

EURODYN 2020

XI International Conference
on Structural Dynamics



23-26 November 2020
Streamed from Athens, Greece

EASD
European Association
for Structural Dynamics

Nodal-Boundary Finite-Element Method for Periodic Solutions of the Signorini Problem



Presenting Author: David Urman
Second Author: Mathias Legrand

Non-Smooth Modal Analysis (NSMA)

- Objective: resonant frequencies and energies of structures in unilateral contact
- Requires finding **periodic solutions** to autonomous non-smooth systems

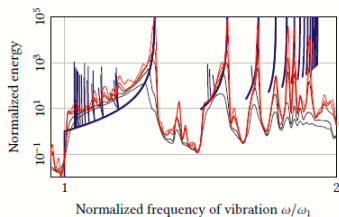
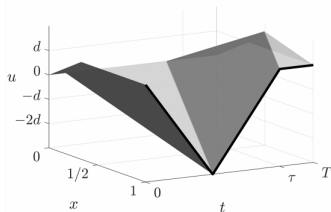
(a) Forced Response Diagram and modes¹(b) Periodic solution²

Figure 1: Modal spectrum of bar in unilateral contact and periodic solution

Current Works

- NSMA for constant-area bar using Wave-FEM¹, and analytically²
- Methods does not apply for varying-area bar or multi-dimensional case

¹C. Yoong. Nonsmooth modal analysis of a finite linear elastic bar subject to unilateral contact constraint. PhD thesis

²D. Urman and M. Legrand. Nonlinear modal analysis of the bar in unilateral contact via analytical weak-solutions to the wave equation.

Problem Statement

- Governing Equation

$$\rho u_{tt} = (\sigma(u(x, t)))_x \quad (1)$$

$$\sigma(u(x, t)) = A(x)E u_x \quad (2)$$

- Cantilever conditions

$$u(0, t) = 0 \quad (3)$$

- Signorini conditions³

- ▶ Free phase

$$u(L, t) \leq g, \quad \sigma(u(x, t))|_{x=L} = 0 \quad (4)$$

- ▶ Contact phase

$$\sigma(u(x, t))|_{x=L} \leq 0, \quad u(L, t) = g \quad (5)$$

- Initial conditions and periodicity

$$u(x, 0) = u_0(x) \quad u_t(x, 0) = v_0(x) \quad (6)$$

$$u(x, 0) = u(x, T) \quad u_t(x, 0) = u_t(x, T) \quad (7)$$

Figure 2: motion of bar in unilateral contact

³V.A. Yastrebov. Numerical Methods in Contact Mechanics

Finite Element Analysis with Newton Impact Law

- Displacement approximation via nodal displacements \mathbf{u}

$$u(x, t) \approx \sum_{i=1}^N \phi_i(x) u_i(t), \quad \mathbf{u} = (u_1(t) \quad u_2(t) \quad \dots \quad u_N(t))^T \quad (8)$$

- Test functions $\mathbf{w} = (w_1(t) \quad w_2(t) \quad \dots \quad w_N(t))^T$
- FEM weak-formulation of cantilever bar: for all \mathbf{w}

$$\mathbf{w}^T (\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u}) - w_N \lambda = 0 \quad (9)$$

λ : boundary-force.

- Application of Signorini condition \Rightarrow ill-posed problem⁴
- Signorini conditions require an impact law for well-posedness

$$u_N(t) = g \Rightarrow \dot{u}_N(t^+) = -e \dot{u}_N(t^-) \quad (10)$$

e : coefficient of restitution ($0 \leq e \leq 1$).

⁴D. Doyen, A. Ern, and S. Piperno. Time-integration schemes for the finite element dynamic Signorini problem

Newton Impact Law and Signorini Condition

- $e = 0$: Theoretical Sticking Phase (✓) with dissipation of energy (✗).
- $e = 1$: Chattering (✗) with conservation of energy (✓).

(a) $e = 0$

(b) $e = 1$

Figure 3: FEM bar with impact law, $N = 20$, quadratic polynomials

Motivation for Nodal Boundary Method

- **Issue:** Impact law $e = 0$ annihilates periodic solutions.
- **Proposed solution:** Determination of $\dot{\mathbf{u}}_N$ via **FEM stress approximation:** $\sigma_N(\mathbf{u})$.

Proposed Methodology

- FEM: weak form and stress approximation

$$\mathbf{w}^T(\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u}) - w_N \sigma_N(\mathbf{u}) = 0 \quad (11)$$

$$\sigma_N(\mathbf{u}) = EA(L) \sum_{i=1}^N \phi_{i,x}(L) u_i \quad (12)$$

- Nodal boundary method (NBM): explicitly determine u_N at free/contact phase

$$\text{Contact phase: } \sigma_N(\mathbf{u}) \leq 0, \quad u_N = g, \quad w_N = 0 \quad (13)$$

$$\text{Free phase: } u_N \leq g, \quad \sigma_N(\mathbf{u}) = 0, \quad \sigma_N(\mathbf{w}) = 0 \quad (14)$$

- Restriction $\sigma_N(\mathbf{u}) = 0$ defines motion u_N during free phase
- Restriction $\sigma_N(\mathbf{w}) = 0$ allows energy conservation during free phase

Example of Nodal Boundary Method on linear shape functions and $N = 2$

- FEM on cantilever bar

$$(w_1 \quad w_2) \left(\begin{bmatrix} 4/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{bmatrix} 16 & -8 \\ -8 & 8 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) + w_2 \sigma_2(\mathbf{u}) = 0 \quad (15)$$

$$\sigma_2(\mathbf{u}) = \phi_{1,x}(1)u_1 + \phi_{2,x}(1)u_2 = 2(u_2 - u_1) \quad (16)$$

Determination of Free Motion

- Signorini conditions for free motion

$$u_2 < g, \quad \sigma_2(\mathbf{u}) = 2(u_2 - u_1) = 0 \text{ and } \sigma_2(\mathbf{w}) = 2(w_2 - w_1) = 0 \quad (17)$$

- The constraints (17) are satisfied by setting

$$u_2 \leftarrow u_1 \text{ and } w_2 \leftarrow w_1 \quad (18)$$

- Constraints (18) are plugged into ODE (15). Apply for all $w_1 \Rightarrow$ ODE in u_1

$$8/3\ddot{u}_1 + 8u_1 = 0, \quad u_1 < g \quad (19)$$

Example continued: Determination of Contact Motion

- Signorini conditions for contact motion

$$2(u_2 - u_1) \leq 0, \quad u_2 = g \Rightarrow \ddot{u}_2 = 0 \text{ and } w_2 = 0 \quad (20)$$

- Constraints (20) are plugged into Finite Element ODE (15) \Rightarrow ODE in u_1

$$4/3\ddot{u}_1 + 16u_1 - 8g = 0, \quad u_1 \geq g \quad (21)$$

ODE by Nodal Boundary Method

- Displacement of contact node, u_2 , does not participate in ODE
- Displacement u_2 and stress $\sigma_2(\mathbf{u})$ are defined by u_1

$$u_2 = \begin{cases} g & u_1 \geq g \\ u_1 & u_1 < g \end{cases}, \quad \sigma_2(\mathbf{u}) = \begin{cases} 2(g - u_1) & u_1 \geq g \\ 0 & u_1 < g \end{cases} \quad (22)$$

- Non-contact boundary node u_1 abides non smooth ODE

$$\begin{cases} 4/3\ddot{u}_1 + 16u_1 - 8g = 0 & u_1 \geq g, \text{ contact phase} \\ 8/3\ddot{u}_1 + 8u_1 = 0 & u_1 < g, \text{ free phase} \end{cases} \quad (23)$$

Example Continued: Properties of Nodal Boundary Method

- Theoretical sticking at contact in the continuous time framework.
- Energy can be regained and does not always dissipate.

(a) NBM

(b) $e = 0$

(c) $e = 1$

Figure 4: FEM bar with impact law or NBM, $N = 2$, linear polynomials

Nodal Boundary Method: General Procedure

1. Apply $\sigma_N(\mathbf{u}) = 0$ to determine u_N during free motion

$$\sigma_N(\mathbf{u}) = \sum_{i=1}^N \phi_{i,x}(L)u_i = 0 \Rightarrow u_N = - \sum_{i=1}^{N-1} \frac{\phi_{i,x}(L)}{\phi_{N,x}(L)} u_i \equiv S(\mathbf{u}) \quad (24)$$

2. $S(\mathbf{u})$ does not depend on u_N
3. Apply $\sigma_N(\mathbf{w}) = 0$ to determine w_N : $w_N = S(\mathbf{w})$
4. Restate Signorini conditions in terms of $S(\mathbf{u})$

$$S(\mathbf{u}) \geq g, \quad u_N = g, \quad w_N = 0, \quad \sigma_N(\mathbf{u}) = \phi_{N,x}(L)(g - S(\mathbf{u})) \quad (25)$$

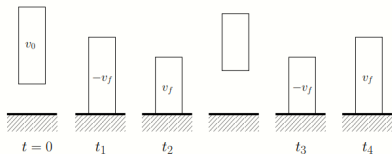
$$S(\mathbf{u}) < g, \quad u_N = S(\mathbf{u}), \quad w_N = S(\mathbf{w}), \quad \sigma_N(\mathbf{u}) = 0 \quad (26)$$

5. Apply Signorini conditions to weak form of FEM.

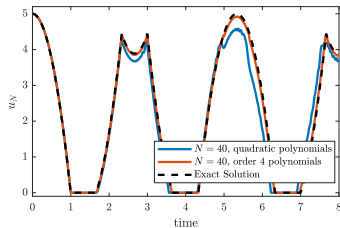
$$\begin{cases} \mathbf{M}_f \ddot{\mathbf{u}}_r + \mathbf{K}_f \mathbf{u}_r = 0 & S(\mathbf{u}_r) < g, \text{ Free motion} \\ \mathbf{M}_c \ddot{\mathbf{u}}_r + \mathbf{K}_c \mathbf{u}_r = g\mathbf{f}_c & S(\mathbf{u}_r) \geq g, \text{ Contact motion} \end{cases} \quad (27)$$

subscript r - removed term u_N .

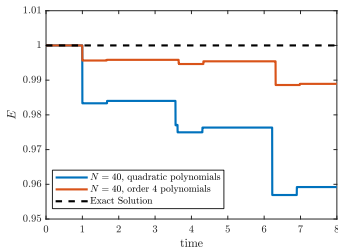
Case-study: The Bouncing Bar



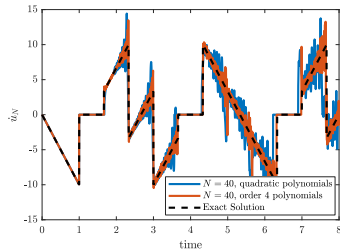
(a) Bouncing bar problem⁴



(b) Contact Boundary Displacement



(c) Total Energy



(d) Contact Boundary Velocity

Figure 5: Bouncing Bar with Nodal Boundary method

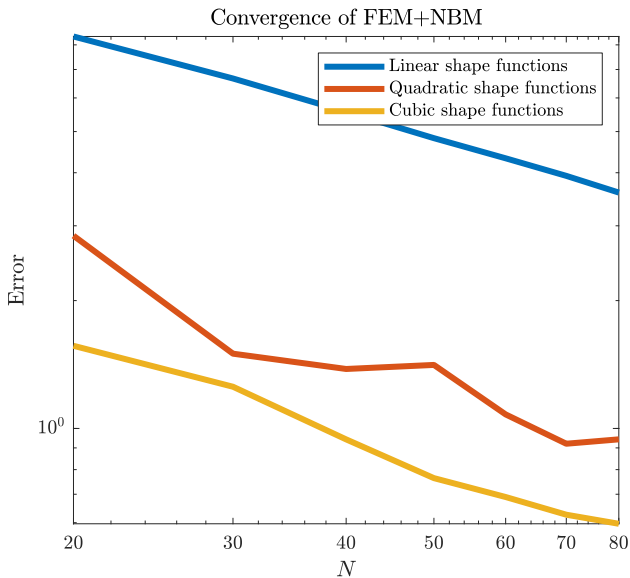


Figure 6: Convergence plot of bouncing bar

Bar with Varying Area: Periodic Solutions

- Periodic solutions to the bar with varying area $A(x) = 1 - x/5$ and length $L = 1$.
- Find periodic solutions to nodal boundary method ODE of the bar:

$$\mathcal{L}[\mathbf{u}_r] = \begin{cases} \mathbf{M}_f \ddot{\mathbf{u}}_r + \mathbf{K}_f \mathbf{u}_r = 0 & S(\mathbf{u}_r) < g, \\ \mathbf{M}_c \ddot{\mathbf{u}}_r + \mathbf{K}_c \mathbf{u}_r = g \mathbf{f}_c & S(\mathbf{u}_r) \geq g, \end{cases} \quad (28)$$

Harmonic Balance Method⁵(HBM)

- Trial solution $\mathbf{u}_r \approx \mathbf{U}$:

$$U_i = \sum_{j=0}^H C_{ij} \cos(j\omega t) \quad (29)$$

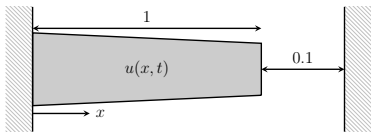
- Minimization of residual $\mathcal{L}[\mathbf{U}(t, \omega, \mathbf{C})]$ via projection onto cosine series

$$F_i(\omega, \mathbf{C}) = \int_0^{2\pi/\omega} \cos(i\omega t) \mathcal{L}[\mathbf{U}(t, \omega, \mathbf{C})] dt = 0 \quad (30)$$

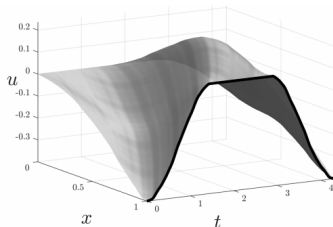
where \mathbf{F} is a system of non-smooth equations in \mathbf{C} and ω .

⁵W.-J.Kim, N.C.Perkins. Harmonic balance/Galerkin method for non-smooth dynamic systems

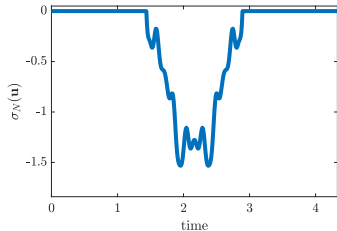
Results: Periodic solution of the varying area bar problem



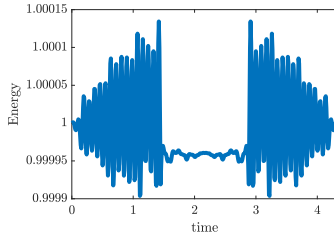
(a) Varying Area Bar



(b) Periodic Solution, $N = 100$, $H = 50$, cubic polynomials



(c) Contact Force



(d) Total Energy

Figure 7: Periodic solution to Varying Area Bar

HBM Convergence in H

- Difference between HBM solution and accurate ODE solution for specific H

$$\text{Error} = \int_0^{2\pi/\omega} \|\mathbf{U}(t) - \mathbf{u}_r(t)\|_2 dt \quad (31)$$

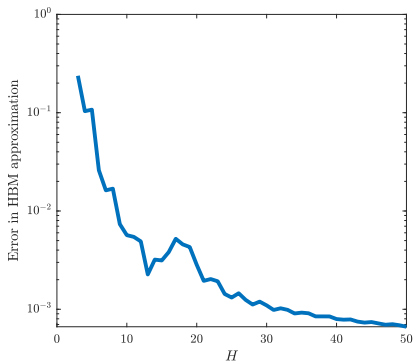


Figure 8: Convergence of HBM in H for $N = 10$

HBM Convergence in N

- Difference in HBM variables \mathbf{C} and ω with respect to previous N

$$\text{Error} = \|\mathbf{C}_N - \mathbf{C}_{N-1}\| + \|\omega_N - \omega_{N-1}\| \quad (32)$$

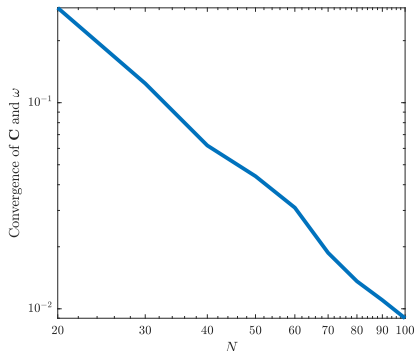


Figure 9: Convergence of HBM in N for $H = 50$

Nodal Boundary method: Conclusions

- Novel method for Signorini complementarity condition with theoretical sticking
- Instead of impact law, motion of u_N (and \dot{u}_N) dictated by stress approximation $\sigma_N(\mathbf{u})$
- Resulting ODE exhibits periodic solutions
- Periodic solutions found for previously unsolved case: varying area bar in unilateral contact

Future Work

- Continuation of periodic solutions to find non-smooth modes
- Analytical proofs for convergence
- Extension of the method in 2D and 3D

Contact Information

Email: david.urman@mail.mcgill.ca