for some  $y'_1 \in \Sigma^*$  and  $y'_2 \in \Sigma^{|y|}$ , so that

$$\begin{aligned} x'uy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|} &\Leftrightarrow \mu(x'_2uy'_1) \in F \\ &\Leftrightarrow \mu(x'_2vy'_1) \in F \\ &\Leftrightarrow x'vy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|}. \end{aligned}$$

Hence,  $x'uy' \in xLy$  if and only if  $x'vy' \in xLy$  for all  $x', y' \in \Sigma^*$  such that  $|x'| \geq |x|$  and  $|y'| \geq |y|$ , so that, by definition of the stability index  $s$  of  $\varphi$ , we have  $\varphi(x'uy') = \varphi(x'vy')$  for all  $x', y' \in \Sigma^s$ . Thus, it follows that  $\varphi \in \mathbf{EV}$ .

By hypothesis, this means that  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ , that is, the syntactic semigroup of  $xLy$  belongs to  $\mathbf{V} \vee \mathbf{LI}$ . Therefore,  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ .

As it is true for any  $L \in \mathcal{L}(\mathbf{V})$  over some alphabet  $\Sigma$  and any  $x, y \in \Sigma^*$ , we can conclude that  $\mathbf{V}$  verifies criterion (A).  $\square$

But criterion (A) is also a sufficient condition for the converse of Lemma 4.1 to be true, as we will now prove.

**Lemma 4.3.** Let  $\mathbf{V}$  be a variety of monoids that verifies criterion (A). Then for any stamp  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid, we have  $\varphi \in \mathbf{EV}$  if and only if  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ .

*Proof.* Let  $\varphi: \Sigma^* \rightarrow M$  for  $\Sigma$  an alphabet and  $M$  a finite monoid be a stamp.

Assume first that  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ . Then, by Lemma 4.1, we have that  $\varphi \in \mathbf{EV}$ .



Assume now that  $\varphi \in \mathbf{EV}$ . Given  $s$  its stability index, this means there exists a stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

As  $\mathbf{V}$  verifies criterion (A), for any  $m \in N$  and  $x, y \in \Sigma^s$ , the syntactic morphism  $\eta_{m,x,y}: \Sigma^* \rightarrow M_{m,x,y}$  of the language  $x\mu^{-1}(m)y$  verifies  $\eta_{m,x,y}(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ . Let  $E = N \times \Sigma^s \times \Sigma^s$  and let us now define the unique stamp  $\varphi': \Sigma^* \rightarrow S'^1$  such that  $\varphi'(a) = \prod_{(m,x,y) \in E} \eta_{m,x,y}(a)$  for all  $a \in \Sigma$  and  $S'$  is the subsemigroup of  $\prod_{(m,x,y) \in E} \eta_{m,x,y}(\Sigma^+)$  generated by  $\{\prod_{(m,x,y) \in E} \eta_{m,x,y}(a) \mid a \in \Sigma\}$ . It is obvious that  $S'$  belongs to  $\mathbf{V} \vee \mathbf{LI}$ .

Let  $w, w' \in \Sigma^+$  such that  $\varphi'(w) = \varphi'(w')$ : this means that  $\eta_{m,x,y}(w) = \eta_{m,x,y}(w')$  for all  $m \in N$  and  $x, y \in \Sigma^s$ . We are going to show that  $\varphi(w) = \varphi(w')$ , by considering two different cases.

- Either it is the case that  $|w| \geq 2s$  or  $|w'| \geq 2s$ . Assume without loss of generality that  $|w| \geq 2s$ . We thus have that  $w = xuy$  for some  $x, y \in \Sigma^s$  and  $u \in \Sigma^*$ . This means that  $xuy \in x\mu^{-1}(\mu(u))y$ , but since  $\eta_{\mu(u),x,y}(w) = \eta_{\mu(u),x,y}(w')$ , we also have that  $w' \in x\mu^{-1}(\mu(u))y$ . Hence,  $w' = xvy$  with  $v \in \Sigma^*$  verifying  $\mu(v) = \mu(u)$ . So, to conclude,  $\varphi(w) = \varphi(xuy) = \varphi(xvy) = \varphi(w')$ .
- Or it is the case that  $|w| < 2s$  and  $|w'| < 2s$ . Assume without loss of generality that  $|w'| \leq |w| < 2s$ . Take an arbitrary  $z \in \Sigma^{2s-|w|}$ : we thus have that  $wz = xy$  for some  $x, y \in \Sigma^s$ , so that  $wz \in x\mu^{-1}(1)y$  where we denote by 1 the identity of  $N$ . But since  $\eta_{1,x,y}(w) = \eta_{1,x,y}(w')$ , it must be that  $w'z \in x\mu^{-1}(1)y$ . However, the sole member of  $x\mu^{-1}(1)y$  of length at most  $2s$  is  $xy$ , so we have that  $w'z = wz$  and hence necessarily that  $w = w'$ . This entails that  $\varphi(w) = \varphi(w')$ .

Hence, for all  $w, w' \in \Sigma^+$ , we have that  $\varphi'(w) = \varphi'(w') \Rightarrow \varphi(w) = \varphi(w')$ .

Let  $S = \varphi(\Sigma^+)$ . Let  $\rho: S' \rightarrow \Sigma^+$  be an arbitrary mapping from  $S'$  to  $\Sigma^+$  such that  $\varphi'(\rho(s')) = s'$  for all  $s' \in S'$ . Define the mapping  $\alpha: S' \rightarrow S$  by  $\alpha(s') = \varphi(\rho(s'))$  for all  $s' \in S'$ .

Let  $s'_1, s'_2 \in S'$ . We have that  $\alpha(s'_1 s'_2) = \varphi(\rho(s'_1 s'_2))$ . Now,  $\varphi'(\rho(s'_1 s'_2)) = s'_1 s'_2 = \varphi'(\rho(s'_1))\varphi'(\rho(s'_2)) = \varphi'(\rho(s'_1)\rho(s'_2))$ . Hence,

$$\varphi(\rho(s'_1 s'_2)) = \varphi(\rho(s'_1)\rho(s'_2)) = \varphi(\rho(s'_1))\varphi(\rho(s'_2)) = \alpha(s'_1)\alpha(s'_2) .$$

So  $\alpha$  is a morphism. Moreover, it is surjective because for all  $t \in S$ , there exists  $w \in \Sigma^+$  such that  $\varphi(w) = t$  and thus  $\alpha(\varphi'(w)) = \varphi(\rho(\varphi'(w))) = \varphi(w)$  since  $\varphi'(\rho(\varphi'(w))) = \varphi'(w)$ .

This allows us to conclude that  $S$  divides  $S'$  and, thus, that  $\varphi(\Sigma^+) \in \mathbf{V} \vee \mathbf{LI}$ .

This concludes the proof of the lemma.  $\square$

The following proposition summarises the results of this section.

**Proposition 4.4.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $(\mathbf{V} \vee \mathbf{LI})_{\text{ne}} \subseteq \mathbf{EV}$  and there is equality if and only if  $\mathbf{V}$  verifies criterion (A).*

## 5 Applications

In this last section, we use the link between essentially- $\mathbf{V}$  stamps and  $\mathbf{V} \vee \mathbf{LI}$  to reprove some characterisations of joins between  $\mathbf{LI}$  and some well-known varieties of monoids in terms of identities.

We first give an equivalent formulation of criterion (A) that solely depends on the variety of monoids  $\mathbf{V}$ , without explicit reference to  $\mathbf{LI}$ .

**Proposition 5.1.** *Let  $\mathbf{V}$  be a variety of monoids. Then  $\mathbf{V}$  satisfies criterion (A) if and only if for any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and  $x, y \in \Sigma^*$ , there exist  $k, l \in \Sigma^*$  such that for all  $u \in \Sigma^k, v \in \Sigma^l$ , there exists some language  $K \in \mathcal{L}(\mathbf{V})$  over alphabet  $\Sigma$  verifying  $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$ .*

*Proof.* Let us first observe that given any alphabet  $\Sigma$ , given any language  $K$  on that alphabet and given any two words  $x, y \in \Sigma^*$ , we have that  $x(x^{-1}Ky^{-1})y = x\Sigma^*y \cap K$  and  $x^{-1}(xKy)y^{-1} = K$ .

**Inclusion from right to left.** Assume that for any  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and  $x, y \in \Sigma^*$ , there exist  $k, l \in \Sigma^*$  such that for all  $u \in \Sigma^k, v \in \Sigma^l$ , there exists some language  $K \in \mathcal{L}(\mathbf{V})$  over alphabet  $\Sigma$  verifying  $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$ .

This implies, by our observation at the beginning of the proof, that

$$x(u\Sigma^*v \cap L)y = xu(u^{-1}Lv^{-1})vy = xu((xu)^{-1}K(vy)^{-1})vy = xu\Sigma^*vy \cap K$$

for all  $u \in \Sigma^k, v \in \Sigma^l$ . Using Proposition 2.1, we thus have that  $x(u\Sigma^*v \cap L)y \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  for all  $u \in \Sigma^k, v \in \Sigma^l$ . Moreover, since we have that the set of words of  $L$  of length at least  $k+l$  is

$$\Sigma^{k+l} \cap L = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (u\Sigma^*v \cap L)$$

and since

$$L = (\Sigma^{\geq k+l} \cap L) \cup F$$

where  $F$  is a finite set of words on  $\Sigma$  of length less than  $k+l$ , we have that

$$xLy = x((\Sigma^{\geq k+l} \cap L) \cup F)y = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} x(u\Sigma^*v \cap L)y \cup xFy.$$

We can thus conclude that  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  by using Proposition 2.1 as well as the fact that  $xFy \in \mathcal{L}(\mathbf{LI})(\Sigma)$  and because  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$  is closed under unions.

**Inclusion from left to right.** Assume that  $\mathbf{V}$  satisfies criterion (A). Take some  $L \in \mathcal{L}(\mathbf{V})(\Sigma)$  for some alphabet  $\Sigma$  and take  $x, y \in \Sigma^*$ . By hypothesis, we know that  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ .

By Proposition 2.1, this means that  $xLy$  is a Boolean combination of languages in  $\mathcal{L}(\mathbf{V})(\Sigma) \cup \mathcal{L}(\mathbf{LI})(\Sigma)$ . Further, this implies that  $xLy$  can be written as the union of intersections of languages of  $\mathcal{L}(\mathbf{V})(\Sigma)$  and  $\mathcal{L}(\mathbf{LI})(\Sigma)$  or their complements, which in turn implies, by closure of  $\mathcal{L}(\mathbf{V})(\Sigma)$  and  $\mathcal{L}(\mathbf{LI})(\Sigma)$  under Boolean operations, that  $xLy$  can be written as a finite union of languages

of the form  $K \cap (U\Sigma^*V \cup W)$  with  $K \in \mathcal{L}(\mathbf{V})(\Sigma)$  and  $U, V, W \subseteq \Sigma^*$  finite. Since any word in  $xLy$  must be of length at least  $|xy|$  and have  $x$  as a prefix and  $y$  as a suffix, we can assume that any language  $K \cap (U\Sigma^*V \cup W)$  appearing in a finite union as described above verifies that  $U \subseteq x\Sigma^*$ , that  $V \subseteq \Sigma^*y$  and that  $W \subseteq x\Sigma^*y$ . Now, if we take  $k, l \in \mathbb{N}$  big enough, we thus have that

$$xLy = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (K_{u,v} \cap xu\Sigma^*vy) \cup F$$

where  $K_{u,v} \in \mathcal{L}(\mathbf{V})(\Sigma)$  for all  $u \in \Sigma^k, v \in \Sigma^l$  and  $F \subseteq \Sigma^{<|xy|+k+l}$ . Hence, for all  $u \in \Sigma^k, v \in \Sigma^l$ , we have

$$\begin{aligned} u^{-1}Lv^{-1} &= u^{-1}(x^{-1}(xLy)y^{-1})v^{-1} \\ &= (xu)^{-1} \left( \bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (K_{u',v'} \cap xu'\Sigma^*v'y) \cup F \right) (vy)^{-1} \\ &= \bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (xu)^{-1} \left( xu'((xu')^{-1}K_{u',v'}(v'y)^{-1})v'y \right) (vy)^{-1} \cup \\ &\quad (xu)^{-1}F(vy)^{-1} \\ &= (xu)^{-1}K_{u,v}(vy)^{-1}, \end{aligned}$$

observing that  $(xu)^{-1}K(vy)^{-1} = \emptyset$  for any  $K \subseteq \Sigma^*$  such that  $K \cap xu\Sigma^*vy = \emptyset$ .  $\square$

We now prove the announced characterisations of joins between **LI** and some well-known varieties of monoids in terms of identities.

**Theorem 5.2.** *We have the following.*

1.  $\langle \mathbf{R} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{ER} = \llbracket x^\omega y(ab)^\omega azt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$ .
2.  $\langle \mathbf{L} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EL} = \llbracket x^\omega yb(ab)^\omega zt^\omega = x^\omega y(ab)^\omega zt^\omega \rrbracket_{\text{ne}}$ .
3.  $\langle \mathbf{J} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EJ} = \llbracket x^\omega y(ab)^\omega zt^\omega = x^\omega y(ba)^\omega zt^\omega, a^{\omega+1} = a^\omega \rrbracket_{\text{ne}}$ .
4.  $\langle \mathbf{H} \vee \mathbf{LI} \rangle_{\text{ne}} = \mathbf{EH}$  for any variety of groups **H**.

*Proof.* In each case, we prove that the variety of monoids under consideration satisfies criterion (A) using Proposition 5.1. We then use Proposition 4.4.

**Proof of 1.** It is well-known that given an alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{R})(\Sigma)$  contains all languages that are disjoint unions of languages that are of the form  $A_0^*a_1A_1^* \cdots a_kA_k^*$  where  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma^*$  and  $a_i \notin A_{i-1}$  for all  $i \in [k]$  (see [6, Chapter 4, Theorem 3.3]).

Let  $\Sigma$  be an alphabet and take some language  $A_0^*a_1A_1^* \cdots a_kA_k^*$  where  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma^*$  and  $a_i \notin A_{i-1}$  for all  $i \in [k]$ . Take  $x, y \in \Sigma^*$ . Observe that  $y$  can be uniquely written as  $y = zt$  where  $z, t \in \Sigma^*$  and either  $|t| = 0$  and  $z \in A_k^*$ , or  $|t| > 0$  and  $t_1 \notin A_k$ . We have

$$\begin{aligned} &A_0^*a_1A_1^* \cdots a_kA_k^* \\ &= x^{-1} \left( xA_0^*a_1A_1^* \cdots a_kA_k^*t \cap \bigcap_{v \in A_k^{<|z|}} (xA_0^*a_1A_1^* \cdots a_kA_k^*vt)^{\mathbb{G}} \right) y^{-1} \end{aligned}$$

where  $x A_0^* a_1 A_1^* \cdots a_k A_k^* t \cap \bigcap_{v \in A_k^{\leq |z|}} (x A_0^* a_1 A_1^* \cdots a_k A_k^* t)^{\mathbf{G}}$  does belong to the set  $\mathcal{L}(\mathbf{R})(\Sigma)$  because the latter is closed under Boolean operations and by construction of  $z$  and  $t$ . We can conclude that for each  $L \in \mathcal{L}(\mathbf{R})(\Sigma)$  and  $x, y \in \Sigma^*$ , there exists  $K \in \mathcal{L}(\mathbf{R})(\Sigma)$  such that  $L = x^{-1} K y^{-1}$  by using the characterisation of  $\mathcal{L}(\mathbf{R})(\Sigma)$  and the fact that  $x^{-1} K_1 y^{-1} \cup x^{-1} K_2 y^{-1} = x^{-1} (K_1 \cup K_2) y^{-1}$  for all  $K_1, K_2 \in \Sigma^*$ .

**Proof of 2.** It is also well-known that given an alphabet  $\Sigma$ , the set  $\mathcal{L}(\mathbf{L})(\Sigma)$  contains all languages that are disjoint unions of languages that are of the form  $A_0^* a_1 A_1^* \cdots a_k A_k^*$  where  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,  $A_0, A_1, \dots, A_k \subseteq \Sigma^*$  and  $a_i \notin A_i$  for all  $i \in [k]$  (see [6, Chapter 4, Theorem 3.4]). The proof is then dual to the previous case.

**Proof of 3.** Given an alphabet  $\Sigma$ , for each  $k \in \mathbb{N}$ , we define the equivalence relation  $\sim_k$  on  $\Sigma^*$  by  $u \sim_k v$  for  $u, v \in \Sigma^*$  whenever  $u$  and  $v$  have the same set of subwords of length at most  $k$ . This relation is a congruence of finite index on  $\Sigma^*$ . Simon proved [8] that for all any alphabet  $\Sigma$ , a language belongs to  $\mathcal{L}(\mathbf{J})(\Sigma)$  if and only if it is equal to a union of  $\sim_k$ -classes for some  $k \in \mathbb{N}$ .

Let  $\Sigma$  be an alphabet and take some  $L \in \mathcal{L}(\mathbf{J})(\Sigma)$  as well as  $x, y \in \Sigma^*$ . Thus, there exists  $k \in \mathbb{N}$  such that  $L$  is a union of  $\sim_k$ -classes. Define the language  $K = \bigcup_{w \in L} [xwy]_{\sim_{|xy|+k}}$ : it belongs to  $\mathcal{L}(\mathbf{J})(\Sigma)$  by construction. We now show that  $L = x^{-1} K y^{-1}$ , which concludes the proof. Let  $w \in L$ : we have that  $xwy \in [xwy]_{\sim_{|xy|+k}} \subseteq K$ , so that  $w \in x^{-1} K y^{-1}$ . Let conversely  $w \in x^{-1} K y^{-1}$ . This means that  $xwy \in K$ , which implies that there exists  $w' \in L$  such that  $xwy \sim_{|xy|+k} xw'y$ . Actually, it holds that any  $u \in \Sigma^*$  of length at most  $k$  is a subword of  $w$  if and only if it is a subword of  $w'$ , because  $xuy$  is a subword of  $xwy$  if and only if it is a subword of  $xw'y$ . Hence,  $w \sim_k w'$ , which implies that  $w \in L$ .

**Proof of 3.** Consider some variety of groups  $\mathbf{H}$ . Take some language  $L \in \mathcal{L}(\mathbf{H})(\Sigma)$  for an alphabet  $\Sigma$  and let  $x, y \in \Sigma^*$ . Consider the syntactic morphism  $\eta: \Sigma^* \rightarrow M$  of  $L$ : we have that  $M$  is a group in  $\mathbf{H}$ . Define the language  $K = \eta^{-1}(\eta(x)\eta(L)\eta(y))$ . We now show that  $L = x^{-1} K y^{-1}$ , which concludes the proof. Let  $w \in L$ : we have that  $\eta(xwy) \in \eta(x)\eta(L)\eta(y)$ , so that  $w \in x^{-1} K y^{-1}$ . Conversely, let  $w \in x^{-1} K y^{-1}$ . We have that  $xwy \in K$ , which means that  $\eta(xwy) = \eta(x)\eta(w')\eta(y)$  for some  $w' \in L$ , so that  $\eta(w) = \eta(w') \in \eta(L)$ , as any element in  $\mathbf{H}$  is invertible. Thus,  $w \in L$ .  $\square$

## 6 Conclusion

**Acknowledgements.** The author wants to thank Thomas Place, who suggested the link between essentially- $\mathbf{V}$  stamps and  $\mathbf{V} \vee \mathbf{LI}$ , but also Luc Segoufin who started the discussion with Thomas Place and encouraged the author to write the present article.

## References

- [1] Jorge Almeida. *Finite semigroups and universal algebra*, volume 3. World Scientific, 1994.
- [2] José Carlos Costa. Some pseudovariety joins involving locally trivial semigroups. In *Semigroup Forum*, volume 64, pages 12–28. Springer, 2001.
- [3] Samuel Eilenberg. *Automata, Languages, and Machines*, volume A. Academic Press, New York, 1974.
- [4] Samuel Eilenberg. *Automata, Languages, and Machines*, volume B. Academic Press, New York, 1976.
- [5] Nathan Grosshans, Pierre McKenzie, and Luc Segoufin. Tameness and the power of programs over monoids in DA. *CoRR*, abs/2101.07495, 2021. URL: <https://arxiv.org/abs/2101.07495>, arXiv:2101.07495.
- [6] Jean-Éric Pin. *Varieties of formal languages*. North Oxford, London and Plenum, New-York, 1986. (Traduction de Variétés de langages formels).
- [7] Jean-Éric Pin and Howard Straubing. Some results on  $\mathcal{C}$ -varieties. *RAIRO-Theor. Inf. Appl.*, 39(1):239–262, 2005.
- [8] Imre Simon. Piecewise testable events. In *Automata Theory and Formal Languages, 2nd GI Conference*, volume 33 of *Lecture Notes in Computer Science*, pages 214–222. Springer, 1975.
- [9] Marc Zeitoun. On the join of two pseudovarieties. *Semigroups, Automata and Languages*, eds. J. Almeida, GMS Gomes, and PV Silva. World Scientific, pages 281–288, 1996.