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LARGE MASS RIGIDITY FOR A LIQUID DROP MODEL IN 2D WITH KERNELS OF FINITE MOMENTS

BENOIT MERLET AND MARC PEGON

ABSTRACT. Motivated by Gamow's liquid drop model in the large mass regime, we consider an isoperimetric problem in which the standard perimeter $P(E)$ is replaced by $P(E) - \gamma P_\varepsilon(E)$, with $0 < \gamma < 1$ and P_ε a nonlocal energy such that $P_\varepsilon(E) \rightarrow P(E)$ as ε vanishes. We prove that unit area minimizers are disks for $\varepsilon > 0$ small enough.

More precisely, we first show that in dimension 2, connected minimizers are necessarily convex, provided that ε is small enough. In turn, this implies that minimizers have nearly circular boundaries, that is, their boundary is a small Lipschitz perturbation of the circle. Then, using a Fuglede-type argument, we prove that (in arbitrary dimension $n \geq 2$) the unit ball in \mathbb{R}^n is the unique unit-volume minimizer of the problem among centered nearly spherical sets. As a consequence, up to translations, the unit disk is the unique minimizer.

This isoperimetric problem is equivalent to a generalization of the liquid drop model for the atomic nucleus introduced by Gamow, where the nonlocal repulsive potential is given by a radial, sufficiently integrable kernel. In that formulation, our main result states that if the first moment of the kernel is smaller than an explicit threshold, there exists a critical mass m_0 such that for any $m > m_0$, the disk is the unique minimizer of area m up to translations. This is in sharp contrast with the usual case of Riesz kernels, where the problem does not admit minimizers above a critical mass.

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1. INTRODUCTION

Given a positive, radial, measurable kernel $G : \mathbb{R}^n \rightarrow [0, +\infty)$ with finite first moment (that is, $|x|G(x) \in L^1(\mathbb{R}^n)$), we consider the nonlocal perimeter functional P_G (see e.g. [10, 6]) defined on measurable sets $E \subseteq \mathbb{R}^n$ by

$$P_G(E) := 2 \iint_{E \times (\mathbb{R}^n \setminus E)} G(x-y) \, dx \, dy = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\mathbf{1}_E(x) - \mathbf{1}_E(y)| G(x-y) \, dx \, dy. \quad (1.1)$$

Here $\mathbf{1}_E$ denotes the indicator function of E .

For $\varepsilon > 0$, we introduce the rescaled kernel $G_\varepsilon(x) := \varepsilon^{-(n+1)} G(\varepsilon^{-1}x)$, $x \in \mathbb{R}^n$. As will be justified later, the first moment of G is fixed to an explicit dimensional constant (see (H2)) so that $P_{G_\varepsilon}(E)$ converges to $P(E)$ as ε vanishes. Given $\gamma \in (0, 1)$ and $\varepsilon > 0$, we study the minimization problem

$$\min \left\{ P(E) - \gamma P_{G_\varepsilon}(E) : |E| = |B_1| \right\}, \quad (\mathcal{P}_{\gamma, \varepsilon})$$

over sets of finite perimeter E in \mathbb{R}^n , where $|E|$ denotes the volume of E (which we often call its mass), that is, its Lebesgue measure, and B_1 is the open unit ball of \mathbb{R}^n .

Let us emphasize the competition between the two terms. The perimeter is an attractive term minimized by balls under volume constraint. On the contrary, if G is radially decreasing, due to the negative sign, the nonlocal term is maximized by balls.¹ This competition makes the minimization problem nontrivial, even when it comes to existence of minimizers.

Problem $(\mathcal{P}_{\gamma,\varepsilon})$ is closely linked with variations of Gamow's liquid drop model for the atomic nucleus. Indeed, if in addition we assume that G is integrable in \mathbb{R}^n , then we may write

$$\frac{1}{2} P_{G_\varepsilon}(E) = \|G_\varepsilon\|_{L^1(\mathbb{R}^n)} |E| - \iint_{E \times E} G(x-y) \, dx \, dy, \quad (1.2)$$

so that, changing variables, E is a minimizer of $(\mathcal{P}_{\gamma,\varepsilon})$ if and only if the set $F := \varepsilon^{-1}E$ is a minimizer of

$$\min \left\{ P(F) + \iint_{F \times F} \tilde{G}(x-y) : |F| = m_\varepsilon \right\}, \quad (1.3)$$

where $m_\varepsilon := \varepsilon^{-n}|B_1|$ and $\tilde{G} := 2\gamma G$. When $n = 3$ and $\tilde{G}(x) = \frac{1}{8\pi} \frac{1}{|x|}$, this is Gamow's liquid drop model (see [11] for a general overview); note however that in that case, the minimized functional cannot be rewritten as the difference between the perimeter and a nonlocal perimeter, since \tilde{G} is not integrable at infinity. As a prototypical model for various physical systems involving the competition between short-range attractive forces and long-range repulsive ones, generalizations of this model have gained increasing interest during the past decade, in particular generalizations in higher dimensions, where the Coulomb potential is replaced with Riesz potentials, that is, $\tilde{G}(x) = |x|^{\alpha-n}$, $\alpha \in (0, n)$. In particular, it was shown that for every Riesz kernel, in the small mass regime, the unique minimizer of the liquid drop model (1.3) is the ball, up to translations (see [23, 24, 22, 7, 14]). Conversely, for $\alpha \in [n-2, n)$, the problem admits no minimizer above a critical mass (see [7, 23, 24, 26, 15, 17]; see also [16]). More general kernels of Riesz-type were studied e.g. in [9, 29], where the unit ball is shown to be the unique minimizer in the small mass regime.

Although the small mass regime has been extensively studied, the literature on large mass minimizers for Gamow-type problems is still sparse, since existence is rather unexpected in that case, and is usually only recovered by adding an extra attractive potential, such as in [2, 1, 20], or by adding a density to the perimeter, as in [3], where the authors show that if the density is a power-law growing sufficiently fast at infinity, then minimizers always exist, and are balls for large masses. It is worth mentioning that in the case of general kernels *with compact support*, the author of [32] shows that minimizers exist *for all masses*.

Between Riesz kernels, which are not integrable at infinity, where (1.3) does not admit minimizers above a critical mass, and compactly supported kernels, where (1.3) always admits minimizers, it is natural to wonder what happens with non-compactly supported but reasonably decaying kernels, such as Bessel kernels. These kernels behave as Riesz potentials near the origin, but decrease exponentially at infinity. They were suggested in [25] as a replacement for Riesz kernels for modeling diblock copolymers when long-range interactions are partially screened by fluctuations in the background nuclear fluid density. The study of the liquid drop model in the *large mass regime* for integrable kernels with finite first moment (such as Bessel kernels), which is equivalent to the study of $(\mathcal{P}_{\gamma,\varepsilon})$ when ε is small, has been started by the second author in [31]. The existence of minimizers for any $\gamma \in (0, 1)$ was established therein for ε small enough, as well as the convergence of minimizers to the unit ball as ε vanishes. It was conjectured there that the ball is actually the unique minimizer up to translations, for ε small enough. In this paper, we give a positive answer to this conjecture in dimension $n = 2$ under reasonable assumptions on the first moment of G and the second moment of ∇G , which are still satisfied by Bessel kernels. The conjecture remains open in higher dimensions.

Let us introduce the ‘‘critical energies’’

$$\mathcal{E}_{G_\varepsilon} := P - P_{G_\varepsilon} = \mathcal{F}_{1,G_\varepsilon},$$

and define the energies

$$\mathcal{F}_{\gamma,G_\varepsilon}(E) := (1-\gamma)P(E) + \gamma\mathcal{E}_{G_\varepsilon}(E) = P(E) - \gamma P_{G_\varepsilon}(E). \quad (1.4)$$

¹In fact, even if G is not radially decreasing, G_ε ‘‘concentrates’’ near the origin when ε is small, heuristically making $(-P_{G_\varepsilon})$ a repulsive term whenever G is not identically equal to 0.

Although the paper mostly deals with the “subcritical case” $\gamma < 1$, we focus on the critical energies in [Section 3.1](#), and show that they decrease by convexification. Finally, for any $k \in \mathbb{N} \setminus \{0\}$, we denote by

$$I_G^k := \int_{\mathbb{R}^n} |x|^k |\partial_r^{k-1} G(x)| \, dx$$

the k -th moment of the $(k-1)$ -th radial derivative of the kernel G , whenever it is well-defined.

In the paper, starting from [Section 2](#), **we shall always implicitly assume** that the kernel G satisfies the following general assumptions:

- (H1) G is nonnegative and radial, that is, there exists a measurable function $g : (0, +\infty) \rightarrow [0, +\infty)$ such that $G(x) = g(|x|)$ for a.e. $x \in \mathbb{R}^n$;
- (H2) the first moment of G is finite and set to be

$$I_G^1 = \frac{1}{\mathbf{K}_{1,n}},$$

where, for any $\nu \in \mathbb{S}^{n-1}$, the constant $\mathbf{K}_{1,n}$ is defined by

$$\mathbf{K}_{1,n} = \int_{\mathbb{S}^{n-1}} |\sigma \cdot \nu| \, d\mathcal{H}_\sigma^{n-1}.$$

Starting from [Section 4](#), we may explicitly use the extra assumption:

- (H3) $G \in W_{\text{loc}}^{1,1}(\mathbb{R}^n \setminus \{0\})$, $I_G^2 < \infty$, and $g'(r) = O(r^{-(n+1)})$ at infinity.

Those assumptions are in particular satisfied by the Bessel kernels $\mathcal{B}_{\kappa,\alpha}$, that is, fundamental solutions of the operators $(I - \kappa\Delta)^{\frac{\alpha}{2}}$, for $\kappa, \alpha > 0$ (see e.g. [\[31, § 3.2\]](#) or [\[21\]](#) for their definition and properties).

Even though our main result is in dimension $n = 2$, where we prove that the unit disk is the only minimizer up to translations, provided that ε is small enough, note that the intermediate results of [Section 3.4](#) and [Section 4](#) are obtained in arbitrary dimension. That is, convex minimizers are nearly spherical sets, and the unit ball of \mathbb{R}^n is the unique minimizer among nearly spherical sets whenever ε is small enough.

Note also that the kernel G is assumed to be radial but not necessarily radially nonincreasing, as is often the case. Let us emphasize that, contrarily to the small mass regime for Riesz-type potentials, here the nonlocal perimeter term does not vanish in the limit but rather converges to a fraction of the standard perimeter.

We shall now state the main result of the paper.

Theorem A (Minimality of the unit disk). *Assume $n = 2$ and G satisfies (H1) to (H3). Then there exists $\varepsilon_* = \varepsilon_*(G, \gamma) > 0$, such that, for every $\varepsilon < \varepsilon_*$, the unit disk is the unique minimizer of $(\mathcal{P}_{\gamma,\varepsilon})$, up to translations and Lebesgue-negligible sets.*

In terms of Gamow’s problem [\(1.3\)](#), this means that if $I_{\tilde{G}} < \frac{2}{\mathbf{K}_{1,2}} = \pi$ and \tilde{G} is in addition integrable, above a critical mass $m_* > 0$, the unique minimizer of area $m > m_*$ is the disk, up to translations. In the particular case of the Bessel kernels $\tilde{G} = \mathcal{B}_{\kappa,\alpha}$ with $\kappa, \alpha > 0$, such a critical mass exists whenever

$$\kappa < \pi \left(\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{1+\alpha}{2})} \right)^2,$$

(see [\[31, Corollary 3.9 & Proposition 3.10\]](#)).

The proof of [Theorem A](#) decomposes as follows. First we establish:

Theorem 1 (2D minimizers are convex). *Assume $n = 2$ and G satisfies (H1) and (H2). Then there exists $\varepsilon_1 = \varepsilon_1(G, \gamma) > 0$ such that, for every $0 < \varepsilon < \varepsilon_1$, $(\mathcal{P}_{\gamma,\varepsilon})$ admits a minimizer, and every minimizer is convex, up to a Lebesgue-negligible set.*

The existence of minimizers for small ε was shown in [\[31\]](#), where the second author also proved that they are necessarily connected whenever ε is small enough. The idea for proving the convexity of minimizers is to study the critical energy on the real line, and show that it decreases by convexification and by expansion of segments, so that, by a slicing argument, the critical energy of a *connected set in dimension 2* decreases after convexification. As a consequence, since the perimeter of a connected set is also reduced by convexification, so is $\mathcal{F}_{\gamma,G_\varepsilon}$. This slicing argument is specific to the dimension 2, where a line intersects a connected set if and only if it intersects its convex hull. This fails in higher dimension.

Note that this is not enough to conclude that minimizers are convex. Indeed, although for every minimizer $E_\varepsilon \subset \mathbb{R}^2$ with ε small enough, we have $\mathcal{F}_{\gamma, G_\varepsilon}(\text{co}(E_\varepsilon)) \leq \mathcal{F}_{\gamma, G_\varepsilon}(E_\varepsilon)$, where $\text{co}(E_\varepsilon)$ denotes the convex hull of E_ε , the volume of $\text{co}(E_\varepsilon)$ is larger than $|B_1|$ if E_ε is not convex. However, using the fact that a minimizer E_ε is already close to the unit ball by [31] and the convexity of $\text{co}(E_\varepsilon)$, we prove that, if E_ε is not convex, scaling down $\text{co}(E_\varepsilon)$ to make its volume equal to $|B_1|$ strictly decreases the energy $\mathcal{F}_{\gamma, G_\varepsilon}$, which contradicts the minimality of E_ε .

The convexity of minimizers E_ε allows us to improve the convergence of ∂E_ε towards ∂B_1 as ε goes to 0, from the previously known Hausdorff convergence to Lipschitz convergence. We deduce that minimizers are nearly spherical sets, whose definition is given just below.

Definition 1 (Nearly spherical sets). *For $t \in (0, \frac{1}{2})$, we say that $E \subseteq \mathbb{R}^n$ is a centered t -nearly spherical set if*

$$\int_E x \, dx = 0,$$

and if there exists $u \in \text{Lip}(\mathbb{S}^{n-1})$ with $\|u\|_{L^\infty(\mathbb{S}^{n-1})} + \|\nabla_\tau u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$ such that

$$\partial E = \left\{ (1 + tu(x))x : x \in \mathbb{S}^{n-1} \right\}.$$

In dimension $n = 2$, we use the terminology “ t -nearly circular set” for “ t -nearly spherical set”.

Theorem 2 (2D minimizers have nearly circular boundaries; see Proposition 3.11). *Assume $n = 2$ and G satisfies (H1) and (H2). There exist $\varepsilon_2 = \varepsilon_2(G, \gamma) > 0$ and a function $t : (0, \varepsilon_2) \rightarrow [0, \frac{1}{2})$ depending only on G and γ such that*

- $t(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$;
- for every $\varepsilon < \varepsilon_2$, any minimizer E of $(\mathcal{P}_{\gamma, \varepsilon})$ is, up to a translation and a Lebesgue-negligible set, a centered $t(\varepsilon)$ -nearly circular set.

Theorem 2 is a direct consequence of Theorem 1 and the uniform convergence of minimizers already shown in [31], using the geometric fact that the normal vectors to the boundary of a convex set lying between two balls B_r and B_R , $r < 1 < R$ converge to those of the unit sphere as $r, R \rightarrow 1$ (see [30, 18]).

We end the proof by showing that for ε and t small enough, any centered t -spherical minimizer of $(\mathcal{P}_{\gamma, \varepsilon})$ is the unit ball. This last result is not specific to dimension $n = 2$.

Theorem 3 (Minimality of the unit ball among nearly spherical sets; see Proposition 4.2). *Assume that G satisfies (H1) to (H3). Then there exist $t_* = t_*(n, G, \gamma) > 0$ and $\varepsilon_3 = \varepsilon_3(n, G, \gamma) > 0$ such that, for every $t < t_*$, if E is a t -nearly spherical set, then we have*

$$\mathcal{F}_{\gamma, \varepsilon}(B_1) \leq \mathcal{F}_{\gamma, \varepsilon}(E), \quad \forall 0 < \varepsilon < \varepsilon_3,$$

and the inequality is strict if $E \neq B_1$ (in the sense that they differ by a set positive measure).

The proof relies on a Fuglede-type result [18] for the nonlocal perimeter P_{G_ε} of centered nearly spherical sets E_t : more precisely, by a Taylor expansion we control $P_{G_\varepsilon}(E_t) - P_{G_\varepsilon}(B_1)$ from above in terms of the $H^1(\mathbb{S}^{n-1})$ norm of the perturbation u describing E_t (see Lemma 4.1).

Theorem A is then an immediate consequence of Theorems 1 to 3.

Outline of the paper. The structure of the paper follows the strategy of the proof. In Section 2 we recall some useful results from [31] on minimizers of $(\mathcal{P}_{\gamma, \varepsilon})$ and some facts on nonlocal perimeters. In Section 3, we show that 2D minimizers are convex and thus nearly circular sets for small ε , that is, Theorem 1 and Theorem 2, where the latter is a consequence of Proposition 3.11. Finally, Section 4 is dedicated to the proof of Theorem 3, which is a consequence of Proposition 4.2.

Notation.

Operations on sets. For any set $E \subseteq \mathbb{R}^n$, $E^c := \mathbb{R}^n \setminus E$ denotes its complement, $\text{co}(E)$ its convex hull (that is, the intersection of all convex sets containing E), and $|E|$ its Lebesgue measure, whenever E is measurable. We write $E \Delta F$ for the symmetric difference of E and F , and $E \sqcup F$ for the union of E and F whenever they are disjoint.

Hausdorff measures. We denote by \mathcal{H}^k the k -dimensional Hausdorff measure in \mathbb{R}^n . When integrating w.r.t. the measure \mathcal{H}^k in a variable x , we use the notation $d\mathcal{H}_x^k$ instead of the more standard but less compact $d\mathcal{H}^k(x)$.

Balls and spheres. We denote by $B_r(x)$ the open ball in \mathbb{R}^n of radius r centered at x . For brevity, we write B_r when x is the origin. The volume of B_1 is $\omega_n := |B_1| = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$, and the area of the unit sphere \mathbb{S}^{n-1} is $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$, which we also write $|\mathbb{S}^{n-1}|$ for simplicity.

Sets of finite perimeter. We denote by $\text{BV}(\mathbb{R}^n)$ the space of functions with bounded variation in \mathbb{R}^n . For any $f \in \text{BV}(\mathbb{R}^n)$ we let $|Df|$ be its total variation measure, and set $[f]_{\text{BV}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |Df|$. For a set of finite perimeter E in \mathbb{R}^n , we let $\mathbf{1}_E \in \text{BV}(\mathbb{R}^n)$ be its characteristic function (i.e., $\mathbf{1}_E(x) = 1$ if $x \in E$ and 0 otherwise), and define its perimeter by $P^n(E) := \int_{\mathbb{R}^n} |D\mathbf{1}_E|$. When there can be no confusion, we may drop the superscript and simply write $P(E)$ for the perimeter functional in \mathbb{R}^n . We denote by $\mu_E := D\mathbf{1}_E$ the Gauss–Green measure associated with the set of finite perimeter E , and by $\nu_E(x)$ the outer unit normal of ∂^*E at x , where ∂^*E stands for the reduced boundary of E . We refer to e.g. [13, Chapter 5] or [27] for further details on functions of bounded variations and sets of finite perimeter.

2. PRELIMINARIES

From now on we assume that G satisfies (H1) and (H2).

For a general nonnegative radial kernel K with finite first moment, we have the following control of P_K by the perimeter, as an immediate consequence of [31, Proposition 3.1] and of the second expression of the nonlocal perimeter given by (1.1).

Proposition 2.1. *If $K : \mathbb{R}^n \rightarrow [0, +\infty)$ is a nonnegative radial kernel with finite first moment, for every set of finite perimeter E in \mathbb{R}^n , we have*

$$P_K(E) \leq \mathbf{K}_{1,n} I_K^1 P(E).$$

In particular, by (H1) and (H2), we have

$$P_{G_\varepsilon}(E) \leq P(E), \quad \forall \varepsilon > 0.$$

We also have the following convergence result, which is a consequence of [12] and our choice of I_G^1 .

Proposition 2.2. *For any set of finite perimeter E in \mathbb{R}^n , we have*

$$P_{G_\varepsilon}(E) \xrightarrow{\varepsilon \rightarrow 0} P(E). \quad (2.1)$$

We will use the following computation obtained in [31, Lemma 3.5], which clarifies the behavior of the nonlocal perimeter under scaling.

Lemma 2.3. *For any set of finite perimeter $E \subseteq \mathbb{R}^n$, the function $t \mapsto P_{G_\varepsilon}(tE)$ is locally Lipschitz continuous in $(0, +\infty)$, and for almost every t , we have*

$$\frac{d}{dt} [P_{G_\varepsilon}(tE)] = \frac{n}{t} P_{G_\varepsilon}(tE) - \frac{1}{t} \tilde{P}_{G_\varepsilon}(tE),$$

where $\tilde{P}_{G_\varepsilon}(E)$ is defined by

$$\tilde{P}_{G_\varepsilon}(E) := 2 \int_E \int_{\partial^*E} G_\varepsilon(x-y) (y-x) \cdot \nu_E(y) d\mathcal{H}_y^{n-1} dx. \quad (2.2)$$

Let us remark that in [31], G is assumed to be in addition integrable in \mathbb{R}^n , but Lemma 2.3 can be deduced by approximating G with maps $G_k : x \mapsto \chi_k(|x|)G(x) \in L^1(\mathbb{R}^n)$. Indeed, let $\chi_k \in C^\infty(\mathbb{R}^+, [0, 1])$ be cutoff functions with $\chi_k(r) = 0$ for $r \leq \frac{1}{k}$, $\chi_k(r) = 1$ for $r \geq \frac{2}{k}$, so that $I_{G_k}^1 \leq I_G^1$, $x \mapsto |x|G_k(x)$ converges to $x \mapsto |x|G(x)$ in $L^1(\mathbb{R}^n)$, and notice that

$$|P_G(E) - P_{G_k}(E)| \leq P(E) \int_{\mathbb{R}^n} |x| |(G - G_k)(x)| dx,$$

and

$$|\tilde{P}_G(E) - \tilde{P}_{G_k}(E)| \leq 2P(E) \int_{\mathbb{R}^n} |x| |(G - G_k)(x)| dx.$$

In order to study the minimality of the unit ball among nearly spherical sets, we will use the following Bourgain-Brezis-Mironescu-type result (see [8]) for approximating the H^1 seminorm on the sphere by nonlocal seminorms.

Lemma 2.4. *Let us define the $(n-1)$ -dimensional approximation of identity $(\eta_\varepsilon)_{\varepsilon>0}$ by*

$$\eta(t) := 2t^2 g(t), \quad \text{and} \quad \eta_\varepsilon(t) := \varepsilon^{-(n-1)} \eta(\varepsilon^{-1}t), \quad \forall t > 0, \forall \varepsilon > 0.$$

When $n = 2$, we assume in addition that g is such that the family $(\eta_\varepsilon)_{\varepsilon>0}$ satisfies

$$\sup_{r \in (R, 2)} \eta_\varepsilon(r) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \forall R \in (0, 2). \quad (2.3)$$

Then for any $u \in H^1(\mathbb{S}^{n-1})$, we have

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{(u(x) - u(y))^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \leq (1 + q_\eta(\varepsilon)) \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1},$$

where $q_\eta(\varepsilon)$ vanishes as ε goes to 0, and depends only on n and G . In addition, for any $u \in H^1(\mathbb{S}^{n-1})$,

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{(u(x) - u(y))^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1}.$$

Proof. One easily checks that assumptions (H1) and (H2) ensure that the family $(\eta_\varepsilon)_{\varepsilon>0}$ is a $(n-1)$ -dimensional approximation of identity, up to multiplication by the constant $\mathbf{K}_{2,n-1} = \frac{1}{n-1}$, i.e.,

$$(i) \quad |\mathbb{S}^{n-2}| \int_0^{+\infty} \eta_\varepsilon(r) r^{n-2} dr = \frac{1}{\mathbf{K}_{2,n-1}}; \quad (ii) \quad \lim_{\varepsilon \rightarrow 0} \int_\delta^{+\infty} \eta_\varepsilon(r) r^{n-2} dr = 0, \quad \forall \delta > 0.$$

These properties (together with (2.3) when $n = 2$) allow us to apply [31, Propositions A.1 & A.4], which gives the result. \square

Remark 2.5. If (H3) stands true, then the condition (2.3) is satisfied, in particular when $n = 2$. Indeed, with (H2), (H3) implies that $g \in W_{\text{loc}}^{1,1}(0, +\infty)$ and the functions $t \mapsto t^n g(t)$ and $t \mapsto t^{n+1} g'(t)$ are integrable on $(0, +\infty)$. In addition, integrating the function $(t^{n+1} g(t))'$ between r and R , we have the relation

$$R^{n+1} g(R) - r^{n+1} g(r) = (n+1) \int_r^R t^n g(t) dt + \int_r^R t^{n+1} g'(t) dt.$$

Since $t^n g(t)$ and $t^{n+1} g'(t)$ are integrable on $(0, +\infty)$, this implies that $r^{n+1} g(r)$ has a limit in 0^+ and at infinity. By the integrability of $t^n g(t)$ on $(0, +\infty)$, those limits are necessarily 0. In particular, $r^{n+1} g(r) \xrightarrow{r \rightarrow \infty} 0$, so that

$$\eta_\varepsilon(r) = r^2 \varepsilon^{-(n+1)} g(\varepsilon^{-1}r) = r^{1-n} (\varepsilon^{-1}r)^{-(n+1)} g(\varepsilon^{-1}r)$$

vanishes uniformly on $(R, 2)$ as $\varepsilon \rightarrow 0$, for every $R \in (0, 2)$.

Eventually, gathering results from [31, Theorems A and B] (see also Theorem 4.16 therein), we have existence and convergence results for minimizers of $(\mathcal{P}_{\gamma, \varepsilon})$. We also know that minimizers are connected for small ε . Here connectedness is to be understood in a measure-theoretic sense for sets of finite perimeter, often referred to as indecomposability, as defined below (see [4]).

Definition 2.6. We say that a set of finite perimeter E is decomposable if there exist two sets of finite perimeter E_1 and E_2 such that $E = E_1 \sqcup E_2$, $|E_1| > 0$, $|E_2| > 0$ and $P(E) = P(E_1) + P(E_2)$. Naturally, we say that a set of finite perimeter is indecomposable if it is not decomposable.

Let us remark that by [4, Theorem 2], the notion of connectedness and indecomposability coincide whenever E is an open set of finite perimeter such that $\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial^* E)$.

Theorem 2.7. *There exist $\varepsilon_0 = \varepsilon_0(n, G, \gamma)$ and a function $\delta = \delta(n, G, \gamma) : (0, +\infty) \rightarrow (0, \frac{1}{4})$ vanishing in 0^+ such that the following holds. For every $0 < \varepsilon < \varepsilon_0$, $(\mathcal{P}_{\gamma, \varepsilon})$ admits a minimizer. In addition, any such minimizer E_ε is indecomposable, and up to a translation and a Lebesgue-negligible set, it satisfies*

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+\delta(\varepsilon)}. \quad (2.4)$$

In dimension $n = 2$, any minimizer E_ε with $0 < \varepsilon < \varepsilon_0$ is Lebesgue-equivalent to a connected set which still satisfies (2.4).

Proof. In [31], the kernel G is assumed to be integrable in \mathbb{R}^n . However, it is actually only required for the two following reasons: first, to be able to write (1.2) and obtain the equivalence with the Gamow-type minimization problem (1.3); second, by this equivalence, to deduce that minimizers of $(\mathcal{P}_{\gamma,\varepsilon})$ are so-called quasi-minimizers of the perimeter, and thus are (non-uniformly in ε) $C^{1,\frac{1}{2}}$ -regular outside a “small” singular set. Here, we do not need the equivalence with (1.3) nor the *a priori* regularity of minimizers. In the end, apart from the $C^{1,\frac{1}{2}}$ partial regularity of minimizers, all the conclusions of [31, Theorems A and B] follow. More precisely, there exists $\varepsilon_0 = \varepsilon_0(n, G, \gamma)$ such that, for any $0 < \varepsilon < \varepsilon_0$, $(\mathcal{P}_{\gamma,\varepsilon})$ admits a minimizer. In addition, any such minimizer E_ε is indecomposable and, up to a translation and Lebesgue-negligible set, it satisfies

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+\delta(\varepsilon)},$$

where $\delta : (0, +\infty) \rightarrow (0, \frac{1}{4})$ is a function depending only on n, G , and γ vanishing in 0^+ .

To conclude, there remains to show that in dimension $n = 2$, E_ε is equivalent to a connected set with (2.4), that is, to link the indecomposability of E_ε with the topological notion of connectedness. It is not a trivial question, at least without (weak) regularity results on minimizers. However, [4, Theorem 8] shows that in dimension 2, $\tilde{E}_\varepsilon := \dot{E}_\varepsilon^M \setminus \partial^S E_\varepsilon$ is connected, where \dot{E}_ε^M is the measure-theoretic interior of E_ε , and

$$\partial^S E_\varepsilon := \left\{ x \in \mathbb{R}^2 : \limsup_{r \rightarrow 0^+} \frac{\mathcal{H}^1(\partial^* E_\varepsilon \cap B_r(x))}{r} > 0 \right\}.$$

Since $\mathcal{H}^1(\partial^S E_\varepsilon \setminus \partial^* E_\varepsilon) = 0$ and $\mathcal{L}^2(E_\varepsilon \Delta \dot{E}_\varepsilon^M) = 0$, we have $\mathcal{L}^2(E_\varepsilon \Delta \tilde{E}_\varepsilon) = 0$, and since $B_{1-\delta(\varepsilon)} \subseteq \dot{E}_\varepsilon^M \subseteq B_{1+\delta(\varepsilon)}$ and $\partial^S E_\varepsilon \subseteq \overline{B}_{1+\delta(\varepsilon)} \setminus B_{1-\delta(\varepsilon)}$, \tilde{E}_ε satisfies (2.4). \square

Remark 2.8. In higher dimensions $n \geq 3$, one could show as well that any minimizer of $(\mathcal{P}_{\gamma,\varepsilon})$ is equivalent to a connected set for small ε , without assuming that $G \in L^1(\mathbb{R}^n)$. Indeed, proceeding e.g. as in [28, Lemma 5.6], it is possible to obtain *uniform* (w.r.t. ε) density estimates for minimizers, and with those to deduce that any minimizer is equivalent to an open set E_ε such $\partial E_\varepsilon = \text{spt } \mu_\varepsilon$. The indecomposability of E_ε then implies connectedness by [4].

3. MINIMIZERS ARE NEARLY CIRCULAR SETS IN DIMENSION 2

3.1. Decrease of the critical energy by convexification. We can recover the nonlocal perimeter of a measurable set $E \subseteq \mathbb{R}^n$ by integrating the 1-dimensional nonlocal perimeter of all 1-dimensional slices of E in a given direction, and averaging over all the directions.

Proposition 3.1. *For any measurable set $E \subseteq \mathbb{R}^{n-1}$, we have*

$$\begin{aligned} P_{G_\varepsilon}(E) &= \int_{\mathbb{S}^{n-1}} \int_{\{\sigma\}^\perp} \left(\iint_{E_{\sigma,y} \times (\mathbb{R} \setminus E_{\sigma,y})} |s-t| g_\varepsilon(|s-t|) ds dt \right) d\mathcal{H}_y^{n-1} d\mathcal{H}_\sigma^{n-1} \\ &= \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \left(\int_{\{\sigma\}^\perp} P_{\rho_\varepsilon}^1(E_{\sigma,y}) d\mathcal{H}_y^{n-1} \right) d\mathcal{H}_\sigma^{n-1}, \end{aligned}$$

where

$$E_{\sigma,y} := \left\{ s \in \mathbb{R} : y + s\sigma \in E \right\}, \quad (3.1)$$

ω_{n-1} is the volume of the unit ball in \mathbb{R}^{n-1} , and $P_{\rho_\varepsilon}^1$ is the 1-dimensional nonlocal perimeter in \mathbb{R} associated with the kernel ρ_ε defined by $\rho_\varepsilon(r) := \omega_{n-1} |r|^{n-1} g_\varepsilon(|r|)$ for $r \in \mathbb{R} \setminus \{0\}$, that is,

$$P_{\rho_\varepsilon}^1(J) := \iint_{\mathbb{R} \times \mathbb{R}} |\mathbf{1}_J(s) - \mathbf{1}_J(t)| \rho_\varepsilon(s-t) ds dt,$$

for every measurable set $J \subseteq \mathbb{R}$.

Proof. By the change of variable $y = x + r\sigma$ with fixed x and Fubini's theorem, we have

$$\begin{aligned} P_{G_\varepsilon}(E) &= 2 \iint_{E \times E^c} G(x-y) dx dy \\ &= 2 \int_{\mathbb{S}^{n-1}} \int_0^{+\infty} \int_E \mathbf{1}_{E^c}(x+r\sigma) r^{n-1} g_\varepsilon(r) dx dr d\mathcal{H}_\sigma^1 \\ &= \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_E \mathbf{1}_{E^c}(x+r\sigma) \rho_\varepsilon(r) dx dr d\mathcal{H}_\sigma^1, \end{aligned}$$

where we have used the definition of ρ_ε for the last equality. Then, for σ fixed, let us make the change of variable $x = y + s\sigma$, where $y = \pi_{\{\sigma\}^\perp}(x)$ is the orthogonal projection of x on $\{\sigma\}^\perp$. This yields

$$P_{G_\varepsilon}(E) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \int_{\{\sigma\}^\perp} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_E(s\sigma + y) \mathbf{1}_{E^c}((s+r)\sigma + y) \rho_\varepsilon(r) \, ds \, dr \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1}.$$

Finally, using Fubini's theorem and making the change of variable $t = s + r$, where s is fixed, we obtain, by definition of $E_{\sigma,y}$ and $P_{\rho_\varepsilon}^1$,

$$\begin{aligned} P_{G_\varepsilon}(E) &= \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\{\sigma\}^\perp} \int_{E_{\sigma,y}} \int_{\mathbb{R} \setminus E_{\sigma,y}} \rho_\varepsilon(s-t) \, ds \, dt \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1} \\ &= \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\{\sigma\}^\perp} P_{\rho_\varepsilon}^1(E_{\sigma,y}) \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1}. \end{aligned}$$

This concludes the proof. \square

Remark 3.2. Recall that by [31, Lemma 3.13], $\mathbf{K}_{1,n} = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n+1}{2})}$, so that, by (H2), the kernel ρ_ε satisfies

$$\int_{\mathbb{R}} |t| \rho_\varepsilon(t) \, dt = 2\omega_{n-1} \int_0^\infty t^n g_\varepsilon(t) \, dt = \frac{2\omega_{n-1} I_G^1}{|\mathbb{S}^{n-1}|} = \frac{2\omega_{n-1}}{\mathbf{K}_{1,n} |\mathbb{S}^{n-1}|} = 1 = \frac{1}{\mathbf{K}_{1,1}}.$$

Hence by Proposition 2.1, we have

$$\mathcal{E}_{\rho_\varepsilon}^1(E) = P(E) - P_{\rho_\varepsilon}^1(E) \geq 0$$

for every measurable set $E \subseteq \mathbb{R}$.

Similarly, as a straightforward consequence of [5, Theorem 3.103], for any set of finite perimeter $E \subseteq \mathbb{R}^{n-1}$, $E_{\sigma,y}$ is a one-dimensional set of finite perimeter for \mathcal{H}^{n-1} -a.e. σ and y , and we have

$$P(E) = \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\{\sigma\}^\perp} P^1(E_{\sigma,y}) \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1},$$

where $P^1(E_{\sigma,y}) = \mathcal{H}^0(\partial E_{\sigma,y})$ is the standard perimeter in dimension 1. Hence, we have the following representation of the critical energy $\mathcal{E}_{G_\varepsilon}$.

Corollary 3.3. *For any set of finite perimeter $E \subseteq \mathbb{R}^n$, we have*

$$\mathcal{E}_{G_\varepsilon}(E) = \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\{\sigma\}^\perp} \mathcal{E}_{\rho_\varepsilon}^1(E_{\sigma,y}) \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1},$$

where $\mathcal{E}_{\rho_\varepsilon}^1 := P^1 - P_{\rho_\varepsilon}^1$, and $E_{\sigma,y}$ is given by (3.1).

We give a simple expression of the one-dimensional critical energy of a segment $(a, b) \subseteq \mathbb{R}$.

Lemma 3.4. *For every $a, b \in \mathbb{R}$ such that $a < b$, we have*

$$\mathcal{E}_{\rho_\varepsilon}^1((a, b)) = 4 \int_{-\infty}^a \int_b^{+\infty} \rho_\varepsilon(s-t) \, ds \, dt. \quad (3.2)$$

In particular $\mathcal{E}_{\rho_\varepsilon}^1((a, b))$ decreases as the interval grows. In addition,

$$\mathcal{E}_{\rho_\varepsilon}^1(\emptyset) = \mathcal{E}_{\rho_\varepsilon}^1(\mathbb{R}) = \mathcal{E}_{\rho_\varepsilon}^1((b, +\infty)) = \mathcal{E}_{\rho_\varepsilon}^1((-\infty, a)) = 0. \quad (3.3)$$

Proof. By a change of variable and Fubini's theorem, for any $a \in \mathbb{R}$, we have

$$P_{\rho_\varepsilon}^1((-\infty, a)) = 2 \int_{-\infty}^a \int_a^{+\infty} \rho_\varepsilon(t-s) \, ds \, dt = 2 \int_0^{+\infty} \rho(t) \left(\int_0^t ds \right) dt = 2 \int_0^{+\infty} t \rho(t) \, dt = 1.$$

Similarly, $P_{\rho_\varepsilon}^1((b, +\infty)) = 1$, and since $P^1((-\infty, a)) = P^1((b, +\infty)) = 1$, (3.3) follows. Next,

$$\begin{aligned} 2 &= P^1((a, b)) = P^1((-\infty, a)) + P^1((b, +\infty)) \\ &= 2 \int_{-\infty}^a \int_a^{+\infty} \rho_\varepsilon(s-t) \, ds \, dt + 2 \int_{-\infty}^b \int_b^{+\infty} \rho_\varepsilon(s-t) \, ds \, dt, \end{aligned} \quad (3.4)$$

for every $a, b \in \mathbb{R}$ such that $a < b$. We also have

$$P_{\rho_\varepsilon}^1((a, b)) = 2 \int_{-\infty}^a \int_a^b \rho_\varepsilon(s-t) \, ds \, dt + 2 \int_b^{+\infty} \int_a^b \rho_\varepsilon(s-t) \, ds \, dt. \quad (3.5)$$

Subtracting (3.5) from (3.4), we obtain (3.2). The identities (3.3) are obvious. \square

For a general set of finite perimeter in \mathbb{R} , we have the following expression of the critical energy.

Lemma 3.5. *For any set $E \subset \mathbb{R}$ which is a finite disjoint union of open intervals, let $\{C_i\}_{i \in \{1, \dots, N\}}$ be the connected components of E^c . Then we have*

$$\mathcal{E}_{\rho_\varepsilon}^1(E) = 2 \sum_{\substack{1 \leq i, j \leq N \\ i \neq j}} \iint_{C_i \times C_j} \rho_\varepsilon(s-t) \, ds \, dt + \sum_{i=1}^N \mathcal{E}_{\rho_\varepsilon}^1(C_i), \quad (3.6)$$

where, for the intervals C_i , $\mathcal{E}_{\rho_\varepsilon}^1(C_i)$ is given by Lemma 3.4.

Let us point out that E^c may have up to two unbounded connected components, and their critical energy is 0 by Lemma 3.4. As a consequence of Lemmas 3.4 and 3.5, the 1D critical energy decreases by convexification and the energy of a nonempty open segment is a decreasing function of its length.

Corollary 3.6. *Let $E \subseteq J \subseteq \mathbb{R}$ with J an interval and $\mathcal{L}^1(E) > 0$, then $\mathcal{E}_{\rho_\varepsilon}^1(J) \leq \mathcal{E}_{\rho_\varepsilon}^1(E)$.*

Proof. Let $E \subseteq \mathbb{R}$ with $\mathcal{L}^1(E) > 0$, and let J be an interval containing E . Let $a' \leq a < b \leq b' \in \mathbb{R} \cup \{\pm\infty\}$ such that $\text{co}(E) = (a, b)$ and $J = (a', b')$. If $a' = -\infty$ or $b' = +\infty$, $\mathcal{E}_{\rho_\varepsilon}^1(J) = 0 \leq \mathcal{E}_{\rho_\varepsilon}^1(E)$ by Lemma 3.4 and Remark 3.2, so the result holds true. If E does not have finite perimeter, then $\mathcal{E}_{\rho_\varepsilon}^1(E) = +\infty$, and the result holds true as well. Thus, let us assume that E is a bounded set of finite perimeter. In particular, up to a negligible set, E is the disjoint union of k open intervals with $k \geq 1$ since $\mathcal{L}^1(E) > 0$. In addition, since E is bounded, E^c has two unbounded components $C_1 = (-\infty, a)$ and $C_{k+1} = (b, +\infty)$ (up to renumbering). By Lemmas 3.4 and 3.5, we have

$$\mathcal{E}_{\rho_\varepsilon}^1(E) \geq 4 \int_{C_1 \times C_{k+1}} \rho_\varepsilon(s-t) \, ds \, dt = \mathcal{E}_{\rho_\varepsilon}^1((a, b)) \geq \mathcal{E}_{\rho_\varepsilon}^1(J).$$

□

Proof of Lemma 3.5. Let

$$E = \bigsqcup_{i=1}^k (a_i, b_i),$$

with $-\infty \leq a_1 < b_1 < \dots < a_k < b_k \leq +\infty$, so that, setting $b_0 := -\infty$ and $a_{k+1} := +\infty$, the connected components of E^c are given by

$$C_i = (b_i, a_{i+1}), \quad \forall i \in \{0, \dots, k\}.$$

Beware that C_0 and C_k are either empty or unbounded. If $a_1 = -\infty$, $C_0 = \emptyset$, and if $b_k = +\infty$, $C_k = \emptyset$, in which cases it is an abuse to consider them connected components of E^c , but they do not contribute to the terms in (3.6). Omitting the integrand $\rho_\varepsilon(s-t) \, ds \, dt$ for the sake of readability, let us write

$$P_{\rho_\varepsilon}^1(E) = 2 \sum_{i=0}^k \int_E \int_{C_i} = \sum_{i=0}^k \left(2 \int_{C_i} \int_{\{t \in E : t > a_{i+1}\}} + 2 \int_{C_i} \int_{\{t \in E : t < b_i\}} \right), \quad (3.7)$$

with the convention $\{t < -\infty\} = \{t > +\infty\} = \emptyset$. By (3.4), we have

$$P^1(E) = \sum_{i=1}^k \left(2 \int_{-\infty}^{a_i} \int_{a_i}^{+\infty} + 2 \int_{-\infty}^{b_i} \int_{b_i}^{+\infty} \right). \quad (3.8)$$

Note that this holds even if $a_1 = -\infty$ or $b_k = +\infty$. Let us define

$$R_i := 2 \int_{-\infty}^{a_{i+1}} \int_{a_{i+1}}^{+\infty} - 2 \int_{C_i} \int_{\{t \in E : t > a_{i+1}\}}, \quad \forall i \in \{0, \dots, k\},$$

and similarly

$$L_i := 2 \int_{-\infty}^{b_i} \int_{b_i}^{+\infty} - 2 \int_{C_i} \int_{\{t \in E : t < b_i\}}, \quad \forall i \in \{0, \dots, k\}.$$

Notice that $R_k = L_0 = 0$. We observe that by definition of L_i , R_i and (3.7), (3.8),

$$\mathcal{E}_{\rho_\varepsilon}^1(E) = \sum_{i=0}^k (L_i + R_i). \quad (3.9)$$

Writing

$$\int_{-\infty}^{a_{i+1}} \int_{a_{i+1}}^{+\infty} = \int_{-\infty}^{b_i} \int_{a_{i+1}}^{+\infty} + \int_{C_i} \int_{a_{i+1}}^{+\infty},$$

for $i \in \{0, \dots, k\}$, using [Lemma 3.4](#) we have

$$R_i = 2 \left(\int_{-\infty}^{b_i} \int_{a_{i+1}}^{+\infty} + \int_{C_i} \int_{a_{i+1}}^{+\infty} \right) - 2 \int_{C_i} \int_{\{t \in E : t > a_{i+1}\}} = 2 \int_{C_i} \int_{\{t \in E^c : t > a_{i+1}\}} + \frac{1}{2} \mathcal{E}_{\rho_\varepsilon}^1(C_i). \quad (3.10)$$

and similarly,

$$L_i = 2 \int_{C_i} \int_{\{t \in E^c : t < b_i\}} + \frac{1}{2} \mathcal{E}_{\rho_\varepsilon}^1(C_i). \quad (3.11)$$

The two previous equations hold even if C_0, C_k are empty or unbounded. Inserting [\(3.10\)](#) and [\(3.11\)](#) into [\(3.9\)](#) yields

$$\begin{aligned} \mathcal{E}_{\rho_\varepsilon}^1(E) &= 2 \sum_{i=0}^k \left(\int_{C_i} \int_{E^c \cap \{t > a_{i+1}\}} + \int_{C_i} \int_{E^c \cap \{t < b_i\}} \right) + \sum_{i=0}^k \mathcal{E}_{\rho_\varepsilon}^1(C_i) \\ &= 2 \sum_{i=0}^k \sum_{j \neq i} \int_{C_i} \int_{C_j} + \sum_{i=0}^k \mathcal{E}_{\rho_\varepsilon}^1(C_i), \end{aligned}$$

which concludes the proof. \square

We easily deduce from [Lemma 3.5](#) and the slicing decomposition of the critical energy stated in [Corollary 3.3](#) that in dimension $n = 2$, the critical energy of a connected set decreases by convexification.

Proposition 3.7. *If $E \subseteq \mathbb{R}^2$ is a bounded, connected set of finite perimeter, then*

$$\mathcal{E}_{G_\varepsilon}(\text{co}(E)) \leq \mathcal{E}_{G_\varepsilon}(E), \quad \forall \varepsilon > 0.$$

Proof. First, recall that a bounded convex set is a set of finite perimeter, since it is Lebesgue-equivalent to an open set with Lipschitz boundary, thus $\text{co}(E)$ is a set of finite perimeter. Then, by [Corollary 3.3](#) we have

$$\mathcal{E}_{G_\varepsilon}(E) = \frac{1}{4} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \mathcal{E}_{\rho_\varepsilon}^1(E_{\sigma,t}) dt d\mathcal{H}_\sigma^1, \quad \text{and} \quad \mathcal{E}_{G_\varepsilon}(\text{co}(E)) = \frac{1}{4} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \mathcal{E}_{\rho_\varepsilon}^1(F_{\sigma,t}) dt d\mathcal{H}_\sigma^1, \quad (3.12)$$

where

$$E_{\sigma,t} := \left\{ s \in \mathbb{R} : t\sigma^\perp + s\sigma \in E \right\}, \quad \text{and} \quad F_{\sigma,t} := \left\{ s \in \mathbb{R} : t\sigma^\perp + s\sigma \in \text{co}(E) \right\}.$$

Since E is connected, for every $\sigma \in \mathbb{S}^1$ and $t \in \mathbb{R}$, the slice $F_{\sigma,t}$ is empty if and only if $E_{\sigma,t}$ is empty (this is the argument which is valid in dimension 2 only). In addition, since E and $\text{co}(E)$ are bounded sets of finite perimeter in \mathbb{R}^2 , for \mathcal{H}^1 -a.e. $\sigma \in \mathbb{S}^1$ and \mathcal{L}^1 -a.e. $t \in \mathbb{R}$, $E_{\sigma,t}$ and $F_{\sigma,t}$ are bounded sets of finite perimeter in \mathbb{R} , and for every nonempty slice, $F_{\sigma,t}$ is an interval s.t. $E_{\sigma,t} \subseteq F_{\sigma,t}$ (since $F_{\sigma,t}$ is a slice of a convex set). Hence, by [Lemmas 3.4](#) and [3.5](#), for \mathcal{H}^1 -a.e. $\sigma \in \mathbb{S}^1$ and \mathcal{L}^1 -a.e. $t \in \mathbb{R}$, there holds

$$\mathcal{E}_{\rho_\varepsilon}^1(F_{\sigma,t}) \leq \mathcal{E}_{\rho_\varepsilon}^1(E_{\sigma,t}).$$

In view of [\(3.12\)](#), this concludes the proof. \square

3.2. Convexity of minimizers in 2D. In this part, we shall use for the sake of brevity the abbreviations $\mathcal{F}_{\gamma,\varepsilon} := \mathcal{F}_{\gamma,G_\varepsilon}$ and $\mathcal{E}_\varepsilon := \mathcal{E}_{G_\varepsilon}$.

Consider a connected minimizer $E_\varepsilon \subseteq \mathbb{R}^2$ of $(\mathcal{P}_{\gamma,\varepsilon})$. Recall that $\mathcal{F}_{\gamma,\varepsilon} = (1 - \gamma)P + \gamma\mathcal{E}_\varepsilon$, thus by [Proposition 3.7](#), $\mathcal{F}_{\gamma,\varepsilon}(\text{co}(E_\varepsilon)) \leq \mathcal{F}_{\gamma,\varepsilon}(E_\varepsilon)$. However, if E_ε is not convex, $|\text{co}(E_\varepsilon)| > |E_\varepsilon|$, so $\text{co}(E_\varepsilon)$ is not a valid competitor for $(\mathcal{P}_{\gamma,\varepsilon})$. Recalling that $B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+\delta(\varepsilon)}$ for small ε , we have

$$B_{1-\delta(\varepsilon)} \subseteq \text{co}(E_\varepsilon) \subseteq B_{1+\delta(\varepsilon)},$$

and defining $t_\varepsilon := \sqrt{\frac{|B_1|}{|\text{co}(E_\varepsilon)|}}$, we see that $t_\varepsilon \text{co}(E_\varepsilon)$ is a valid competitor with $(1 + \delta(\varepsilon))^{-1} < t_\varepsilon < 1$. Let us show that $\mathcal{F}_{\gamma,\varepsilon}(t_\varepsilon \text{co}(E_\varepsilon)) < \mathcal{F}_{\gamma,\varepsilon}(\text{co}(E_\varepsilon))$. This follows from the following result, which we prove further below.

Lemma 3.8. *There exists $\bar{\varepsilon}_1 = \bar{\varepsilon}_1(G, \gamma) > 0$ such that the following holds. If $E \subseteq \mathbb{R}^2$ is a convex set such that*

$$B_{1-\delta(\varepsilon)} \subseteq E \subseteq B_{1+\delta(\varepsilon)},$$

with $0 < \varepsilon < \bar{\varepsilon}_1$, and where δ is the function given by [Theorem 2.7](#), then

$$\mathcal{F}_{\gamma, \varepsilon}(tE) < \mathcal{F}_{\gamma, \varepsilon}(E), \quad \forall t \in (\frac{1}{2}, 1).$$

As a consequence, we obtain that minimizers of $(\mathcal{P}_{\gamma, \varepsilon})$ are necessarily convex for small ε in dimension $n = 2$, that is [Theorem 1](#):

Proof of [Theorem 1](#). Let $E_\varepsilon \subseteq \mathbb{R}^2$ be a minimizer of $(\mathcal{P}_{\gamma, \varepsilon})$ with $0 < \varepsilon < \varepsilon_1 \leq \varepsilon_0$, where $\varepsilon_0(G, \gamma)$ is given by [Theorem 2.7](#). Thus, up to a Lebesgue-negligible set and a translation, E_ε is connected and satisfies

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq \text{co}(E_\varepsilon) \subseteq B_{1+\delta(\varepsilon)}.$$

By contradiction, let us assume that E_ε is not convex, so that $|\text{co}(E_\varepsilon)| > |B_1|$. We have $|t_\varepsilon \text{co}(E_\varepsilon)| = |B_1|$ with $t_\varepsilon := \sqrt{\frac{|B_1|}{|\text{co}(E_\varepsilon)|}}$ satisfying $\frac{1}{2} \leq (1 + \delta(\varepsilon))^{-1} < t_\varepsilon < 1$, provided that $\varepsilon_1 = \varepsilon_1(G, \gamma)$ is chosen small enough. If in addition $\varepsilon_1 < \bar{\varepsilon}_1(G, \gamma)$, where $\bar{\varepsilon}_1$ is given by [Lemma 3.8](#), we find $\mathcal{E}_{\gamma, \varepsilon}(t_\varepsilon \text{co}(E_\varepsilon)) < \mathcal{E}_{\gamma, \varepsilon}(\text{co}(E_\varepsilon))$. Now, since E_ε is connected, there holds $\mathcal{E}_{\gamma, \varepsilon}(\text{co}(E_\varepsilon)) \leq \mathcal{E}_{\gamma, \varepsilon}(E_\varepsilon)$ by [Corollary 3.6](#), hence $\mathcal{E}_{\gamma, \varepsilon}(t_\varepsilon \text{co}(E_\varepsilon)) < \mathcal{E}_{\gamma, \varepsilon}(E_\varepsilon)$. This contradicts the minimality of E_ε , whence E_ε is convex. \square

3.3. Proof of [Lemma 3.8](#). In order to prove that $\mathcal{F}_{\gamma, \varepsilon}(\text{co}(E_\varepsilon))$ (strictly) decreases by scaling down $\text{co}(E_\varepsilon)$ by t_ε for small ε , we need to estimate the term $\tilde{P}_{G_\varepsilon}(t_\varepsilon \text{co}(E_\varepsilon))$ appearing when applying [Lemma 2.3](#). While it is not so difficult to see that for a fixed set E with C^1 boundary, $\tilde{P}_{G_\varepsilon}(E)$ converges to $P(E)$ as ε vanishes, here we do not have uniform regularity estimates on minimizers of $(\mathcal{P}_{\gamma, \varepsilon})^2$. However, we can make use of the convexity of $\text{co}(E_\varepsilon)$ and the fact that it lies between two balls whose radii are close to 1 to prove the following.

Lemma 3.9. *For any $\alpha \in (0, 1)$, there exist positive constants $r = r(\alpha)$ and $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(G, \gamma, \alpha)$ such that the following holds. If $E \subseteq \mathbb{R}^2$ is a convex set such that*

$$B_{1-\delta(\varepsilon)} \subseteq E \subseteq B_{1+\delta(\varepsilon)},$$

with $0 < \varepsilon < \bar{\varepsilon}_2$, and where the function δ is given by [Theorem 2.7](#), then we have

$$\left\{ y \in B_r(x) : \frac{x-y}{|x-y|} \cdot \nu_E(x) \geq \alpha \right\} \subseteq E, \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial E. \quad (3.13)$$

Proof. We proceed in two steps.

Step 1. We show that the normal vector $\nu_E(x)$ converges uniformly for \mathcal{H}^1 -a.e. $x \in \partial E$ to $\nu_{B_1}(x)$ as ε vanishes, in the sense that

$$\left| \nu_E(x) - \frac{x}{|x|} \right| \leq 2\sqrt{\frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}}, \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial E. \quad (3.14)$$

The simple geometric argument is well-known (see e.g. [\[30, 18\]](#)), but we state it here for the reader's convenience. Let $\theta_\varepsilon \in (0, \pi)$ be the angular diameter of the ball $B_{1-\delta(\varepsilon)}$ from any point $x \in \partial B_{1+\delta(\varepsilon)}$, that is, θ_ε is the non-oriented angle between the two lines passing by $x \in \partial B_{1+\delta(\varepsilon)}$ and tangent to $B_{1-\delta(\varepsilon)}$. Then we have $\sin(\frac{\theta_\varepsilon}{2}) = \frac{1-\delta(\varepsilon)}{1+\delta(\varepsilon)}$. Let $x \in \partial E$, and let $\varphi \in (0, \frac{\pi}{2})$ be the non-oriented angle between x and some tangent line to E at x . Since E is convex and $B_{1-\delta(\varepsilon)} \subseteq E$, the tangent cone to E at x cannot intersect $B_{1-\delta(\varepsilon)}$, and since $|x| \leq 1 + \delta(\varepsilon)$, we find

$$\sin \varphi \geq \sin\left(\frac{\theta_\varepsilon}{2}\right) \frac{1-\delta(\varepsilon)}{1+\delta(\varepsilon)}.$$

By convexity of E , for \mathcal{H}^1 -a.e. $x \in \partial E$, there is a unique tangent line to E at x , and writing

$$\left| \nu_E(x) - \frac{x}{|x|} \right|^2 = 2(1 - \sin(\varphi)) = \frac{4\delta(\varepsilon)}{1+\delta(\varepsilon)}$$

²The lack of uniform regularity estimates for minimizers of $(\mathcal{P}_{\gamma, \varepsilon})$ as ε goes to 0 is the main obstacle for proving that the unit ball is, up to translations, the unique minimizer for ε small enough. With sufficient regularity, the arguments of [Section 4](#) would yield the result.

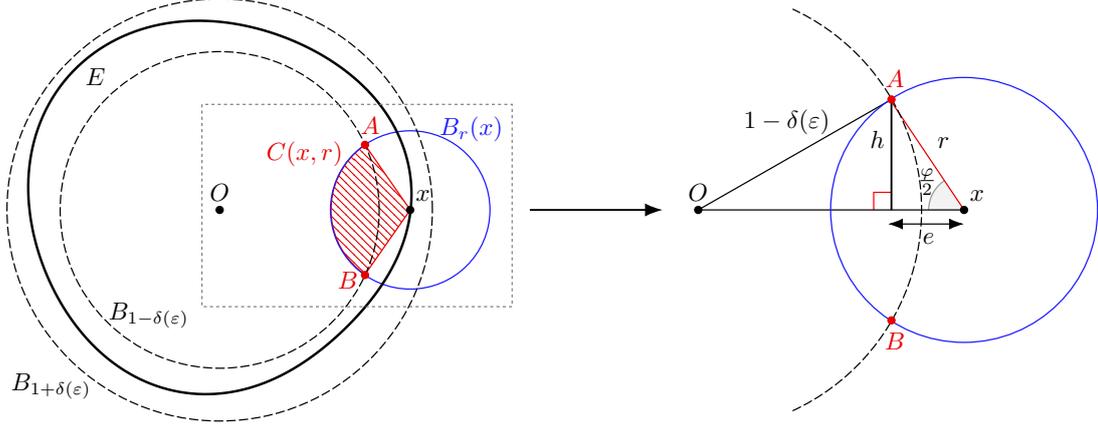


FIGURE 1. The situation in Step 2 of the proof of Lemma 3.9. The red-dashed area is $C(x, r)$.

we deduce (3.14).

Step 2. Let $\alpha \in (0, 1)$, and let $r(\alpha)$, $\bar{\varepsilon}_2(G, \gamma, \alpha)$ to be fixed later. Then let $E \subseteq \mathbb{R}^2$ be as in the statement of the lemma, with $0 < \varepsilon < \bar{\varepsilon}_2$. As a consequence of Step 1, we may assume that for any ε small enough depending only on G , γ and α , we have

$$\left| \frac{x-y}{|x-y|} \cdot \left(\nu_E(x) - \frac{x}{|x|} \right) \right| \leq \frac{\alpha}{2}, \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial E, \text{ and for every } y \in \mathbb{R}^n,$$

so that proving (3.13) amounts to showing that for $r(\alpha)$ and $\bar{\varepsilon}_2(G, \gamma, \alpha)$ small enough,

$$\left\{ y \in B_r(x) : \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \geq \frac{\alpha}{2} \right\} \subseteq E, \quad \forall x \in \partial E. \quad (3.15)$$

We assume that $r \in (0, \frac{1}{2})$ and then that $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(G, \gamma, r) \in (0, \frac{r}{2})$ is such that $\delta(\varepsilon) < \frac{1}{4}$ whenever $0 < \varepsilon < \bar{\varepsilon}_2$. Let $x = |x|(\cos \psi, \sin \psi) \in \partial E$ with $\psi \in [0, 2\pi)$. Since $x \in B_{1+\delta(\varepsilon)} \setminus B_{1-\delta(\varepsilon)}$, the set $\partial B_r(x) \cap B_{1-\delta(\varepsilon)}$ is made of exactly two points A and B . Let us denote by $\varphi(x, r) \in (0, \pi)$ the angle between the segments $[Ax]$ and $[Bx]$. Then let us introduce the circular sector $C(x, r)$ of $B_r(x)$ delimited by A and B (see Figure 1), that is

$$\begin{aligned} C(x, r) &:= \left\{ x - s(\cos(\theta + \psi), \sin(\theta + \psi)) : 0 < s < r, |\theta| \leq \frac{\varphi(x, r)}{2} \right\} \\ &= \left\{ y \in B_r(x) : \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \geq \cos\left(\frac{\varphi(x, r)}{2}\right) \right\}. \end{aligned} \quad (3.16)$$

We claim that for $\bar{\varepsilon}_2$ small enough, $\sin\left(\frac{\varphi(x, r)}{2}\right) \geq \sqrt{1-r^2}$. The situation is as in Figure 1, where we introduce the lengths h and e . We have

$$\begin{cases} h^2 + e^2 = r^2 \\ (|x| - e)^2 + h^2 = (1 - \delta(\varepsilon))^2 \end{cases},$$

which gives, after computation,

$$h^2 = r^2 \left[1 - \frac{1}{4|x|^2} \left(r^2 + 2(|x|^2 - (1 - \delta(\varepsilon))^2) + \frac{1}{r^2} (|x|^2 - (1 - \delta(\varepsilon))^2)^2 \right) \right].$$

Recalling that $1 - \delta(\varepsilon) \leq |x| < 1 + \delta(\varepsilon)$ and $r < \frac{1}{2}$, this implies

$$h^2 \geq r^2 - \frac{r^4}{4(1 + \delta(\varepsilon))^2} \left[1 + \frac{4\delta(\varepsilon)}{r^2} + \left(\frac{4\delta(\varepsilon)}{r^2} \right)^2 \right] \geq r^2 (1 - r^2),$$

provided $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(G, \gamma, r)$ is chosen small enough. Thus, the angle $\varphi(r, x)$ satisfies

$$\sin\left(\frac{\varphi(r, x)}{2}\right) = \frac{h}{r} \geq \sqrt{1-r^2},$$

which proves the claim. Choosing $r = r(\alpha)$ and $\bar{\varepsilon}_2(G, \gamma, r)$ small enough, we deduce $\cos\left(\frac{\varphi(r, x)}{2}\right) \leq \frac{\alpha}{2}$. By convexity of E and the fact that $B_{1-\delta(\varepsilon)} \subseteq E$, we have $C(x, r) \subseteq E$, hence (3.15) holds, in view of (3.16). This concludes the proof. \square

Then we can estimate $\tilde{P}_{G_\varepsilon}(E)$ when $E \subseteq \mathbb{R}^2$ is a convex set lying between the disks $B_{1-\delta(\varepsilon)}$ and $B_{1+\delta(\varepsilon)}$, for any ε small enough.

Lemma 3.10. *For any $\tau > 0$, there exists $\bar{\varepsilon}_3 = \bar{\varepsilon}_3(G, \gamma, \tau) > 0$ such that the following holds. If $E \subseteq \mathbb{R}^2$ is a convex set such that*

$$B_{1-\delta(\varepsilon)} \subseteq E \subseteq B_{1+\delta(\varepsilon)},$$

with $0 < \varepsilon < \bar{\varepsilon}_3$, where the δ function is the one from Theorem 2.7, then we have

$$(1 - \tau)P(E) \leq \tilde{P}_{G_\varepsilon}(E) \leq P(E),$$

where $\tilde{P}_{G_\varepsilon}(E)$ is defined by (2.2).

Proof. Let E be as in the statement of the lemma, with $0 < \varepsilon < \bar{\varepsilon}_3(G, \gamma, \tau)$, where $\bar{\varepsilon}_3$ is to be fixed later. Recall that since E is a bounded convex set, it is a set of finite perimeter and its topological boundary is \mathcal{H}^1 -equivalent to ∂^*E . We proceed in two steps.

Step 1. Upper bound. First, note that by convexity of E , we have

$$(x - y) \cdot \nu_E(x) \geq 0, \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial E, \text{ and every } y \in E,$$

and, defining for \mathcal{H}^1 -a.e. $x \in \partial E$,

$$H_x := \left\{ y \in \mathbb{R}^n : (x - y) \cdot \nu_E(x) > 0 \right\},$$

it holds

$$\tilde{P}_{G_\varepsilon}(E) \leq 2 \int_{\partial E} \int_{H_x} G_\varepsilon(x - y) (x - y) \cdot \nu_E(x) \, d\mathcal{H}_x^1 \, dy.$$

Then, by a change of variable and the coarea formula, we find

$$\begin{aligned} \tilde{P}_{G_\varepsilon}(E) &\leq 2 \int_{\partial E} \int_{\{z \in \mathbb{R}^2 : z \cdot \nu_E(x) > 0\}} G_\varepsilon(z) (z \cdot \nu_E(x)) \, dz \, d\mathcal{H}_x^1 \\ &= 2 \int_{\partial E} \int_0^{+\infty} g_\varepsilon(t) t^2 \left(\int_{\{\sigma \in \mathbb{S}^1 : \sigma \cdot \nu_E(x) > 0\}} \sigma \cdot \nu_E(x) \, d\mathcal{H}_\sigma^1 \right) \, dt \, d\mathcal{H}_x^1 \\ &= \int_{\partial E} \int_0^{+\infty} g_\varepsilon(t) t^2 \left(\int_{\mathbb{S}^1} |\sigma \cdot \nu_E(x)| \, d\mathcal{H}_\sigma^1 \right) \, dt \, d\mathcal{H}_x^1 \\ &= \mathbf{K}_{1,2} \int_{\partial E} |\mathbb{S}^1| \int_0^{+\infty} g_\varepsilon(t) t^2 \, dt \, d\mathcal{H}_x^1 = I_G^1 \mathbf{K}_{1,2} P(E) = P(E), \end{aligned}$$

where we also used the definition of $\mathbf{K}_{1,2}$ for the third equality, and (H2) for the last one.

Step 2. Lower bound. Let us first recall that by assumption (H2),

$$P(E) = 4 \int_{\partial E} \int_0^{+\infty} t^2 g_\varepsilon(t) \, dt \, d\mathcal{H}^1 = 2 \int_{\partial E} \int_0^{+\infty} t^2 g_\varepsilon(t) \int_{\{\sigma \in \mathbb{S}^1 : \sigma \cdot e > 0\}} (\sigma \cdot e) \, d\mathcal{H}_\sigma^1 \, dt \, d\mathcal{H}^1, \quad (3.17)$$

for all $e \in \mathbb{S}^1$. Let $\alpha \in (0, 1)$ to be chosen later, and $\theta_0 := \arcsin(\alpha)$. Then let $r = r(\alpha) > 0$ and $\bar{\varepsilon}_2(G, \gamma, \alpha)$ given by Lemma 3.9. Assume that $\bar{\varepsilon}_3 < \bar{\varepsilon}_2$. Let us write

$$\begin{aligned} \tilde{P}_{G_\varepsilon}(E) &= 2 \int_{\partial E} \int_{E \cap \{|x-y| < r\}} G_\varepsilon(x - y) (x - y) \cdot \nu_E(x) \, dy \, d\mathcal{H}_x^1 \\ &\quad + 2 \int_{\partial E} \int_{E \cap \{|x-y| \geq r\}} G_\varepsilon(x - y) (x - y) \cdot \nu_E(x) \, dx \, d\mathcal{H}_x^1 =: \tilde{P}_{G_\varepsilon}^{(1)}(E) + \tilde{P}_{G_\varepsilon}^{(2)}(E). \end{aligned}$$

Since $\varepsilon < \bar{\varepsilon}_2$, by Lemma 3.9, we have

$$\begin{aligned} \tilde{P}_{G_\varepsilon}^{(1)}(E) &\geq 2 \int_{\partial E} \int_{\{y \in B_r(x) : \frac{x-y}{|x-y|} \cdot \nu_E(x) \geq \alpha\}} G_\varepsilon(x - y) (x - y) \cdot \nu_E(x) \, dy \, d\mathcal{H}_x^1 \\ &= 2 \int_{\partial E} \int_0^{+\infty} g_\varepsilon(t) t^2 \int_{\{\sigma \in \mathbb{S}^1 : \alpha \leq \sigma \cdot \nu_E(x) \leq 1\}} \sigma \cdot \nu_E(x) \, d\mathcal{H}_\sigma^1 \, dt \, d\mathcal{H}_x^1. \end{aligned}$$

We compute

$$\int_{\{\sigma \in \mathbb{S}^1 : \alpha \leq \sigma \cdot \nu_E(x) \leq 1\}} \sigma \cdot \nu_E(x) \, d\mathcal{H}_\sigma^1 = 2 \int_0^{\arccos \alpha} \cos \theta \, d\theta = 2 \sin(\arccos \alpha) = \sqrt{1 - \alpha^2}.$$

Together with (3.17), this leads to

$$\tilde{P}_{G_\varepsilon}^{(1)}(E) \geq \sqrt{1 - \alpha^2} P(E).$$

Choosing α small enough, depending only on τ , we then find

$$\tilde{P}_{G_\varepsilon}^{(1)}(E) \geq \left(1 - \frac{\tau}{2}\right) P(E).$$

Since $\alpha = \alpha(\tau)$, we have $r = r(\tau)$, and $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(G, \gamma, \tau)$. Now with r fixed, up to choosing $\bar{\varepsilon}_3 < \bar{\varepsilon}_2$ even smaller if needed, there holds

$$|\tilde{P}_{G_\varepsilon}^{(2)}(E)| \leq P(E) \int_{B_r^c} |x| G_\varepsilon(x) \, dx = P(E) \int_{B_{r/\varepsilon}^c} |x| G(x) \, dx \leq \frac{\tau}{2} P(E),$$

since the first moment of G is finite. Hence, from our choice of r and $\bar{\varepsilon}_3$, we obtain

$$\tilde{P}_{G_\varepsilon}(E) \geq (1 - \tau) P(E),$$

which concludes the proof. \square

We are now in position to prove Lemma 3.8.

Proof of Lemma 3.8. Let $\tau > 0$ to be chosen later. Recall that by Lemma 2.3, for almost every t , we have

$$\frac{d}{dt} [P_{G_\varepsilon}(tE)] = \frac{2}{t} P_{G_\varepsilon}(tE) - \frac{1}{t} \tilde{P}_{G_\varepsilon}(tE), \quad (3.18)$$

where $\tilde{P}_{G_\varepsilon}$ is defined by (2.2). Changing variables, we deduce

$$\begin{aligned} \frac{1}{t} \tilde{P}_{G_\varepsilon}(tE) &= 2t^3 \int_E \int_{\partial E} G_\varepsilon(t(x-y))(y-x) \cdot \nu_E(y) \\ &= 2 \int_E \int_{\partial E} G_{(t^{-1}\varepsilon)}(x-y)(y-x) \cdot \nu_E(y) \, d\mathcal{H}_y^1 \, dx = \tilde{P}_{G_{t^{-1}\varepsilon}}(E). \end{aligned}$$

Let $\tau > 0$ such that $\gamma(1 + \tau) < 1$, and let $\varepsilon < \bar{\varepsilon}_3(G, \gamma, \tau)/2$, where $\bar{\varepsilon}_3$ is given by Lemma 3.10. By Lemma 3.10, we have

$$\frac{1}{t} \tilde{P}_{G_\varepsilon}(tE) \geq (1 - \tau) P(E).$$

With (3.18) and Proposition 2.1, this leads to

$$\frac{d}{dt} [P_{G_\varepsilon}(tE)] \leq \frac{2}{t} P(tE) - (1 - \tau) P(E) = (1 + \tau) P(E).$$

Hence, for every $t \in (\frac{1}{2}, 1)$, by our choice of τ , it follows

$$\mathcal{F}_{\gamma, \varepsilon}(E) - \mathcal{F}_{\gamma, \varepsilon}(tE) = (1 - t) P(E) - \gamma \int_t^1 \frac{d}{ds} [P_{G_\varepsilon}(sE)] \, ds \geq (1 - t) P(E) [1 - \gamma(1 + \tau)] > 0.$$

This proves the lemma. \square

3.4. Convex minimizers are nearly spherical sets. In arbitrary dimension, it is classical to improve Hausdorff convergence of the boundary of minimizers to Lipschitz convergence once convexity is established. Here, we show that convex minimizers of $(\mathcal{P}_{\gamma, \varepsilon})$ are centered $t(\varepsilon)$ -nearly spherical sets (see Definition 1), up to a translation, where the function t vanishes in 0^+ .

Proposition 3.11 (Convex minimizers are nearly spherical sets). *There exists $\bar{\varepsilon}_4 = \bar{\varepsilon}_4(n, G, \gamma) > 0$ such that the following holds. If $E_\varepsilon \subseteq \mathbb{R}^n$ is a convex minimizer of $(\mathcal{P}_{\gamma, \varepsilon})$ with $0 < \varepsilon < \bar{\varepsilon}_4$, then, up to a translation, we have*

$$\partial E_\varepsilon = \left\{ (1 + u_\varepsilon(x))x : x \in \mathbb{S}^{n-1} \right\}$$

and

$$\int_{E_\varepsilon} x \, dx = 0,$$

with $u_\varepsilon \in \text{Lip}(\mathbb{S}^{n-1})$, $\|u_\varepsilon\|_{L^\infty(\mathbb{S}^{n-1})} \leq C'\delta(\varepsilon)$ and $\|\nabla_\tau u_\varepsilon\|_{L^\infty(\mathbb{S}^{n-1})} \leq C''\delta(\varepsilon)^{\frac{1}{2}}$, where $\delta = \delta(n, G, \gamma)$ is the function of [Theorem 2.7](#) vanishing in 0^+ , and $C', C'' > 0$ only depends on n .

Proof. In the proof, we write $\|\cdot\|_\infty$ for $\|\cdot\|_{L^\infty(\mathbb{S}^{n-1})}$. Let E_ε be a convex minimizer $(\mathcal{P}_{\gamma, \varepsilon})$ with $0 < \varepsilon < \bar{\varepsilon}_4$, where $\bar{\varepsilon}_4$ is to be fixed later. If $\bar{\varepsilon}_4 < \varepsilon_0$, where ε_0 is given by [Theorem 2.7](#), up to a translation and a negligible set, E_ε lies between the balls $B_{1-\delta(\varepsilon)}$ and $B_{1+\delta(\varepsilon)}$. By convexity, this implies that the set E_ε itself (without the addition or the subtraction of a negligible set) satisfies, up to a translation,

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq \bar{B}_{1+\delta(\varepsilon)}.$$

Setting $y_\varepsilon := -\frac{1}{|B_1|} \int_{E_\varepsilon} x \, dx$, we have

$$\int_{y_\varepsilon + E_\varepsilon} x \, dx = y_\varepsilon |E_\varepsilon| + \int_{E_\varepsilon} x \, dx = y_\varepsilon |B_1| + \int_{E_\varepsilon} x \, dx = 0.$$

Notice that

$$-\int_{E_\varepsilon} x \, dx = -\int_{B_{1+\delta(\varepsilon)}} x \, dx + \int_{B_{1+\delta(\varepsilon)} \setminus E_\varepsilon} x \, dx = \int_{B_{1+\delta(\varepsilon)} \setminus E_\varepsilon} x \, dx$$

thus

$$\left| \int_{E_\varepsilon} x \, dx \right| \leq (1 + \delta(\varepsilon)) |B_{1+\delta(\varepsilon)} \setminus E_\varepsilon| \leq (1 + \delta(\varepsilon)) [(1 + \delta(\varepsilon))^n - (1 - \delta(\varepsilon))^n] |B_1| \leq C(n)\delta(\varepsilon)$$

provided that $\bar{\varepsilon}_4$ is small enough depending only on n, G and γ . Hence, up to translating E_ε by y_ε , we may assume that it satisfies

$$B_{1-C'\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+C'\delta(\varepsilon)}, \quad (3.19)$$

with $C' := C(n) + 1$ and is centered, that is,

$$\int_{E_\varepsilon} x \, dx = 0.$$

By convexity of E_ε , for every $x \in \mathbb{S}^{n-1}$, there is a unique point of intersection $p_\varepsilon(x) = t_\varepsilon(x)x$ of $\{tx : t > 0\}$ and ∂E_ε , and by (3.19), $|p_\varepsilon(x) - x| \leq C'\delta(\varepsilon)$. The map $p_\varepsilon : \mathbb{S}^{n-1} \rightarrow \partial E_\varepsilon$ is obviously onto, so that, setting $u_\varepsilon(x) = t_\varepsilon(x) - 1$, we have

$$\partial E_\varepsilon = \left\{ (1 + u_\varepsilon(x))x : x \in \mathbb{S}^{n-1} \right\},$$

and $\|u_\varepsilon\|_\infty \leq C'\delta(\varepsilon)$. In addition, the fact that E_ε is convex implies $u_\varepsilon \in \text{Lip}(\mathbb{S}^{n-1})$. Moreover, for any $x \in \mathbb{S}^{n-1}$, the distance between the normal vector of ∂E_ε (which exists for \mathcal{H}^{n-1} -almost every $x \in \partial E_\varepsilon$) at $p_\varepsilon(x)$ and the vector x (which is the normal vector of \mathbb{S}^{n-1} at x) is controlled by $\|p_\varepsilon - \text{Id}\|_\infty = \|u_\varepsilon\|_\infty$, in view of Step 1 of the proof of [Lemma 3.9](#) (or simply by [[30](#), Corollary 1]), which gives a control of $\|\nabla_\tau u_\varepsilon\|_\infty$ by $\|u_\varepsilon\|_\infty$. More precisely, by [[18](#), Inequality (**)], we have

$$\|\nabla_\tau u_\varepsilon\|_\infty \leq 2 \left(\frac{1 + \|u_\varepsilon\|_\infty}{1 - \|u_\varepsilon\|_\infty} \right) \|u_\varepsilon\|_\infty^{\frac{1}{2}} \leq C''\delta(\varepsilon)^{\frac{1}{2}},$$

where we used the inequality $\|u_\varepsilon\|_\infty \leq C'\delta(\varepsilon) \leq \frac{1}{2}$ for the last inequality, provided that $\bar{\varepsilon}_4(n, G, \gamma)$ is chosen small, and defined $C'' := 6\sqrt{C'}$. This concludes the proof. \square

Observe that [Theorem 2](#) is an immediate corollary of [Theorem 1](#) and [Proposition 3.11](#).

4. MINIMALITY OF THE UNIT BALL AMONG NEARLY SPHERICAL SETS

This section is devoted to the proof of the minimality of the unit ball of \mathbb{R}^n among t -nearly spherical sets, for small t and ε .

4.1. A Fuglede-type result for the nonlocal perimeter. The minimality of the unit ball among nearly spherical sets relies on a bound (in our case, an upper bound) on the quantity $P_{G_\varepsilon}(E_t) - P_{G_\varepsilon}(B_1)$ for a centered t -nearly spherical set E_t with $\partial E_t = \{(1 + tu(x)) : x \in \mathbb{S}^{n-1}\}$, in terms of the L^2 norms of u and $\nabla_\tau u$ on the sphere. In the case of the local perimeter, this kind of control is well-known and is originally due to B. Fuglede (see [18, Theorem 1.2]), who proved

$$\frac{t^2}{10} \left(\|u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) \leq \frac{P(E_t) - P(B_1)}{P(B_1)} \leq \frac{3}{5} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2,$$

provided that t is small enough, depending only on n . One can find a more precise lower bound in [19, Proof of Theorem 3.1, eq. (3.8)]. Similar results were obtained for so-called fractional perimeters P_s as well as for Riesz potentials in [14] (see Theorems 2.1 and 8.1 therein), where the quantities are bounded in terms of the L^2 norm and fractional Sobolev seminorms on the sphere. Our computations are inspired by the ones in [14], however, due to the general form of the kernel G , they are more involved and quite tricky at times.

In addition, when ε vanishes, the quadratic terms from the Taylor expansion of $P_{G_\varepsilon}(E_t) - P_{G_\varepsilon}(B_1)$ compensate exactly those of $P(B_1) - P(E_t)$, so that the constant γ must be smaller than 1, and the expansion needs to be pushed to the third order to be able to conclude. Let us point out that similarly to fractional perimeters, fractional Sobolev-type seminorms associated with the kernel G_ε (and its derivatives) naturally appear in the expansion. However, those converge to the H^1 seminorm as ε vanishes, which is the quantity we need to compare them to (when expanding the local perimeter): that is why we chose to control $P_{G_\varepsilon}(E_t) - P_{G_\varepsilon}(B_1)$ directly in terms of the H^1 seminorm of u on \mathbb{S}^{n-1} .

Lemma 4.1. *Assume that G satisfies (H1), (H2) and (H3). There exist positive constants $\bar{\varepsilon}_5 = \bar{\varepsilon}_5(n, G)$ and $t_0 = t_0(n)$ such that the following holds. If E_t is a centered t -nearly spherical set with $0 < t < t_0$, then for any $0 < \varepsilon < \bar{\varepsilon}_5$, we have*

$$P_{G_\varepsilon}(E_t) \leq P_{G_\varepsilon}(B_1) + \frac{t^2}{2} \left((1 + Cq_\eta(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) + Ct^3 \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \quad (4.1)$$

where $C = C(n, G)$, q_η is given by Lemma 2.4, and $q_\eta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. In the proof, unless stated otherwise, C denotes a positive constant depending only on n and possibly changing from line to line. For the sake of brevity we write B for the open unit ball in \mathbb{R}^n , ∂B for the unit sphere \mathbb{S}^{n-1} , and $\|\cdot\|_\infty$ for $\|\cdot\|_{L^\infty(\mathbb{S}^{n-1})}$. Let $0 < \varepsilon < \bar{\varepsilon}_5$, where $\bar{\varepsilon}_5 = \bar{\varepsilon}_5(n, G)$ is to be fixed later.

Let $t_0 = t_0(n) < \frac{1}{8}$ to be fixed later, and let E_t be a centered t -nearly spherical set such that $\partial E_t = \{(1 + tu(x))x : x \in \mathbb{S}^{n-1}\}$ with $0 < t < t_0$. We proceed in 3 steps.

Step 1. We rewrite $P_{G_\varepsilon}(E_t)$ in a more convenient form, introducing two terms that we will control from above in the next steps. Using polar coordinates, we have

$$P_{G_\varepsilon}(E_t) = 2 \iint_{\partial B \times \partial B} \int_0^{1+tu(x)} \int_{1+tu(y)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}.$$

By symmetry of G , we see that

$$P_{G_\varepsilon}(E_t) = \iint_{\partial B \times \partial B} \left(\int_0^{1+tu(x)} \int_{1+tu(y)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho + \int_0^{1+tu(y)} \int_{1+tu(x)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho \right) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}.$$

Using the identity

$$\int_0^b \int_a^{+\infty} + \int_0^a \int_b^{+\infty} = \int_a^b \int_a^b + \int_0^a \int_a^{+\infty} + \int_0^b \int_b^{+\infty},$$

and the symmetry of G yet again, we find

$$\begin{aligned} P_{G_\varepsilon}(E_t) &= \iint_{\partial B \times \partial B} \int_{1+tu(y)}^{1+tu(x)} \int_{1+tu(y)}^{1+tu(x)} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\quad + 2 \iint_{\partial B \times \partial B} \int_0^{1+tu(x)} \int_{1+tu(x)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}. \end{aligned}$$

Changing variables, we deduce

$$\begin{aligned} &P_{G_\varepsilon}(E_t) \\ &= t^2 \iint_{\partial B \times \partial B} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} (1+ta)^{n-1} (1+tb)^{n-1} G_\varepsilon(x-y+t(ax-by)) da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\quad + 2 \iint_{\partial B \times \partial B} \int_0^{1+tu(x)} \int_{1+tu(x)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &=: t^2 \varphi_\varepsilon(t) + \psi_\varepsilon(t). \end{aligned} \tag{4.2}$$

This concludes Step 1.

Step 2. Estimation of $\varphi_\varepsilon(t)$. Note that by [Lemma 2.4](#) and [Remark 2.5](#), we have

$$\begin{aligned} \varphi_\varepsilon(0) &= \iint_{\partial B \times \partial B} (u(x) - u(y)) G_\varepsilon(x-y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= \iint_{\partial B \times \partial B} \frac{(u(x) - u(y))^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\leq (1 + Cq_\eta(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2. \end{aligned} \tag{4.3}$$

Then let us write,

$$\begin{aligned} &\varphi_\varepsilon(t) - \varphi_\varepsilon(0) \\ &= \iint_{\partial B \times \partial B} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} \left((1+ta)^{n-1} (1+tb)^{n-1} G_\varepsilon(x-y+t(ax-by)) \right. \\ &\quad \left. - G_\varepsilon(x-y) \right) da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= \iint_{\partial B \times \partial B} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} \left((1+ta)^{n-1} (1+tb)^{n-1} - 1 \right) G_\varepsilon(x-y) da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\quad + \iint_{\partial B \times \partial B} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} (1+ta)^{n-1} (1+tb)^{n-1} \left(G_\varepsilon(x-y+t(ax-by)) - G_\varepsilon(x-y) \right) \\ &\quad da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &=: I_1(t) + I_2(t). \end{aligned} \tag{4.4}$$

On one hand, since $\|u\|_\infty \leq 1$, on the domain of integration we have

$$|(1+ta)^{n-1} (1+tb)^{n-1} - 1| \leq Ct \|u\|_\infty \leq Ct.$$

Thus

$$\begin{aligned} I_1(t) &\leq Ct \iint_{\partial B \times \partial B} \int_{u(x)}^{u(y)} \int_{u(x)}^{u(y)} G_\varepsilon(x-y) da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= Ct \iint_{\partial B \times \partial B} (u(x) - u(y))^2 G_\varepsilon(x-y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= Ct \iint_{\partial B \times \partial B} \frac{(u(x) - u(y))^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}, \end{aligned}$$

where η_ε is defined as in [Lemma 2.4](#) so that $g_\varepsilon(r) = \frac{\eta_\varepsilon(r)}{2r^2}$. In view of [Lemma 2.4](#) and [Remark 2.5](#) again, it follows

$$I_1(t) \leq Ct(1 + q_\eta(\varepsilon)) \int_{\partial B} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} \leq Ct \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2, \tag{4.5}$$

for any ε small enough (depending only on n and G).

Let us now bound the term $I_2(t)$. Integrating on a line (recall that g is absolutely continuous in $(0, +\infty)$)

by (H3), and using the inequality $(1 + ta)^{n-1}(1 + tb)^{n-1} \leq 2^{2(n-1)}$ for any $0 \leq |a|, |b| \leq \|u\|_\infty \leq 1$, since $t < t_0 < 1$, we find

$$I_2(t) \leq Ct \iint_{\partial B \times \partial B} \int_{u(x)}^{u(y)} \int_{u(x)}^{u(y)} \int_0^1 |\nabla G_\varepsilon(x - y + st(ax - by)) \cdot (ax - by)| \, ds \, da \, db \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1}. \quad (4.6)$$

Observe that in (4.6), if the term $(x - y)$ inside ∇G_ε were not perturbed by $st(ax - by)$, using the inequality

$$|ax - by|^2 = (a - b)^2 + ab|x - y|^2 \leq (1 + \|\nabla_\tau u\|_\infty^2) |x - y|^2 \leq 2|x - y|^2 \quad (4.7)$$

on the domain of integration, we would have

$$I_2(t) \leq Ct \iint_{\partial B \times \partial B} (u(x) - u(y))^2 |\nabla G_\varepsilon(x - y)| |x - y| \, ds \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1},$$

which we could estimate in terms of $\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2$ by applying directly Lemma 2.4 with $t \mapsto t|g'(t)|$ in place of g . Here, to deal with this perturbation, we apply the technical Lemma A.2, by showing that the right-hand side of (4.6) is bounded by a term of the form

$$Ct \int_0^1 \iint_{\partial B \times \partial B} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{stu}(x, y, r, \rho)|) \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \, ds,$$

where

$$\Phi_{stu}(x, y, r, \rho) := (x - y) + st \left[(ru(x) + (1 - r)u(y))x - (\rho u(y) + (1 - \rho)u(x))y \right] \quad (4.8)$$

is a small perturbation of $(x - y)$, and k_ε is a kernel defined further below. Let us remark that, since G is radial, we have $\nabla G_\varepsilon(x) = g'_\varepsilon(|x|) \frac{x}{|x|}$, thus

$$\nabla G_\varepsilon(x - y + st(ax - by)) \cdot (ax - by) = g'_\varepsilon(|x - y + st(ax - by)|) \frac{\frac{(a+b)}{2}|x - y|^2 + st|ax - by|^2}{|x - y + st(ax - by)|}. \quad (4.9)$$

Then notice that, on the domain of integration, we have

$$\begin{aligned} |x - y + st(ax - by)|^2 &= s^2 t^2 (a - b)^2 + (1 + sta)(1 + stb)|x - y|^2 \\ &\geq (1 + sta)(1 + stb)|x - y|^2 - s^2 t^2 (u(x) - u(y))^2 \\ &\geq \frac{9}{16}|x - y|^2 - \frac{1}{16}\|\nabla_\tau u\|_\infty^2 |x - y|^2 \geq \frac{1}{2}|x - y|^2, \end{aligned} \quad (4.10)$$

since $t < t_0 < \frac{1}{4}$. By (4.7), (4.10), and the fact that $t < t_0 < \frac{1}{8}$, on the domain of integration we deduce

$$\frac{(a+b)}{2}|x - y|^2 + st|ax - by|^2 \leq |x - y|^2 + \frac{1}{4}|x - y|^2 \leq 4|x - y + st(ax - by)|^2.$$

Hence, with (4.9), it follows

$$|\nabla G_\varepsilon(x - y + st(ax - by)) \cdot (ax - by)| \leq C k_\varepsilon(|x - y + st(ax - by)|), \quad (4.11)$$

where we have set $k(r) := r|g'(r)|$ and $k_\varepsilon(r) := \varepsilon^{-(n+1)}k(\varepsilon^{-1}r)$ for all $r > 0$ and $\varepsilon > 0$. For $x, y \in \partial B$ fixed, making the changes of variables $a = u(y) + r(u(x) - u(y))$, and $b = u(x) + \rho(u(y) - u(x))$ in (4.6) yields, with (4.8) and (4.11),

$$I_2(t) \leq Ct \int_0^1 \iint_{\partial B \times \partial B} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{stu}(x, y, r, \rho)|) \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \, ds.$$

By (H3), $I_k^1 = I_G^2 < \infty$ and $k(r) = O(r^{-(n+1)})$ at infinity, so the family $(k_\varepsilon)_{\varepsilon > 0}$ satisfies the assumptions of Lemma A.2. Thus choosing $t_0 < t_1(n)$ and $\bar{\varepsilon}_5 < \bar{\varepsilon}_6$, where $t_1(n)$ and $\bar{\varepsilon}_6(n, G)$ are given by Lemma A.2, we have

$$I_2(t) \leq Ct \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2,$$

for a constant C depending on n and G . Combining this with (4.3) to (4.5), we deduce

$$\varphi_\varepsilon(t) \leq (1 + Cq_\eta(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct \left(\|u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \quad (4.12)$$

for some $C = C(n, G)$, which concludes this step.

Step 3. Estimation of $\psi_\varepsilon(t)$. By Fubini's theorem, we have

$$\psi_\varepsilon(t) = 2 \int_{\partial B} \int_0^{1+tu(x)} \int_{1+tu(x)}^{+\infty} r^{n-1} \rho^{n-1} \left(\int_{\partial B} G_\varepsilon(rx - \rho y) d\mathcal{H}_y^{n-1} \right) dr d\rho d\mathcal{H}_x^{n-1}. \quad (4.13)$$

Notice that for any rotation $R \in SO(n)$, $|rx - \rho(R^{-1}y)| = |r(Rx) - \rho y|$, so that, for any $\sigma \in \partial B$, making the change of variable $y' = R^{-1}y$ with R a rotation mapping x to σ , by symmetry of G , we find

$$\int_{\partial B} G_\varepsilon(rx - \rho y) d\mathcal{H}_y^{n-1} = \int_{\partial B} G_\varepsilon(r\sigma - \rho y) d\mathcal{H}_y^{n-1}.$$

Hence, the above integral does not depend on $x \in \partial B$, and by averaging over ∂B , we obtain

$$\int_{\partial B} G_\varepsilon(rx - \rho y) d\mathcal{H}_y^{n-1} = \frac{1}{P(B)} \iint_{\partial B \times \partial B} G_\varepsilon(r\sigma - \rho y) d\mathcal{H}_\sigma^{n-1} d\mathcal{H}_y^{n-1}.$$

Inserting this into (4.13) and using Fubini's theorem once again yields

$$\begin{aligned} \psi_\varepsilon(t) &= \frac{1}{P(B)} \int_{\partial B} \left(2 \iint_{\partial B \times \partial B} \int_0^{1+tu(x)} \int_{1+tu(x)}^{+\infty} r^{n-1} \rho^{n-1} G_\varepsilon(r\sigma - \rho y) dr d\rho d\mathcal{H}_\sigma^{n-1} d\mathcal{H}_y^{n-1} \right) d\mathcal{H}_x^{n-1} \\ &= \frac{1}{P(B)} \int_{\partial B} \left(2 \iint_{B_{1+tu(x)} \times B_{1+tu(x)}^c} G_\varepsilon(y - z) dy dz \right) d\mathcal{H}_x^{n-1} \\ &= \frac{1}{P(B)} \int_{\partial B} P_{G_\varepsilon}(B_{1+tu(x)}) d\mathcal{H}_x^{n-1}. \end{aligned}$$

Then, by Proposition 2.1, we deduce

$$\psi_\varepsilon(t) \leq \frac{1}{P(B)} \int_{\partial B} P(B_{1+tu(x)}) = \int_{\partial B} (1 + tu(x))^{n-1} d\mathcal{H}_x^{n-1}. \quad (4.14)$$

Since $t < t_0 < \frac{1}{8}$ and $\|u\|_\infty \leq 1$, we have

$$(1 + tu(x))^{n-1} \leq 1 + (n-1)tu(x) + (n-1)(n-2)\frac{t^2}{2}u(x)^2 + Ct|u(x)|^3,$$

thus, by (A.1) of Lemma A.1, from (4.14) it follows

$$\begin{aligned} \psi_\varepsilon(t) &\leq \int_{\partial B} (1 + tu(x))^{n-1} d\mathcal{H}_x^{n-1} \\ &\leq P(B) + \frac{t^2}{2} \left((n-1)(n-2) - (n-1)^2 \right) \int_{\partial B} u^2 d\mathcal{H}^{n-1} + t^3 \int_{\partial B} |u|^3 d\mathcal{H}^{n-1} \\ &= P(B) - (n-1)\frac{t^2}{2} \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + t^3 \int_{\partial B} |u|^3 d\mathcal{H}^{n-1}. \end{aligned} \quad (4.15)$$

This concludes Step 3.

Conclusion. Eventually, gathering (4.2), (4.12) and (4.15), we find

$$\begin{aligned} P_{G_\varepsilon}(E_t) &\leq P_{G_\varepsilon}(B) + \frac{t^2}{2} \left((1 + Cq_\eta(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) \\ &\quad + Ct^3 \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \end{aligned}$$

for some $C = C(n, G)$, provided $t_0 = t_0(n)$ is chosen small enough, which concludes the proof. \square

4.2. Minimality of the unit ball. In order to take advantage of Lemma 4.1, one should have that for a centered t -nearly spherical set E_t such that $\partial E_t = \{(1 + tu(x))x : x \in \mathbb{S}^{n-1}\}$, the quantity

$$(1 + Cq_\eta(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2$$

from (4.1) controls $\|u\|_{H^1(\mathbb{S}^{n-1})}^2$ for small t and ε . This is the purpose of Lemma A.1 in appendix.

With Lemma 4.1 and Fuglede's result for the local perimeter, we deduce a lower bound for $\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1)$.

Proposition 4.2. *Assume that G satisfies (H1) to (H3). Then there exist positive constants t_* and ε_3 depending only on n , G and γ such that the following holds. If E_t is a centered t -nearly spherical set such that $\partial E_t = \{(1 + tu(x))x : x \in \mathbb{S}^{n-1}\}$ with $0 < t < t_*$, then for every $0 < \varepsilon < \varepsilon_3$, we have*

$$\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1) \geq \frac{t^2}{16}(1 - \gamma) \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right).$$

Proof. Assume $0 < \varepsilon < \varepsilon_3$, and $0 < t < t_*$, where $t_* = t_*(n)$ and $\varepsilon_3 = \varepsilon_3(n, G, \gamma)$ will be fixed later. If E_t is a centered t -nearly spherical set and $t_* < t_1$ as well, where $t_1 = t_1(n)$ is given by Lemma A.1, we have in particular

$$\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \leq 4 \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1)\|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right). \quad (4.16)$$

Then, by [19, Proof of Theorem 3.1, eq. (3.8)] (which gives a more precise lower bound of $P(E_t) - P(B)$ than [18, Theorem 1.2]) and (4.16), up to choosing t_* even smaller but depending only on n , we have

$$P(E_t) \geq P(B_1) + \frac{t^2}{2} \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1)\|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) - Ct^3 \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right).$$

for some positive constant $C = C(n)$. Then, assuming that $t_* < t_0(n)$ as well, and $\varepsilon < \bar{\varepsilon}_5(n, G)$, where t_0 and $\bar{\varepsilon}_5$ are given by Lemma 4.1, we find

$$\begin{aligned} & \mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1) \\ & \geq \frac{t^2}{2} \left[(1 - \gamma) \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1)\|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) - C'q_\eta(\varepsilon)\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 \right] \\ & \quad - C't^3 \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \end{aligned} \quad (4.17)$$

for some positive constant $C' = C'(n, G)$, with $q_\eta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. From (4.16) and (4.17), it follows

$$\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1) \geq \frac{1}{4} \left(\frac{t^2}{2} \left((1 - \gamma) - C'q_\eta(\varepsilon) \right) - (C + C')t^3 \right) \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right).$$

Eventually, choosing ε_3, t_* small enough depending only on n, G and γ , for any $0 < \varepsilon < \varepsilon_3$ and $0 < t < t_*$, we have

$$\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1) \geq \frac{t^2}{16}(1 - \gamma) \left(\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right),$$

which proves the result. \square

An immediate consequence of Proposition 4.2 is that the unit ball is the only minimizer, up to translations, of $(\mathcal{P}_{\gamma, \varepsilon})$, among t -nearly spherical sets whenever $t < t_*(n, G, \gamma)$ and $\varepsilon < \varepsilon_3(n, G, \gamma)$, that is, Theorem 3. In dimension $n = 2$, by Theorems 2 and 3, setting $\varepsilon_* := \min(\varepsilon_2, \varepsilon_3)$, we readily obtain Theorem A.

APPENDIX A. ADDITIONAL COMPUTATIONS FOR SECTION 4

In the following lemma, we establish some general inequalities on functions $u : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ describing centered nearly-spherical sets. For this, we need to recall a few basic facts and notation on spherical harmonics. For $k \geq 0$, we denote by \mathcal{S}_k the subspace of spherical harmonics of degree k (i.e., restrictions to \mathbb{S}^{n-1} of polynomials of degree k in \mathbb{R}^n), which is a finite-dimensional vector space of degree $d(k)$. Let $(Y_k^i)_{i \in \{1, \dots, d(k)\}}$ be an orthonormal basis of \mathcal{S}_k for the standard scalar product of $L^2(\mathbb{S}^{n-1})$. When there can be no confusion, we write Y_k for a generic vector of the basis of \mathcal{S}_k . It is well-known that the family $(Y_k^i)_{i \in \{1, \dots, d(k)\}}$ is a Hilbert basis of $L^2(\mathbb{S}^{n-1})$ which diagonalizes the Laplace-Beltrami operator on the sphere, and the eigenvalue associated with Y_k^i is $l_k := k(k+n-2)$, for all $i \in \{1, \dots, d(k)\}$. We recall that $d(0) = 1$, $d(1) = n$, and that the Y_1^i may be chosen colinear to $x \mapsto x_i$, for example $Y_1^i = |B_1|^{-\frac{1}{2}}x_i$ for all $i \in \{1, \dots, n\}$.

Lemma A.1. *There exist $t_1 = t_1(n) > 0$ and $C = C(n) > 0$ such that the following holds. If E_t is a centered t -nearly spherical set such that $\partial E_t = \{(1 + tu(x))x : x \in \mathbb{S}^{n-1}\}$ with $0 < t < t_1$, then we have*

$$\left| t \int_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} + (n-1) \frac{t^2}{2} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1} \right| \leq Ct^3 \int_{\mathbb{S}^{n-1}} |u|^3 \, d\mathcal{H}^{n-1}, \quad (A.1)$$

and

$$\frac{1}{2} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1)\|u\|_{L^2(\mathbb{S}^{n-1})}^2 \geq \frac{1}{2} \|u\|_{L^2(\mathbb{S}^{n-1})}^2. \quad (A.2)$$

Proof. Since $|E_t| = |B|$, we have

$$n|E_t| = \int_{\partial B} (1 + tu(x))^n d\mathcal{H}^{n-1} = \int_{\partial B} 1 d\mathcal{H}^{n-1} = n|B|.$$

Thus, writing

$$(1 + tu(x))^n - \left(1 + ntu(x) + \frac{(n-1)}{2}t^2u(x)^2\right) = \sum_{k=3}^n \binom{n}{k} t^k u(x)^k,$$

we deduce (A.1), choosing e.g. $t_1 = 1$ and C depending only on n . There remains to show (A.2). We decompose u in spherical harmonics

$$u = \sum_{k=0}^{+\infty} \sum_{i=1}^{d(k)} a_k^i(u) Y_k^i, \quad (\text{A.3})$$

so that

$$\|u\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{k=0}^{+\infty} \sum_{i=1}^{d(k)} a_k^i(u)^2, \quad \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{k=1}^{+\infty} \sum_{i=1}^{d(k)} l_k a_k^i(u)^2.$$

Recall that $d(0) = 1$, Y_0^1 is constant, $d(1) = n$, and that Y_1^i is colinear to $x \mapsto x_i$. Since $l_1 = n - 1$ and $l_k \geq 2n$ for $k \geq 2$, it follows

$$\begin{aligned} & \frac{1}{2} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \\ & \geq \|u\|_{L^2(\mathbb{S}^{n-1})}^2 - \frac{(n+1)}{2} \left(\sum_{i=1}^n a_1^i(u)^2 \right) - (n-1) a_0^1(u)^2. \end{aligned} \quad (\text{A.4})$$

On one hand,

$$a_0^1(u) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} u d\mathcal{H}^{n-1},$$

so that by (A.1), we have

$$|a_0^1(u)| \leq \frac{2t}{|\mathbb{S}^{n-1}|} \|u\|_{L^2(\mathbb{S}^{n-1})} \leq \frac{1}{2\sqrt{n-1}} \|u\|_{L^2(\mathbb{S}^{n-1})}, \quad (\text{A.5})$$

up to choosing $t_1 = t_1(n)$ small enough, and using $t < t_1$ and $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$. On the other hand, the barycenter condition

$$\int_{E_t} x dx = 0$$

gives

$$\int_{\mathbb{S}^{n-1}} x_i (1 + tu(x))^n d\mathcal{H}_x^{n-1} = 0, \quad \forall i \in \{1, \dots, n\}.$$

Using the binomial formula again and $Y_1^i(x) = |B_1|^{-\frac{1}{2}} x_i$, we obtain, for $i \in \{1, \dots, n\}$,

$$a_1^i(u) = \frac{1}{\sqrt{|B_1|}} \int_{\mathbb{S}^{n-1}} x_i u(x) d\mathcal{H}_x^{n-1} = -\frac{1}{n\sqrt{|B_1|}} \sum_{k=2}^n \binom{n}{k} t^k \int_{\mathbb{S}^{n-1}} x_i u(x)^k d\mathcal{H}_x^{n-1}.$$

Next, using $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$, Cauchy-Schwarz inequality, and $n|B_1| = |\mathbb{S}^{n-1}|$, we get

$$|a_i(u)| \leq \frac{2^n}{n} t^2 \|u\|_{L^2(\mathbb{S}^{n-1})}.$$

Whence, choosing t_1 even smaller, but still depending only on n , we may assume

$$\sum_{i=1}^n a_1^i(u)^2 \leq \frac{1}{2(n+1)} \|u\|_{L^2(\mathbb{S}^{n-1})}^2. \quad (\text{A.6})$$

Gathering (A.4) to (A.6), we find

$$\frac{1}{2} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \geq \frac{1}{2} \|u\|_{L^2(\mathbb{S}^{n-1})}^2,$$

which proves (A.2) and concludes the proof. \square

We establish a technical lemma to control terms of the form

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1},$$

by the $H^1(\mathbb{S}^{n-1})$ norm of u , where $\Phi_{tu}(x, y, r, \rho)$ is a small perturbation of $(x - y)$, and k_ε are suitable rescalings of a nonnegative kernel.

Lemma A.2. *Let $k : (0, +\infty) \rightarrow [0, +\infty)$ be a measurable function such that $k(r) \leq C_0 r^{-(n+1)}$ on $(R_0, +\infty)$ for some positive constants C_0, R_0 , and*

$$I_k^1 := \int_{\mathbb{R}^n} |x| k(|x|) dx < +\infty.$$

Let us define the rescaling $k_\varepsilon(r) := \varepsilon^{-(n+1)} k(\varepsilon^{-1}r)$, $r, \varepsilon > 0$. For $v \in \text{Lip}(\mathbb{S}^{n-1})$, we define a map $\Phi_v : (\mathbb{S}^{n-1})^2 \times (0, 1)^2$ by

$$\Phi_v(x, y, r, \rho) = (x - y) + \left[(rv(x) + (1-r)v(y))x - (\rho v(y) + (1-\rho)v(x))y \right]. \quad (\text{A.7})$$

Then there exist positive constants $\bar{t}_1 = \bar{t}_1(n)$, $\bar{C}_0 = \bar{C}_0(n, C_0, I_k^1)$ and $\bar{\varepsilon}_6 = \bar{\varepsilon}_6(n, R_0)$ such that the following holds. For any $u \in \text{Lip}(\mathbb{S}^{n-1})$ with $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$, $\|\nabla_\tau u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$, and any $t \in (0, \bar{t}_1)$, we have

$$\begin{aligned} & \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \leq \bar{C}_0 \left(\int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} |u|^2 d\mathcal{H}^{n-1} \right). \end{aligned} \quad (\text{A.8})$$

Proof. Let $\bar{t}_1 > 0$ to be chosen later. In the proof, we assume that $0 < t < \bar{t}_1$. We work in local coordinates, proceeding in two steps.

Step 1. Let us denote by $D_r(x)$ the open ball of radius r centered at x in \mathbb{R}^{n-1} . We first show that for any $\tilde{u} \in \text{Lip}(D_2)$ such that $\|\tilde{u}\|_{L^\infty(D_2)} + \|\nabla \tilde{u}\|_{L^\infty(D_2)} \leq M$, we have

$$\int_{D_2} \int_{D_2} \int_0^1 \int_0^1 (\tilde{u}(x) - \tilde{u}(y))^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dr d\rho dx dy \leq C \int_{D_2} |\nabla \tilde{u}|^2 dx, \quad \forall r, \rho \in (0, 1), \quad (\text{A.9})$$

for all $\varepsilon > 0$, provided $t < \bar{t}_1$, and \bar{t}_1 is chosen small enough depending only on n and M , and $C = C(n, I_k^1)$. Here, by a slight abuse of notation, Φ_v denotes the map from $(\mathbb{R}^{n-1})^2 \times (0, 1)^2$ whose expression is given by (A.7). For $v \in \text{Lip}(D_2)$, let us define the map $E_v : (\mathbb{R}^{n-1})^2 \times (0, 1)^2$ by

$$E_v(x, y, r, \rho) := (rv(x) + (1-r)v(y))x - (\rho v(y) + (1-\rho)v(x))y,$$

so that $\Phi_v(x, y, r, \rho) = x - y + E_v(x, y, r, \rho)$. Due to the L^∞ bound on \tilde{u} and $\nabla \tilde{u}$, we easily see that the maps $E_{t\tilde{u}}$ converge uniformly to 0 on $(D_2)^2 \times (0, 1)^2$ as t vanishes. In fact, we have

$$\begin{aligned} |E_{t\tilde{u}}(x, y, r, \rho)| & \leq t (2\|\tilde{u}\|_{L^\infty(D_2)} + \|\nabla \tilde{u}\|_{L^\infty(D_2)}) |x - y| \\ & \leq 2M\bar{t}_1 |x - y| \end{aligned} \quad \forall x, y \in D_2, \forall r, \rho \in (0, 1). \quad (\text{A.10})$$

In particular, choosing $\bar{t}_1 < \frac{1}{4M}$,

$$\frac{|x - y|}{2} \leq |\Phi_{t\tilde{u}}(x, y, r, \rho)| \leq 2|x - y|, \quad \forall x, y \in D_2, \forall r, \rho \in (0, 1). \quad (\text{A.11})$$

Integrating on lines, using Cauchy-Schwarz inequality and Fubini's theorem, we have

$$\begin{aligned} & \int_{D_2} \int_{D_2} \int_0^1 \int_0^1 (\tilde{u}(x) - \tilde{u}(y))^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dr d\rho dx dy \\ & \leq \int_0^1 \int_0^1 \int_{D_2} \int_{D_2} |\nabla \tilde{u}(x + s(y - x))|^2 |x - y|^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dx dy ds dr d\rho. \end{aligned} \quad (\text{A.12})$$

Now let us focus on the integral on $D_2 \times D_2$, fixing $s \in (0, \frac{1}{2})$ and $r, \rho \in (0, 1)$. By (A.11), our choice of \bar{t}_1 , and the fact that $0 < t < \bar{t}_1$, it follows

$$\begin{aligned} & \int_{D_2} \int_{D_2} |\nabla \tilde{u}(x + s(y-x))|^2 |x-y|^2 k_\varepsilon(|\Phi_{tu}^\sim(x, y, r, \rho)|) dx dy \\ & \leq 4 \int_{D_2} \int_{D_2} |\nabla \tilde{u}(x + s(y-x))|^2 |\Phi_{tu}^\sim(x, y, r, \rho)|^2 k_\varepsilon(|\Phi_{tu}^\sim(x, y, r, \rho)|) dx dy. \end{aligned} \quad (\text{A.13})$$

We wish to make the change of variables

$$\Psi_{tu, r, \rho, s}^\sim(x, y) = (\Phi_{tu}^\sim(x, y, r, \rho), (1-s)x + sy) =: (x', y').$$

First, let us remark that $\Psi_{tu, r, \rho, s}^\sim$ is an injection whenever $\bar{t}_1 < \frac{1}{4M}$. Indeed,

$$\begin{aligned} \Psi_{tu, r, \rho, s}^\sim(x_1, y_1) = \Psi_{tu, r, \rho, s}^\sim(x_2, y_2) & \iff \begin{cases} x_1 - x_2 = y_1 - y_2 + E_{tu}^\sim(x_2, y_2) - E_{tu}^\sim(x_1, y_1) \\ (1-s)(x_1 - x_2) = s(y_2 - y_1) \end{cases} \\ & \iff \begin{cases} x_1 - x_2 = s [E_{tu}^\sim(x_2, y_2) - E_{tu}^\sim(x_1, y_1)] \\ (1-s)(x_1 - x_2) = s(y_2 - y_1). \end{cases} \end{aligned} \quad (\text{A.14})$$

We compute

$$\begin{aligned} & E_{tu}^\sim(x_2, y_2) - E_{tu}^\sim(x_1, y_1) \\ & = t \left[(r\tilde{u}(x_2) + (1-r)\tilde{u}(y_2))x_2 - (r\tilde{u}(x_1) + (1-r)\tilde{u}(y_2))x_1 \right] \\ & \quad - t \left[(\rho\tilde{u}(y_2) + (1-\rho)\tilde{u}(x_2))y_2 - (\rho\tilde{u}(y_1) + (1-\rho)\tilde{u}(x_1))y_1 \right] \\ & = t \left[(r\tilde{u}(x_2) + (1-r)\tilde{u}(y_2))(x_2 - x_1) + (r(\tilde{u}(x_2) - \tilde{u}(x_1)) + (1-r)(\tilde{u}(y_1) - \tilde{u}(y_2)))x_1 \right] \\ & \quad - t \left[(\rho\tilde{u}(y_2) + (1-\rho)\tilde{u}(x_2))(y_2 - y_1) + (\rho(\tilde{u}(y_2) - \tilde{u}(y_1)) + (1-\rho)(\tilde{u}(x_2) - \tilde{u}(x_1)))y_1 \right], \end{aligned}$$

thus, using the inequality $\|\tilde{u}\|_{L^\infty(D_2)} + \|\nabla \tilde{u}\|_{L^\infty(D_2)} \leq M$ and the fact that $x_1, y_1 \in D_2$, we find

$$\begin{aligned} |E_{tu}^\sim(x_2, y_2) - E_{tu}^\sim(x_1, y_1)| & \leq t (\|\tilde{u}\|_{L^\infty(D_2)} + 2\|\nabla \tilde{u}\|_{L^\infty(D_2)}) (|x_2 - x_1| + |y_2 - y_1|) \\ & \leq 2M\bar{t}_1 (|x_2 - x_1| + |y_2 - y_1|). \end{aligned} \quad (\text{A.15})$$

If $(1-s)(x_1 - x_2) = s(y_2 - y_1)$, (A.15) implies

$$s|E_{tu}^\sim(x_2, y_2) - E_{tu}^\sim(x_1, y_1)| \leq 2M\bar{t}_1|x_2 - x_1|,$$

and plugging this into (A.14), we see that $\Psi_{tu, r, \rho, s}^\sim$ is injective if $\bar{t}_1 < \frac{1}{4M}$. Then, a simple computation gives

$$D\Psi_{tu, r, \rho, s}^\sim(x, y) = \begin{pmatrix} A(x, y) & -B(x, y) \\ (1-s)I_n & sI_n \end{pmatrix},$$

where I_n denotes the identity matrix in \mathbb{R}^n , and for almost every $x, y \in D_2$,

$$A(x, y) := I_n + t \left[(r\tilde{u}(x) + (1-r)\tilde{u}(y))I_n + (rx - (1-\rho)y) \otimes \nabla \tilde{u}(x) \right]$$

and

$$B(x, y) := I_n + t \left[(\rho\tilde{u}(y) + (1-\rho)\tilde{u}(x))I_n + (\rho y - (1-r)x) \otimes \nabla \tilde{u}(y) \right].$$

Since $\det D\Psi_{tu, r, \rho, s}^\sim = \det(sA + (1-s)B)$ and

$$sA(x, y) + (1-s)B(x, y) = I_n + t \left[I_n + s(rx - (1-\rho)y) \otimes \nabla \tilde{u}(x) + (1-s)((1-r)x - \rho y) \otimes \nabla \tilde{u}(y) \right],$$

we see that, choosing \bar{t}_1 even smaller if needed, depending only on n and M , we have

$$|\det D\Psi_{tu, r, \rho, s}^\sim(x, y)| \geq \frac{1}{2}.$$

Hence, making the change of variables $(x', y') = \Psi_{\tilde{t}u, r, \rho, s}(x, y)$ in (A.13) yields

$$\begin{aligned} & \int_{D_2} \int_{D_2} |\nabla \tilde{u}(x + s(y - x))|^2 |x - y|^2 k_\varepsilon(|\Phi_{\tilde{t}u}(x, y, r, \rho)|) dx dy \\ & \leq 8 \iint_{\Psi_{\tilde{t}u, r, \rho, s}(D_2 \times D_2)} |\nabla \tilde{u}(y')|^2 |x'|^2 k_\varepsilon(|x'|) dx' dy' \\ & \leq 8 \left(\int_{D_2} |\nabla \tilde{u}(y)|^2 dy \right) \left(\int_0^8 t^n k_\varepsilon(t) dt \right), \end{aligned} \quad (\text{A.16})$$

where we used the fact that $\Psi_{\tilde{t}u, r, \rho, s}(D_2 \times D_2) \subseteq D_8 \times D_2$ for the last inequality, in view of (A.11). Plugging (A.16) into (A.12) and making the change of variable $t' = t/\varepsilon$ gives (A.9), by the assumptions on k , which concludes this step.

Step 2. We split the domain of integration in the left-hand side of (A.8) in $\{x, y \in \mathbb{S}^{n-1} : |x - y| > \frac{1}{4}\}$ and $\{x, y \in \mathbb{S}^{n-1} : |x - y| \leq \frac{1}{4}\}$. We first treat the contribution of distant pairs (x, y) . Choosing \bar{t}_1 small enough depending only on n , we have

$$|\Phi_{tu}(x, y, r, \rho)| \geq \frac{|x - y|}{2}, \quad \forall x, y \in \mathbb{S}^{n-1}, \forall r, \rho \in (0, 1),$$

so that, by the squared triangle inequality and by symmetry,

$$\begin{aligned} & \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ \{|x-y| > \frac{1}{4}\}}} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \leq 2 \int_{\mathbb{S}^{n-1}} |u(x)|^2 \left(\int_{\substack{\mathbb{S}^{n-1} \\ \{|x-y| > \frac{1}{4}\}}} \int_0^1 \int_0^1 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_y^{n-1} \right) d\mathcal{H}_x^{n-1} \\ & \leq 2|\mathbb{S}^{n-1}| \left(\int_{\mathbb{S}^{n-1}} |u|^2 d\mathcal{H}^{n-1} \right) \left(\sup_{r > \frac{1}{8}} k_\varepsilon(r) \right). \end{aligned} \quad (\text{A.17})$$

Then, choosing $\bar{\varepsilon}_6 = \frac{1}{8R_0}$, and using that $k(r) \leq C_0 r^{-(n+1)}$ for all $r \in (R_0, +\infty)$, for any $0 < \varepsilon < \bar{\varepsilon}_6$, (A.17) implies

$$\begin{aligned} & \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ \{|x-y| > \frac{1}{4}\}}} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \leq C \int_{\mathbb{S}^{n-1}} |u|^2 d\mathcal{H}^{n-1}, \end{aligned} \quad (\text{A.18})$$

for some $C = C(n, C_0)$.

There remains to estimate the integral over the domain $\mathcal{M} := \{x, y \in \mathbb{S}^{n-1} : |x - y| < \frac{1}{4}\}$. For this, we cover \mathcal{M} by a finite number $N(n)$ of $\mathcal{M}_i := \mathbb{S}_+^{n-1}(x_i) \times \mathbb{S}_+^{n-1}(x_i)$, where for each $i \in \{1, \dots, N\}$, $x_i \in \mathbb{S}^{n-1}$, and $\mathbb{S}_+^{n-1}(x_i)$ denotes the hemisphere with center x_i . Using the stereographic projection Π_i with respect to $-x_i$, we map $\mathbb{S}_+^{n-1}(x_i)$ to $D_2 \subseteq \mathbb{R}^{n-1}$. By the changes of variables $\xi = \Pi_i(x)$, $\zeta = \Pi_i(y)$, setting $\tilde{u}_i := u \circ \Pi_i^{-1}$, we have $\|\tilde{u}_i\|_{L^\infty(D_2)} + \|\nabla \tilde{u}_i\|_{L^\infty(D_2)} \leq C(n)$ since $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$ and $\|\nabla_\tau u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$. Applying Step 1 with $\tilde{u} = \tilde{u}_i$, for any $\varepsilon > 0$, we obtain

$$\begin{aligned} & \iint_{\mathcal{M}_i} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \leq C \iint_{D_2 \times D_2} \int_0^1 \int_0^1 (\tilde{u}_i(\xi) - \tilde{u}_i(\zeta))^2 k_\varepsilon(|\Phi_{\tilde{t}u_i}(\xi, \zeta, r, \rho)|) dr d\rho d\xi d\zeta \\ & \leq C \int_{D_2} |\nabla \tilde{u}_i|^2 dx \leq C \int_{\mathbb{S}_+^{n-1}(x_i)} |\nabla_\tau u|^2 d\mathcal{H}^{n-1}, \end{aligned}$$

whenever $t < \bar{t}_1$, where $\bar{t}_1 = \bar{t}_1(n)$ and $C = C(n, I_k^1)$. Summing these estimates over $i \in \{1, \dots, N(n)\}$ and using (A.18), we conclude the proof. \square

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REFERENCES

- [1] S. ALAMA, L. BRONSARD, R. CHOKSI, I. TOPALOGLU : Droplet breakup in the liquid drop model with background potential, *Commun. Contemp. Math.* **21.03** (2018), pp. 1850022.
- [2] S. ALAMA, L. BRONSARD, R. CHOKSI, I. TOPALOGLU : Ground-states for the liquid drop and TFDW models with long-range attraction, *Journal of Mathematical Physics* **58.10** (2017), pp. 103503.
- [3] S. ALAMA, L. BRONSARD, I. TOPALOGLU, A. ZUNIGA : A nonlocal isoperimetric problem with density perimeter, *Calc. Var. Partial Differ. Equ.* **60.1** (2021), pp. 1.
- [4] L. AMBROSIO, V. CASELLES, S. MASNOU, J.-M. MOREL : Connected components of sets of finite perimeter and applications to image processing, *J. Eur. Math. Soc.* **3.1** (2001), pp. 39–92.
- [5] L. AMBROSIO, N. FUSCO, D. PALLARA : *Functions of bounded variation and free discontinuity problems*, Oxford mathematical monographs, Clarendon Press, (2000).
- [6] J. BERENDSEN, V. PAGLIARI : On the asymptotic behaviour of nonlocal perimeters, *ESAIM: COCV* **25** (2019), pp. 48.
- [7] M. BONACINI, R. CRISTOFERI : Local and global minimality results for a nonlocal isoperimetric problem on \mathbb{R}^N , *SIAM J. Math. Anal.* **46.4** (2014), pp. 2310–2349.
- [8] J. BOURGAIN, H. BREZIS, P. MIRONESCU : Another look at Sobolev spaces, *Optimal Control and Partial Differential Equations (In honor of Professor Alain Bensoussan's 60th anniversary)*, ed. by J. MENALDI, E. ROFMAN, A. SULEM, (2001), pp. 439–455.
- [9] D. CARAZZATO, N. FUSCO, A. PRATELLI : Minimality of balls in the small volume regime for a general Gamow type functional, 2020, URL: <http://arxiv.org/abs/2009.03599>.
- [10] A. CESARONI, M. NOVAGA : The isoperimetric problem for nonlocal perimeters, *Discrete Contin. Dyn. Syst. - S* **11.3** (2018), pp. 425–440.
- [11] R. CHOKSI, C. B. MURATOV, I. TOPALOGLU : An old problem resurfaces nonlocally: Gamow's liquid drops inspire today's research and applications, *Notices Amer. Math. Soc.* **64.11** (2017), pp. 1275–1283.
- [12] J. DÁVILA : On an open question about functions of bounded variation, *Calc. Var. Partial Differ. Equ.* **15.4** (2002), pp. 519–527.
- [13] L. C. EVANS, R. F. GARIEPY : *Measure theory and fine properties of functions*, Revised edition, Textbooks in mathematics, CRC Press, Taylor & Francis Group, (2015).
- [14] A. FIGALLI, N. FUSCO, F. MAGGI, V. MILLOT, M. MORINI : Isoperimetry and stability properties of balls with respect to nonlocal energies, *Commun. Math. Phys.* **336.1** (2015), pp. 441–507.
- [15] R. L. FRANK, R. KILLIP, P. T. NAM : Nonexistence of large nuclei in the liquid drop model, *Lett Math Phys* **106.8** (2016), pp. 1033–1036.
- [16] R. L. FRANK, E. H. LIEB : A compactness lemma and Its application to the existence of minimizers for the liquid drop model, *SIAM J. Math. Anal.* **47.6** (2015), pp. 4436–4450.
- [17] R. L. FRANK, P. T. NAM : Existence and nonexistence in the liquid drop model, 2021, URL: <http://arxiv.org/abs/2101.02163>.
- [18] B. FUGLEDE : Stability in the isoperimetric problem for convex or nearly spherical domains in \mathbb{R}^n , *Trans. Amer. Math. Soc.* **314.2** (1989), pp. 619–638.
- [19] N. FUSCO : The quantitative isoperimetric inequality and related topics, *Bull. Math. Sci.* **5.3** (2015), pp. 517–607.
- [20] F. GÉNÉRAU, E. OUDET : Large volume minimizers of a nonlocal isoperimetric problem: theoretical and numerical approaches, *SIAM J. Math. Anal.* **50.3** (2018), pp. 3427–3450.
- [21] L. GRAFAKOS : *Classical Fourier Analysis*, vol. 249, Graduate Texts in Mathematics, Springer New York, (2014).
- [22] V. JULIN : Isoperimetric problem with a coulomb repulsive term, *Indiana Univ. Math. J.* **63.1** (2014), pp. 77–89.
- [23] H. KNÜPFER, C. B. MURATOV : On an isoperimetric problem with a competing nonlocal term I: the planar case, *Comm. Pure Appl. Math.* **66.7** (2013), pp. 1129–1162.
- [24] H. KNÜPFER, C. B. MURATOV : On an isoperimetric problem with a competing nonlocal term II: the general case, *Comm. Pure Appl. Math.* **67.12** (2014), pp. 1974–1994.

- [25] H. KNÜPFER, C. B. MURATOV, M. NOVAGA : Low density phases in a uniformly charged liquid, *Commun. Math. Phys.* **345.1** (2016), pp. 141–183.
- [26] J. LU, F. OTTO : Nonexistence of a minimizer for Thomas-Fermi-Dirac-von Weizsäcker model, *Comm. Pure Appl. Math.* **67.10** (2014), pp. 1605–1617.
- [27] F. MAGGI : *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*, Cambridge University Press, (2012).
- [28] C. B. MURATOV, T. M. SIMON : A nonlocal isoperimetric problem with dipolar repulsion, *Commun. Math. Phys.* **372.3** (2019), pp. 1059–1115.
- [29] M. NOVAGA, A. PRATELLI : Minimisers of a general Riesz-type Problem, 2020, URL: <http://arxiv.org/abs/2007.02107>.
- [30] R. OSSERMAN : A strong form of the isoperimetric inequality in \mathbb{R}^n , *Complex Variables, Theory and Application: An International Journal* **9.2-3** (1987), pp. 241–249.
- [31] M. PEGON : Large mass minimizers for isoperimetric problems with integrable nonlocal potentials, *Nonlinear Analysis* **211** (2021), pp. 112395.
- [32] S. RIGOT : Ensembles quasi-minimaux avec contrainte de volume et rectifiabilité uniforme, *Mémoires Soc. Math. France* **1** (2000), pp. 1–104.

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