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# THE ISOPERIMETRIC PROBLEM FOR THE PERIMETER MINUS A FRACTION OF A NONLOCAL APPROXIMATE PERIMETER IN 2D

BENOIT MERLET AND MARC PEGON

ABSTRACT. We prove that unit area minimizers are disks for an isoperimetric problem in which the standard perimeter  $P(E)$  is replaced by  $P(E) - \gamma P_\varepsilon(E)$ , where  $\gamma \in (0, 1)$  and  $P_\varepsilon$  is a nonlocal energy such that  $P_\varepsilon(E) \rightarrow P(E)$  as  $\varepsilon$  vanishes.

More precisely, we show that in dimension 2, connected minimizers are necessarily convex, provided that  $\varepsilon$  is small enough. In turn, this implies that minimizers have nearly circular boundaries, that is, their boundary is a small Lipschitz perturbation of the circle. Then, using a Fuglede-type argument, we prove that in arbitrary dimension  $n \geq 2$ , the unit ball in  $\mathbb{R}^n$  is the unique unit-volume minimizer of the problem among nearly spherical sets. As a consequence, up to translations, the unit disk is the unique minimizer.

This isoperimetric problem is equivalent to a generalization of the so-called liquid drop model for the atomic nucleus introduced by Gamow, where the nonlocal repulsive potential is given by a radial kernel sufficiently integrable kernel. In that formulation our result states that above a critical mass  $m_0$ , the disk of area  $m > m_0$  is the unique minimizer of area  $m$ , up to translations.

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## 1. INTRODUCTION

Given a positive, radial, integrable kernel  $G : \mathbb{R}^n \rightarrow [0, +\infty)$  with finite first moment (that is,  $|x|G(x) \in L^1(\mathbb{R}^n)$ ), we consider the so-called nonlocal perimeter functional  $P_G$  (see e.g. [9, 5]) defined on measurable subsets  $E \subseteq \mathbb{R}^n$  by

$$P_G(E) := 2 \iint_{E \times (\mathbb{R}^n \setminus E)} G(x - y) \, dx \, dy = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\mathbf{1}_E(x) - \mathbf{1}_E(y)| G(x - y) \, dx \, dy. \quad (1.1)$$

Here  $\mathbf{1}_E$  denotes the indicator function of  $E$ .

For  $\varepsilon > 0$ , we introduce the rescaled kernel  $G_\varepsilon(x) := \varepsilon^{-(n+1)} G(\varepsilon^{-1}x)$ ,  $x \in \mathbb{R}^n$ . As will be justified later, the first moment of  $G$  is fixed to an explicit dimensional constant (see (H2)) so that  $P_{G_\varepsilon}(E)$  converges to  $P(E)$  as  $\varepsilon$  vanishes. Given  $\gamma \in (0, 1)$  and  $\varepsilon > 0$ , we study the minimization problem

$$\min \left\{ P(E) - \gamma P_{G_\varepsilon}(E) : |E| = |B_1| \right\}, \quad (\mathcal{P}_{\gamma, \varepsilon})$$

over sets of finite perimeter  $E$  in  $\mathbb{R}^n$ , where  $|E|$  denotes the volume of  $E$ , that is, its Lebesgue measure, and  $B_1$  is the open unit ball of  $\mathbb{R}^n$ .

Let us emphasize the competition between the two terms. The perimeter is an attractive term minimized by balls under volume constraint. On the contrary, if  $G$  is radially decreasing, it is easy to

see that, due to the negative sign, the nonlocal term is maximized by balls.<sup>1</sup> This competition makes the minimization problem nontrivial, even when it comes to existence of minimizers.

Let us introduce the “critical energies”

$$\mathcal{E}_{G_\varepsilon} := P - P_{G_\varepsilon} = \mathcal{F}_{1, G_\varepsilon},$$

and define the energies

$$\mathcal{F}_{\gamma, G_\varepsilon}(E) := (1 - \gamma)P(E) + \gamma\mathcal{E}_{G_\varepsilon}(E) = P(E) - \gamma P_{G_\varepsilon}(E). \quad (1.2)$$

Although the paper mostly deals with the “subcritical case”  $\gamma < 1$ , we focus on the critical energies in [Section 3.1](#), and show that they decrease by convexification. Finally, for any  $k \in \mathbb{N} \setminus \{0\}$ , we denote by

$$I_G^k := \int_{\mathbb{R}^n} |x|^k |\partial_r^{k-1} G(x)| dx$$

the  $k$ -th moment of the  $(k - 1)$ -th radial derivative of the kernel  $G$ , whenever it is well defined.

In the paper, starting from [Section 2](#), we shall always implicitly assume that the kernel  $G$  satisfies the following general assumptions:

- (H1)  $G$  is nonnegative and radial, that is, there exists a measurable function  $g : (0, +\infty) \rightarrow [0, +\infty)$  such that  $G(x) = g(|x|)$  for a.e.  $x \in \mathbb{R}^n$ ;
- (H2) the first moment of  $G$  is finite and set to be

$$I_G^1 = \frac{1}{\mathbf{K}_{1,n}},$$

where, for any  $\nu \in \mathbb{S}^{n-1}$ , the constant  $\mathbf{K}_{1,n}$  is defined by

$$\mathbf{K}_{1,n} = \int_{\mathbb{S}^{n-1}} |\sigma \cdot \nu| d\mathcal{H}_\sigma^{n-1}.$$

Starting from [Section 4](#), we may explicitly use the extra assumption

- (H3)  $G \in W_{\text{loc}}^{3,1}(\mathbb{R}^n \setminus \{0\})$  and the moments  $I_G^k$  are finite for  $k \in \{2, 3, 4\}$ .

Note that those assumptions are in particular satisfied by so-called Bessel kernels, that is, fundamental solutions of the operators  $(I - \kappa\Delta)^{\frac{\alpha}{2}}$ , for  $\alpha, \kappa > 0$  (see e.g. [\[28, § 3.2\]](#) or [\[18\]](#) for their definition and properties).

We are interested in the asymptotic behavior of  $(\mathcal{P}_{\gamma,\varepsilon})$  as  $\varepsilon$  vanishes, especially in the planar case  $n = 2$ , where we prove that the unit disk is the only minimizer, up to translations, provided that  $\varepsilon$  is small enough. Note however that the intermediate results of [Section 3.4](#) and [Section 4](#) are obtained in arbitrary dimension, that is, convex minimizers are so-called nearly spherical sets, and the unit ball of  $\mathbb{R}^n$  is the unique minimizer among nearly spherical sets whenever  $\varepsilon$  is small enough.

Let us elaborate on the motivation for studying problem  $(\mathcal{P}_{\gamma,\varepsilon})$ . In fact, it is closely linked with variations of Gamow’s liquid drop model for the atomic nucleus. Indeed, if in addition we assume that  $G$  is integrable in  $\mathbb{R}^n$ , then we may write

$$\frac{1}{2} P_{G_\varepsilon}(E) = \|G_\varepsilon\|_{L^1(\mathbb{R}^n)} |E| - \iint_{E \times E} G(x - y) dx dy, \quad (1.3)$$

so that, changing variables, we see that  $E$  is a minimizer of  $(\mathcal{P}_{\gamma,\varepsilon})$  if and only if the set  $F := \varepsilon^{-1}E$  is a minimizer of

$$\min \left\{ P(F) + \iint_{F \times F} \tilde{G}(x - y) : |F| = m_\varepsilon \right\}, \quad (1.4)$$

where  $m_\varepsilon := \varepsilon^{-n}|B_1|$  and  $\tilde{G} := 2G$ . When  $n = 3$  and  $\tilde{G}(x) = \frac{1}{8\pi|x|}$ , this is Gamow’s liquid drop model (see [\[10\]](#) for a general overview); note however that in that case, the minimized functional cannot be rewritten as the difference between the perimeter and a nonlocal perimeter, since  $G$  is not integrable near the origin. As a prototypical model for various physical systems involving the competition between short-range attractive forces and long-range repulsive ones, generalizations of this model have gained increasing interest during the past decade, in particular generalizations in higher dimensions, where the Coulomb potential is replaced with Riesz potentials, that is,  $\tilde{G}(x) = |x|^{\alpha-n}$ ,  $\alpha \in (0, n)$ . Without being exhaustive, it was shown that for every Riesz kernel, there exists a critical mass  $m_0 > 0$  such that, for every  $m < m_0$ , the ball of volume  $m$  is, up to translations, the unique minimizer of the liquid drop

<sup>1</sup>In fact, even if  $G$  is not radially decreasing,  $G_\varepsilon$  “concentrates” near the origin when  $\varepsilon$  is small, heuristically making  $(-P_{G_\varepsilon})$  a repulsive term whenever  $G$  does not vanish identically.

model (1.4) (see [20, 21, 19, 6, 13]). Conversely, for  $\alpha \in [n - 2, n)$ , there exists a critical mass  $m_1 > 0$  such that, for every  $m > m_1$ , the problem admits no minimizer (see [6, 20, 21, 23, 14]). More general kernels of Riesz-type were then studied e.g. in [8, 26], where the unit ball is shown to be the unique minimizer in the small mass regime. The case of general compactly supported kernels has also been addressed in [29], where the author shows existence of minimizers *for all masses*.

Between Riesz kernels, which are not integrable at infinity, where (1.4) does not admit minimizers above a critical mass, and compactly supported kernels, where (1.4) always admits minimizers, it is natural to wonder what happens with non-compactly supported but reasonably decaying kernels, such as Bessel kernels. These kernels behave as Riesz potentials near the origin, but decrease exponentially at infinity. They were suggested in [22] as a replacement for Riesz kernels for modeling diblock copolymers when long-range interactions are partially screened by fluctuations in the background nuclear fluid density. The study of the liquid drop model in the large mass regime with kernels satisfying assumptions (H1) and (H2) (such as Bessel kernels), which is equivalent to the study of  $(\mathcal{P}_{\gamma,\varepsilon})$  when  $\varepsilon$  is small, has been started by the second author in [28]. The existence of minimizers for any  $\gamma \in (0, 1)$  was established therein for  $\varepsilon$  small enough, as well as the convergence of minimizers to the unit ball as  $\varepsilon$  vanishes. It was conjectured there that the ball is actually the unique minimizer up to translations, for  $\varepsilon$  small enough.

In this paper, we give a positive answer to this conjecture in dimension  $n = 2$ , with the extra assumption (H3), which is still satisfied by Bessel kernels. The conjecture remains open in higher dimensions. Note that the kernel  $G$  is assumed to be radial but not necessarily radially nonincreasing, as is often the case. Let us also emphasize that, contrarily to the small mass regime for Riesz-type potentials, where the nonlocal repulsive term vanishes compared to the attractive perimeter term, here the nonlocal perimeter term does not vanish but rather localizes to a fraction of the standard perimeter. Although the small mass regime has been extensively studied, the literature on large mass minimizers for Gamow-type problem is still sparse, and existence is usually recovered by adding an extra attractive potential such as in [2, 1, 17].

We shall now state the main result of the paper.

**Theorem A** (Minimality of the unit disk). *Assume  $n = 2$  and  $G$  satisfies (H1) to (H3). Then there exists  $\varepsilon_* = \varepsilon_*(G, \gamma) > 0$ , such that, for every  $\varepsilon < \varepsilon_*$ , the unit disk is the unique minimizer of  $(\mathcal{P}_{\gamma,\varepsilon})$ , up to translations.*

The proof decomposes as follows. First we establish:

**Theorem 1** (2D minimizers are convex). *Assume  $n = 2$  and  $G$  satisfies (H1) and (H2). Then there exists  $\varepsilon_1 = \varepsilon_1(G, \gamma) > 0$  such that, for every  $0 < \varepsilon < \varepsilon_1$ ,  $(\mathcal{P}_{\gamma,\varepsilon})$  admits a minimizer, and every minimizer is convex, up to a Lebesgue-negligible set.*

The existence of minimizers for small  $\varepsilon$  was shown in [28], where the author also proved that they are necessarily connected whenever  $\varepsilon$  is small enough. The idea for proving the convexity of minimizers is to study the critical energy on the real line, and show that it decreases by convexification and by expansion of segments, so that, by a slicing argument, the critical energy of a *connected set in dimension 2* decreases after convexification. As a consequence, since the perimeter of a connected set is also reduced by convexification, so is  $\mathcal{F}_{\gamma,G_\varepsilon}$ . This slicing argument is specific to the dimension 2, where a line intersects a connected set if and only if it intersects its convex hull. This fails in higher dimension.

Note that this is not enough to conclude that minimizers are convex. Indeed, although for every minimizer  $E \subset \mathbb{R}^2$  with  $\varepsilon$  small enough, we have  $\mathcal{F}_{\gamma,G_\varepsilon}(\text{co}(E)) \leq \mathcal{F}_{\gamma,G_\varepsilon}(E)$ , where  $\text{co}(E)$  denotes the convex hull of  $E$ , the volume of  $\text{co}(E)$  is larger than  $|B_1|$  if  $E$  is not convex. However, using the fact that a minimizer  $E$  is already close to the unit ball by [28] and the convexity of  $\text{co}(E)$ , we prove that, if  $E$  is not convex, scaling down  $\text{co}(E)$  to make its volume equal to  $|B_1|$  strictly decreases  $\mathcal{F}_{\gamma,G_\varepsilon}(E)$ , which contradicts the minimality.

The convexity of minimizers  $E_\varepsilon$  allows us to improve the convergence of  $\partial E_\varepsilon$  towards  $\partial B_1$  as  $\varepsilon$  goes to 0, from the previously known Hausdorff convergence to Lipschitz convergence. We deduce that minimizers are nearly spherical sets, whose definition we give just below.

**Definition 1** (Nearly spherical sets). *For  $t \in (0, \frac{1}{2})$ , we say that  $E$  is a centered  $t$ -nearly spherical set if*

$$\int_E x \, dx = 0,$$

and if there exists  $u \in \text{Lip}(\mathbb{S}^{n-1})$  with  $\|u\|_{L^\infty(\mathbb{S}^{n-1})} + \|\nabla_\tau u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$  such that

$$\partial E = \left\{ (1 + tu(x))x : x \in \mathbb{S}^{n-1} \right\}.$$

In dimension  $n = 2$ , we use the terminology “ $t$ -nearly circular set” for “ $t$ -nearly spherical set”.

**Theorem 2** (2D minimizers have nearly circular boundaries; see [Proposition 3.11](#)). *Assume  $n = 2$  and  $G$  satisfies (H1) and (H2). There exist  $\varepsilon_2 = \varepsilon_2(G, \gamma) > 0$  and a function  $t : (0, \varepsilon_2) \rightarrow [0, \frac{1}{2})$  depending only on  $G$  and  $\gamma$  vanishing in 0 such that, for every  $\varepsilon < \varepsilon_2$ , any minimizer  $E$  of  $(\mathcal{P}_{\gamma, \varepsilon})$  is, up to translation, a centered  $t(\varepsilon)$ -nearly circular set.*

[Theorem 2](#) is a direct consequence of [Theorem 1](#) and the uniform convergence of minimizers already shown in [\[28\]](#), using the geometric fact that the normal vectors to a convex set lying between two balls  $B_r$  and  $B_R$ ,  $r \leq 1 \leq R$  converge to those of the unit ball as  $r, R \rightarrow 1$  (see [\[27, 15\]](#)).

We end the proof by showing that for  $\varepsilon$  and  $t$  small enough, any centered  $t$ -spherical minimizer of  $(\mathcal{P}_{\gamma, \varepsilon})$  is the unit ball. This last result is not specific to dimension  $n = 2$ .

**Theorem 3** (Minimality of the unit ball among nearly spherical sets; see [Proposition 4.3](#)). *Assume that  $G$  satisfies (H1) to (H3). Then there exist  $t_* = t_*(n, G, \gamma) > 0$  and  $\varepsilon_3 = \varepsilon_3(n, G, \gamma) > 0$  such that, for every  $t < t_*$ , if  $E$  is a  $t$ -nearly spherical set, then we have*

$$\mathcal{F}_{\gamma, \varepsilon}(B_1) \leq \mathcal{F}_{\gamma, \varepsilon}(E), \quad \forall 0 < \varepsilon < \varepsilon_3,$$

and the inequality is strict if  $E \neq B_1$ .

The proof relies on a Fuglede-type result [\[15\]](#) for the nonlocal perimeter  $P_{G_\varepsilon}$  of nearly spherical sets  $E_t$ : more precisely, by a Taylor expansion we control  $P_{G_\varepsilon}(E_t) - P_{G_\varepsilon}(B_1)$  from above in terms of the  $H^1(\mathbb{S}^{n-1})$  norm of the perturbation  $u$  describing  $E_t$  (see [Lemma 4.2](#)).

[Theorem A](#) is then an immediate consequence of [Theorems 1 to 3](#).

**Outline of the paper.** The structure of the paper follows the strategy of the proof. In [Section 2](#) we recall some useful results from [\[28\]](#) on minimizers of  $(\mathcal{P}_{\gamma, \varepsilon})$  and some facts on nonlocal perimeters. In [Section 3](#), we show that 2D minimizers are convex and thus nearly circular sets for small  $\varepsilon$ , that is, [Theorem 1](#) and [Theorem 2](#), where the latter is a consequence of [Proposition 3.11](#). Finally, [Section 4](#) is dedicated to the proof of [Theorem 3](#), which is a consequence of [Proposition 4.3](#).

### Notation.

*Operations on sets.* For any set  $E \subseteq \mathbb{R}^n$ ,  $E^c := \mathbb{R}^n \setminus E$  denotes its complement,  $\text{co}(E)$  its convex hull (that is, the intersection of all convex sets containing  $E$ ), and  $|E|$  its Lebesgue measure, whenever  $E$  is measurable. We write  $E \sqcup F$  for the union of two disjoint sets.

*Hausdorff measures.* We denote by  $\mathcal{H}^k$  the  $k$ -dimensional Hausdorff measure in  $\mathbb{R}^n$ . When integrating w.r.t. the measure  $\mathcal{H}^k$  in a variable  $x$ , we use the notation  $d\mathcal{H}_x^k$  instead of the more standard but less compact  $d\mathcal{H}^k(x)$ .

*Balls and spheres.* We denote by  $B_r(x)$  the open ball in  $\mathbb{R}^n$  of radius  $r$  centered at  $x$ . For brevity, we write  $B_r$  when  $x$  is the origin. The volume of  $B_1$  is  $\omega_n := |B_1| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}$ , and the area of the unit sphere  $\mathbb{S}^{n-1}$  is  $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$ , which we also write  $|\mathbb{S}^{n-1}|$  for simplicity.

*Sets of finite perimeter.* We denote by  $\text{BV}(\mathbb{R}^n)$  the space of functions with bounded variation in  $\mathbb{R}^n$ . For any  $f \in \text{BV}(\mathbb{R}^n)$  we let  $|Df|$  be its total variation measure, and set  $[f]_{\text{BV}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |Df|$ . For a set of finite perimeter  $E$  in  $\mathbb{R}^n$ , we let  $\mathbf{1}_E \in \text{BV}(\mathbb{R}^n)$  be its characteristic function (i.e.,  $\mathbf{1}_E(x) = 1$  if  $x \in E$  and 0 otherwise), and define its perimeter by  $P^n(E) := \int_{\mathbb{R}^n} |D\mathbf{1}_E|$ . When there can be no confusion, we may drop the superscript and simply write  $P(E)$  for the perimeter functional in  $\mathbb{R}^n$ . We denote by  $\mu_E := D\mathbf{1}_E$  the Gauss–Green measure associated with the set of finite perimeter  $E$ , and by  $\nu_E(x)$  the outer unit normal of  $\partial^*E$  at  $x$ , where  $\partial^*E$  stands for the reduced boundary of  $E$ . We refer to e.g. [\[12, Chapter 5\]](#) or [\[24\]](#) for further details on functions of bounded variations and sets of finite perimeter.

## 2. PRELIMINARIES

From now on we assume that  $G$  satisfies (H1) and (H2).

For a general nonnegative radial kernel  $K$  with finite first moment, we have the following control of  $P_K$  by the perimeter, as an immediate consequence of [28, Proposition 3.1] and of the second expression of the nonlocal perimeter in (1.1).

**Proposition 2.1.** *If  $K : \mathbb{R}^n \rightarrow [0, +\infty)$  is a nonnegative radial kernel with finite first moment, for every set of finite perimeter  $E$  in  $\mathbb{R}^n$ , we have*

$$P_K(E) \leq \mathbf{K}_{1,n} I_K^1 P(E).$$

In particular, by (H1) and (H2), we have

$$P_{G_\varepsilon}(E) \leq P(E), \quad \forall \varepsilon > 0.$$

We also have the following convergence result, which is a consequence of [11].

**Proposition 2.2.** *For any set of finite perimeter  $E$  in  $\mathbb{R}^n$ , we have*

$$P_{G_\varepsilon}(E) \xrightarrow{\varepsilon \rightarrow 0} P(E).$$

We will use the following computation obtained in [28, Lemma 3.5], which clarifies the behavior of the nonlocal perimeter under scaling.

**Lemma 2.3.** *For any set of finite perimeter  $E \subseteq \mathbb{R}^n$ , the function  $t \mapsto P_{G_\varepsilon}(tE)$  is locally Lipschitz continuous in  $(0, +\infty)$ , and for almost every  $t$ , we have*

$$\frac{d}{dt} [P_{G_\varepsilon}(tE)] = \frac{n}{t} P_{G_\varepsilon}(tE) - \frac{1}{t} \tilde{P}_{G_\varepsilon}(tE),$$

where  $\tilde{P}_{G_\varepsilon}(E)$  is defined by

$$\tilde{P}_{G_\varepsilon}(E) := 2 \int_E \int_{\partial^* E} G_\varepsilon(x-y) (y-x) \cdot \nu_E(y) d\mathcal{H}_y^{n-1} dx. \quad (2.1)$$

Let us remark that in [28],  $G$  is assumed to be in addition integrable in  $\mathbb{R}^n$ , but the proof is already done by approximation with smooth compactly functions, and it adapts by approximating  $x \mapsto |x|G(x)$  with functions  $x \mapsto |x|G_k(x)$  in  $L^1(\mathbb{R}^n)$  s.t.  $|G_k| \leq G$  and noticing that

$$|P_G(E) - P_{G_k}(E)| \leq P(E) \int_{\mathbb{R}^n} |x| |(G - G_k)(x)| dx.$$

In order to study the minimality of the unit ball among nearly spherical sets, we will use the following Bourgain-Brezis-Mironescu-type result (see [7]) for approximating the  $H^1$  seminorm on the sphere by nonlocal seminorms.

**Lemma 2.4.** *Let us define the  $(n-1)$ -dimensional approximation of identity  $(\eta_\varepsilon)_{\varepsilon>0}$  by*

$$\eta(t) := 2t^2 g(t), \quad \text{and} \quad \eta_\varepsilon(t) := \varepsilon^{-(n-1)} \eta(\varepsilon^{-1}t), \quad \forall t > 0, \quad \forall \varepsilon > 0.$$

When  $n = 2$ , we assume in addition that  $g$  is such that the family  $(\eta_\varepsilon)_{\varepsilon>0}$  satisfies

$$\sup_{r \in (R, 2)} \eta_\varepsilon(r) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \forall R \in (0, 2). \quad (2.2)$$

Then for any  $u \in H^1(\mathbb{S}^{n-1})$ , we have

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{(u(x) - u(y))^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \leq (1 + q_\eta(\varepsilon)) \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1},$$

where  $q_\eta(\varepsilon)$  vanishes as  $\varepsilon$  goes to 0, and depends only on  $n$  and  $G$ . In addition, for any  $u \in H^1(\mathbb{S}^{n-1})$ ,

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{(u(x) - u(y))^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1}.$$

*Proof.* One easily checks that assumptions (H1) and (H2) ensure that the family  $(\eta_\varepsilon)_{\varepsilon>0}$  is a  $(n-1)$ -dimensional approximation of identity, up to multiplication by the constant  $\mathbf{K}_{2,n-1} = \frac{1}{n-1}$ , i.e.,

$$(i) \quad |\mathbb{S}^{n-2}| \int_0^{+\infty} \eta_\varepsilon(r) r^{n-2} dr = \frac{1}{\mathbf{K}_{2,n-1}}; \quad (ii) \quad \lim_{\varepsilon \rightarrow 0} \int_\delta^{+\infty} \eta_\varepsilon(r) r^{n-2} dr = 0, \quad \forall \delta > 0.$$

These properties (together with (2.2) when  $n = 2$ ) allow us to apply [28, Propositions A.1 & A.4], which gives the result.  $\square$

*Remark 2.5.* If (H3) stands true, then the condition (2.2) is satisfied, in particular when  $n = 2$ . Indeed, by Remark 4.1 below, we see that (H3) implies that  $r^{n+1}g(r) \xrightarrow{r \rightarrow \infty} 0$ , so that

$$\eta_\varepsilon(r) = r^2 \varepsilon^{-(n+1)} g(\varepsilon^{-1}r) = r^{1-n} (\varepsilon^{-1}r)^{-(n+1)} g(\varepsilon^{-1}r)$$

vanishes uniformly on  $(R, 2)$  as  $\varepsilon \rightarrow 0$ , for every  $R \in (0, 2)$ .

Eventually, gathering results from [28, Theorems A and B] (see also Theorem 4.16 therein), we have existence and convergence results for minimizers of  $(\mathcal{P}_{\gamma,\varepsilon})$ . We also know that minimizers are connected for small  $\varepsilon$ . Here connectedness is to be understood in a measure-theoretic sense for sets of finite perimeter, often referred to as indecomposability, as defined below (see [3]).

**Definition 2.6.** We say that a set of finite perimeter  $E$  is decomposable if there exist two sets of finite perimeter  $E_1$  and  $E_2$  such that  $E = E_1 \sqcup E_2$ ,  $|E_1| > 0$ ,  $|E_2| > 0$  and  $P(E) = P(E_1) + P(E_2)$ . Naturally, we say that a set of finite perimeter is indecomposable if it is not decomposable.

Let us remark that by [3, Theorem 2], the notion of connectedness and indecomposability coincide whenever  $E$  is an open set of finite perimeter such that  $\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial^* E)$ .

**Theorem 2.7.** *There exist  $\varepsilon_0 = \varepsilon_0(n, G, \gamma)$  and a function  $\delta = \delta(n, G, \gamma) : (0, +\infty) \rightarrow (0, \frac{1}{4})$  vanishing in 0 such that the following holds. For every  $0 < \varepsilon < \varepsilon_0$ ,  $(\mathcal{P}_{\gamma,\varepsilon})$  admits a minimizer. In addition, any such minimizer  $E_\varepsilon$  is indecomposable, and up to a translation and a Lebesgue-negligible set, it satisfies*

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+\delta(\varepsilon)}. \quad (2.3)$$

*In dimension  $n = 2$ , any minimizer  $E_\varepsilon$  with  $0 < \varepsilon < \varepsilon_0$  is Lebesgue-equivalent to a connected set which still satisfies (2.3).*

*Proof.* In [28], the kernel  $G$  is assumed to be integrable in  $\mathbb{R}^n$ . However, it is actually only required for the two following reasons: first, to be able to write (1.3) and obtain the equivalence with the Gamow-type minimization problem (1.4); second, by this equivalence, to deduce that minimizers of  $(\mathcal{P}_{\gamma,\varepsilon})$  are so-called quasi-minimizers of the perimeter, and thus are (non-uniformly in  $\varepsilon$ )  $C^{1,\frac{1}{2}}$ -regular outside a “small” singular set. Here, we do not need the equivalence with (1.4) nor the a priori regularity of minimizers. In the end, apart from the  $C^{1,\frac{1}{2}}$  partial regularity of minimizers, all the conclusions of [28, Theorems A and B] follow. More precisely, there exists  $\varepsilon_0 = \varepsilon_0(n, G, \gamma)$  such that, for any  $0 < \varepsilon < \varepsilon_0$ ,  $(\mathcal{P}_{\gamma,\varepsilon})$  admits a minimizer. In addition, any such minimizer  $E_\varepsilon$  is indecomposable and, up to a translation and Lebesgue-negligible set, it satisfies

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+\delta(\varepsilon)},$$

where  $\delta : (0, +\infty) \rightarrow (0, \frac{1}{4})$  is a function depending only on  $n$ ,  $G$ , and  $\gamma$  vanishing in 0.

To conclude, there remains to show that in dimension  $n = 2$ ,  $E_\varepsilon$  is equivalent to a connected set with (2.3), that is, to link the indecomposability of  $E_\varepsilon$  with the topological notion of connectedness. It is not a trivial question, at least without (weak) regularity results on minimizers. However, [3, Theorem 8] shows that in dimension 2,  $\tilde{E}_\varepsilon := \mathring{E}_\varepsilon^M \setminus \partial^S E_\varepsilon$  is connected, where  $\mathring{E}_\varepsilon^M$  is the measure-theoretic interior of  $E_\varepsilon$ , and

$$\partial^S E_\varepsilon := \left\{ x \in \mathbb{R}^2 : \limsup_{r \rightarrow 0^+} \frac{\mathcal{H}^1(\partial^* E_\varepsilon \cap B_r(x))}{r} > 0 \right\}.$$

Since  $\mathcal{H}^1(\partial^S E_\varepsilon \setminus \partial^* E_\varepsilon) = 0$  and  $\mathcal{L}^2(E_\varepsilon \triangle \mathring{E}_\varepsilon^M) = 0$ , we have  $|E_\varepsilon \triangle \tilde{E}_\varepsilon| = 0$ , and since  $B_{1-\delta(\varepsilon)} \subseteq \mathring{E}_\varepsilon^M \subseteq B_{1+\delta(\varepsilon)}$  and  $\partial^S E_\varepsilon \subseteq \overline{B}_{1+\delta(\varepsilon)} \setminus B_{1-\delta(\varepsilon)}$ ,  $\tilde{E}_\varepsilon$  satisfies (2.3).  $\square$

*Remark 2.8.* In higher dimensions  $n \geq 3$ , one could show as well that any minimizer of  $(\mathcal{P}_{\gamma,\varepsilon})$  is equivalent to a connected set for small  $\varepsilon$ , without assuming that  $G \in L^1(\mathbb{R}^n)$ . Indeed, proceeding e.g. as in [25, Lemma 5.6], it is possible to obtain uniform (w.r.t.  $\varepsilon$ ) density estimates for minimizers, and with those to deduce that any minimizer is equivalent to an open set  $E_\varepsilon$  such  $\partial E_\varepsilon = \text{spt } \mu_\varepsilon$ . The indecomposability of  $E_\varepsilon$  then implies connectedness by [3].

## 3. MINIMIZERS ARE NEARLY CIRCULAR SETS IN DIMENSION 2

**3.1. Decrease of the critical energy by convexification.** We can recover the nonlocal perimeter of a measurable set  $E \subseteq \mathbb{R}^n$  by integrating the 1-dimensional nonlocal perimeter of all 1-dimensional sections of  $E$  in a given direction, and averaging over all the directions.

**Proposition 3.1.** *For any measurable set  $E \subseteq \mathbb{R}^{n-1}$ , we have*

$$\begin{aligned} P_{G_\varepsilon}(E) &= \int_{\mathbb{S}^{n-1}} \int_{(\text{span}(\sigma))^\perp} \left( \iint_{E_{\sigma,y} \times (\mathbb{R} \setminus E_{\sigma,y})} |s-t| g_\varepsilon(|s-t|) \, ds \, dt \right) \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1} \\ &= \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \left( \int_{(\text{span}(\sigma))^\perp} P_{\rho_\varepsilon}^1(E_{\sigma,y}) \, d\mathcal{H}_y^{n-1} \right) \, d\mathcal{H}_\sigma^{n-1}, \end{aligned}$$

where

$$E_{\sigma,y} := \left\{ s \in \mathbb{R} : y + s\sigma \in E \right\}, \quad (3.1)$$

$\omega_{n-1}$  is the volume of the unit ball in  $\mathbb{R}^{n-1}$ , and  $P_{\rho_\varepsilon}^1$  is the 1-dimensional nonlocal perimeter in  $\mathbb{R}$  associated with the kernel  $\rho_\varepsilon$  defined by  $\rho_\varepsilon(r) := \omega_{n-1} |r|^{n-1} g_\varepsilon(|r|)$  for  $\mathcal{L}^1$ -a.e.  $r \in \mathbb{R}$ , that is,

$$P_{\rho_\varepsilon}^1(J) := \iint_{\mathbb{R} \times \mathbb{R}} |\mathbf{1}_J(s) - \mathbf{1}_J(t)| \rho_\varepsilon(s-t) \, ds \, dt,$$

for every measurable set  $J \subseteq \mathbb{R}$ .

*Proof.* By the change of variable  $y = x + r\sigma$  with fixed  $x$  and Fubini's theorem, we have

$$\begin{aligned} P_{G_\varepsilon}(E) &= 2 \iint_{E \times E^c} G(x-y) \, dx \, dy \\ &= 2 \int_{\mathbb{S}^{n-1}} \int_0^{+\infty} \int_E \mathbf{1}_{E^c}(x+r\sigma) r^{n-1} g_\varepsilon(r) \, dx \, dr \, d\mathcal{H}_\sigma^1 \\ &= \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_E \mathbf{1}_{E^c}(x+r\sigma) \rho_\varepsilon(r) \, dx \, dr \, d\mathcal{H}_\sigma^1, \end{aligned}$$

where we have used the definition of  $\rho_\varepsilon$  for the last equality. Then, for  $\sigma$  fixed, let us make the change of variable  $x = y + s\sigma$ , where  $y = \pi_{(\text{span}(\sigma))^\perp}(x)$  is the orthogonal projection of  $x$  on  $(\text{span}(\sigma))^\perp$ . This yields

$$P_{G_\varepsilon}(E) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \int_{(\text{span}(\sigma))^\perp} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_E(s\sigma + y) \mathbf{1}_{E^c}((s+r)\sigma + y) \rho_\varepsilon(r) \, ds \, dr \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1}.$$

Finally, using Fubini's theorem and making the change of variable  $t = s + r$ , where  $s$  is fixed, we obtain, by definition of  $E_{\sigma,y}$  and  $P_{\rho_\varepsilon}^1$ ,

$$\begin{aligned} P_{G_\varepsilon}(E) &= \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{(\text{span}(\sigma))^\perp} \int_{E_{\sigma,y}} \int_{\mathbb{R} \setminus E_{\sigma,y}} \rho_\varepsilon(s-t) \, ds \, dt \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1} \\ &= \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{(\text{span}(\sigma))^\perp} P_{\rho_\varepsilon}^1(E_{\sigma,y}) \, d\mathcal{H}_y^{n-1} \, d\mathcal{H}_\sigma^{n-1}. \end{aligned}$$

This concludes the proof.  $\square$

*Remark 3.2.* Recall that by [28, Lemma 3.13],  $\mathbf{K}_{1,n} = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n+1}{2})}$ , so that, by (H2), the kernel  $\rho_\varepsilon$  satisfies

$$\int_{\mathbb{R}} |t| \rho_\varepsilon(t) \, dt = 2\omega_{n-1} \int_0^\infty t^n g_\varepsilon(t) \, dt = \frac{2\omega_{n-1} I_G^1}{|\mathbb{S}^{n-1}|} = \frac{2\omega_{n-1}}{\mathbf{K}_{1,n} |\mathbb{S}^{n-1}|} = 1 = \frac{1}{\mathbf{K}_{1,1}}.$$

Hence by Proposition 2.1, we have

$$\mathcal{E}_{\rho_\varepsilon}^1(E) = P(E) - P_{\rho_\varepsilon}^1(E) \geq 0$$

for every measurable set  $E \subseteq \mathbb{R}$ .

Similarly, as a straightforward consequence of [4, Theorem 3.103], for any set of finite perimeter  $E \subseteq \mathbb{R}^{n-1}$ ,  $E_{\sigma,y}$  is a one-dimensional set of finite perimeter for  $\mathcal{H}^{n-1}$ -a.e.  $\sigma$  and  $y$ , and we have

$$P(E) = \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{(\text{span}(\sigma))^\perp} P^1(E_{\sigma,y}) d\mathcal{H}_y^{n-1} d\mathcal{H}_\sigma^{n-1},$$

where  $P^1(E_{\sigma,y}) = \mathcal{H}^0(\partial E_{\sigma,y})$  is the standard perimeter in dimension 1. Hence we have the following representation of the critical energy  $\mathcal{E}_{G_\varepsilon}$ .

**Corollary 3.3.** *For any set of finite perimeter  $E \subseteq \mathbb{R}^n$ , we have*

$$\mathcal{E}_{G_\varepsilon}(E) = \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \int_{(\text{span}(\sigma))^\perp} \mathcal{E}_{\rho_\varepsilon}^1(E_{\sigma,y}) d\mathcal{H}_y^{n-1} d\mathcal{H}_\sigma^{n-1},$$

where  $\mathcal{E}_{\rho_\varepsilon}^1 := P^1 - P_{\rho_\varepsilon}^1$ , and  $E_{\sigma,y}$  is given by (3.1).

For a segment  $(a,b) \subseteq \mathbb{R}$ , we have the following simple expression of the one-dimensional critical energy.

**Lemma 3.4.** *For every  $a, b \in \mathbb{R}$  such that  $a < b$ , we have*

$$\mathcal{E}_{\rho_\varepsilon}^1((a,b)) = 4 \int_{-\infty}^a \int_b^{+\infty} \rho_\varepsilon(s-t) ds dt. \quad (3.2)$$

In particular  $\mathcal{E}_{\rho_\varepsilon}^1(a,b)$  decreases as the interval grows. In addition,

$$\mathcal{E}_{\rho_\varepsilon}^1((b,+\infty)) = \mathcal{E}_{\rho_\varepsilon}^1((-\infty,a)) = 0. \quad (3.3)$$

*Proof.* By a change of variable and Fubini's theorem, for any  $a \in \mathbb{R}$ , we have

$$P_{\rho_\varepsilon}^1((-\infty,a)) = 2 \int_{-\infty}^a \int_a^{+\infty} \rho_\varepsilon(t-s) ds dt = 2 \int_0^{+\infty} \rho(t) \left( \int_0^t ds \right) dt = 2 \int_0^{+\infty} t\rho(t) dt = 1.$$

Similarly,  $P_{\rho_\varepsilon}^1((b,+\infty)) = 1$ , and since  $P^1((-\infty,a)) = P^1((b,+\infty)) = 1$ , (3.3) follows. Next,

$$\begin{aligned} 2 &= P^1((a,b)) = P^1((-\infty,a)) + P^1((b,+\infty)) \\ &= 2 \int_{-\infty}^a \int_a^{+\infty} \rho_\varepsilon(s-t) ds dt + 2 \int_{-\infty}^b \int_b^{+\infty} \rho_\varepsilon(s-t) ds dt, \end{aligned} \quad (3.4)$$

for every  $a, b \in \mathbb{R}$  such that  $a < b$ . We also have

$$P_{\rho_\varepsilon}^1((a,b)) = 2 \int_{-\infty}^a \int_a^b \rho_\varepsilon(s-t) ds dt + 2 \int_b^{+\infty} \int_a^b \rho_\varepsilon(s-t) ds dt. \quad (3.5)$$

Subtracting (3.5) from (3.4), we obtain (3.2).  $\square$

For a general set of finite perimeter in  $\mathbb{R}$ , we have the following expression of the critical energy.

**Lemma 3.5.** *For any set  $E \subseteq \mathbb{R}$  which is the finite disjoint union of open intervals, let  $\{C_i\}_{i \in \{1, \dots, N\}}$  be the connected components of  $E^c$ . Then we have*

$$\mathcal{E}_{\rho_\varepsilon}^1(E) = 2 \sum_{\substack{1 \leq i, j \leq N \\ i \neq j}} \iint_{C_i \times C_j} \rho_\varepsilon(s-t) ds dt + \sum_{i=1}^N \mathcal{E}_{\rho_\varepsilon}^1(C_i), \quad (3.6)$$

where, for the intervals  $C_i$ ,  $\mathcal{E}_{\rho_\varepsilon}^1(C_i)$  is given by Lemma 3.4.

Let us point out that  $E^c$  may have up to two unbounded connected components, and their critical energy is 0 by Lemma 3.4. As a consequence of Lemmas 3.4 and 3.5, the 1D critical energy decreases by convexification and by increase of length.

**Corollary 3.6.** *Let  $E \subseteq J \subseteq \mathbb{R}$  with  $J$  an interval and  $\mathcal{L}^1(E) > 0$ , then  $\mathcal{E}_{\rho_\varepsilon}^1(J) \leq \mathcal{E}_{\rho_\varepsilon}^1(E)$ .*

*Proof.* Let  $E \subseteq \mathbb{R}$  with  $\mathcal{L}^1(E) > 0$ , and let  $J$  be an interval containing  $E$ . Let  $a' \leq a < b \leq b' \in \mathbb{R} \cup \{\pm\infty\}$  such that  $\text{co}(E) = (a,b)$  and  $J = (a',b')$ . If  $a' = -\infty$  or  $b' = +\infty$ ,  $\mathcal{E}_{\rho_\varepsilon}^1(J) = 0 \leq \mathcal{E}_{\rho_\varepsilon}^1(E)$  by Lemma 3.4 and Remark 3.2, so the result holds true. If  $E$  does not have finite perimeter, then  $\mathcal{E}_{\rho_\varepsilon}^1(E) = +\infty$ , and the result holds true as well. Thus, let us assume that  $E$  is a bounded set of finite perimeter. In particular, up to a negligible set,  $E$  is the disjoint union of  $k$  open intervals with  $k \geq 1$

since  $\mathcal{L}^1(E) > 0$ . In addition, since  $E$  is bounded,  $E^c$  has two unbounded components  $C_1 = (-\infty, a)$  and  $C_{k+1} = (b, +\infty)$  (up to renumbering). By [Lemmas 3.4](#) and [3.5](#), we have

$$\mathcal{E}_{\rho_\varepsilon}^1(E) \geq 4 \int_{C_1 \times C_{k+1}} \rho_\varepsilon(s-t) ds dt = \mathcal{E}_{\rho_\varepsilon}^1((a, b)) \geq \mathcal{E}_{\rho_\varepsilon}^1(J).$$

□

*Proof of [Lemma 3.5](#).* Let

$$E = \bigsqcup_{i=1}^k (a_i, b_i),$$

with  $-\infty \leq a_1 < b_1 < \dots < a_k < b_k \leq +\infty$ , so that, setting  $b_0 := -\infty$  and  $a_{k+1} := +\infty$ , the connected components of  $E^c$  are given by

$$C_i = (b_i, a_{i+1}), \quad \forall i \in \{0, \dots, k\}.$$

Beware that  $C_0$  and  $C_k$  are either empty or unbounded. If  $a_1 = -\infty$ ,  $C_0 = \emptyset$ , and if  $b_k = +\infty$ ,  $C_k = \emptyset$ , in which cases it is an abuse to consider them connected components of  $E^c$ , but they do not contribute to the terms in [\(3.6\)](#). Omitting the integrand  $\rho_\varepsilon(s-t) ds dt$  for the sake of readability, let us write

$$P_{\rho_\varepsilon}^1(E) = 2 \sum_{i=0}^k \int_E \int_{C_i} = \sum_{i=0}^k \left( 2 \int_{C_i} \int_{\{t \in E : t > a_{i+1}\}} + 2 \int_{C_i} \int_{\{t \in E : t < b_i\}} \right), \quad (3.7)$$

with the convention  $\{t < -\infty\} = \{t > +\infty\} = \emptyset$ . By [\(3.4\)](#), we have

$$P^1(E) = \sum_{i=1}^k \left( 2 \int_{-\infty}^{a_i} \int_{a_i}^{+\infty} + 2 \int_{-\infty}^{b_i} \int_{b_i}^{+\infty} \right). \quad (3.8)$$

Note that this holds even if  $a_1 = -\infty$  or  $b_k = +\infty$ . Let us define

$$R_i := 2 \int_{-\infty}^{a_{i+1}} \int_{a_{i+1}}^{+\infty} - 2 \int_{C_i} \int_{\{t \in E : t > a_{i+1}\}}, \quad \forall i \in \{0, \dots, k\},$$

and similarly

$$L_i := 2 \int_{-\infty}^{b_i} \int_{b_i}^{+\infty} - 2 \int_{C_i} \int_{\{t \in E : t < b_i\}}, \quad \forall i \in \{0, \dots, k\}.$$

Notice that  $R_k = L_0 = 0$ . We observe that by definition of  $L_i$ ,  $R_i$  and [\(3.7\)](#), [\(3.8\)](#),

$$\mathcal{E}_{\rho_\varepsilon}^1(E) = \sum_{i=0}^k (L_i + R_i). \quad (3.9)$$

Writing

$$\int_{-\infty}^{a_{i+1}} \int_{a_{i+1}}^{+\infty} = \int_{-\infty}^{b_i} \int_{a_{i+1}}^{+\infty} + \int_{C_i} \int_{a_{i+1}}^{+\infty},$$

for  $i \in \{0, \dots, k\}$ , using [Lemma 3.4](#) we have

$$R_i = 2 \left( \int_{-\infty}^{b_i} \int_{a_{i+1}}^{+\infty} + \int_{C_i} \int_{a_{i+1}}^{+\infty} \right) - 2 \int_{C_i} \int_{\{t \in E : t > a_{i+1}\}} = 2 \int_{C_i} \int_{\{t \in E^c : t > a_{i+1}\}} + \frac{1}{2} \mathcal{E}_{\rho_\varepsilon}^1(C_i). \quad (3.10)$$

and similarly,

$$L_i = 2 \int_{C_i} \int_{\{t \in E^c : t < b_i\}} + \frac{1}{2} \mathcal{E}_{\rho_\varepsilon}^1(C_i). \quad (3.11)$$

The two previous equations hold even if  $C_0$ ,  $C_k$  are empty or unbounded. Inserting [\(3.10\)](#) and [\(3.11\)](#) into [\(3.9\)](#) yields

$$\begin{aligned} \mathcal{E}_{\rho_\varepsilon}^1(E) &= 2 \sum_{i=0}^k \left( \int_{C_i} \int_{E^c \cap \{t > a_{i+1}\}} + \int_{C_i} \int_{E^c \cap \{t < b_i\}} \right) + \sum_{i=0}^k \mathcal{E}_{\rho_\varepsilon}^1(C_i) \\ &= 2 \sum_{i=0}^k \sum_{j \neq i} \int_{C_i} \int_{C_j} + \sum_{i=0}^k \mathcal{E}_{\rho_\varepsilon}^1(C_i), \end{aligned}$$

which concludes the proof. □

We easily deduce from [Lemma 3.5](#) and the slicing decomposition of the critical energy stated in [Corollary 3.3](#) that in dimension  $n = 2$ , the critical energy of a connected set decreases by convexification.

**Proposition 3.7.** *If  $E \subseteq \mathbb{R}^2$  is a bounded, connected set of finite perimeter, then*

$$\mathcal{E}_{G_\varepsilon}(\text{co}(E)) \leq \mathcal{E}_{G_\varepsilon}(E), \quad \forall \varepsilon > 0.$$

*Proof.* First, recall that a bounded convex set is a set of finite perimeter, since it is Lebesgue-equivalent to an open set with Lipschitz boundary, thus  $\text{co}(E)$  is a set of finite perimeter. Then, by [Corollary 3.3](#) we have

$$\mathcal{E}_{G_\varepsilon}(\text{co}(E)) = \frac{1}{4} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \mathcal{E}_{\rho_\varepsilon}^1(E_{\sigma,t}) dt d\mathcal{H}_\sigma^1, \quad \text{and} \quad \mathcal{E}_{G_\varepsilon}(\text{co}(E)) = \frac{1}{4} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \mathcal{E}_{\rho_\varepsilon}^1(F_{\sigma,t}) dt d\mathcal{H}_\sigma^1,$$

where

$$E_{\sigma,t} := \left\{ s \in \mathbb{R} : t\sigma^\perp + s\sigma \in E \right\}, \quad \text{and} \quad F_{\sigma,t} := \left\{ s \in \mathbb{R} : t\sigma^\perp + s\sigma \in \text{co}(E) \right\}.$$

Since  $E$  is connected, for every  $\sigma \in \mathbb{S}^1$  and  $t \in \mathbb{R}$ , the slice  $F_{\sigma,t}$  is empty if and only if  $E_{\sigma,t}$  is empty: this is the argument which is valid in dimension 2 only. In addition, since  $E$  and  $\text{co}(E)$  are bounded sets of finite perimeter in  $\mathbb{R}^2$ , for  $\mathcal{H}^1$ -a.e.  $\sigma \in \mathbb{S}^1$  and  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}$ ,  $E_{\sigma,t}$  and  $F_{\sigma,t}$  are bounded sets of finite perimeter in  $\mathbb{R}$ , and for every nonempty slice,  $F_{\sigma,t}$  is an interval s.t.  $E_{\sigma,t} \subseteq F_{\sigma,t}$ , since  $F_{\sigma,t}$  is a slice of the convex hull of  $E$ . Hence, by [Lemmas 3.4](#) and [3.5](#), for  $\mathcal{H}^1$ -a.e.  $\sigma \in \mathbb{S}^1$  and  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}$ , there holds

$$\mathcal{E}_{\rho_\varepsilon}^1(F_{\sigma,t}) \leq \mathcal{E}_{\rho_\varepsilon}^1(E_{\sigma,t}).$$

The formula of [Corollary 3.3](#) concludes the proof.  $\square$

**3.2. Convexity of minimizers in 2D.** In this part, we shall use for the sake of brevity the abbreviations  $\mathcal{F}_{\gamma,\varepsilon} := \mathcal{F}_{\gamma,G_\varepsilon}$  and  $\mathcal{E}_\varepsilon := \mathcal{E}_{G_\varepsilon}$ .

Consider a connected minimizer  $E_\varepsilon \subseteq \mathbb{R}^2$  of  $(\mathcal{P}_{\gamma,\varepsilon})$ . Recall that  $\mathcal{F}_{\gamma,\varepsilon} = (1 - \gamma)P + \gamma\mathcal{E}_\varepsilon$ , thus by [Proposition 3.7](#),  $\mathcal{F}_{\gamma,\varepsilon}(\text{co}(E_\varepsilon)) \leq \mathcal{F}_{\gamma,\varepsilon}(E_\varepsilon)$ . However, if  $E_\varepsilon$  is not convex,  $|\text{co}(E_\varepsilon)| > |E_\varepsilon|$ , so  $\text{co}(E_\varepsilon)$  is not a valid competitor for  $(\mathcal{P}_{\gamma,\varepsilon})$ . Recalling that  $B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+\delta(\varepsilon)}$  for small  $\varepsilon$ , we have

$$B_{1-\delta(\varepsilon)} \subseteq \text{co}(E_\varepsilon) \subseteq B_{1+\delta(\varepsilon)},$$

and defining  $t_\varepsilon := \sqrt{\frac{|B_1|}{|\text{co}(E_\varepsilon)|}}$ , we see that  $t_\varepsilon \text{co}(E_\varepsilon)$  is a valid competitor with  $(1 + \delta(\varepsilon))^{-1} < t_\varepsilon < 1$ . Let us show that  $\mathcal{F}_{\gamma,\varepsilon}(t_\varepsilon \text{co}(E_\varepsilon)) < \mathcal{F}_{\gamma,\varepsilon}(\text{co}(E_\varepsilon))$ . This follows from the following result, which we prove further below.

**Lemma 3.8.** *There exists  $\bar{\varepsilon}_1 = \bar{\varepsilon}_1(G, \gamma) > 0$  such that the following holds. If  $E \subseteq \mathbb{R}^2$  is a convex set such that*

$$B_{1-\delta(\varepsilon)} \subseteq E \subseteq B_{1+\delta(\varepsilon)},$$

*with  $0 < \varepsilon < \bar{\varepsilon}_1$  and where  $\delta$  is the function given by [Theorem 2.7](#), then*

$$\mathcal{F}_{\gamma,\varepsilon}(tE) < \mathcal{F}_{\gamma,\varepsilon}(E), \quad \forall t \in \left(\frac{1}{2}, 1\right).$$

As a consequence, we obtain that minimizers of  $(\mathcal{P}_{\gamma,\varepsilon})$  are necessarily convex for small  $\varepsilon$  in dimension  $n = 2$ , that is [Theorem 1](#):

*Proof of [Theorem 1](#).* Let  $E_\varepsilon \subseteq \mathbb{R}^2$  be a minimizer of  $(\mathcal{P}_{\gamma,\varepsilon})$  with  $0 < \varepsilon < \varepsilon_1 \leq \varepsilon_0$ , where  $\varepsilon_0(G, \gamma)$  is given by [Theorem 2.7](#). Thus, up to a Lebesgue-negligible set and a translation,  $E_\varepsilon$  is connected and satisfies

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq \text{co}(E_\varepsilon) \subseteq B_{1+\delta(\varepsilon)}.$$

By contradiction, let us assume that  $E_\varepsilon$  is not convex, so that  $|\text{co}(E_\varepsilon)| > |B_1|$ . We have  $|t_\varepsilon \text{co}(E_\varepsilon)| = |B_1|$  with  $t_\varepsilon := \sqrt{\frac{|B_1|}{|\text{co}(E_\varepsilon)|}}$  satisfying  $\frac{1}{2} \leq (1 + \delta(\varepsilon))^{-1} < t_\varepsilon < 1$ , provided that  $\varepsilon_1 = \varepsilon_1(G, \gamma)$  is chosen small enough. If in addition  $\varepsilon_1 < \bar{\varepsilon}_1(G, \gamma)$ , where  $\bar{\varepsilon}_1$  is given by [Lemma 3.8](#), we find  $\mathcal{E}_{\gamma,\varepsilon}(t_\varepsilon \text{co}(E_\varepsilon)) < \mathcal{E}_{\gamma,\varepsilon}(\text{co}(E_\varepsilon))$ . Now, since  $E_\varepsilon$  is connected, there holds  $\mathcal{E}_{\gamma,\varepsilon}(\text{co}(E_\varepsilon)) \leq \mathcal{E}_{\gamma,\varepsilon}(E_\varepsilon)$  by [Corollary 3.6](#), hence  $\mathcal{E}_{\gamma,\varepsilon}(t_\varepsilon \text{co}(E_\varepsilon)) < \mathcal{E}_{\gamma,\varepsilon}(E_\varepsilon)$ . This contradicts the minimality of  $E_\varepsilon$ , whence  $E_\varepsilon$  is convex.  $\square$

**3.3. Proof of Lemma 3.8.** In order to prove that  $\mathcal{F}_{\gamma,\varepsilon}(\text{co}(E_\varepsilon))$  (strictly) decreases by scaling down  $\text{co}(E_\varepsilon)$  by  $t_\varepsilon$  for small  $\varepsilon$ , we need to estimate the term  $\widetilde{P}_{G_\varepsilon}(t_\varepsilon \text{co}(E_\varepsilon))$  appearing when applying Lemma 2.3. While it is not so difficult to see that for a fixed set  $E$  with  $C^1$  boundary,  $\widetilde{P}_{G_\varepsilon}(E)$  converges to  $P(E)$  as  $\varepsilon$  vanishes, here we do not have uniform regularity estimates on minimizers of  $(\mathcal{P}_{\gamma,\varepsilon})^2$ . However, we can make use of the convexity of  $\text{co}(E_\varepsilon)$  and the fact that it lies between two balls whose radii are close to 1 to prove the following.

**Lemma 3.9.** *For any  $\alpha \in (0, 1)$ , there exist positive constants  $r = r(\alpha)$  and  $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(G, \gamma, \alpha)$  such that the following holds. If  $E \subseteq \mathbb{R}^2$  is a convex set such that*

$$B_{1-\delta(\varepsilon)} \subseteq E \subseteq B_{1+\delta(\varepsilon)},$$

with  $0 < \varepsilon < \bar{\varepsilon}_2$ , where the function  $\delta$  is given by Theorem 2.7, then we have

$$\left\{ y \in B_r(x) : \frac{x-y}{|x-y|} \cdot \nu_E(x) \geq \alpha \right\} \subseteq E, \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial E. \quad (3.12)$$

*Proof.* We proceed in two steps.

*Step 1.* We show that the normal vector  $\nu_E$  converges uniformly for  $\mathcal{H}^1$ -a.e.  $x \in \partial E$  to  $\nu_{B_1}$  as  $\varepsilon$  vanishes, in the sense that

$$\left| \nu_E(x) - \frac{x}{|x|} \right| \leq 2\sqrt{\frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}}, \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial E. \quad (3.13)$$

The simple geometric argument is well known (see e.g. [27, 15]), but we state it here for the reader's convenience. Let  $\theta_\varepsilon \in (0, \pi)$  be the angular diameter of the ball  $B_{1-\delta(\varepsilon)}$  from any point  $x \in \partial B_{1+\delta(\varepsilon)}$ , that is,  $\theta_\varepsilon$  is the non-oriented angle between the two lines passing by  $x \in \partial B_{1+\delta(\varepsilon)}$  and tangent to  $\partial B_{1-\delta(\varepsilon)}$ . Then we have  $\theta_\varepsilon = 2 \arcsin\left(\frac{1-\delta(\varepsilon)}{1+\delta(\varepsilon)}\right)$ . Let  $x \in \partial E$ , and let  $\varphi \in (0, \frac{\pi}{2})$  be the non-oriented angle between  $x$  and a tangent line to  $E$  at  $x$ . Since  $E$  is convex and  $B_{1-\delta(\varepsilon)} \subseteq E$ , the tangent cone to  $E$  at  $x$  cannot intersect  $B_{1-\delta(\varepsilon)}$ , and since  $|x| \leq 1 + \delta(\varepsilon)$ , we find

$$\varphi \leq \frac{\theta_\varepsilon}{2} = \arcsin\left(\frac{1-\delta(\varepsilon)}{1+\delta(\varepsilon)}\right).$$

By convexity of  $E$ , for  $\mathcal{H}^1$ -a.e.  $x \in \partial E$ , there is a unique tangent line to  $E$  at  $x$ , and writing

$$\left| \nu_E(x) - \frac{x}{|x|} \right|^2 = 2(1 - \sin(\varphi)),$$

we deduce (3.13).

*Step 2.* Let  $\alpha \in (0, 1)$ , and let  $r(\alpha)$ ,  $\bar{\varepsilon}_2(G, \gamma, \alpha)$  to be fixed later. Then let  $E \subseteq \mathbb{R}^2$  be as in the statement of the lemma, with  $0 < \varepsilon < \bar{\varepsilon}_2$ . As a consequence of Step 1, proving (3.12) amounts to showing that for  $r(\alpha)$  and  $\bar{\varepsilon}_2(G, \gamma, \alpha)$  small enough,

$$\left\{ y \in B_r(x) : \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \geq \frac{\alpha}{2} \right\} \subseteq E, \quad \forall x \in \partial E. \quad (3.14)$$

We assume that  $r \in (0, \frac{1}{2})$  and then that  $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(G, \gamma, r) \in (0, \frac{r}{2})$  is such that  $\delta(\varepsilon) < \frac{1}{4}$  whenever  $0 < \varepsilon < \bar{\varepsilon}_2$ . Let  $x = |x|(\cos \psi, \sin \psi) \in \partial E$  with  $\psi \in [0, 2\pi)$ . Since  $x \in B_{1+\delta(\varepsilon)} \setminus B_{1-\delta(\varepsilon)}$ , the set  $\partial B_r(x) \cap B_{1-\delta(\varepsilon)}$  is made of exactly two points  $A$  and  $B$ . Let us denote by  $\varphi(x, r) \in (0, \pi)$  the angle between the segments  $[Ax]$  and  $[Bx]$ . Then let us introduce the circular sector  $C(x, r)$  of  $B_r(x)$  delimited by  $A$  and  $B$  (see Figure 1), that is

$$\begin{aligned} C(x, r) &:= \left\{ x - s(\cos(\theta + \psi), \sin(\theta + \psi)) : 0 < s < r, |\theta| \leq \frac{\varphi(x, r)}{2} \right\} \\ &= \left\{ y \in B_r(x) : \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \geq \cos\left(\frac{\varphi(x, r)}{2}\right) \right\}. \end{aligned} \quad (3.15)$$

<sup>2</sup>The lack of uniform regularity estimates for minimizers of  $(\mathcal{P}_{\gamma,\varepsilon})$  as  $\varepsilon$  goes to 0 is the main obstacle for proving that the unit ball is, up to translations, the unique minimizer for  $\varepsilon$  small enough. With sufficient regularity, the arguments of Section 4 would yield the result.

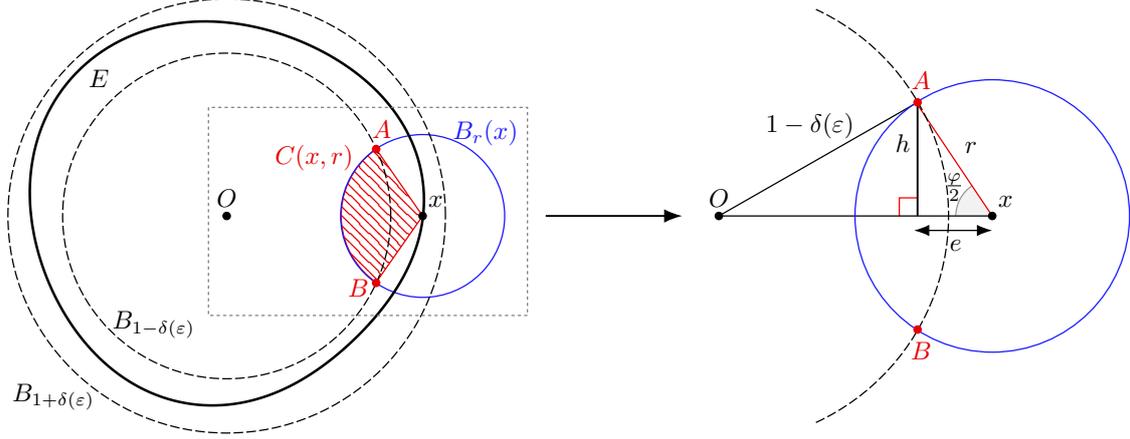


FIGURE 1. The situation in Step 2 of the proof of Lemma 3.9. The red-dashed area is  $C(x, r)$ .

We claim that for  $\bar{\varepsilon}_2$  small enough,  $\varphi(x, r) \geq 2 \arcsin \sqrt{1 - r^2}$ . The situation is as in Figure 1, where we introduce the lengths  $h$  and  $e$ . We have

$$\begin{cases} h^2 + e^2 & = r^2 \\ (|x| - e)^2 + h^2 & = (1 - \delta(\varepsilon))^2 \end{cases},$$

which gives, after computation,

$$h^2 = r^2 \left[ 1 - \frac{1}{4|x|^2} \left( r^2 + 2(|x|^2 - (1 - \delta(\varepsilon))^2) + \frac{1}{r^2} (|x|^2 - (1 - \delta(\varepsilon))^2)^2 \right) \right].$$

Recalling that  $1 - \delta(\varepsilon) \leq |x| < 1 + \delta(\varepsilon)$  and  $r < \frac{1}{2}$ , this implies

$$h^2 \geq r^2 - \frac{r^4}{4(1 + \delta(\varepsilon))^2} \left[ 1 + \frac{4\delta(\varepsilon)}{r^2} + \left( \frac{4\delta(\varepsilon)}{r^2} \right)^2 \right] \geq r^2 (1 - r^2),$$

provided  $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(G, \gamma, r)$  is chosen small enough. Thus, the angle  $\varphi(r, x)$  satisfies

$$\sin \left( \frac{\varphi(r, x)}{2} \right) = \frac{h}{r} \geq \sqrt{1 - r^2},$$

which proves the claim. Choosing  $r = r(\alpha)$  and  $\bar{\varepsilon}_2(G, \gamma, r)$  small enough, we deduce  $\cos \left( \frac{\varphi(r, x)}{2} \right) \leq \frac{\alpha}{2}$ . By convexity of  $E$  and the fact that  $B_{1-\delta(\varepsilon)} \subseteq E$ , we have  $C(x, r) \subseteq E$ , hence (3.14) holds, in view of (3.15). This concludes the proof.  $\square$

Then we can estimate  $\tilde{P}_{G_\varepsilon}(E)$  when  $E$  is a convex set lying between the disks  $B_{1-\delta(\varepsilon)}$  and  $B_{1+\delta(\varepsilon)}$ , for small  $\varepsilon$ .

**Lemma 3.10.** *For any  $\tau > 0$ , there exists  $\bar{\varepsilon}_3 = \bar{\varepsilon}_3(G, \gamma, \tau) > 0$  such that the following holds. If  $E \subseteq \mathbb{R}^2$  is a convex set such that*

$$B_{1-\delta(\varepsilon)} \subseteq E \subseteq B_{1+\delta(\varepsilon)},$$

with  $0 < \varepsilon < \bar{\varepsilon}_3$ , where the  $\delta$  function is the one from Theorem 2.7, then we have

$$(1 - \tau)P(E) \leq \tilde{P}_{G_\varepsilon}(E) \leq P(E),$$

where  $\tilde{P}_{G_\varepsilon}(E)$  is defined by (2.1).

*Proof.* Let  $E$  be as in the statement of the lemma, with  $0 < \varepsilon < \bar{\varepsilon}_3(G, \gamma, \tau)$ , where  $\bar{\varepsilon}_3$  is to be fixed later. Recall that since  $E$  is a bounded convex set, it is a set of finite perimeter and its topological boundary is  $\mathcal{H}^1$ -equivalent to  $\partial^*E$ . We proceed in two steps.

*Step 1.* We show that  $\tilde{P}_{G_\varepsilon}(E) \leq P(E)$ . First, note that by convexity of  $E$ , we have

$$(x - y) \cdot \nu_E(x) \geq 0, \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial E, \forall y \in E,$$

and, defining for  $\mathcal{H}^1$ -a.e.  $x \in \partial E$ ,

$$H_x := \left\{ y \in \mathbb{R}^n : (x - y) \cdot \nu_E(x) > 0 \right\},$$

it holds

$$\tilde{P}_{G_\varepsilon}(E) \leq \int_{\partial E} \int_{H_x} G_\varepsilon(x - y)(x - y) \cdot \nu_E(x) \, d\mathcal{H}_x^1 \, dy.$$

Then, by a change of variable and the coarea formula, we find

$$\begin{aligned} \tilde{P}_{G_\varepsilon}(E) &\leq 2 \int_{\partial E} \int_{\{z \in \mathbb{R}^2 : z \cdot \nu_E(x) > 0\}} G_\varepsilon(z)(z \cdot \nu_E(x)) \, dz \, d\mathcal{H}_x^1 \\ &= 2 \int_{\partial E} \int_0^{+\infty} g_\varepsilon(t)t^2 \left( \int_{\{\sigma \in \mathbb{S}^1 : \sigma \cdot \nu_E(x) > 0\}} \sigma \cdot \nu_E(x) \, d\mathcal{H}_\sigma^1 \right) dt \, d\mathcal{H}_x^1 \\ &= \int_{\partial E} \int_0^{+\infty} g_\varepsilon(t)t^2 \left( \int_{\mathbb{S}^1} |\sigma \cdot \nu_E(x)| \, d\mathcal{H}_\sigma^1 \right) dt \, d\mathcal{H}_x^1 \\ &= \mathbf{K}_{1,2} \int_{\partial E} |\mathbb{S}^1| \int_0^{+\infty} g_\varepsilon(t)t^2 \, dt \, d\mathcal{H}_x^1 = I_G^1 \mathbf{K}_{1,2} P(E) = P(E), \end{aligned}$$

where we also used the definition of  $\mathbf{K}_{1,2}$  for the third equality, and (H2) for the last one.

*Step 2. Conclusion.* Let  $\alpha \in (0, 1)$  to be chosen later, and  $\theta_0 := \arcsin(\alpha)$ . Then let  $r = r(\alpha) > 0$  and  $\bar{\varepsilon}_2(G, \gamma, \alpha)$  given by Lemma 3.9. Assume that  $\bar{\varepsilon}_3 < \bar{\varepsilon}_2$ . Let us write

$$\begin{aligned} \tilde{P}_{G_\varepsilon}(E) &= 2 \int_{\partial E} \int_{E \cap \{|x-y| < r\}} G_\varepsilon(x - y)(x - y) \cdot \nu_E(x) \, dy \, d\mathcal{H}_x^1 \\ &\quad + 2 \int_{\partial E} \int_{E \cap \{|x-y| \geq r\}} G_\varepsilon(x - y)(x - y) \cdot \nu_E(x) \, dx \, d\mathcal{H}_x^1 =: \tilde{P}_{G_\varepsilon}^{(1)}(E) + \tilde{P}_{G_\varepsilon}^{(2)}(E). \end{aligned}$$

Since  $\varepsilon < \bar{\varepsilon}_2$ , by Lemma 3.9, we have

$$\begin{aligned} \tilde{P}_{G_\varepsilon}^{(1)}(E) &\geq 2 \int_{\partial E} \int_{\{y \in B_r(x) : \frac{x-y}{|x-y|} \cdot \nu_E(x) \geq \alpha\}} G_\varepsilon(x - y)(x - y) \cdot \nu_E(x) \, dy \, d\mathcal{H}_x^1 \\ &= 2 \int_{\partial E} \int_0^{+\infty} g_\varepsilon(t)t^2 \int_{\{\sigma \in \mathbb{S}^1 : \alpha \leq \sigma \cdot \nu_E(x) \leq 1\}} \sigma \cdot \nu_E(x) \, d\mathcal{H}_\sigma^1 \, dt \, d\mathcal{H}_x^1 \\ &\geq P(E) - 2 \int_{\partial E} \int_0^{+\infty} g_\varepsilon(t)t^2 \int_0^{2\theta_0} (\cos(\theta), \sin(\theta)) \cdot \nu_E(x) \, d\theta \, dt \, d\mathcal{H}_x^1. \end{aligned}$$

Here we used the fact that, by assumption (H2),

$$P(E) = 2 \int_{\partial E} \int_0^{+\infty} t^2 g_\varepsilon(t) \int_{\{\sigma \in \mathbb{S}^1 : \sigma \cdot e > 0\}} (\sigma \cdot e) \, d\mathcal{H}_\sigma^1 \, dt \, d\mathcal{H}_x^1, \quad \forall e \in \mathbb{S}^1.$$

We bound the remaining term by using  $|(\cos(\theta), \sin(\theta)) \cdot \nu_E(x)| \leq 1$  and (H2) again. We get

$$\tilde{P}_{G_\varepsilon}^{(1)}(E) \geq \left(1 - \frac{2I_G^1 \theta_0}{\pi}\right) P(E) = \left(1 - \frac{2\theta_0}{\mathbf{K}_{1,2}\pi}\right) P(E).$$

Choosing  $\alpha$  (and thus  $\theta_0$ ) small enough, depending only on  $\tau$ , we then find

$$\tilde{P}_{G_\varepsilon}^{(1)}(E) \geq \left(1 - \frac{\tau}{2}\right) P(E).$$

Since  $\alpha = \alpha(\tau)$ , we have  $r = r(\tau)$ , and  $\bar{\varepsilon}_2 = \bar{\varepsilon}_2(G, \gamma, \tau)$ . Now that with  $r$  now fixed, up to choosing  $\bar{\varepsilon}_3 < \bar{\varepsilon}_2$  even smaller if needed, there holds

$$|\tilde{P}_{G_\varepsilon}^{(2)}(E)| \leq P(E) \int_{B_r^c} |x| G_\varepsilon(x) \, dx = P(E) \int_{B_{r/\varepsilon}^c} |x| G(x) \, dx \leq \frac{\tau}{2} P(E),$$

since the first moment of  $G$  is finite. Hence, from our choice of  $r$  and  $\bar{\varepsilon}_3$ , we obtain

$$\tilde{P}_{G_\varepsilon}(E) \geq (1 - \tau)P(E),$$

which concludes the proof.  $\square$

We are now in position to prove Lemma 3.8.

*Proof of Lemma 3.8.* Let  $\tau > 0$  to be chosen later. Recall that by Lemma 2.3, for almost every  $t$ , we have

$$\frac{d}{dt} [P_{G_\varepsilon}(tE)] = \frac{2}{t} P_{G_\varepsilon}(tE) - \frac{1}{t} \tilde{P}_{G_\varepsilon}(tE),$$

where  $\tilde{P}_{G_\varepsilon}$  is defined by (2.1). Changing variables, we have

$$\begin{aligned} \frac{1}{t} \tilde{P}_{G_\varepsilon}(tE) &= 2t^3 \int_E \int_{\partial E} G_\varepsilon(t(x-y))(y-x) \cdot \nu_E(y) \\ &= 2 \int_E \int_{\partial E} G_{(t^{-1}\varepsilon)}(x-y)(y-x) \cdot \nu_E(y) d\mathcal{H}_y^1 dx = \tilde{P}_{G_{t^{-1}\varepsilon}}(E). \end{aligned}$$

Let  $\tau > 0$  such that  $\gamma(1+\tau) < 1$ , and let  $\varepsilon < \bar{\varepsilon}_3(G, \gamma, \tau)/2$ , where  $\bar{\varepsilon}_3$  is given by Lemma 3.10. By Lemma 3.10, we have

$$\frac{1}{t} \tilde{P}_{G_\varepsilon}(tE) \geq (1-\tau)P(E).$$

With Proposition 2.1, this leads to

$$\frac{d}{dt} [P_{G_\varepsilon}(tE)] \leq \frac{2}{t} P(tE) - (1-\tau)P(E) = (1+\tau)P(E).$$

Hence, for every  $t \in (\frac{1}{2}, 1)$ , it follows

$$\mathcal{F}_{\gamma, \varepsilon}(E) - \mathcal{F}_{\gamma, \varepsilon}(tE) = (1-t)P(E) - \gamma \int_t^1 \frac{d}{ds} [P_{G_\varepsilon}(sE)] ds \geq (1-t)P(E)[1-\gamma(1+\tau)] > 0.$$

This proves the lemma.  $\square$

**3.4. Convex minimizers are nearly spherical sets.** In arbitrary dimension, it is classical to improve Hausdorff convergence of the boundary of minimizers to Lipschitz convergence once convexity is established. Here, we show that convex minimizers of  $(\mathcal{P}_{\gamma, \varepsilon})$  are centered  $t(\varepsilon)$ -nearly spherical sets (see Definition 1), up to a translation, where the function  $t$  vanishes in  $0^+$ .

**Proposition 3.11** (Convex minimizers are nearly spherical sets). *There exists  $\bar{\varepsilon}_4 = \bar{\varepsilon}_4(n, G, \gamma) > 0$  such that the following holds. If  $E_\varepsilon \subseteq \mathbb{R}^n$  is a convex minimizer of  $(\mathcal{P}_{\gamma, \varepsilon})$  with  $0 < \varepsilon < \bar{\varepsilon}_4$ , then, up to a translation, we have*

$$\partial E_\varepsilon = \left\{ (1 + u_\varepsilon(x))x : x \in \mathbb{S}^{n-1} \right\}$$

and

$$\int_{E_\varepsilon} x dx = 0,$$

with  $u_\varepsilon \in \text{Lip}(\mathbb{S}^{n-1})$ ,  $\|u_\varepsilon\|_{L^\infty(\mathbb{S}^{n-1})} \leq \delta(\varepsilon)$  and  $\|\nabla_\tau u_\varepsilon\|_{L^\infty(\mathbb{S}^{n-1})} \leq 14\delta(\varepsilon)^{\frac{1}{2}}$ , where  $\delta = \delta(n, G, \gamma)$  is the function of Theorem 2.7 vanishing in 0.

*Proof.* In the proof, we write  $\|\cdot\|_\infty$  for  $\|\cdot\|_{L^\infty(\mathbb{S}^{n-1})}$ . Let  $E_\varepsilon$  be a minimizer  $(\mathcal{P}_{\gamma, \varepsilon})$  with  $0 < \varepsilon < \bar{\varepsilon}_4$ , where  $\bar{\varepsilon}_4$  is to be fixed later. If  $\bar{\varepsilon}_4 < \varepsilon_0$ , where  $\varepsilon_0$  is given by Theorem 2.7, up to a translation and a negligible set,  $E_\varepsilon$  lies between the balls  $B_{1-\delta(\varepsilon)}$  and  $B_{1+\delta(\varepsilon)}$ . By convexity, this implies that the set  $E_\varepsilon$  itself (without the addition or the subtraction of a negligible set) satisfies, up to a translation,

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq \bar{B}_{1+\delta(\varepsilon)}.$$

Setting  $y_\varepsilon := -\frac{1}{|B_1|} \int_{E_\varepsilon} x dx$ , we have

$$\int_{y_\varepsilon + E_\varepsilon} x dx = y_\varepsilon |E_\varepsilon| + \int_{E_\varepsilon} x dx = y_\varepsilon |B_1| + \int_{E_\varepsilon} x dx = 0.$$

Notice that

$$-\int_{E_\varepsilon} x dx = -\int_{B_{1+\delta(\varepsilon)}} x dx + \int_{B_{1+\delta(\varepsilon)} \setminus E_\varepsilon} x dx = \int_{B_{1+\delta(\varepsilon)} \setminus E_\varepsilon} x dx$$

thus

$$\left| \int_{E_\varepsilon} x dx \right| \leq (1 + \delta(\varepsilon)) |B_{1+\delta(\varepsilon)} \setminus E_\varepsilon| \leq 2\delta(\varepsilon)$$

provided that  $\bar{\varepsilon}_4$  is small enough depending only on  $n, G$  and  $\gamma$ . Hence, up to translating  $E_\varepsilon$  by  $y_\varepsilon$ , we may assume that it satisfies

$$B_{1-3\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+3\delta(\varepsilon)}, \quad (3.16)$$

and is centered, that is,

$$\int_{E_\varepsilon} x \, dx = 0.$$

By convexity of  $E_\varepsilon$ , for every  $x \in \mathbb{S}^{n-1}$ , there is a unique point of intersection  $p_\varepsilon(x) = t_\varepsilon(x)x$  of  $\{tx : t > 0\}$  and  $\partial E_\varepsilon$ , and by (3.16),  $|p_\varepsilon(x) - x| \leq 3\delta(\varepsilon)$ . The map  $p_\varepsilon : \mathbb{S}^{n-1} \rightarrow \partial E_\varepsilon$  is obviously onto, so that, setting  $u_\varepsilon(x) = t_\varepsilon(x) - 1$ , we have

$$\partial E_\varepsilon = \left\{ (1 + u_\varepsilon(x))x : x \in \mathbb{S}^{n-1} \right\},$$

and  $\|u_\varepsilon\|_\infty \leq 3\delta(\varepsilon)$ . In addition, the fact that  $E_\varepsilon$  is convex implies  $u_\varepsilon \in \text{Lip}(\mathbb{S}^{n-1})$ . Moreover, for any  $x \in \mathbb{S}^{n-1}$ , the distance between the normal vector of  $\partial E_\varepsilon$  (which exists for  $\mathcal{H}^{n-1}$ -almost every  $x \in \partial E_\varepsilon$ ) at  $p_\varepsilon(x)$  and the vector  $x$  (which is the normal vector of  $\mathbb{S}^{n-1}$  at  $x$ ) is controlled by  $\|p_\varepsilon - \text{Id}\|_\infty = \|u_\varepsilon\|_\infty$ , in view of Step 1 of the proof of Lemma 3.9 (or simply by [27, Corollary 1]), which gives a control of  $\|\nabla_\tau u_\varepsilon\|_\infty$  by  $\|u_\varepsilon\|_\infty$ . More precisely, by [15, Inequality (\*\*)], we have

$$\|\nabla_\tau u_\varepsilon\|_\infty \leq 2 \left( \frac{1 + \|u_\varepsilon\|_\infty}{1 - \|u_\varepsilon\|_\infty} \right) \|u_\varepsilon\|_\infty^{\frac{1}{2}} \leq 14\delta(\varepsilon)^{\frac{1}{2}},$$

where we used the inequality  $\|u_\varepsilon\|_\infty \leq \delta(\varepsilon) \leq \frac{1}{4}$  for the last inequality. This concludes the proof.  $\square$

Observe that Theorem 2 is an immediate corollary of Theorem 1 and Proposition 3.11.

#### 4. MINIMALITY OF THE UNIT BALL AMONG NEARLY SPHERICAL SETS

This section is devoted to the proof of the minimality of the unit ball of  $\mathbb{R}^n$  among  $t$ -nearly spherical sets, for small  $t$  and  $\varepsilon$ . First, let us point out some consequences of the extra assumption (H3).

*Remark 4.1.* Note that with (H2), (H3) is equivalent to saying that  $g \in W_{\text{loc}}^{3,1}(0, +\infty)$  and the functions  $t \mapsto t^{n+i} \left| \frac{d^i g}{dt^i}(t) \right|$ , are integrable on  $(0, +\infty)$ , for  $i \in \{0, 1, 2, 3\}$ . In addition, integrating the function  $(t^{n+1}g(t))'$  between  $r$  and  $R$ , we have the relation

$$R^{n+1}g(R) - r^{n+1}g(r) = (n+1) \int_r^R t^n g(t) \, dt + \int_r^R t^{n+1} g'(t) \, dt.$$

Since  $t^n g(t)$  and  $t^{n+1} g'(t)$  are integrable on  $(0, +\infty)$ , this implies that  $t^{n+1} g(t)$  has a limit in  $0^+$  and at infinity. By the integrability of  $t^n g(t)$  on  $(0, +\infty)$ , those limits are necessarily 0. Hence, letting  $r \rightarrow 0$  and  $R \rightarrow \infty$  in the previous relation yields

$$\int_0^{+\infty} t^{n+1} g'(t) \, dt = -(n+1) \int_0^{+\infty} t^n g(t) \, dt.$$

Proceeding similarly for  $(t^{n+2}g'(t))'$ , we also find

$$\int_0^{+\infty} t^{n+2} g''(t) \, dt = -(n+2) \int_0^{+\infty} t^{n+1} g'(t) \, dt = (n+1)(n+2) \int_0^{+\infty} t^n g(t) \, dt.$$

**4.1. A Fuglede-type result for the nonlocal perimeter.** The minimality of the unit ball among nearly spherical sets relies on a bound (in our case, an upper bound) on the quantity  $P_{G_\varepsilon}(E_t) - P_{G_\varepsilon}(B_1)$  for a centered  $t$ -nearly spherical set  $E_t$  with  $\partial E_t = \{(1 + tu(x)) : x \in \mathbb{S}^{n-1}\}$ , in terms of the  $L^2$  norms of  $u$  and  $\nabla_\tau u$  on the sphere. In the case of the local perimeter, this kind of control is well known and is originally due to B. Fuglede (see [15, Theorem 1.2]), who proved

$$\frac{t^2}{10} \left( \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) \leq \frac{P(E_t) - P(B_1)}{P(B_1)} \leq \frac{3}{5} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2,$$

provided that  $t$  is small enough, depending only on  $n$ . One can find a more precise lower bound in [16, Proof of Theorem 3.1, eq. (3.8)]. Similar results were obtained for so-called fractional perimeters  $P_s$  as well as for Riesz potentials in [13] (see Theorems 2.1 and 8.1 therein), where the quantities are bounded in terms of the  $L^2$  norm and fractional Sobolev seminorms on the sphere. Our computations are inspired by the ones in [13], however, due to the general form of the kernel  $G$ , they are more involved and quite tricky at times.

In addition, when  $\varepsilon$  vanishes, the quadratic terms from the Taylor expansion of  $P_{G_\varepsilon}(E_t) - P_{G_\varepsilon}(B_1)$  compensate exactly those of  $P(B_1) - P(E_t)$ , so that the constant  $\gamma$  must be smaller than 1, and the

expansion needs to be pushed to the third order to be able to conclude. Let us point out that similarly to fractional perimeters, fractional Sobolev-type seminorms associated with the kernel  $G_\varepsilon$  (and its derivatives) naturally appear in the expansion. However, those converge to the  $H^1$  seminorm as  $\varepsilon$  vanishes, which is the quantity we need to compare them to (when expanding the local perimeter): that is why we chose to control  $P_{G_\varepsilon}(E_t) - P_{G_\varepsilon}(B_1)$  directly in terms of the  $H^1$  seminorm of  $u$  on  $\mathbb{S}^{n-1}$ .

**Lemma 4.2.** *Assume that  $G$  satisfies (H1), (H2) and (H3). There exist positive constants  $\bar{\varepsilon}_5 = \bar{\varepsilon}_5(n, G)$  and  $t_0 = t_0(n)$  such that the following holds. If  $E_t$  is a centered  $t$ -nearly spherical set with  $0 < t < t_0$ , then for any  $0 < \varepsilon < \bar{\varepsilon}_5$ , we have*

$$P_{G_\varepsilon}(E_t) \leq P_{G_\varepsilon}(B_1) + \frac{t^2}{2} \left( (1 + Cq_n(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - ((n-1) - e_G(\varepsilon)) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) + Ct^3 \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \quad (4.1)$$

where  $C = C(n, G)$ ,  $q_n$  is given by Lemma 2.4,  $q_n(\varepsilon) \rightarrow 0$  as  $\varepsilon \downarrow 0$ , and  $e_G : (0, +\infty) \rightarrow [0, +\infty)$  is a function vanishing in  $0^+$  depending only on  $n$  and  $G$ .

*Proof.* In the proof, unless stated otherwise,  $C$  denotes a positive constant depending only on  $n$  and possibly changing from line to line. For the sake of brevity we write  $B$  for the open unit ball in  $\mathbb{R}^n$ ,  $\partial B$  for the unit sphere  $\mathbb{S}^{n-1}$ , and  $\|\cdot\|_\infty$  for  $\|\cdot\|_{L^\infty(\mathbb{S}^{n-1})}$ . Let  $0 < \varepsilon < \bar{\varepsilon}_5$ , where  $\bar{\varepsilon}_5 = \bar{\varepsilon}_5(n, G)$  is to be fixed later.

Let  $t_0 = t_0(n) < \frac{1}{8}$  to be fixed later, and let  $E_t$  be a centered  $t$ -nearly spherical set such that  $\partial E_t = \{(1 + tu(x))x : x \in \mathbb{S}^{n-1}\}$  with  $0 < t < t_0$ . We proceed in 3 steps.

*Step 1.* We rewrite  $P_{G_\varepsilon}(E_t)$  in a more convenient form, introducing two terms that we will bound from above in the next steps. Using polar coordinates, we have

$$P_{G_\varepsilon}(E_t) = 2 \iint_{\partial B \times \partial B} \int_0^{1+tu(x)} \int_{1+tu(y)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}.$$

By symmetry of  $G$ , we see that

$$P_{G_\varepsilon}(E_t) = \iint_{\partial B \times \partial B} \left( \int_0^{1+tu(x)} \int_{1+tu(y)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho + \int_0^{1+tu(y)} \int_{1+tu(x)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho \right) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}.$$

Using the identity

$$\int_0^b \int_a^{+\infty} + \int_0^a \int_b^{+\infty} = \int_a^b \int_a^b + \int_0^a \int_a^{+\infty} + \int_0^b \int_b^{+\infty},$$

and the symmetry of  $G$  yet again, we find

$$P_{G_\varepsilon}(E_t) = \iint_{\partial B \times \partial B} \int_{1+tu(y)}^{1+tu(x)} \int_{1+tu(y)}^{1+tu(x)} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} + 2 \iint_{\partial B \times \partial B} \int_0^{1+tu(x)} \int_{1+tu(x)}^{+\infty} G_\varepsilon(rx - \rho y) r^{n-1} \rho^{n-1} dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}. \quad (4.2)$$

A straightforward computation leads to the identity

$$|rx - \rho y|^2 = (r - \rho)^2 + r\rho|x - y|^2, \quad \forall x, y \in \mathbb{S}^{n-1}, \forall r, \rho > 0.$$

Defining

$$f_\varepsilon(|x - y|, r, \rho) := r^{n-1} \rho^{n-1} G_\varepsilon(rx - \rho y) = r^{n-1} \rho^{n-1} g_\varepsilon \left( ((r - \rho)^2 + r\rho|x - y|^2)^{\frac{1}{2}} \right),$$

using identity (4.2) and changing variables, we find

$$\begin{aligned}
P_{G_\varepsilon}(E_t) &= t^2 \iint_{\partial B \times \partial B} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} f_\varepsilon(|x-y|, 1+ta, 1+tb) da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&+ 2 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} (1+tu(x))^2 f_\varepsilon(|x-y|, (1+tu(x))r, (1+tu(x))\rho) dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&=: t^2 \varphi_\varepsilon(t) + \psi_\varepsilon(t).
\end{aligned} \tag{4.3}$$

This concludes Step 1.

*Step 2. Estimation of  $\varphi_\varepsilon(t)$ .* Note that by Lemma 2.4 and Remark 2.5, we have

$$\begin{aligned}
\varphi_\varepsilon(0) &= \iint_{\partial B \times \partial B} (u(x) - u(y)) G_\varepsilon(x-y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&= \iint_{\partial B \times \partial B} \frac{(u(x) - u(y))^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&\leq (1 + Cq_\eta(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2.
\end{aligned} \tag{4.4}$$

Then, let us write,

$$\begin{aligned}
&\varphi_\varepsilon(t) - \varphi_\varepsilon(0) \\
&= \iint_{\partial B \times \partial B} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} \left( (1+ta)^{n-1} (1+tb)^{n-1} G_\varepsilon(x-y + t(ax-by)) \right. \\
&\quad \left. - G_\varepsilon(x-y) \right) da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&= \iint_{\partial B \times \partial B} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} \left( (1+ta)^{n-1} (1+tb)^{n-1} - 1 \right) G_\varepsilon(x-y) da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&\quad + \iint_{\partial B \times \partial B} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} (1+ta)^{n-1} (1+tb)^{n-1} \left( G_\varepsilon(x-y + t(ax-by)) - G_\varepsilon(x-y) \right) \\
&\quad da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&=: I_1(t) + I_2(t)
\end{aligned} \tag{4.5}$$

On one hand, since  $\|u\|_\infty \leq 1$ , we have on the domain of integration,

$$|(1+ta)^{n-1} (1+tb)^{n-1} - 1| \leq Ct \|u\|_\infty \leq Ct.$$

Thus

$$\begin{aligned}
I_1(t) &\leq Ct \iint_{\partial B \times \partial B} \int_{(u(x), u(y))} \int_{(u(x), u(y))} G_\varepsilon(x-y) da db d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&= Ct \iint_{\partial B \times \partial B} (u(x) - u(y))^2 G_\varepsilon(x-y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&= Ct \iint_{\partial B \times \partial B} \frac{(u(x) - u(y))^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1},
\end{aligned}$$

where  $\eta_\varepsilon$  is defined as in Lemma 2.4 so that  $g_\varepsilon(r) = \frac{\eta_\varepsilon(r)}{2r^2}$ . In view of Lemma 2.4 and Remark 2.5, it follows

$$\begin{aligned}
I_1(t) &\leq Ct(1 + q_\eta(\varepsilon)) \int_{\partial B} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} \\
&\leq Ct \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2,
\end{aligned} \tag{4.6}$$

for any  $\varepsilon$  small enough (depending only on  $n$  and  $G$ ).

Let us now bound the term  $I_2(t)$ . Integrating on a line (recall that  $g$  is absolutely continuous in  $(0, +\infty)$  by (H3)), and using the inequality  $(1+ta)^{n-1} (1+tb)^{n-1} \leq 2$  for any  $0 < a, b \leq \|u\|_\infty \leq 1$ , provided that  $t_0 = t_0(n)$  is chosen small enough, we find

$$I_2(t) \leq 2t \iint_{\partial B \times \partial B} \left| \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} \int_0^1 |\nabla G_\varepsilon(x-y + st(ax-by)) \cdot (ax-by)| ds da db \right| d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}. \tag{4.7}$$

Observe that in (4.7), if the term  $(x - y)$  inside  $\nabla G_\varepsilon$  were not perturbed by  $st(ax - by)$ , we would have

$$I_2(t) \leq 4t \iint_{\partial B \times \partial B} (u(x) - u(y))^2 |\nabla G_\varepsilon(x - y)| |x - y| ds d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1},$$

which we could estimate in terms of  $\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2$  by applying directly Lemma 2.4 with the kernel  $x \mapsto |\nabla G_\varepsilon(x)| |x|^3$ . Here, to deal with this perturbation, we apply the technical Lemma A.2, by showing that the right-hand side of (4.7) is bounded by a term of the form

$$Ct \int_0^1 \iint_{\partial B \times \partial B} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon^{(1)}(|\Phi_{stu}(x, y, r, \rho)|) dr d\rho dx dy ds,$$

where

$$\Phi_{stu}(x, y, r, \rho) := (x - y) + st \left[ (ru(x) + (1 - r)u(y))x - (\rho u(y) + (1 - \rho)u(x))y \right] \quad (4.8)$$

is a small perturbation of  $(x - y)$ , and  $k_\varepsilon^{(1)}$  is a kernel defined further below. Let us remark that, since  $G$  is radial, we have  $\nabla G_\varepsilon(x) = g'_\varepsilon(|x|) \frac{x}{|x|}$ , thus

$$\nabla G_\varepsilon(x - y + st(ax - by)) \cdot (ax - by) = g'_\varepsilon(|x - y + st(ax - by)|) \frac{\frac{(a+b)}{2}|x - y|^2 + st|ax - by|^2}{|x - y + st(ax - by)|}. \quad (4.9)$$

Then notice that, on the domain of integration, we have

$$\begin{aligned} |x - y + st(ax - by)|^2 &= s^2 t^2 (a - b)^2 + (1 + sta)(1 + stb)|x - y|^2 \\ &\geq (1 + sta)(1 + stb)|x - y|^2 - s^2 t^2 (u(x) - u(y))^2 \\ &\geq \frac{9}{16}|x - y|^2 - \frac{1}{16}\|\nabla_\tau u\|_\infty^2 |x - y|^2 \geq \frac{1}{2}|x - y|^2, \end{aligned} \quad (4.10)$$

provided  $t_0 < \frac{1}{4}$ . By (4.10), and the inequality  $|ax - by|^2 \leq 2|x - y|^2$  on the domain of integration, for  $t_0$  small enough depending only on  $n$ , we have

$$\frac{(a+b)}{2}|x - y|^2 + st|ax - by|^2 \leq 8|x - y + st(ax - by)|^2.$$

Hence, with (4.9), it follows

$$|\nabla G_\varepsilon(x - y + st(ax - by)) \cdot (ax - by)| \leq C k_\varepsilon^{(1)}(|x - y + st(ax - by)|), \quad (4.11)$$

where we have set  $k^{(1)}(r) := r|g'(r)|$  and  $k_\varepsilon^{(1)}(r) := \varepsilon^{-(n+1)}k^{(1)}(\varepsilon^{-1}r)$  for all  $r > 0$  and  $\varepsilon > 0$ . Making the changes of variables  $a = u(x) + r(u(y) - u(x))$ , and  $b = u(y) + \rho(u(x) - u(y))$  in (4.7) yield, with (4.8) and (4.11),

$$I_2(t) \leq Ct \int_0^1 \iint_{\partial B \times \partial B} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon^{(1)}(|\Phi_{stu}(x, y, r, \rho)|) dr d\rho dx dy ds.$$

It is easy to see that by (H3) and Remark 4.1, the family  $(k_\varepsilon^{(1)})_{\varepsilon > 0}$  satisfies the assumptions of Lemma A.2. Thus choosing  $t_0 < t_1(n)$  and  $\bar{\varepsilon}_5 < \bar{\varepsilon}_6$ , where  $t_1(n)$  and  $\bar{\varepsilon}_6(n, G)$  are given by Lemma A.2, we have

$$I_2(t) \leq Ct \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2, \quad (4.12)$$

for a constant  $C$  depending on  $n$  and  $G$ . Combining (4.12) with (4.4) to (4.6), we deduce

$$\varphi_\varepsilon(t) \leq (1 + Cq_n(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct \left( \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \quad (4.13)$$

for some  $C = C(n, G)$ , which concludes this step.

*Step 3. Estimation of  $\psi_\varepsilon(t)$ .* We have

$$\begin{aligned}
\psi_\varepsilon(t) &= 2 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} (1+tu(x))^{2n} r^{n-1} \rho^{n-1} G_\varepsilon((1+tu(x))(rx-\rho y)) \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\
&= 2 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} (1+tu(x))^{2n} r^{n-1} \rho^{n-1} G_\varepsilon(rx-\rho y) \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\
&\quad + 2 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} (1+tu(x))^{2n} r^{n-1} \rho^{n-1} [G_\varepsilon((1+tu(x))(rx-\rho y)) \\
&\quad \quad \quad - G_\varepsilon(rx-\rho y)] \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\
&=: I_3(t) + I_4(t).
\end{aligned} \tag{4.14}$$

Notice that

$$\begin{aligned}
&2 \int_{\partial B} \int_0^1 \int_1^{+\infty} r^{n-1} \rho^{n-1} G_\varepsilon(rx-\rho y) \, dr \, d\rho \, d\mathcal{H}_y^{n-1} \\
&= \frac{2}{P(B)} \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} r^{n-1} \rho^{n-1} G_\varepsilon(rx-\rho y) \, dr \, d\rho \, d\mathcal{H}_y^{n-1} = \frac{P_{G_\varepsilon}(B)}{P(B)}
\end{aligned}$$

does not depend on  $x \in \partial B$ , by symmetry of  $G_\varepsilon$ , hence

$$I_3(t) = \frac{P_{G_\varepsilon}(B)}{P(B)} \int_{\partial B} (1+tu(x))^{2n} \, d\mathcal{H}_x^{n-1}.$$

By a Taylor expansion, this gives

$$\begin{aligned}
I_3(t) &\leq P_{G_\varepsilon}(B) + \frac{P_{G_\varepsilon}(B)}{P(B)} \left( 2nt \int_{\partial B} u \, d\mathcal{H}^{n-1} + n(2n-1)t^2 \int_{\partial B} u^2 \, d\mathcal{H}^{n-1} \right. \\
&\quad \left. + Ct^3 \int_{\partial B} |u|^3 \, d\mathcal{H}^{n-1} \right).
\end{aligned} \tag{4.15}$$

Whence, recalling that  $P_{G_\varepsilon}(B) \leq P(B)$ , (4.15) gives

$$I_3(t) \leq P_{G_\varepsilon}(B) + n^2 t^2 \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct^3 \int_{\partial B} |u|^3 \, d\mathcal{H}^{n-1}. \tag{4.16}$$

As for  $I_4(t)$ , integrating on a line and splitting it into two parts, we write

$$\begin{aligned}
I_4(t) &= 2t \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} \int_0^1 u(x)(1+tu(x))^{2n} r^{n-1} \rho^{n-1} \nabla G_\varepsilon((1+stu(x))(rx-\rho y)) \\
&\quad \quad \quad \cdot (rx-\rho y) \, ds \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\
&= 2t \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} u(x)(1+tu(x))^{2n} r^{n-1} \rho^{n-1} \nabla G_\varepsilon(rx-\rho y) \cdot (rx-\rho y) \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\
&\quad + 2t \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} \int_0^1 u(x)(1+tu(x))^{2n} r^{n-1} \rho^{n-1} [\nabla G_\varepsilon((1+stu(x))(rx-\rho y)) \\
&\quad \quad \quad - \nabla G_\varepsilon(rx-\rho y)] \cdot (rx-\rho y) \, ds \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\
&=: I_5(t) + I_6(t).
\end{aligned} \tag{4.17}$$

Using the symmetry of the kernel  $x \mapsto \nabla G(x) \cdot x$ , we find

$$\begin{aligned}
I_5(t) &= \frac{1}{P(B)} \left( t \int_{\partial B} u(z)(1+tu(z))^{2n} \, d\mathcal{H}_z^{n-1} \right) \\
&\quad \left( 2 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} r^{n-1} \rho^{n-1} \nabla G_\varepsilon(rx-\rho y) \cdot (rx-\rho y) \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \right) \\
&= \frac{1}{P(B)} \left( t \int_{\partial B} u(z)(1+tu(z))^{2n} \, d\mathcal{H}_z^{n-1} \right) \left( 2 \iint_{B \times B^c} \nabla G_\varepsilon(x-y) \cdot (x-y) \, dx \, dy \right).
\end{aligned} \tag{4.18}$$

Noticing that  $\operatorname{div}_y((x-y)G_\varepsilon(x-y)) = -nG_\varepsilon(x-y) - \nabla G_\varepsilon(x-y) \cdot (x-y)$  and integrating by parts on  $B^c$ , we have

$$2 \iint_{B \times B^c} \nabla G_\varepsilon(x-y) \cdot (x-y) \, dx \, dy = - \left( n P_{G_\varepsilon}(B) + \tilde{P}_{G_\varepsilon}(B) \right),$$

where  $\tilde{P}_{G_\varepsilon}(B)$  is defined by (2.1). On the one hand, by Proposition 2.2,  $P_{G_\varepsilon}(B)$  converges to  $P(B)$  as  $\varepsilon$  vanishes, and on the other hand, as mentioned in the beginning of Section 3.2, it is not difficult to see that  $\tilde{P}_{G_\varepsilon}(B)$  also converges to  $P(B)$  (the situation is similar but much simpler than in the proof of Lemma 3.10). Thus,

$$e_G^{(1)}(\varepsilon) := 2 \iint_{B \times B^c} \nabla G_\varepsilon(x-y) \cdot (x-y) \, dx \, dy + (n+1)P(B) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.19)$$

Expanding  $(1+tu(z))^{2n}$  and using the Taylor expansion (A.1) from Lemma A.1, we find

$$\begin{aligned} t \int_{\partial B} u(z)(1+tu(z))^{2n} \, d\mathcal{H}_z^{n-1} &\geq t \int_{\partial B} u \, d\mathcal{H}^{n-1} + 2nt^2 \int_{\partial B} u^2 \, d\mathcal{H}^{n-1} - Ct^3 \int_{\partial B} |u|^3 \, d\mathcal{H}^{n-1} \\ &\geq t^2 \left( 2n - \frac{(n-1)}{2} \right) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 - Ct^3 \int_{\partial B} |u|^3 \, d\mathcal{H}^{n-1} \\ &= t^2 \left( \frac{3n+1}{2} \right) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 - Ct^3 \int_{\partial B} |u|^3 \, d\mathcal{H}^{n-1}. \end{aligned} \quad (4.20)$$

Hence, gathering (4.18) to (4.20) yields

$$\begin{aligned} I_5(t) &\leq - \left[ (n+1) - \frac{e_G^{(1)}(\varepsilon)}{P(B)} \right] \left[ t^2 \left( \frac{3n+1}{2} \right) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 - Ct^3 \int_{\partial B} |u|^3 \, d\mathcal{H}^{n-1} \right] \\ &\leq -t^2 \left( \frac{1}{2}(3n+1)(n+1) - Ce_G^{(1)}(\varepsilon) \right) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct^3 \int_{\partial B} |u|^3 \, d\mathcal{H}^{n-1}, \end{aligned} \quad (4.21)$$

provided that  $\bar{\varepsilon}_5 = \bar{\varepsilon}_5(n, G)$  is chosen small enough.

Then for  $I_6(t)$ , recalling that  $\nabla G_\varepsilon(x) \cdot x = g'_\varepsilon(|x|)|x|$  and that  $g'_\varepsilon$  is absolutely continuous by (H3), integrating on a line, we obtain

$$\begin{aligned} I_6(t) &= 2t^2 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} \int_0^1 \int_0^1 su(x)^2 (1+tu(x))^{2n} r^{n-1} \rho^{n-1} g''_\varepsilon((1+ptu(x))(rx-\rho y)) \\ &\quad |rx-\rho y|^2 \, dp \, ds \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\ &= I_7(t) + I_8(t), \end{aligned} \quad (4.22)$$

where

$$I_7(t) := 2t^2 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} \int_0^1 \int_0^1 su(x)^2 (1+tu(x))^{2n} r^{n-1} \rho^{n-1} g''_\varepsilon(rx-\rho y) |rx-\rho y|^2 \, dp \, ds \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1}$$

and

$$I_8(t) := 2t^2 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} \int_0^1 \int_0^1 su(x)^2 (1+tu(x))^{2n} r^{n-1} \rho^{n-1} |rx-\rho y|^2 \left[ g''_\varepsilon((1+ptu(x))(rx-\rho y)) - g''_\varepsilon(rx-\rho y) \right] \, dp \, ds \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1}.$$

Now, by symmetry, similarly to  $I_5(t)$ , we have

$$I_7(t) = \frac{1}{P(B)} \left( \frac{t^2}{2} \int_{\partial B} u(z)^2 (1+tu(z))^{2n} \, d\mathcal{H}_z^{n-1} \right) \left( 2 \iint_{B \times B^c} g''_\varepsilon(|x-y|)|x-y|^2 \, dx \, dy \right). \quad (4.23)$$

Setting  $k^{(2)}(r) := g(r) - rg'(r)$  for  $r \in (0, +\infty)$ , and then defining the kernels  $K^{(2)}(x) := k^{(2)}(|x|)$  for  $x \in \mathbb{R}^n \setminus \{0\}$  and the rescalings  $K_\varepsilon^{(2)} := \varepsilon^{-(n+1)}K^{(2)}(\varepsilon^{-1} \cdot)$ ,  $\varepsilon > 0$ , notice that

$$2 \iint_{B \times B^c} g''_\varepsilon(|x-y|)|x-y|^2 \, dx \, dy = -2 \iint_{B \times B^c} \nabla K_\varepsilon^{(2)}(x-y) \cdot (x-y) \, dx \, dy.$$

Note that by (H3),  $K^{(2)}$  has finite first moment. However, unlike  $G$ , it is not necessarily nonnegative. Still, integrating by parts on  $B^c$  as we did before for  $I_5(t)$ , we find

$$2 \iint_{B \times B^c} g_\varepsilon''(|x-y|)|x-y|^2 dx dy = n P_{K_\varepsilon^{(2)}}(B) + \tilde{P}_{K_\varepsilon^{(2)}}(B). \quad (4.24)$$

It is easy to see that we may split  $K^{(2)}$  and  $K_\varepsilon^{(2)}$  into their negative and positive parts  $K^{(2)} = K^{(2)+} + K^{(2)-}$  with  $K^{(2)-} := \max(-K^{(2)}, 0)$  and  $K^{(2)+} := \max(K^{(2)}, 0)$ , so that  $P_{K_\varepsilon^{(2)}} = P_{K_\varepsilon^{(2)+}} - P_{K_\varepsilon^{(2)-}}$ . Then applying Proposition 2.2 to each nonnegative term, we find

$$P_{K_\varepsilon^{(2)}}(B) = P_{K_\varepsilon^{(2)+}}(B) - P_{K_\varepsilon^{(2)-}}(B) \xrightarrow{\varepsilon \rightarrow 0} \mathbf{K}_{1,n}(I_{K^{(2)+}}^1 - I_{K^{(2)-}}^1) P(B) = \mathbf{K}_{1,n} P(B) \int_{\mathbb{R}^n} |x| K^{(2)}(x) dx.$$

Similarly, we have

$$\tilde{P}_{K_\varepsilon^{(2)}}(B) \xrightarrow{\varepsilon \rightarrow 0} \mathbf{K}_{1,n} P(B) \int_{\mathbb{R}^n} |x| K^{(2)}(x) dx.$$

Hence, recalling (4.24), and using that, in view of Remark 4.1,

$$\int_{\mathbb{R}^n} |x| K^{(2)}(x) dx = |\mathbb{S}^{n-1}| \int_0^{+\infty} t^n g(t) - t^{n+1} g'(t) dt = (n+2) I_G^1 = \frac{n+2}{\mathbf{K}_{1,n}},$$

it follows

$$e_G^{(2)}(\varepsilon) := 2 \iint_{B \times B^c} g_\varepsilon''(|x-y|)|x-y|^2 dx dy - (n+1)(n+2)P(B) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.25)$$

Inserting inequality (A.1) from Lemma A.1 and (4.25) into (4.23) then yields

$$I_7(t) \leq \frac{1}{2} \left( (n+1)(n+2) + \frac{e_G^{(2)}(\varepsilon)}{P(B)} \right) t^2 \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct^3 \int_{\partial B} |u|^3 d\mathcal{H}^{n-1}. \quad (4.26)$$

Eventually, we turn to  $I_8(t)$ . By (H3),  $g_\varepsilon''$  is absolutely continuous on  $(0, +\infty)$ , so that, integrating on lines and using  $(1+tu(x))^{2n} \leq 2$ , we find

$$\begin{aligned} I_8(t) &\leq Ct^3 \iint_{\partial B \times \partial B} \int_0^1 \int_1^{+\infty} \int_0^1 \int_0^1 \int_0^1 |u(x)|^3 s p r^{n-1} \rho^{n-1} |g_\varepsilon'''((1+qpst)(rx-\rho y))| |rx-\rho y|^3 \\ &\quad dq dp ds dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\leq Ct^3 \left( \int_{\partial B} |u|^3 d\mathcal{H}^{n-1} \right) \left[ \int_0^1 \int_0^1 \int_0^1 \left( \iint_{B \times B^c} |g_\varepsilon'''((1+qpst)(x-y))| |x-y|^3 dx dy \right) dq dp ds \right], \end{aligned} \quad (4.27)$$

where we used the symmetry of  $(x, y) \mapsto |g_\varepsilon'''(x-y+pst(rx-\rho y))| |rx-\rho y|^3$  and Fubini's theorem for the last inequality. Notice that, for any  $q, p, s \in (0, 1)$ ,  $t \in (0, t_0)$ ,

$$\begin{aligned} \iint_{B \times B^c} |g_\varepsilon'''((1+qpst)(x-y))| |x-y|^3 dx dy &\leq 2 \iint_{B \times B^c} |g_\varepsilon'''((1+qpst)(x-y))| |(1+qpst)(x-y)|^3 dx dy \\ &= P_{K_{(1+qpst)\varepsilon}^{(3)}}(B), \end{aligned} \quad (4.28)$$

up to choosing  $t_0(n)$  small enough, where we have set  $k^{(3)}(r) := r^3 |g'''(r)|$ ,  $\forall r > 0$ , defined the kernel  $K^{(3)} := k^{(3)}(|x|)$ ,  $\forall x \in \mathbb{R}^n \setminus \{0\}$ , and the rescalings  $K_\varepsilon^{(3)} := \varepsilon^{-(n+1)} K(\varepsilon^{-1} \cdot)$ . Note that  $I_{K^{(3)}}^1 = I_G^4 < \infty$  by (H3), so that, by Proposition 2.1, we have  $P_{K_{(1+qpst)\varepsilon}^{(3)}}(B) \leq \mathbf{K}_{1,n} I_G^4 P(B)$ . Hence, from (4.27) and (4.28), we obtain

$$I_8(t) \leq Ct^3 \|u\|_{L^2(\mathbb{S}^{n-1})}^2, \quad (4.29)$$

where the constant  $C$  here depends on  $n$  and  $G$ . Gathering (4.22), (4.26) and (4.29), we find

$$I_6(t) \leq \frac{t^2}{2} \left( (n+1)(n+2) + C e_G^{(2)}(\varepsilon) \right) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct^3 \int_{\partial B} |u|^3 d\mathcal{H}^{n-1},$$

where  $C = C(n, G)$ , and thus, with (4.17) and (4.21), setting  $e_G := C(e_G^{(1)} + e_G^{(2)})$ ,

$$\begin{aligned} I_4(t) &\leq \frac{t^2}{2} ((n+1)(n+2) - (3n+1)(n+1) + Ce_G(\varepsilon)) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct^3 \int_{\partial B} |u|^3 d\mathcal{H}^{n-1} \\ &= \frac{t^2}{2} (-2n^2 - n + 1 + Ce_G(\varepsilon)) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct^3 \int_{\partial B} |u|^3 d\mathcal{H}^{n-1}. \end{aligned}$$

In turn, combining this estimate with (4.14) and (4.16), we deduce

$$\begin{aligned} \psi_\varepsilon(t) &\leq P_{G_\varepsilon}(B) + \frac{t^2}{2} (2n^2 - 2n^2 - n + 1 + e_G(\varepsilon)) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct^3 \int_{\partial B} |u|^3 d\mathcal{H}^{n-1} \\ &= P_{G_\varepsilon}(B) - (n-1 - e_G(\varepsilon)) \frac{t^2}{2} \|u\|_{L^2(\mathbb{S}^{n-1})}^2 + Ct^3 \int_{\partial B} |u|^3 d\mathcal{H}^{n-1}, \end{aligned} \quad (4.30)$$

where  $C$  depends only on  $n$  and  $G$ , which concludes this step. Eventually, gathering (4.3), (4.13) from Step 2 and (4.30), we deduce

$$\begin{aligned} P_{G_\varepsilon}(E_t) &\leq P_{G_\varepsilon}(B) + \frac{t^2}{2} \left( (1 + Cq_\eta(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - ((n-1) - e_G(\varepsilon)) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) \\ &\quad + Ct^3 \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \end{aligned} \quad (4.31)$$

for some  $C = C(n, G)$ , provided  $t_0 = t_0(n)$  is chosen small enough, which concludes the proof.  $\square$

**4.2. Minimality of the unit ball.** In order to take advantage of Lemma 4.2, one should have that for a centered  $t$ -nearly spherical set  $E_t$  such that  $\partial E_t = \{(1 + tu(x))x : x \in \mathbb{S}^{n-1}\}$ , the quantity

$$(1 + Cq_\eta(\varepsilon)) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - ((n-1) - e_G(\varepsilon)) \|u\|_{L^2(\mathbb{S}^{n-1})}^2$$

from (4.1) controls  $\|u\|_{H^1(\mathbb{S}^{n-1})}^2$  for small  $t$  and  $\varepsilon$ . This is the purpose of Lemma A.1 in appendix.

With Lemma 4.2 and Fuglede's result for the local perimeter, we deduce a lower bound for  $\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1)$ .

**Proposition 4.3.** *Assume that  $G$  satisfies (H1) to (H3). Then there exist positive constants  $t_*$  and  $\varepsilon_3$  depending only on  $n, G$  and  $\gamma$  such that the following holds. If  $E_t$  is a centered  $t$ -nearly spherical set such that  $\partial E_t = \{(1 + tu(x))x : x \in \mathbb{S}^{n-1}\}$  with  $0 < t < t_*$ , then for every  $0 < \varepsilon < \varepsilon_3$ , we have*

$$\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1) \geq \frac{t^2}{16} (1 - \gamma) \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right).$$

*Proof.* Assume  $0 < \varepsilon < \varepsilon_3$ , and  $0 < t < t_*$ , where  $t_* = t_*(n)$  and  $\varepsilon_3 = \varepsilon_3(n, G, \gamma)$  will be fixed later. If  $E_t$  is a centered  $t$ -nearly spherical set and  $t_* < t_1$  as well, where  $t_1 = t_1(n)$  is given by Lemma A.1, we have in particular

$$\|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \leq 4 \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right). \quad (4.32)$$

Then, by [16, Proof of Theorem 3.1, eq. (3.8)] (which gives a more precise lower bound of  $P(E_t) - P(B)$  than [15, Theorem 1.2]) and (4.32), up to choosing  $t_*$  even smaller but depending only on  $n$ , we have

$$P(E_t) \geq P(B_1) + \frac{t^2}{2} \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) - Ct^3 \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right).$$

Then, assuming that  $t_* < t_0(n)$  as well, and  $\varepsilon < \bar{\varepsilon}_5(n, G)$ , where  $t_0$  and  $\bar{\varepsilon}_5$  are given by Lemma 4.2, we find

$$\begin{aligned} &\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1) \\ &\geq \frac{t^2}{2} \left[ (1 - \gamma) \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right) - Cq_\eta(\varepsilon) \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - e_G(\varepsilon) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right] \\ &\quad - Ct^3 \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \end{aligned} \quad (4.33)$$

with  $e_G(\varepsilon), q_\eta(\varepsilon) \rightarrow 0$  as  $\varepsilon \downarrow 0$ . From (4.32) and (4.33), it follows

$$\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1) \geq \frac{1}{4} \left( \frac{t^2}{2} ((1 - \gamma) - Cq_\eta(\varepsilon) - Ce_G(\varepsilon)) - Ct^3 \right) \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right).$$

Eventually, choosing  $\varepsilon_3, t_*$  small enough, for  $0 < \varepsilon < \varepsilon_3, 0 \leq t \leq t_*$ , we have

$$\mathcal{F}_{\gamma, G_\varepsilon}(E_t) - \mathcal{F}_{\gamma, G_\varepsilon}(B_1) \geq \frac{t^2}{16}(1 - \gamma) \left( \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right),$$

which proves the result.  $\square$

An immediate consequence of [Proposition 4.3](#) is that the unit ball is the only minimizer, up to translations, of  $(\mathcal{P}_{\gamma, \varepsilon})$ , among  $t$ -nearly spherical sets whenever  $t < t_*(n, G, \gamma)$  and  $\varepsilon < \varepsilon_3(n, G, \gamma)$ , that is, [Theorem 3](#). In dimension  $n = 2$ , by [Theorems 2](#) and [3](#), setting  $\varepsilon_* := \min(\varepsilon_2, \varepsilon_3)$ , we readily obtain [Theorem A](#).

#### APPENDIX A. ADDITIONAL COMPUTATIONS FOR [SECTION 4](#)

In the following lemma, we establish some general inequalities on functions  $u : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  describing centered nearly-spherical sets. For this, we need to recall a few basic facts and notation on spherical harmonics. For  $k \geq 0$ , we denote by  $\mathcal{S}_k$  the subspace of spherical harmonics of degree  $k$  (i.e., restrictions to  $\mathbb{S}^{n-1}$  of polynomials of degree  $k$  in  $\mathbb{R}^n$ ), which is a finite-dimensional vector space of degree  $d(k)$ . Let  $(Y_k^i)_{i \in \{1, \dots, d(k)\}}$  be an orthonormal basis of  $\mathcal{S}_k$  for the standard scalar product of  $L^2(\mathbb{S}^{n-1})$ . When there can be no confusion, we write  $Y_k$  for a generic vector of the basis of  $\mathcal{S}_k$ . It is well known that the family  $(Y_k^i)_{\substack{i \in \{1, \dots, d(k)\} \\ k \in \mathbb{N}}}$  is a Hilbert basis of  $L^2(\mathbb{S}^{n-1})$  which diagonalizes the Laplace-Beltrami operator on the sphere, and the eigenvalue associated with  $Y_k^i$  is  $l_k := l_k = k(k + n - 2)$ , for all  $i \in \{1, \dots, d(k)\}$ . We recall that  $d(0) = 1, d(1) = n$ , and that the  $Y_1^i$  may be chosen colinear to  $x \mapsto x_i$ , for example  $Y_1^i = |B_1|^{-\frac{1}{2}} x_i, i \in \{1, \dots, n\}$ .

**Lemma A.1.** *There exist  $t_1 = t_1(n) > 0$  and  $C = C(n) > 0$  such that the following holds. If  $E_t$  is a centered  $t$ -nearly spherical set such that  $\partial E_t = \{(1 + tu(x))x : x \in \mathbb{S}^{n-1}\}$  with  $0 < t < t_1$ , then we have*

$$\left| t \int_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} + (n-1) \frac{t^2}{2} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1} \right| \leq Ct^3 \int_{\mathbb{S}^{n-1}} |u|^3 \, d\mathcal{H}^{n-1}, \quad (\text{A.1})$$

and

$$\frac{1}{2} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \geq \frac{1}{2} \|u\|_{L^2(\mathbb{S}^{n-1})}^2. \quad (\text{A.2})$$

*Proof.* Since  $|E_t| = |B|$ , we have

$$n|E_t| = \int_{\partial B} (1 + tu(x))^n \, d\mathcal{H}^{n-1} = \int_{\partial B} 1 \, d\mathcal{H}^{n-1} = n|B|.$$

Thus, writing

$$(1 + tu(x))^n - \left( 1 + ntu(x) + \frac{(n-1)}{2} t^2 u(x)^2 \right) = \sum_{k=3}^n \binom{n}{k} t^k u(x)^k,$$

we deduce [\(A.1\)](#), choosing e.g.  $t_1 = 1$  and  $C$  depending only on  $n$ . There remains to show [\(A.2\)](#). We decompose  $u$  in spherical harmonics

$$u = \sum_{k=0}^{+\infty} \sum_{i=1}^{d(k)} a_k^i(u) Y_k^i, \quad (\text{A.3})$$

so that

$$\|u\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{k=0}^{+\infty} \sum_{i=1}^{d(k)} a_k^i(u)^2, \quad \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{k=1}^{+\infty} \sum_{i=1}^{d(k)} l_k a_k^i(u)^2.$$

Recall that  $d(0) = 1, Y_0^1$  is constant,  $d(1) = n$ , and that  $Y_1^i$  is colinear to  $x \mapsto x_i$ . Since  $l_1 = n - 1$  and  $l_k \geq 2n$  for  $k \geq 2$ , it follows

$$\begin{aligned} & \frac{1}{2} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \\ & \geq \|u\|_{L^2(\mathbb{S}^{n-1})}^2 - \frac{(n+1)}{2} \left( \sum_{i=1}^n a_1^i(u)^2 \right) - (n-1) a_0^1(u)^2. \end{aligned} \quad (\text{A.4})$$

On one hand,

$$a_0^1(u) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1},$$

so that by (A.1), we have

$$|a_0^1(u)| \leq \frac{2t}{|\mathbb{S}^{n-1}|} \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \leq \frac{1}{2\sqrt{n-1}} \|u\|_{L^2(\mathbb{S}^{n-1})}, \quad (\text{A.5})$$

up to choosing  $t_1 = t_1(n)$  small enough, and using  $t < t_1$  and  $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$ . On the other hand, the barycenter condition

$$\int_{E_t} x \, dx = 0$$

gives

$$\int_{\mathbb{S}^{n-1}} x_i (1 + tu(x))^n \, d\mathcal{H}_x^{n-1} = 0, \quad \forall i \in \{1, \dots, n\}.$$

Using the binomial formula again and  $Y_1^i(x) = |B_1|^{-\frac{1}{2}} x_i$ , we obtain, for  $i \in \{1, \dots, n\}$ ,

$$a_1^i(u) = \frac{1}{\sqrt{|B_1|}} \int_{\mathbb{S}^{n-1}} x_i u(x) \, d\mathcal{H}_x^{n-1} = -\frac{1}{n\sqrt{|B_1|}} \sum_{k=2}^n \binom{n}{k} t^k \int_{\mathbb{S}^{n-1}} x_i u(x)^k \, d\mathcal{H}_x^{n-1}.$$

Next, using  $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$ , Cauchy-Schwarz inequality, and  $n|B_1| = |\mathbb{S}^{n-1}|$ , we get

$$|a_i(u)| \leq \frac{2^n}{n} t^2 \|u\|_{L^2(\mathbb{S}^{n-1})}.$$

Whence, choosing  $t_1$  even smaller, but still depending only on  $n$ , we may assume

$$\sum_{i=1}^n a_1^i(u)^2 \leq \frac{1}{2(n+1)} \|u\|_{L^2(\mathbb{S}^{n-1})}^2. \quad (\text{A.6})$$

Gathering (A.4) to (A.6), we find

$$\frac{1}{2} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 - (n-1) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \geq \frac{1}{2} \|u\|_{L^2(\mathbb{S}^{n-1})}^2,$$

which proves (A.2) and concludes the proof.  $\square$

We establish a technical lemma to control terms of the form

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1},$$

by the  $H^1(\mathbb{S}^{n-1})$  norm of  $u$ , where  $\Phi_{tu}(x, y, r, \rho)$  is a small perturbation of  $(x - y)$ , and  $k_\varepsilon$  are suitable rescalings of a nonnegative kernel.

**Lemma A.2.** *Let  $k : (0, +\infty) \rightarrow [0, +\infty)$  be a measurable function such that  $k(r) = O(r^{-(n+1)})$  at infinity and*

$$\int_0^{+\infty} r^n k(r) \, dr < \infty.$$

*Let us define the rescaling  $k_\varepsilon(r) := \varepsilon^{-(n+1)} k(\varepsilon^{-1}r)$ ,  $r, \varepsilon > 0$ . For  $v \in \text{Lip}(\mathbb{S}^{n-1})$ , we define a map  $\Phi_v : (\mathbb{S}^{n-1})^2 \times (0, 1)^2$  by*

$$\Phi_v(x, y, r, \rho) = (x - y) + \left[ (rv(x) + (1-r)v(y))x - (\rho v(y) + (1-\rho)v(x))y \right]. \quad (\text{A.7})$$

*Then there exist positive constants  $\bar{t}_1 = \bar{t}_1(n)$ ,  $\bar{C}_0 = \bar{C}_0(n, k)$  and  $\bar{\varepsilon}_6 = \bar{\varepsilon}_6(n, k)$  such that the following holds. For any  $u \in \text{Lip}(\mathbb{S}^{n-1})$  with  $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$ ,  $\|\nabla_\tau u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$ , and any  $t \in (0, \bar{t}_1)$ , we have*

$$\begin{aligned} & \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) \, dr \, d\rho \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\ & \leq \bar{C}_0 \left( \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 \, d\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1} \right). \end{aligned} \quad (\text{A.8})$$

*Proof.* Let  $\bar{t}_1 > 0$  to be chosen later, and let  $0 < t < \bar{t}_1$ . We work in local coordinates, proceeding in two steps.

*Step 1.* Let us denote by  $D_r(x)$  the open ball of radius  $r$  in  $\mathbb{R}^{n-1}$ . We first show that for any  $\tilde{u} \in \text{Lip}(D_2)$  s.t.  $\|\tilde{u}\|_{W^{1,\infty}(D_2)} \leq M$ , we have

$$\int_{D_2} \int_{D_2} \int_0^1 \int_0^1 (\tilde{u}(x) - \tilde{u}(y))^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dr d\rho dx dy \leq C \int_{D_2} |\nabla \tilde{u}|^2 dx, \quad \forall r, \rho \in (0, 1), \quad (\text{A.9})$$

for all  $\varepsilon > 0$ , provided  $t < \bar{t}_1$ , and  $\bar{t}_1$  is chosen small depending only on  $M$ , and  $C = C(n, k)$ . Here, by a slight abuse of notation,  $\Phi_v$  denotes the map from  $(\mathbb{R}^{n-1})^2 \times (0, 1)^2$  whose expression is given by (A.7). Due to the  $W^{1,\infty}$  bound on  $\tilde{u}$ , we easily see that the maps  $\Phi_{t\tilde{u}}$  are Lipschitz continuous and converge uniformly to the map  $\Phi(x, y, r, \rho) = x - y$  on  $D_2^2 \times (0, 1)^2$  as  $t$  vanishes. In fact, up to choosing  $\bar{t}_1 = \bar{t}_1(M)$  small enough, we have

$$|\Phi_{t\tilde{u}}(x, y, r, \rho) - (x - y)| \leq \frac{1}{4}, \quad \forall x, y \in D_2, \forall r, \rho \in (0, 1), \quad (\text{A.10})$$

and

$$|\Phi_{t\tilde{u}}(x, y_1, r, \rho) - \Phi_{t\tilde{u}}(x, y_2, r, \rho) - (y_2 - y_1)| \leq \frac{|y_2 - y_1|}{4}, \quad \forall x, y_1, y_2 \in D_2, \forall r, \rho \in (0, 1). \quad (\text{A.11})$$

In particular, for any  $y \in D_2$ , and  $r, \rho \in (0, 1)$  fixed,  $\Psi_{x,r,\rho}^1 : y \mapsto \Phi_{t\tilde{u}}(x, y, r, \rho)$  is bi-Lipschitz, and similarly, for any  $x \in D_2$ ,  $r, \rho \in (0, 1)$ , so is  $\Psi_{y,r,\rho}^2 : x \mapsto \Phi_{t\tilde{u}}(x, y, r, \rho)$ . Integrating on lines, using Cauchy-Schwarz inequality and Fubini's theorem, we have

$$\begin{aligned} & \int_{D_2} \int_{D_2} \int_0^1 \int_0^1 (\tilde{u}(x) - \tilde{u}(y))^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dr d\rho dx dy \\ & \leq \int_0^1 \int_0^1 \int_{D_2} \int_{D_2} |\nabla \tilde{u}(x + s(y - x))|^2 |x - y|^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dx dy ds dr d\rho. \end{aligned} \quad (\text{A.12})$$

Now let us focus on the integral on  $D_2 \times D_2$ , fixing  $s \in (0, \frac{1}{2})$  and  $r, \rho \in (0, 1)$ . Recall that  $|x - y|^2 \leq C|\Phi_{t\tilde{u}}(x, y, r, \rho)|^2$  for a universal constant  $C$  by (A.10), that the map  $\Psi_{x,r,\rho}^2$  is bi-Lipschitz, injective and by (A.11), its Jacobian determinant is universally bounded from below. Thus making the change of variable  $z = \Phi_{t\tilde{u}}(x, y, r, \rho) = \Psi_{x,r,\rho}^2(y)$  (for  $x$  fixed), we find

$$\begin{aligned} & \int_{D_2} \int_{D_2} |\nabla \tilde{u}(x + s(y - x))|^2 |x - y|^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dx dy \\ & \leq C \int_{D_2} \left( \int_{D_2} |\nabla \tilde{u}(x + s(y - x))|^2 |\Phi_{t\tilde{u}}(x, y, r, \rho)|^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dy \right) dx \\ & \leq C \int_{D_2} \left( \int_{\Psi_{x,r,\rho}^2(D_2)} |\nabla \tilde{u}(x + s((\Psi_{x,r,\rho}^2)^{-1}(z) - x))|^2 |z|^2 k_\varepsilon(|z|) dz \right) dx, \end{aligned}$$

for some universal constant  $C$ . By (A.10),  $\Psi_{x,r,\rho}^2(D_2) \subseteq D_{\frac{5}{2}}(x)$ , so that by Fubini's theorem, we obtain

$$\begin{aligned} & \int_{D_2} \int_{D_2} |\nabla \tilde{u}(x + s(y - x))|^2 |x - y|^2 k_\varepsilon(|\Phi_{t\tilde{u}}(x, y, r, \rho)|) dx dy \\ & \leq C \int_{D_{\frac{9}{2}}} \left( \int_{D_2 \cap \{x : z \in \Psi_{x,r,\rho}^2(D_2)\}} |\nabla \tilde{u}(x + s((\Psi_{x,r,\rho}^2)^{-1}(z) - x))|^2 dx \right) |z|^2 k_\varepsilon(|z|) dz \quad (\text{A.13}) \\ & = C \int_{D_{\frac{9}{2}}} \left( \int_{D_2 \cap (-\Psi_{z,\rho,r}^1(D_2))} |\nabla \tilde{u}(x - s(\Psi_{z,\rho,r}^1(x) + x))|^2 dx \right) |z|^2 k_\varepsilon(|z|) dz, \end{aligned}$$

where we used the fact that  $\Phi_{t\tilde{u}}(x, y, r, \rho) = -\Phi_{t\tilde{u}}(y, x, \rho, r)$  for the last equality, which implies  $\Psi_{x,r,\rho}^2(y) = -\Psi_{y,\rho,r}^1(x)$ . By (A.11) and the relation between  $\Psi_{y,\rho,r}^1$  and  $\Psi_{x,r,\rho}^2$  we see that

$$\|((D\Psi_{z,\rho,r}^1)^{-1} + \text{Id}_{\mathbb{R}^{n-1}})(x)\| \leq \frac{1}{4}, \quad \forall x \in D_2.$$

Since  $s \in (0, \frac{1}{2})$ , we deduce that the Jacobian determinant of  $\varphi : x \mapsto x - s((\Psi_{z,\rho,r}^1)^{-1}(x) + x)$  is universally bounded from below. Making the change of variable  $y = \varphi(x)$  in (A.13), it follows

$$\begin{aligned} & \int_{D_2} \int_{D_2} |\nabla \tilde{u}(x + s(y - x))|^2 |x - y|^2 k_\varepsilon(|\Phi_{tu}^\sim(x, y, r, \rho)|) dx dy \\ & \leq C \left( \int_{D_2} |\nabla \tilde{u}(x)|^2 dx \right) \left( \int_{\mathbb{R}^{n-1}} |z|^2 k_\varepsilon(|z|) dz \right). \end{aligned} \quad (\text{A.14})$$

We observe that the last factor is independent of  $\varepsilon$  and finite by assumption. Indeed:

$$\int_{\mathbb{R}^{n-1}} |z|^2 k_\varepsilon(|z|) dz = |\mathbb{S}^{n-2}| \int_0^{+\infty} r^n k(r) dr.$$

Gathering (A.12) and (A.14) yields (A.9), which concludes this step.

*Step 2.* We split the domain of integration in the left-hand side of (A.8) in  $\{x, y \in \mathbb{S}^{n-1} : |x - y| > \frac{1}{4}\}$  and  $\{x, y \in \mathbb{S}^{n-1} : |x - y| \leq \frac{1}{4}\}$ . We first treat the contribution of distant pairs  $(x, y)$ . Choosing  $\bar{t}_1$  small enough depending only on  $n$ , we have

$$|\Phi_{tu}(x, y, r, \rho)| \geq \frac{|x - y|}{2}, \quad \forall x, y \in \mathbb{S}^{n-1}, \forall r, \rho \in (0, 1),$$

so that,

$$\begin{aligned} & \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ \{|x-y| > \frac{1}{4}\}}} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \leq C \int_{\mathbb{S}^{n-1}} |u(x)|^2 \left( \int_{\substack{\mathbb{S}^{n-1} \\ \{|x-y| > \frac{1}{4}\}}} \int_0^1 \int_0^1 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_y^{n-1} \right) d\mathcal{H}_x^{n-1} \quad (\text{A.15}) \\ & \leq C \left( \int_{\mathbb{S}^{n-1}} |u|^2 d\mathcal{H}^{n-1} \right) \left( \sup_{r > \frac{1}{8}} k_\varepsilon(r) \right) \leq C \int_{\mathbb{S}^{n-1}} |u|^2 d\mathcal{H}^{n-1}, \end{aligned}$$

for some  $C = C(n, k)$ , provided that  $\bar{\varepsilon}_6 = \bar{\varepsilon}_6(n, k)$  is chosen small, where we used the fact that  $k(t) = O(t^{-(n+1)})$  at infinity for the last inequality. There remains to bound the integral over the domain  $\mathcal{M} := \{x, y \in \mathbb{S}^{n-1} : |x - y| < \frac{1}{4}\}$ . For this, we cover  $\mathcal{M}$  by a finite number  $N(n)$  of  $\mathcal{M}_i := \mathbb{S}_+^{n-1}(x_i) \times \mathbb{S}_+^{n-1}(x_i)$ , where for each  $i \in \{1, \dots, N\}$ ,  $x_i \in \mathbb{S}^{n-1}$  and  $\mathbb{S}_+^{n-1}(x_i)$  denotes the hemisphere with center  $x_i$ . Using the stereographic projection  $\Pi_i$  with respect to  $-x_i$ , we map  $\mathbb{S}_+^{n-1}(x_i)$  to  $D_2 \subseteq \mathbb{R}^{n-1}$ . By the change of variables  $\xi = \Pi_i(x)$ ,  $\zeta = \Pi_i(y)$ , setting  $\tilde{u}_i := u \circ \Pi_i^{-1}$ , we have  $\|\tilde{u}_i\|_{W^{1,\infty}(D_1)} \leq C(n)$  since  $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$  and  $\|\nabla_\tau u\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$ . Applying Step 1 with  $\tilde{u} = \tilde{u}_i$ , we obtain

$$\begin{aligned} & \iint_{\mathcal{M}_i} \int_0^1 \int_0^1 (u(x) - u(y))^2 k_\varepsilon(|\Phi_{tu}(x, y, r, \rho)|) dr d\rho d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \leq C \iint_{D_2 \times D_2} \int_0^1 \int_0^1 (\tilde{u}_i(\xi) - \tilde{u}_i(\zeta))^2 k_\varepsilon(|\Phi_{tu_i}^\sim(\xi, \zeta, r, \rho)|) dr d\rho d\xi d\zeta \\ & \leq C \int_{D_2} |\nabla \tilde{u}_i|^2 dx \leq C \int_{\mathbb{S}_+^{n-1}(x_i)} |\nabla_\tau u|^2 d\mathcal{H}^{n-1}, \end{aligned}$$

whenever  $t < \bar{t}_1$ , where  $\bar{t}_1 = \bar{t}_1(n)$  and  $C = C(n, k)$ . Summing these estimates over  $i \in \{1, \dots, N(n)\}$  and (A.15) conclude the proof.  $\square$

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