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Vorticity-velocity-pressure formulation for the Stokes problem

FRANÇOIS DUBOIS * □

May 29, 1999 †

Résumé

Nous présentons une nouvelle formulation variationnelle du problème de Stokes de la mécanique des fluides qui permet de prendre en compte des conditions aux limites très générales sur la vitesse, le tourbillon tangentiel ou la pression. Nous montrons que dans un cas particulier de conditions aux limites, cette formulation conduit à un problème mathématiquement bien posé.

Abstract

We present a new variational formulation of Stokes problem of fluid mechanics that allows to take into account very general boundary conditions for velocity, tangential vorticity or pressure. This formulation conducts to a well posed mathematical problem in a family of particular cases.

Key words : fluid mechanics, Stokes equation, mixed finite elements, inf-sup condition, vector field decomposition, boundary layer.

AMS (MOS) classification : 65N30.

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1) PHYSICAL AND NUMERICAL MOTIVATION.

• Let Ω be a bounded connected domain of \mathbb{R}^N ($N = 2$ or 3) with a regular boundary $\partial\Omega = \Gamma$. The Stokes problem modelizes the stationary equilibrium of an incompressible viscous fluid when the velocity u is sufficiently small in order to neglect the nonlinear terms (see e.g. Landau-Lifschitz [LL53]). From a mathematical point of view, this problem is the first step in order to consider the nonlinear Navier-Stokes equations of incompressible fluids, as proposed e.g. by Lions [Li69], Temam [Te77] or Girault-Raviart [GR86]. The Stokes problem can be classically written with primal formulation involving velocity u and pressure p :

$$(1.1) \quad -\nu \Delta u + \nabla p = f \quad \text{in } \Omega$$

$$(1.2) \quad \operatorname{div} u = 0 \quad \text{in } \Omega$$

$$(1.3) \quad u = 0 \quad \text{on } \Gamma$$

where $\nu > 0$ is the kinematic viscosity and f the datum of external forces.

• Our motivation comes from the numerical simulations in computational fluid dynamics. The Marker And Cell (MAC) method proposed by Harlow and Welch [HW65] (see also the C-grid of Arakawa [Ar66]) contains staggered grids relative to velocity and pressure and is still very popular when used in industrial computer softwares as Flow3d of Harper, Hirt and Sicilian [HHS83] or Phoenix developed by Patankar and Spalding [PS72]. This discretization is founded on the use of a cartesian mesh (Figure 1): velocity is defined with the help of fluxes on the faces of the mesh and pressure is supposed to be constant in each cell. We try to generalize these degrees of freedom to arbitrary meshes that respect the usual topological constraints associated with finite elements (see e.g. Ciarlet [Ci78]) and in particular to triangles (Figure 2) or tetrahedra. Some years ago, Nicolaidis [Ni89] has proposed a new interpretation of the MAC-Cgrid method with the help

of dual finite volumes for triangular meshes. An analysis of the MAC scheme as a numerical quadrature for finite elements has also been proposed by Girault and Lopez [GL96].

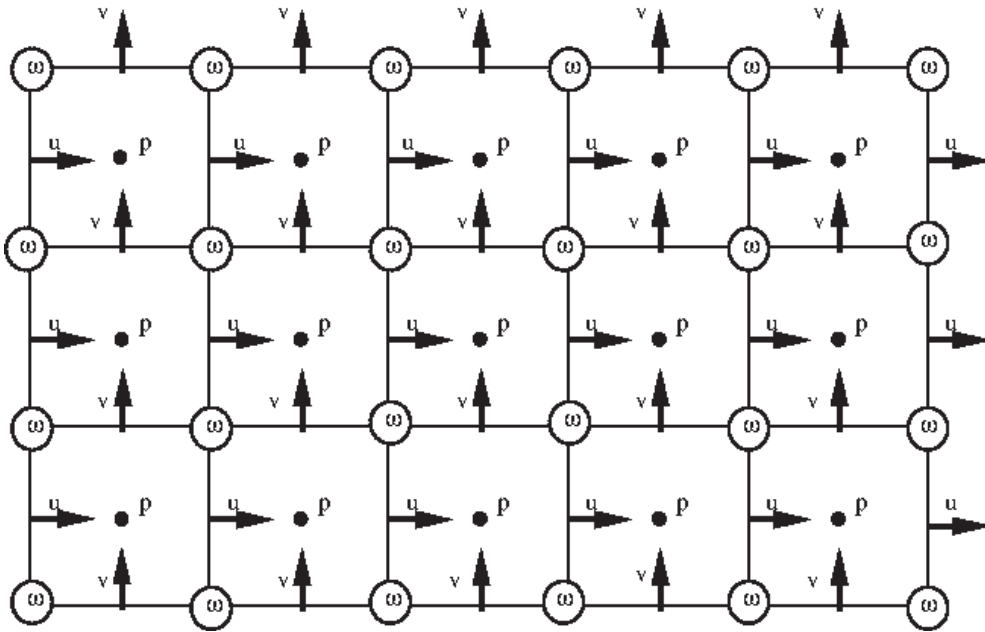


Figure 1. Marker And Cell discretization on a cartesian mesh.

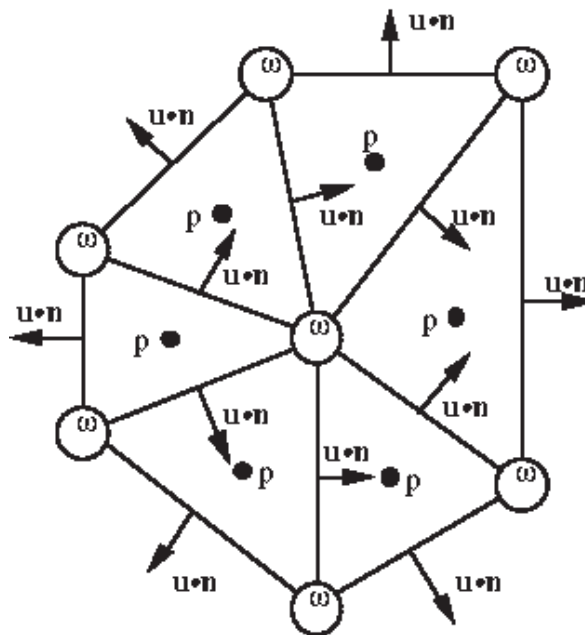


Figure 2. Degrees of freedom on a triangular mesh.

- From the point of view of numerical analysis, this MAC-Cgrid discretization can be seen as the search of an approximation of velocity field conforming in the $H(\text{div}, \Omega)$ Sobolev space with the help of the Raviart-Thomas [RT77] (and Nédélec [Né80] when $N=3$) finite element of degree one. On the other hand the approximation of pressure field in space $L^2(\Omega)$ is associated with discontinuous finite elements of degree zero. But this vision, also adopted by Nicolaidis, is a variational crime for the Stokes problem (1.1)-(1.3), where velocity classically belongs to finite dimensional linear spaces that are included to the Sobolev space $H^1(\Omega)$ (see e.g. Adams [Ad75]). Note also a completely different approach proposed by Ern, Guermond and Quartapelle [EGQ99] for Stokes problem with vorticity and velocity vector fields in \mathbb{R}^3 and associated with a philosophy of classical conforming continuous linear finite elements.

- In this paper we recall the variational formulation that we have previously proposed ([Du92], [Du95]) involving the three fields of vorticity, velocity and pressure. A particularity of this formulation is that boundary conditions can be considered in a very general way ; previous work of Beghe, Conca, Murat and Pironneau [BCMP87] and Girault [Gi88] appears as particular cases of what we obtain and finally boundary condition (1.3) can in our sense be seen as a mixed Dirichlet-Neumann boundary condition. We develop an abstract approach that makes in evidence the technical inf-sup hypotheses which are sufficient to satisfy in order to prove that with this triple formulation, the Stokes problem conducts to a mathematically well posed problem with continuous dependence on the solution from data. These conditions are completely nontrivial for a general tridimensional domain Ω that is bounded, connected, non simply connected and with a non-connected boundary. For proving it, we have been conducted to develop a new general representation theorem for vector fields that generalizes previous results summarized in Bendali, Dominguez and Gallic [BDG85].

2) VORTICITY-VELOCITY-PRESSURE FORMULATION

- The basic idea of our formulation is the same that the one used in stream-function-vorticity formulation (Glowinski [Gl73], Ciarlet-Raviart [CR74], Girault [Gi76] in \mathbb{R}^2 , Nédélec [Né82], Amara-Barucq-Duloué [ABD99] in \mathbb{R}^3) : a solenoidal vector field u (satisfying $\text{div } u = 0$) can a priori be represented as the curl of some stream function ψ : $u = \text{curl } \psi$. For the complete generality of the approach, we have here chosen to do not represent the solenoidal velocity field u with a stream function ψ for multiple reasons : first any representation of the type $u = \text{curl } \psi$ precludes flows with sinks and sources (Foias-Temam [FT78]) and moreover this representation is in the numerical practice restricted to two-dimensional domains

even if, following the idea introduced in [Né82], Roux [Ro84] has done first tentatives in three-dimensional domains with Nédélec's vectorial finite elements [Né80] conforming in space $H(\text{curl}, \Omega)$.

- Recall that if ϵ_{ijk} is the notation for the complete antisymmetric tensor of order 3, (ϵ_{ijk} is equal to 1 if (i, j, k) is a direct permutation of $(1, 2, 3)$, ϵ_{ijk} equal to -1 if the permutation is even and ϵ_{ijk} is null in the other cases), we have

$$\text{div } v = \sum_{j=1}^3 \frac{\partial v_j}{\partial x_j}, \quad v \in H(\text{div}, \Omega)$$

$$(\text{curl } \varphi)_i = \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} \frac{\partial \varphi_k}{\partial x_j}, \quad i = 1, 2, 3, \quad \varphi \in H(\text{curl}, \Omega), \quad \Omega \subset \mathbb{R}^3$$

$$\text{curl } \varphi = \frac{\partial \varphi_1}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_1}, \quad \varphi \in H(\text{curl}, \Omega), \quad \Omega \subset \mathbb{R}^2.$$

With Duvaut-Lions [DL72], Hilbert spaces $H(\text{div}, \Omega)$ and $H(\text{curl}, \Omega)$ are defined by

$$(2.1) \quad H(\text{div}, \Omega) = \{ v \in (L^2(\Omega))^N, \quad \text{div } v \in L^2(\Omega) \}$$

$$(2.2) \quad H(\text{curl}, \Omega) = \{ \varphi \in (L^2(\Omega))^N, \quad \text{curl } \varphi \in (L^2(\Omega))^{2N-3} \}$$

and the associated norms will be denoted by $\| \bullet \|_{\text{div}, \Omega}$ and $\| \bullet \|_{\text{curl}, \Omega}$ and are defined from the L^2 norm $\| \bullet \|_{0, \Omega}$ by the relations

$$(2.3) \quad \| v \|_{\text{div}, \Omega} = \left(\sum_{j=1}^N \| v_j \|_{0, \Omega}^2 + \| \text{div } v \|_{0, \Omega}^2 \right)^{1/2}$$

$$(2.4) \quad \| \varphi \|_{\text{curl}, \Omega} = \left(\sum_{j=1}^N \| \varphi_j \|_{0, \Omega}^2 + \sum_{k=1}^{2N-3} \| (\text{curl } \varphi)_k \|_{0, \Omega}^2 \right)^{1/2}.$$

- In the two-dimensional case, variables ω or φ are scalar valued functions and belong to Sobolev space $H^1(\Omega)$. This space can also be considered in $H(\text{curl}, \Omega')$ for some three-dimensional domain Ω' such that $\Omega \cap (\mathbb{R}^2 \times \{0\}) \subset \Omega'$, with $\omega = (0, 0, \omega_3)$ and $\varphi = (0, 0, \varphi_3)$. The adaptation of these results in the two-dimensional case is classical and we refer to Girault-Raviart [GR86] or our work with Salaün and Salmon [DSS97]. In the following, all notations and formulae are supposed to be correct when Ω is a three-dimensional domain.

- We introduce vorticity :

$$(2.5) \quad \omega = \operatorname{curl} u \quad \text{in } \Omega$$

where $\omega \in \mathbb{R}^{2N-3}$ and we re-write equation (1.1) under the form

$$(2.6) \quad \nu \operatorname{curl} \omega + \nabla p = f \quad \text{in } \Omega.$$

We multiply equation (2.5) (respectively equation (1.1), equation (1.2)) by some test vector function φ in space $H(\operatorname{curl}, \Omega)$ (respectively v in space $H(\operatorname{div}, \Omega)$, q in space $L^2(\Omega)$), we integrate by parts and we denote by (\bullet, \bullet) (respectively $\langle \bullet, \bullet \rangle$) the L^2 scalar product of functions in domain Ω (respectively on the boundary Γ). Then we obtain :

$$(2.7) \quad \omega \in H(\operatorname{curl}, \Omega), \quad u \in H(\operatorname{div}, \Omega), \quad p \in L^2(\Omega)$$

$$(2.8) \quad (\omega, \varphi) - (u, \operatorname{curl} \varphi) = \langle n \times u, \varphi \rangle, \quad \forall \varphi \in H(\operatorname{curl}, \Omega)$$

$$(2.9) \quad \nu (\operatorname{curl} \omega, v) - (p, \operatorname{div} v) = (f, v) - \langle p, v \bullet n \rangle, \quad \forall v \in H(\operatorname{div}, \Omega)$$

$$(2.10) \quad (\operatorname{div} u, q) = 0, \quad \forall q \in L^2(\Omega).$$

• We must give a mathematical sense to the boundary terms $\langle n \times u, \varphi \rangle$ and $\langle p, v \bullet n \rangle$. We first prove a for surfacic traces of $H(\operatorname{div}, \Omega)$ and $H(\operatorname{curl}, \Omega)$ functions (see also Bernardi [Be89], Levillain [Le91] and Amrouche et al [ABDG98]).

Definition 1. Scalar and vectorial functions on the boundary.

Let Γ_1, Γ_2 be a partition of boundary Γ :

$$(2.11) \quad \Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \quad \text{with} \quad \Gamma_1 \cap \Gamma_2 = \emptyset$$

and γ be the trace operator from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$ or from $(H^1(\Omega))^N$ onto $(H^{1/2}(\Gamma))^N$ (see Lions-Magenes [LM68]). We denote by $H_{00}^{1/2}(\Gamma_1)$ and $TH_{00}^{1/2}(\Gamma_1)$ the following spaces of scalar and tangential vector functions that are null on the component Γ_2 of the boundary :

$$(2.12) \quad H_{00}^{1/2}(\Gamma_1) = \{ \gamma \xi, \xi \in H^1(\Omega), \gamma \xi = 0 \text{ on } \Gamma_2 \}$$

$$(2.13) \quad TH_{00}^{1/2}(\Gamma_1) = \{ n \times \gamma \varphi \times n, \varphi \in (H^1(\Omega))^N, \gamma \varphi \times n = 0 \text{ on } \Gamma_2 \}$$

where n is the normal external to the boundary $\partial\Omega$.

Proposition 1. Trace theorem for functions of $H(\operatorname{div}, \Omega)$ and $H(\operatorname{curl}, \Omega)$.

There exists two continuous mappings $\widetilde{\gamma}_\bullet$ and $\widetilde{\gamma}_\times$ satisfying the following conditions

$$(2.14) \quad \widetilde{\gamma}_\bullet : H(\operatorname{div}, \Omega) \rightarrow (H_{00}^{1/2}(\Gamma_1))', \quad \widetilde{\gamma}_\bullet v = v \bullet n \text{ when } v \text{ is regular}$$

$$(2.15) \quad \widetilde{\gamma}_\times : H(\operatorname{curl}, \Omega) \rightarrow (TH_{00}^{1/2}(\Gamma_1))', \quad \widetilde{\gamma}_\times \varphi = \varphi \times n \text{ when } \varphi \text{ is regular}$$

where $(H_{00}^{1/2}(\Gamma_1))'$ (respectively $(TH_{00}^{1/2}(\Gamma_1))'$) is the linear space of continuous linear forms acting on $H_{00}^{1/2}(\Gamma_1)$ (respectively on $TH_{00}^{1/2}(\Gamma_1)$).

Proof of Proposition 1.

• Let $v \in H(\operatorname{div}, \Omega)$ and $\xi \in \mathcal{D}(\bar{\Omega})$ be a regular field null on Γ_2 . We define $\tilde{\gamma} \bullet v$ acting on ξ by the relation

$$(2.16) \quad \langle \tilde{\gamma} \bullet v, \xi \rangle = (v, \nabla \xi) + (\operatorname{div} v, \xi)$$

and we have the following continuity property :

$$(2.17) \quad |\langle \tilde{\gamma} \bullet v, \xi \rangle| \leq 2 \|v\|_{\operatorname{div}, \Omega} \|\xi\|_{1, \Omega} .$$

By Green formula we have clearly $\tilde{\gamma} \bullet v = v \bullet n$ when v is regular. On the other hand, $\langle \tilde{\gamma} \bullet v, \xi \rangle$ depends only on surfacic values $\gamma \xi$ of function ξ since the right hand side of relation (2.16) is null when ξ belongs to space $\mathcal{D}(\Omega)$. So we deduce

$$|\langle \tilde{\gamma} \bullet v, \xi \rangle| \leq 2 \|v\|_{\operatorname{div}, \Omega} \inf_{\zeta \in H^1(\Omega), \gamma \zeta = \gamma \xi} \|\zeta\|_{1, \Omega} .$$

Using lifting theorem of J.L. Lions (see [LM68]), we have

$$\inf_{\zeta \in H^1(\Omega), \gamma \zeta = \gamma \xi} \|\zeta\|_{1, \Omega} \leq C \|\gamma \xi\|_{1/2, \Gamma}$$

for some constant C independent on ξ . We deduce that $\langle \tilde{\gamma} \bullet v, \eta \rangle$ is in fact defined for $\eta \in H_{00}^{1/2}(\Gamma_1)$ and from previous relations we have the estimate

$$(2.18) \quad |\langle \tilde{\gamma} \bullet v, \eta \rangle| \leq C \|v\|_{\operatorname{div}, \Omega} \|\eta\|_{1/2, \Gamma} .$$

The first step (2.14) of the proposition is established.

• The proof is analogous for surfacic traces in $H(\operatorname{curl}, \Omega)$. Let φ be given in $H(\operatorname{curl}, \Omega)$ and $\xi \in \mathcal{D}(\bar{\Omega})^N$ be a regular vector field such that $\xi \times n = 0$ on Γ_2 . We set

$$(2.19) \quad \langle \tilde{\gamma} \times \varphi, \xi \rangle = (\varphi, \operatorname{curl} \xi) - (\operatorname{curl} \varphi, \xi)$$

and we have by Cauchy-Schwarz inequality

$$(2.20) \quad |\langle \tilde{\gamma} \times \varphi, \xi \rangle| \leq 2 \|\varphi\|_{\operatorname{curl}, \Omega} \|\xi\|_{1, \Omega} .$$

The Green formula for regular vector fields in \mathbb{R}^3

$$(2.21) \quad (\varphi, \operatorname{curl} \xi) = (\operatorname{curl} \varphi, \xi) + \langle \varphi \times n, \xi \rangle$$

shows that $\tilde{\gamma} \times \varphi = \varphi \times n$ when φ is regular. Therefore $\langle \tilde{\gamma} \times \varphi, \xi \rangle$ depends only on tangential surfacic values $n \times \xi \times n$ of vector field ξ since $\langle \tilde{\gamma} \times \varphi, \xi \rangle = 0$ if ξ is a vector field such that $\xi \times n$ is null on $\partial\Omega$. We deduce

$$|\langle \tilde{\gamma} \times \varphi, \xi \rangle| \leq 2 \|\varphi\|_{\operatorname{curl}, \Omega} \inf_{\zeta \in (H^1(\Omega))^N, \gamma \zeta = \gamma \xi} \|\zeta\|_{1, \Omega} .$$

Using again the Lions lifting theorem, we get

$$\zeta \in (H^1(\Omega))^N, \gamma\zeta = \gamma\xi \quad \|\zeta\|_{1,\Omega} \leq C \|\gamma\xi\|_{1/2,\Gamma}$$

where C is a constant independent of ξ . By density of traces of $D(\overline{\Omega})^N$ in $TH_{00}^{1/2}(\Gamma_1)$, the previous inequalities remain valid for $\varphi \in H(\text{curl}, \Omega)$ and $\eta \in TH_{00}^{1/2}(\Gamma_1)$. We deduce

$$(2.22) \quad | \langle \widetilde{\gamma} \times \varphi, n \times \eta \times n \rangle | \leq C \|\varphi\|_{\text{curl}, \Omega} \|n \times \eta \times n\|_{1/2, \Gamma} .$$

Then $\widetilde{\gamma} \times \varphi$ is a continuously linear form acting on $TH_{00}^{1/2}(\Gamma_1)$. This establishes the second part (2.15) of Proposition 1. \square

• We give now a mathematical sense to the boundary terms $\langle p, v \bullet n \rangle$ and $\langle n \times u, \varphi \rangle$. We suppose for a while that $v \bullet n$ is null on some part Γ_m of the boundary and that $\varphi \times n$ is also null on an other subset Γ_θ of the boundary *i.e.*

$$(2.23) \quad \widetilde{\gamma} \bullet v = 0 \quad \text{in } (H_{00}^{1/2}(\Gamma_m))'$$

$$(2.24) \quad \widetilde{\gamma} \times \varphi = 0 \quad \text{in } (TH_{00}^{1/2}(\Gamma_\theta))' .$$

It is completely natural to introduce the complements Γ_p and Γ_t of Γ_m and Γ_θ respectively :

$$(2.25) \quad \Gamma = \overline{\Gamma_m} \cup \overline{\Gamma_p} \quad \text{with } \Gamma_m \cap \Gamma_p = \emptyset$$

$$(2.26) \quad \Gamma = \overline{\Gamma_\theta} \cup \overline{\Gamma_t} \quad \text{with } \Gamma_\theta \cap \Gamma_t = \emptyset .$$

Consider now two fields Π_0 and σ_0 in $H_{00}^{1/2}(\Gamma_p)$ and $TH_{00}^{1/2}(\Gamma_t)$ respectively. The boundary conditions

$$(2.27) \quad p = \Pi_0 \quad \text{in space } H_{00}^{1/2}(\Gamma_p)$$

$$(2.28) \quad n \times u \times n = \sigma_0 \quad \text{in space } TH_{00}^{1/2}(\Gamma_t)$$

can be written on the form

$$(2.29) \quad \langle p, v \bullet n \rangle = \langle \Pi_0, \widetilde{\gamma} \bullet v \rangle_{H_{00}^{1/2}(\Gamma_p), (H_{00}^{1/2}(\Gamma_p))'}$$

$$(2.30) \quad \langle n \times u, \varphi \rangle = \langle \sigma_0, \widetilde{\gamma} \times \varphi \rangle_{TH_{00}^{1/2}(\Gamma_t), (TH_{00}^{1/2}(\Gamma_t))'}$$

and the boundary conditions (2.27)-(2.28) are included in the variational formulation. For this reason, we will speak in the following of two independent Neumann boundary conditions on pressure and tangential velocity. In a similar way, two independent Dirichlet boundary conditions for tangential component of vorticity and normal component of velocity naturally appears :

$$(2.31) \quad u \bullet n = g_0 \quad \text{on } \Gamma_m$$

$$(2.32) \quad \omega \times n = \theta_0 \quad \text{on } \Gamma_\theta$$

where g_0 and θ_0 are a priori given in spaces $(H_{00}^{1/2}(\Gamma_m))'$ and $(TH_{00}^{1/2}(\Gamma_\theta))'$ respectively.

• If we restrict to homogeneous Dirichlet boundary conditions, we can introduce appropriate Sobolev spaces :

$$(2.33) \quad W = \left\{ \begin{array}{l} \varphi \in H(\text{curl}, \Omega), \quad \varphi \times n = 0 \text{ on } \Gamma_\theta, \\ \text{i.e. } \widetilde{\gamma} \times \varphi = 0 \text{ in } (TH_{00}^{1/2}(\Gamma_\theta))' \end{array} \right\}$$

$$(2.34) \quad X = \{ v \in H(\text{div}, \Omega), v \bullet n = 0 \text{ on } \Gamma_m, \text{ i.e. } \widetilde{\gamma} \bullet v = 0 \text{ in } (H_{00}^{1/2}(\Gamma_m))' \}$$

$$(2.35) \quad Y = \begin{cases} L^2(\Omega) & \text{if } \text{meas}(\Gamma_p) \neq 0 \\ \{ q \in L^2(\Omega), (q, 1) = 0 \} & \text{if } \text{meas}(\Gamma_p) = 0. \end{cases}$$

We can re-write the Stokes problem (2.7)-(2.10) with non-homogeneous Neumann boundary conditions (2.27)-(2.28) and homogeneous Dirichlet boundary conditions (2.31)-(2.32) (with $g_0 = 0$ and $\theta_0 = 0$) under the following variational form :

$$(2.36) \quad \omega \in W, \quad u \in X, \quad p \in Y$$

$$(2.37) \quad (\omega, \varphi) - (u, \text{curl } \varphi) = \langle \sigma_0, \widetilde{\gamma} \times \varphi \rangle, \quad \forall \varphi \in W$$

$$(2.38) \quad \nu(\text{curl } \omega, v) - (p, \text{div } v) = (f, v) - \langle \Pi_0, \widetilde{\gamma} \bullet v \rangle, \quad \forall v \in X$$

$$(2.39) \quad (\text{div } u, q) = 0, \quad \forall q \in Y.$$

3) AN ABSTRACT RESULT

• In the previous section, we have shown that with the vorticity-velocity-pressure formulation, the Stokes problem takes the form (2.36)-(2.39). This formulation is a "triple mixed" variational formulation and classical results of Babuška [Ba71] or Brezzi [Br74] can not be applied in a straightforward manner. For this reason we have first developed [Du91] a general approach that makes in evidence appropriate inf-sup hypotheses in order to ensure that the Stokes problem is well posed. We present in this section a generalization of this result. Following a remark of Raviart [Ra91], it seems possible to reduce this approach to classical ones with an appropriate choice of product spaces but with a different set of hypotheses.

Theorem 1. Triple mixed variational formulation.

Let Y and Z be two real Hilbert spaces equipped with their scalar product $(\bullet, \bullet)_Y$ and $(\bullet, \bullet)_Z$; the associated norm is respectively denoted by $\| \bullet \|_Y$ and $\| \bullet \|_Z$:

$$(3.1) \quad (q, q)_Y \equiv (q, q)_Y = \| q \|_Y^2, \quad q \in Y$$

$$(3.2) \quad (z, z)_Z \equiv (z, z)_Z = \| z \|_Z^2, \quad z \in Z.$$

• Let $W \subset Z$ and $X \subset Z$ be two subspaces of Z that are moreover equipped with a structure of Hilbert space associated to scalar products $(\bullet, \bullet)_W$ and $(\bullet, \bullet)_X$ and to norms $\|\bullet\|_W$ and $\|\bullet\|_X$ satisfying

$$(3.3) \quad \|\varphi\|_W^2 \equiv (\varphi, \varphi)_W \geq (\varphi, \varphi)_Z, \quad \forall \varphi \in W$$

$$(3.4) \quad \|v\|_X^2 \equiv (v, v)_X \geq (v, v)_Z, \quad \forall v \in X.$$

• We suppose that there exists two continuous mappings $R : W \rightarrow Z$ and $D : X \rightarrow Y$; for each kernel $\ker R$ and $\ker D$ we define two different orthogonal spaces $(\ker R)^\perp$, $(\ker R)^\sqcup$, $(\ker D)^\perp$, $(\ker D)^\sqcup$, and the orthogonal of the range $(\text{Im } R)^\sqcup$ by the following relations

$$(3.5) \quad (\ker R)^\perp = \{\varphi \in W, (\varphi, w)_W = 0, \forall w \in \ker R\}$$

$$(3.6) \quad (\ker R)^\sqcup = \{z \in Z, (z, w)_Z = 0, \forall w \in \ker R\}$$

$$(3.7) \quad (\ker D)^\perp = \{v \in X, (v, x)_X = 0, \forall x \in \ker D\}$$

$$(3.8) \quad (\ker D)^\sqcup = \{z \in Z, (z, x)_Z = 0, \forall x \in \ker D\}$$

$$(3.9) \quad (\text{Im } R)^\sqcup = \{z \in Z, (z, R\varphi)_Z = 0, \forall \varphi \in W\}.$$

• We make the following hypotheses :

$$(3.10) \quad (\ker D)^\sqcup \subset (\text{Im } R)^\sqcup$$

$$(3.11) \quad \exists a > 0, \inf_{q \in Y, q \neq 0} \sup_{v \in X, v \neq 0} \frac{(q, Dv)_Y}{\|q\|_Y \|v\|_X} \geq a$$

$$(3.12) \quad \exists b > 0, \inf_{v \in \ker D, v \neq 0} \sup_{\varphi \in W, \varphi \neq 0} \frac{(v, R\varphi)_Z}{\|v\|_X \|\varphi\|_W} \geq b$$

$$(3.13) \quad \exists d > 0, \forall \varphi \in \ker R, (\varphi, \varphi)_Z \geq d \|\varphi\|_W^2.$$

• Then for each $\sigma = (\lambda, \mu, \nu) \in W' \times X' \times Y'$, the problem

$$(3.14) \quad \omega \in W, \quad u \in X, \quad p \in Y$$

$$(3.15) \quad (\omega, \varphi)_Z + (u, R\varphi)_Z = \langle \lambda, \varphi \rangle, \quad \forall \varphi \in W$$

$$(3.16) \quad (R\omega, v)_Z + (p, Dv)_Y = \langle \mu, v \rangle, \quad \forall v \in X$$

$$(3.17) \quad (Du, q)_Y = \langle \nu, q \rangle, \quad \forall q \in Y$$

has a unique solution $(\omega(\sigma), u(\sigma), p(\sigma)) \in W \times X \times Y$ which continuously depends on datum σ :

$$(3.18) \quad \begin{cases} \exists C > 0, \forall \sigma \in W' \times X' \times Y', \\ \|\omega(\sigma)\|_W + \|u(\sigma)\|_X + \|p(\sigma)\|_Y \leq C \|\sigma\|_{W' \times X' \times Y'}. \end{cases}$$

- We first introduce canonical injections $i : W \rightarrow Z$ and $j : X \rightarrow Z$

$$(3.19) \quad W \ni \varphi \mapsto i\varphi = \varphi \in Z$$

$$(3.20) \quad X \ni v \mapsto jv = v \in Z$$

that are continuous according to inequalities in (3.3) and (3.4). Second, Riesz isomorphisms $k : Z \rightarrow Z'$ and $l : Y \rightarrow Y'$ are defined according to

$$(3.21) \quad Z \ni z \mapsto kz = (Z \ni w \mapsto \langle kz, w \rangle_{Z',Z} = (z, w)_Z \in \mathbb{R}) \in Z'$$

$$(3.22) \quad Y \ni q \mapsto lq = (Y \ni y \mapsto \langle lq, y \rangle_{Y',Y} = (q, y)_Y \in \mathbb{R}) \in Y'$$

and are isometries from Z onto Z' and from Y onto Y' respectively. Moreover, we introduce dual operators $D' : Y' \rightarrow X'$ and $R' : Z' \rightarrow W'$ of D and R respectively by the classical relations

$$(3.23) \quad \langle D'\zeta, x \rangle_{X',X} = \langle \zeta, Dx \rangle_{Y',Y}, \quad \forall \zeta \in Y' \text{ and } x \in X$$

$$(3.24) \quad \langle R'\eta, \varphi \rangle_{W',W} = \langle \eta, R\varphi \rangle_{Z',Z}, \quad \forall \eta \in Z' \text{ and } \varphi \in W.$$

The scalar products in (3.15)-(3.17) can be rewritten in term of previous operators ; we have :

$$(3.25) \quad (\omega, \varphi)_Z = \langle i'k\omega, \varphi \rangle_{W',W}, \quad \forall \omega \in W, \forall \varphi \in W$$

$$(3.26) \quad (u, R\varphi)_Z = \langle R'kj u, \varphi \rangle_{W',W}, \quad \forall u \in X, \forall \varphi \in W$$

$$(3.27) \quad (R\omega, v)_Z = \langle j'kR\omega, v \rangle_{X',X}, \quad \forall \omega \in W, \forall v \in X$$

$$(3.28) \quad (p, Dv)_Y = \langle D'lp, v \rangle_{X',X}, \quad \forall p \in Y, \forall v \in X$$

$$(3.29) \quad (Du, q)_Y = \langle lDu, q \rangle_{Y',Y}, \quad \forall u \in X, \forall q \in Y$$

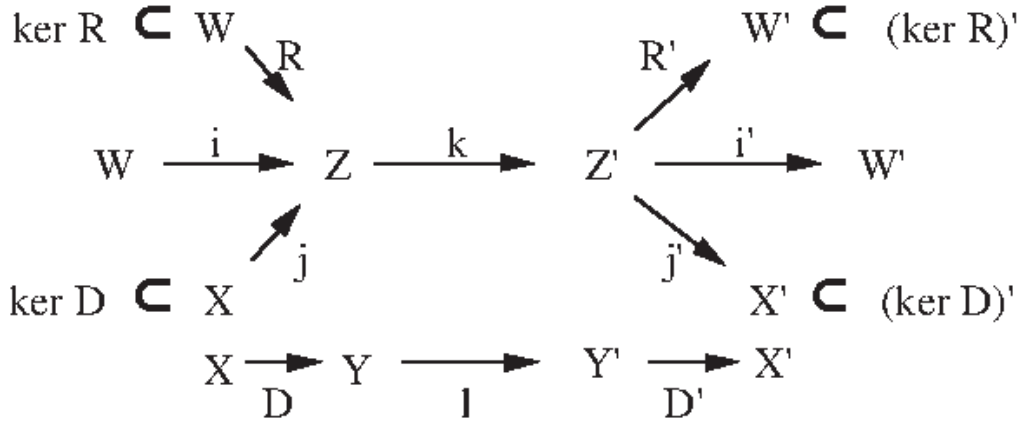


Figure 3. Relations between Hilbert spaces for abstract version of Stokes problem.

Proposition 2. A classical result.

A global map of all the relations evocated above is proposed on Figure 3. The proof of Theorem 1 is based on a classical result derived by Girault and Raviart (see e.g. [GR86]) that we recall.

Let T and M be two Hilbert spaces and $T \times M \ni (t, \mu) \mapsto b(t, \mu) \in \mathbb{R}$ be a continuous bilinear form acting on the product space $T \times M$. We define left kernel V of bilinear form $b(\bullet, \bullet)$ and its polar space V° by the relations

$$(3.30) \quad V = \{t \in T, b(t, \mu) = 0, \forall \mu \in M\}$$

$$(3.31) \quad V^\circ = \{\theta \in T', \langle \theta, t \rangle_{T', T} = 0, \forall t \in V\}$$

and linear operators $B : T \rightarrow M'$ and $B^\circ : M \rightarrow T'$ according to

$$(3.32) \quad T \ni t \mapsto Bt = (M \ni \mu \mapsto \langle Bt, \mu \rangle_{M', M} = b(t, \mu) \in \mathbb{R}) \in M'$$

$$(3.33) \quad M \ni \mu \mapsto B^\circ \mu = (T \ni t \mapsto \langle B^\circ \mu, t \rangle_{T', T} = b(t, \mu) \in \mathbb{R}) \in T'.$$

- The following three conditions are equivalent

$$(3.34) \quad \exists \beta > 0, \quad \inf_{\mu \in M, \mu \neq 0} \sup_{t \in T, t \neq 0} \frac{b(t, \mu)}{\|t\|_T \|\mu\|_M} \geq \beta$$

$$(3.35) \quad B^\circ \in \text{Isom}(M, V^\circ)$$

$$(3.36) \quad B \in \text{Isom}(V^\perp, M')$$

where V^\perp is the orthogonal of kernel V in Hilbert space T .

Proposition 3. Interpretation of hypotheses (3.10)-(3.12).

Under the hypotheses (3.10), (3.11) and (3.12) of Theorem 1, we have the following isomorphisms

$$(3.37) \quad lD \in \text{Isom}((\ker D)^\perp, Y')$$

$$(3.38) \quad D'l \in \text{Isom}(Y, (\ker D)^\circ)$$

$$(3.39) \quad j'kR \in \text{Isom}((\ker R)^\perp, (\ker D)')$$

$$(3.40) \quad R'kj \in \text{Isom}(\ker D, (\ker R)^\circ)$$

where $(\ker D)^\circ \subset Y'$ and $(\ker R)^\circ \subset W'$ are polar spaces of kernels $\ker D$ and $\ker R$ respectively.

Proof of Proposition 3.

- We traduct first the result given by Proposition 1 with the notations proposed for hypothesis (3.11). Due to relations (3.28) and (3.29), we have

$$(3.41) \quad M = Y, \quad T = X, \quad B = lD, \quad B^\circ = D'l \quad \text{and} \quad V = \ker D$$

due to the fact that l is an isomorphism. Then (3.36) implies (3.37) and (3.35) implies (3.38).

• The same mechanism occurs for second inf-sup hypothesis (3.12). We have

$$(3.42) \quad M = \ker D, \quad T = W, \quad B = j'kR, \quad B^\circ = R'kj$$

due to relations (3.26) and (3.27). The evaluation of kernel V can be explicated in this particular case as follows :

$$(3.43) \quad V = \{ \varphi \in W, \forall v \in \ker D, (v, R\varphi)_Z = 0 \}$$

and if φ belongs to V , then $R\varphi$ belongs to $(\ker D)^\perp$ introduced in (3.8). According to hypothesis (3.10) the space $(\ker D)^\perp$ is included in $(\text{Im } R)^\perp$. Then

$$(3.44) \quad R\varphi \in (\text{Im } R) \cap (\text{Im } R)^\perp.$$

Moreover, we have the following classical inclusion :

$$(\text{Im } R) \cap (\text{Im } R)^\perp \subset \text{adh}_Z(\text{Im } R) \cap (\text{adh}_Z(\text{Im } R))^\perp$$

where adh_Z denotes the adherence relatively to topology in space Z . Then $R\varphi$ is null. In consequence,

$$(3.45) \quad V = \ker R$$

and (3.39) and (3.40) are respectively consequences of (3.36) and (3.35). \square

Proof of Theorem 1.

• We introduce operator $A : W \times X \times Y \rightarrow W' \times X' \times Y'$ by the following matrix of operators :

$$(3.46) \quad A = \begin{pmatrix} i'ki & R'kj & 0 \\ j'kR & 0 & D'l \\ 0 & lD & 0 \end{pmatrix}$$

and equations (3.15)-(3.17) can be explicated in the product space $W' \times X' \times Y'$ as follows

$$(3.47) \quad i'ki\omega + R'kj u = \lambda$$

$$(3.48) \quad j'kR\omega + D'l p = \mu$$

$$(3.49) \quad lD u = \nu$$

where $(\omega, u, p) \in W \times X \times Y$ is the unknown and $\sigma = (\lambda, \mu, \nu) \in W' \times X' \times Y'$ is the datum of the problem.

• We first prove now that A is an injective operator. Consider in consequence the particular case $\lambda = 0 \in W', \mu = 0 \in X', \nu = 0 \in Y'$ as right hand sides of equations (3.47)-(3.49). We first take as a test function $v \in \ker D$ inside

equation (3.48) with $\mu = 0$. Then $\langle D'lp, v \rangle = (p, Dv)_Y = 0$ and equation (3.48) implies that $j'kR\omega = 0$ in $(\ker D)'$. Taken into account the isomorphism (3.39), we deduce that $\omega \in \ker R$ and

$$(3.50) \quad R\omega = 0.$$

We report this last property inside equation (3.48). Then isomorphism (3.38) shows that $p = 0$ in Y .

We test equation (3.47) against $\varphi \in \ker R$. Then $\langle R'kj u, \varphi \rangle = (u, R\varphi)_Z$ is null and the particular choice $\varphi = \omega$ (which is allowed due to relation (3.50)) states that $\langle i'ki\omega, \omega \rangle = \|\omega\|_Z^2 = 0$. So $\omega = 0$.

Now equation (3.49) and isomorphism (3.37) show clearly that $u \in \ker D$. Moreover $R'kj u = 0$ due to equation (3.47) considered with $\lambda = 0$. The isomorphism (3.40) shows that $u = 0$. The operator A is injective.

- We establish now that A is a surjective operator. We first look to equation (3.48). The right hand side is equal to $\mu \in X' \subset (\ker D)'$. Then when we apply the left hand side for $v \in \ker D$, we have clearly $\langle D'lp, v \rangle = (p, Dv)_Y = 0$. Then $\mu - D'lp$ belongs to $(\ker D)'$ and does not depend on $p \in Y$. Consequently, the isomorphism (3.39) shows the existence of $\omega_1 \in (\ker R)^\perp$ independent of $p \in Y$ such that

$$(3.51) \quad j'kR\omega_1 = \mu - D'kp \quad \text{in } (\ker D)'.$$

We remark that for each $\omega_2 \in \ker R$, the research of vorticity ω under the form

$$(3.52) \quad \omega = \omega_1 + \omega_2, \quad \omega_1 \in (\ker R)^\perp, \omega_2 \in \ker R$$

satisfies $j'kR\omega = j'kR\omega_1$. Then modulo this (unknown) choice of $\omega_2 \in \ker R$, the equation (3.48) can now be written under the form

$$(3.53) \quad D'lp = \mu - j'kR\omega \equiv \xi.$$

The right hand side does not depend on ω_2 then is completely known and is null on subspace $\ker D$ by construction of ω_2 *i.e.* $\xi \in (\ker D)^\circ$. The isomorphism (3.38) says that there exists some (unique) $p \in Y$ such that equation (3.53) is satisfied. At this step of the proof, equation (3.48) has been entirely solved ; unknowns $p \in Y$ and $\omega_1 \in (\ker R)^\perp$ are fixed and the other component ω_2 of representation (3.53) remains completely free.

The equation (3.49) and the isomorphism (3.37) give a (unique) u_1 lying in $(\ker D)^\perp$ such that $lDu_1 = \nu$ and as previously, we remark that a decomposition of vector field u under the form

$$(3.54) \quad u = u_1 + u_2, \quad u_1 \in (\ker D)^\perp, u_2 \in \ker D$$

is always solution of equation (3.49).

We report the decompositions (3.52) and (3.54) in equation (3.47) that becomes

$$(3.55) \quad i'ki\omega_2 + R'kj u_2 = \lambda - i'ki\omega_1 - R'kj u_1 \equiv \zeta$$

and $\zeta \in W' \subset (\ker R)'$ is completely known. When we test equation (3.55) over $\varphi \in \ker R$, then the second term of left hand side $\langle R'kj u_2, \varphi \rangle_{W',W} = (u_2, R\varphi)_Z$ is null. Then for each choice of $u_2 \in \ker D$, linear form $\eta \equiv \zeta - R'kj u_2$ remains equal to ζ in dual space $(\ker R)'$ and equation (3.55) takes the particular form

$$(3.56) \quad \begin{cases} \omega_2 \in \ker R, \\ (\omega_2, \varphi)_Z = \langle \eta, \varphi \rangle, \quad \forall \varphi \in \ker R. \end{cases}$$

The hypothesis (3.13) show that the Z -scalar product is d-elliptic on subspace $\ker R$. Lax-Milgram lemma [LM54] allows to conclude that equation (3.56) admits a (unique) solution $\omega_2 \in \ker R$. We report this information inside equation (3.55) that can now be written as

$$(3.57) \quad R'kj u_2 = \zeta - i'ki\omega_2$$

and $(\zeta - i'ki\omega_2) \in (\ker R)^\circ$ by construction of variable ω_2 . Then isomorphism (3.40) shows that a (unique) $u_2 \in \ker D$ is solution of problem (3.57). This step achieves the proof of surjectivity of operator A .

- The proof of Theorem 1 is a direct consequence of the fact that operator A is one to one and of Banach isomorphism theorem. \square

4) GENERAL REPRESENTATION OF VECTOR FIELDS.

- Let Ω be a connected bounded domain of \mathbb{R}^N ($N = 2$ or 3) with a smooth boundary $\partial\Omega$. We suppose that $\partial\Omega$ is of class at least \mathcal{C}^2 in order to manage continuous curvatures on $\partial\Omega$, see e.g. Choquet-Bruhat [Ch68]. We suppose that $\partial\Omega$ is decomposed into a partition composed by two subsets Γ_1 and Γ_2 such that

$$(4.1) \quad \partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}, \quad \Gamma_1 \cap \Gamma_2 = \emptyset$$

and we suppose moreover that the intersection

$$(4.2) \quad \Upsilon = \overline{\Gamma_1} \cap \overline{\Gamma_2}$$

is a finite set of points if $N = 2$ or a regular curve pictured on the boundary $\partial\Omega$ if $N = 3$. We denote by $H_0^1(\Omega; \Gamma_1)$ the subspace of Sobolev space $H^1(\Omega)$ composed by scalar functions whose trace is null on Γ_1 or with a null integral if $\text{meas}(\Gamma_1) = 0$:

$$(4.3) \quad H_0^1(\Omega; \Gamma_1) = \begin{cases} \{\zeta \in H^1(\Omega), \gamma\zeta = 0 \text{ on } \Gamma_1\} & \text{if } \text{meas}(\Gamma_1) > 0 \\ \{\zeta \in H^1(\Omega), (\zeta, 1) = 0\} & \text{if } \text{meas}(\Gamma_1) = 0 \end{cases}$$

and $H_0^1(\Omega; \Gamma_1, \Gamma_2)$ is by definition the subspace of $(H^1(\Omega))^N$ composed by vector fields that are tangent on Γ_1 and parallel to the normal direction n on Γ_2 :

$$(4.4) \quad H_0^1(\Omega; \Gamma_1, \Gamma_2) = \{\varphi \in (H^1(\Omega))^N, \tilde{\gamma} \bullet \varphi = 0 \text{ on } \Gamma_1, \tilde{\gamma} \times \varphi = 0 \text{ on } \Gamma_2\}.$$

Proposition 4. Density.

Let $\mathcal{C}^2(\overline{\Omega})$ be the space of regular scalar functions two times continuously derivable on the adherence of domain Ω . Then $\mathcal{C}^2(\overline{\Omega}) \cap H_0^1(\Omega; \Gamma_1)$ is dense in space $H_0^1(\Omega; \Gamma_1)$ and $(\mathcal{C}^2(\overline{\Omega}))^N \cap H_0^1(\Omega; \Gamma_1, \Gamma_2)$ is dense in space $H_0^1(\Omega; \Gamma_1, \Gamma_2)$.

Proof of Proposition 4.

• The first result is clear if $\text{meas}(\Gamma_1) = 0$. If it is not the case, let $\psi \in H_0^1(\Omega; \Gamma_1)$ be a scalar function orthogonal to space $\mathcal{C}^2(\overline{\Omega}) \cap H_0^1(\Omega; \Gamma_1)$ for the H^1 scalar product, *i.e.*

$$(4.5) \quad \begin{cases} \psi \in H_0^1(\Omega; \Gamma_1) \\ (\psi, \varphi) + (\nabla\psi, \nabla\varphi) = 0, \quad \forall \varphi \in \mathcal{C}^2(\overline{\Omega}) \cap H_0^1(\Omega; \Gamma_1). \end{cases}$$

If we restrict to functions φ that belong to space $\mathcal{D}(\Omega)$, we deduce from second line of (4.5) that

$$(4.6) \quad \psi - \Delta\psi = 0 \quad \text{in the sense of distributions}$$

and function $\Delta\psi$ belongs to space $L^2(\Omega)$. Then we multiply relation (4.6) by an arbitrary function $\zeta \in H_0^1(\Omega; \Gamma_1)$ and after integrating by parts, we deduce :

$$(4.7) \quad (\psi, \zeta) + (\nabla\psi, \nabla\zeta) - \langle \frac{\partial\psi}{\partial n}, \gamma\zeta \rangle = 0, \quad \forall \zeta \in H_0^1(\Omega; \Gamma_1).$$

Consider now a given boundary function $\mu \in H_{00}^{1/2}(\Gamma_2)$. Then the function $\tilde{\mu}$ defined by extension of μ to Γ_1 by nullity, *i.e.*

$$(4.8) \quad \tilde{\mu} = \begin{cases} 0 & \text{on } \Gamma_1 \\ \mu & \text{on } \Gamma_2 \end{cases}$$

belongs to space $H^{1/2}(\Gamma)$ and by classical density result (see Lions-Magenes [LM68]) there exists a sequence $\mu_k \in \mathcal{D}(\Gamma_2)$ such that μ_k is converging towards μ in space $H_{00}^{1/2}(\Gamma_2)$ and moreover $\tilde{\mu}_k$ belongs to space $\mathcal{D}(\Gamma)$:

$$(4.9) \quad \mu_k \in \mathcal{D}(\Gamma_2), \quad \mu_k \longrightarrow \mu \text{ in } H_{00}^{1/2}(\Gamma_2), \quad \tilde{\mu}_k \in \mathcal{D}(\Gamma).$$

Consider moreover the solution ζ_k of the Dirichlet Laplace problem

$$(4.10) \quad \begin{cases} -\Delta\zeta = 0 & \text{in } \Omega \\ \gamma\zeta_k = \widetilde{\mu}_k & \text{on } \Gamma. \end{cases}$$

By successive application of regularity theorems [ADN59] and Sobolev embedding injections [Ad75], function ζ_k belongs to space $\mathcal{C}^2(\overline{\Omega}) \cap H_0^1(\Omega; \Gamma_1)$ and can be used in both relations (4.5) and (4.7). We obtain by difference

$$(4.11) \quad \left\langle \frac{\partial\psi}{\partial n}, \widetilde{\mu}_k \right\rangle_{\Gamma} = \left\langle \frac{\partial\psi}{\partial n}, \mu_k \right\rangle_{(H_{00}^{1/2}(\Gamma_2))', H_{00}^{1/2}(\Gamma_2)} = 0;$$

moreover, $\frac{\partial\psi}{\partial n}$ is a continuous linear form acting on space $H_{00}^{1/2}(\Gamma_2)$ and μ_k is converging to μ in this space and μ is arbitrary. Then $\frac{\partial\psi}{\partial n}$ is null on Γ_2 :

$$(4.12) \quad \frac{\partial\psi}{\partial n} = 0 \quad \text{in space } (H_{00}^{1/2}(\Gamma_2))'$$

and due to (4.7), function ψ is solution of the following variational problem :

$$(4.13) \quad \begin{cases} \psi \in H_0^1(\Omega; \Gamma_1) \\ (\psi, \zeta) + (\nabla\psi, \nabla\zeta) = 0, \quad \forall \zeta \in H_0^1(\Omega; \Gamma_1) \end{cases}$$

then is identically equal to zero, that establishes first point of Proposition 4.

• Second result of Proposition 4 concerns vector fields. In analogy with previous case, consider a vector valued function ψ in space $H_0^1(\Omega; \Gamma_1, \Gamma_2)$ and H^1 -orthogonal to vector space $(\mathcal{C}^2(\overline{\Omega}))^N \cap H_0^1(\Omega; \Gamma_1, \Gamma_2)$:

$$(4.14) \quad \begin{cases} \psi \in H_0^1(\Omega; \Gamma_1, \Gamma_2) \\ (\psi, \varphi) + (\nabla\psi, \nabla\varphi) = 0, \quad \forall \varphi \in (\mathcal{C}^2(\overline{\Omega}))^N \cap H_0^1(\Omega; \Gamma_1, \Gamma_2). \end{cases}$$

First taking $\varphi \in (\mathcal{D}(\Omega))^N$, we have $\psi - \Delta\psi = 0$ in the sense of distributions and in consequence $\Delta\psi \in (L^2(\Omega))^N$. For an arbitrary vector field $\zeta \in H_0^1(\Omega; \Gamma_1, \Gamma_2)$, we integrate $(-\Delta\psi, \zeta)$ by parts and obtain as in first step :

$$(4.15) \quad (\psi, \zeta) + (\nabla\psi, \nabla\zeta) - \left\langle \frac{\partial\psi}{\partial n}, \gamma\zeta \right\rangle = 0, \quad \forall \zeta \in H_0^1(\Omega; \Gamma_1, \Gamma_2).$$

The boundary term $\left\langle \frac{\partial\psi}{\partial n}, \gamma\zeta \right\rangle$ is decomposed with the help of tangential and normal components of vector fields :

$$(4.16) \quad \begin{cases} \left\langle \frac{\partial\psi}{\partial n}, \gamma\zeta \right\rangle = \left\langle \frac{\partial}{\partial n} (\psi \times n), \widetilde{\gamma} \times \zeta \right\rangle_{(TH_{00}^{1/2}(\Gamma_1))', TH_{00}^{1/2}(\Gamma_1)} + \\ \left\langle \frac{\partial}{\partial n} (\psi \bullet n), \widetilde{\gamma} \bullet \zeta \right\rangle_{(H_{00}^{1/2}(\Gamma_2))', H_{00}^{1/2}(\Gamma_2)}. \end{cases}$$

Consider in consequence $\xi \in TH_{00}^{1/2}(\Gamma_1)$ and $\mu \in H_{00}^{1/2}(\Gamma_2)$. By density of space of regular functions in Sobolev spaces with exponent $1/2$, there exists two sequences $\xi_k \in (\mathcal{D}(\Gamma_1))^N$ and $\mu_k \in \mathcal{D}(\Gamma_2)$ converging towards ξ and μ in spaces $TH_{00}^{1/2}(\Gamma_1)$ and $H_{00}^{1/2}(\Gamma_2)$ respectively and with notation "tilda" defined by the relation

$$(4.17) \quad \tilde{\xi} = \begin{cases} \xi & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \end{cases}$$

for tangent vector field ξ and by (4.8) for scalar field μ , we have moreover $\tilde{\xi}_k \in (\mathcal{D}(\Gamma))^N$ and $\tilde{\mu}_k \in \mathcal{D}(\Gamma)$:

$$(4.18) \quad \begin{cases} \xi_k \in (\mathcal{D}(\Gamma_1))^N, & \xi_k \longrightarrow \xi \text{ in } TH_{00}^{1/2}(\Gamma_1), & \tilde{\xi}_k \in (\mathcal{D}(\Gamma))^N \\ \mu_k \in \mathcal{D}(\Gamma_2), & \mu_k \longrightarrow \mu \text{ in } H_{00}^{1/2}(\Gamma_2), & \tilde{\mu}_k \in \mathcal{D}(\Gamma). \end{cases}$$

Then we solve the following two auxiliary pure Dirichlet problems for Laplace equation :

$$(4.19) \quad \begin{cases} -\Delta \lambda_k = 0 & \text{in } \Omega \\ \gamma \lambda_k = \tilde{\xi}_k & \text{on } \Gamma. \end{cases}$$

$$(4.20) \quad \begin{cases} -\Delta \chi_k = 0 & \text{in } \Omega \\ \gamma \chi_k = \tilde{\mu}_k n & \text{on } \Gamma \end{cases}$$

and the solution λ_k of (4.19) and χ_k of (4.20) belong to the space $(\mathcal{C}^2(\bar{\Omega}))^N \cap H_0^1(\Omega; \Gamma_1, \Gamma_2)$ thanks to regularity of solution of Laplace equation on a domain with a smooth boundary. We consider the first choice $\zeta = \lambda_k$ inside relations (4.14) and (4.15) and we make the difference between the two equations. It comes :

$$(4.21) \quad \left\langle \frac{\partial}{\partial n} (\psi \times n), \xi_k \right\rangle_{(TH_{00}^{1/2}(\Gamma_1))', TH_{00}^{1/2}(\Gamma_1)} = 0, \quad \forall k \in \mathbb{N}$$

and due to the first line of (4.18) and the fact that ξ is arbitrary, $\frac{\partial}{\partial n} (\psi \times n)$ is null on Γ_1 . We proceed in an analogous way with $\zeta = \chi_k$ and we get using the same argument :

$$(4.22) \quad \left\langle \frac{\partial}{\partial n} (\psi \bullet n), \mu_k \right\rangle_{(H_{00}^{1/2}(\Gamma_2))', H_{00}^{1/2}(\Gamma_2)} = 0, \quad \forall k \in \mathbb{N}.$$

Second line of (4.18) and the fact that μ is arbitrary, we deduce that $\frac{\partial}{\partial n} (\psi \bullet n)$ is null on Γ_2 . Due to (4.16), the boundary term in (4.15) is null and we get :

$$(4.23) \quad \begin{cases} \psi \in H_0^1(\Omega; \Gamma_1, \Gamma_2) \\ (\psi, \zeta) + (\nabla\psi, \nabla\zeta) = 0, \quad \forall \zeta \in H_0^1(\Omega; \Gamma_1, \Gamma_2). \end{cases}$$

Then vector field ψ is null and Proposition 4 is established. \square

Proposition 5. Peetre-Tartar lemma.

Let E_0, E_1, E_2 be three Banach spaces, $A_1 : E_0 \longrightarrow E_1$ and $A_2 : E_0 \longrightarrow E_2$ be two linear continuous mappings such that A_2 is compact and such that

$$(4.24) \quad \exists C > 0, \forall v \in E_0, \|v\|_{E_0} \leq C \left(\|A_1 v\|_{E_1} + \|A_2 v\|_{E_2} \right).$$

Then we have the following two properties :

$$(4.25) \quad \ker A_1 \text{ is finite dimensional, } \text{Im } A_1 \text{ is closed}$$

$$(4.26) \quad \exists C_0 > 0, \forall v \in E_0, \inf_{w \in \ker A_1} \|v + w\|_{E_0} \leq C_0 \|A_1 v\|_{E_1}.$$

- Proposition 5 is classical (see [Pe61], [Ta76]) and this result is necessary for the establishment of Proposition 6.

Proposition 6. An equivalent norm on space $H_0^1(\Omega; \Gamma_1, \Gamma_2)$.

Under the previous general hypotheses done on domain Ω , there exists some constant $C > 0$ such that

$$(4.27) \quad \begin{cases} \|\varphi\|_{1,\Omega} \leq C \left(\|\varphi\|_{0,\Omega}^2 + \|\text{div } \varphi\|_{0,\Omega}^2 + \|\text{curl } \varphi\|_{0,\Omega}^2 \right)^{1/2} \\ \forall \varphi \in H_0^1(\Omega; \Gamma_1, \Gamma_2). \end{cases}$$

Proof of Proposition 6.

- Let (φ, ψ) be a pair of regular vector fields on domain Ω . We first develop the following integral

$$(4.28) \quad (-\Delta\varphi, \psi) = \int_{\Omega} (\text{curl}(\text{curl } \varphi) - \nabla(\text{div } \varphi), \psi) \, d\gamma$$

by integrating by parts each of the three terms of second order. It comes

$$(4.29) \quad \begin{cases} (\nabla\varphi, \nabla\psi) = (\text{curl } \varphi, \text{curl } \psi) + (\text{div } \varphi, \text{div } \psi) + \langle \frac{\partial\varphi}{\partial n}, \psi \rangle_{\Gamma} \\ \quad + \langle \text{curl } \varphi, \psi \times n \rangle_{\Gamma} - \langle \text{div } \varphi, \psi \cdot n \rangle_{\Gamma}. \end{cases}$$

- Following precise geometric work done by Bendali [Be84], it is possible to majorate the boundary terms of relation (4.29) when we set moreover $\psi = \varphi$ and in the two particular cases when $\varphi \cdot n = 0$ or $\varphi \times n = 0$. If φ belongs to space

$(\mathcal{C}^2(\overline{\Omega}))^N \cap H_0^1(\Omega; \Gamma_1, \Gamma_2)$, we get in the first case the existence of a constant C independent of φ such that

$$(4.30) \quad \left\{ \begin{array}{l} \left| \int_{\Gamma_1} \left[\left(\frac{\partial \varphi}{\partial n}, \varphi \right) + (\operatorname{curl} \varphi, \varphi \times n) \right] d\gamma \right| \leq C \int_{\Gamma_1} |\varphi|^2 d\gamma \\ \forall \varphi \in (\mathcal{C}^2(\overline{\Omega}))^N \text{ such that } \varphi \bullet n = 0 \text{ on } \Gamma_1. \end{array} \right.$$

In second case there exists an other constant also named C in order to satisfy

$$(4.31) \quad \left\{ \begin{array}{l} \left| \int_{\Gamma_2} \left[\left(\frac{\partial \varphi}{\partial n}, \varphi \right) + (\operatorname{div} \varphi, \varphi \bullet n) \right] d\gamma \right| \leq C \int_{\Gamma_2} |\varphi|^2 d\gamma \\ \forall \varphi \in (\mathcal{C}^2(\overline{\Omega}))^N \text{ such that } \varphi \times n = 0 \text{ on } \Gamma_2. \end{array} \right.$$

By summation of relations (4.30) and (4.31) inside (4.29), we obtain the estimate

$$(4.32) \quad \left\{ \begin{array}{l} \|\nabla \varphi\|_{0,\Omega}^2 \leq \|\operatorname{div} \varphi\|_{0,\Omega}^2 + \|\operatorname{curl} \varphi\|_{0,\Omega}^2 + C \int_{\partial\Omega} |\varphi|^2 d\gamma \\ \forall \varphi \in (\mathcal{C}^2(\overline{\Omega}))^N \cap H_0^1(\Omega; \Gamma_1, \Gamma_2) \end{array} \right.$$

and by density established at Proposition 4, relation (4.32) holds in space $H_0^1(\Omega; \Gamma_1, \Gamma_2)$ if we replace inside the integral on the boundary $\partial\Omega$ in the right hand side of (4.32), the value φ by the trace $\gamma\varphi \in (H^{1/2}(\Gamma))^N$.

• Definition of the H^1 norm and relation (4.32) show that condition (4.24) of Proposition 5 can be used with the following particular data :

$$(4.33) \quad \left\{ \begin{array}{l} E_0 = H_0^1(\Omega; \Gamma_1, \Gamma_2) \\ E_1 = (L^2(\Omega))^N \times (L^2(\Omega))^N \times (L^2(\Omega))^{2N-3} \\ E_2 = (L^2(\Gamma))^N \end{array} \right.$$

$$(4.34) \quad \left\{ \begin{array}{l} A_1 \varphi = (\varphi, \operatorname{div} \varphi, \operatorname{curl} \varphi) \\ A_2 \varphi = \gamma\varphi. \end{array} \right.$$

The injection $(H^{1/2}(\Gamma))^N \ni \xi \mapsto \xi \in (L^2(\Gamma))^N$ is compact (see e.g. Lions-Magenes [LM68]) then the mapping A_2 is a compact operator and Peetre-Tartar lemma can be applied. First consequence (relation (4.25)) gives no particular information because, following (4.33), we have clearly $\ker A_1 = \{0\}$. On the contrary, taking into account the previous point, relation (4.27) infers

$$(4.35) \quad \left\{ \begin{array}{l} \exists C_0 > 0, \forall \varphi \in H_0^1(\Omega; \Gamma_1, \Gamma_2), \\ \|\varphi\|_{1,\Omega} \leq C_0 (\|\varphi\|_{0,\Omega} + \|\operatorname{div} \varphi\|_{0,\Omega} + \|\operatorname{curl} \varphi\|_{0,\Omega}) \end{array} \right.$$

and after elementary algebraic details, relation (4.35) is equivalent to relation (4.27), that establishes the Proposition. \square

Proposition 7. **Second equivalent norm on space** $H_0^1(\Omega; \Gamma_1, \Gamma_2)$.

Under the same hypotheses concerning domain Ω , the subspace $M^1(\Omega; \Gamma_1, \Gamma_2)$ of $H_0^1(\Omega; \Gamma_1, \Gamma_2)$ obtained by annulation of div and curl operators, *i.e.*

$$(4.36) \quad \left\{ \begin{array}{l} M^1(\Omega; \Gamma_1, \Gamma_2) = \left\{ \varphi \in (H^1(\Omega))^N, \operatorname{div} \varphi = 0, \operatorname{curl} \varphi = 0, \right. \\ \left. \gamma \varphi \cdot n = 0 \text{ on } \Gamma_1, \gamma \varphi \times n = 0 \text{ on } \Gamma_2 \right\} \end{array} \right.$$

is of finite dimension. If $\Pi_{\Gamma_1, \Gamma_2}^1$ denotes the orthogonal projector onto space $M^1(\Omega; \Gamma_1, \Gamma_2)$ relatively to the $L^2(\Omega)^N$ scalar product, there exists a constant $C > 0$ such that

$$(4.37) \quad \left\{ \begin{array}{l} \|\varphi\|_{1, \Omega} \leq C \left(\|\Pi_{\Gamma_1, \Gamma_2}^1 \varphi\|_{0, \Omega}^2 + \|\operatorname{div} \varphi\|_{0, \Omega}^2 + \|\operatorname{curl} \varphi\|_{0, \Omega}^2 \right)^{1/2} \\ \forall \varphi \in H_0^1(\Omega; \Gamma_1, \Gamma_2). \end{array} \right.$$

Proof of Proposition 7.

- We apply again Peetre-Tartar lemma, this time with the particular choice

$$(4.38) \quad \left\{ \begin{array}{l} E_0 = H_0^1(\Omega; \Gamma_1, \Gamma_2) \text{ with the equivalent norm} \\ \|\varphi\|_{E_0} \equiv \left(\|\varphi\|_{0, \Omega}^2 + \|\operatorname{div} \varphi\|_{0, \Omega}^2 + \|\operatorname{curl} \varphi\|_{0, \Omega}^2 \right)^{1/2} \\ E_1 = (L^2(\Omega)) \times (L^2(\Omega))^{2N-3} \\ E_2 = (L^2(\Omega))^N \end{array} \right.$$

$$(4.39) \quad \left\{ \begin{array}{l} A_1 \varphi = (\operatorname{div} \varphi, \operatorname{curl} \varphi) \\ A_2 \varphi = \varphi. \end{array} \right.$$

Due to Rellich theorem and the fact that the domain Ω is bounded, the injection $(H^1(\Omega))^N \hookrightarrow (L^2(\Omega))^N$ is compact, then by composition it is also the case for operator A_2 . We observe also that initial estimate (4.24) in Proposition 5 is exactly the one (4.27) established at Proposition 6. Finally we remark that $\ker A_1 = M^1(\Omega; \Gamma_1, \Gamma_2)$ and the first point of Proposition 7 is established.

- We decompose an arbitrary vector field $\varphi \in H_0^1(\Omega; \Gamma_1, \Gamma_2)$ under the form of two orthogonal fields in $L^2(\Omega)^N$:

$$(4.40) \quad \varphi = \Pi_{\Gamma_1, \Gamma_2}^1 \varphi + \left(\varphi - \Pi_{\Gamma_1, \Gamma_2}^1 \varphi \right)$$

and taking into account (4.27) and (4.36), we have

$$(4.41) \quad \left\{ \begin{array}{l} \inf_{\zeta \in M^1(\Omega; \Gamma_1, \Gamma_2)} \left(\|\varphi - \zeta\|_{0,\Omega}^2 + \|\operatorname{div}(\varphi - \zeta)\|_{0,\Omega}^2 + \right. \\ \left. + \|\operatorname{curl}(\varphi - \zeta)\|_{0,\Omega}^2 \right)^{1/2} = \\ = \left(\|\varphi - \Pi_{\Gamma_1, \Gamma_2}^1 \varphi\|_{0,\Omega}^2 + \|\operatorname{div} \varphi\|_{0,\Omega}^2 + \|\operatorname{curl} \varphi\|_{0,\Omega}^2 \right)^{1/2}. \end{array} \right.$$

Then we have :

$$\begin{aligned} \|\varphi\|_{1,\Omega} &\leq \|\Pi_{\Gamma_1, \Gamma_2}^1 \varphi\|_{1,\Omega} + \|\varphi - \Pi_{\Gamma_1, \Gamma_2}^1 \varphi\|_{1,\Omega} \\ &\leq C \|\Pi_{\Gamma_1, \Gamma_2}^1 \varphi\|_{0,\Omega} + \\ &\quad + C \left(\|\varphi - \Pi_{\Gamma_1, \Gamma_2}^1 \varphi\|_{0,\Omega}^2 + \|\operatorname{div} \varphi\|_{0,\Omega}^2 + \|\operatorname{curl} \varphi\|_{0,\Omega}^2 \right)^{1/2} \\ &\leq C \|\Pi_{\Gamma_1, \Gamma_2}^1 \varphi\|_{0,\Omega} + C \inf_{\zeta \in M^1(\Omega; \Gamma_1, \Gamma_2)} \left(\|\varphi - \zeta\|_{0,\Omega}^2 + \right. \\ &\quad \left. + \|\operatorname{div}(\varphi - \zeta)\|_{0,\Omega}^2 + \|\operatorname{curl}(\varphi - \zeta)\|_{0,\Omega}^2 \right)^{1/2} \quad \text{due to (4.41)} \\ &\leq C \|\Pi_{\Gamma_1, \Gamma_2}^1 \varphi\|_{0,\Omega} + C C_0 \left(\|\operatorname{div} \varphi\|_{0,\Omega} + \|\operatorname{curl} \varphi\|_{0,\Omega} \right) \end{aligned}$$

due to relation (4.26) in Peetre-Tartar lemma and relation (4.37) is established. \square

• We define space $M^0(\Omega; \Gamma_1, \Gamma_2)$ of vector fields φ lying in space $L^2(\Omega)^N$, whose div and curl are identically null and, due to Proposition 1, such that the normal trace $\widetilde{\gamma} \bullet \varphi$ is null in dual space $(H_{00}^{1/2}(\Gamma_1)^N)'$ and tangential trace $\widetilde{\gamma} \times \varphi$ is identically null in dual space $(TH_{00}^{1/2}(\Gamma_2))'$:

$$(4.42) \quad \left\{ \begin{array}{l} M^0(\Omega; \Gamma_1, \Gamma_2) = \left\{ \varphi \in (L^2(\Omega))^N, \operatorname{div} \varphi = 0, \operatorname{curl} \varphi = 0, \right. \\ \left. \widetilde{\gamma} \bullet \varphi = 0 \text{ in } (H_{00}^{1/2}(\Gamma_1)^N)', \widetilde{\gamma} \times \varphi = 0 \text{ in } (TH_{00}^{1/2}(\Gamma_2))' \right\}. \end{array} \right.$$

We observe the inclusion

$$(4.43) \quad M^1(\Omega; \Gamma_1, \Gamma_2) \subset M^0(\Omega; \Gamma_1, \Gamma_2).$$

Proposition 8. Closed partition.

Let Ω be a connected bounded domain of \mathbb{R}^N ($N = 2$ or 3) and (Γ_1, Γ_2) be a partition of its boundary Γ satisfying general hypotheses done at the beginning of section 4. Then space $M^0(\Omega; \Gamma_1, \Gamma_2)$ defined in (4.42) is closed in $L^2(\Omega)^N$ and we denote by $\Pi_{\Gamma_1, \Gamma_2}^0$ the orthogonal projector $(L^2(\Omega))^N \rightarrow M^0(\Omega; \Gamma_1, \Gamma_2)$.

Proof of Proposition 8.

- Let $(\varphi_k)_{k \in \mathbb{N}}$ be a sequence lying in space $M^0(\Omega; \Gamma_1, \Gamma_2)$ and converging in space $L^2(\Omega)^N$ to some function $\varphi \in (L^2(\Omega))^N$. Then sequences $(\operatorname{div} \varphi_k)_{k \in \mathbb{N}}$ and $(\operatorname{curl} \varphi_k)_{k \in \mathbb{N}}$ are converging in the sense of distributions towards $\operatorname{div} \varphi$ and $\operatorname{curl} \varphi$ respectively. We deduce from nullity of previous sequences that we have necessarily $\operatorname{div} \varphi = 0$ and $\operatorname{curl} \varphi = 0$ and $(\varphi_k)_{k \in \mathbb{N}}$ is converging towards φ in space $H(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega)$.

- The normal and tangential traces $\tilde{\gamma} \bullet \varphi$ and $\tilde{\gamma} \times \varphi$ are continuously defined $H(\operatorname{div}, \Omega) \rightarrow (H_{00}^{1/2}(\Gamma_1)^N)'$ and $H(\operatorname{curl}, \Omega) \rightarrow (TH_{00}^{1/2}(\Gamma_2))'$ respectively and sequences $(\tilde{\gamma} \bullet \varphi_k)_{k \in \mathbb{N}}$ and $(\tilde{\gamma} \times \varphi_k)_{k \in \mathbb{N}}$ are identically equal to zero due to the hypothesis $\varphi_k \in M^0(\Omega; \Gamma_1, \Gamma_2)$. Then there are converging to zero in their respective Hilbert spaces and $\tilde{\gamma} \bullet \varphi = 0$, $\tilde{\gamma} \times \varphi = 0$ because $(\varphi_k)_{k \in \mathbb{N}}$ is converging towards φ in space $H(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega)$. Thus we have established that $\varphi \in M^0(\Omega; \Gamma_1, \Gamma_2)$ and Proposition 8 is proven. \square

- In the case where the boundary $\partial\Omega$ is sufficiently regular (or convex, see Amrouche et al [ABDG98]), we know that spaces $M^0(\Omega; \emptyset, \Gamma)$ and $M^1(\Omega; \emptyset, \Gamma)$ on one hand, $M^0(\Omega; \Gamma, \emptyset)$ and $M^1(\Omega; \Gamma, \emptyset)$ on the other hand, are exactly equal (see e.g. Foias-Temam [FT78], Bendali [Be84], Amrouche et al. [ABDG98]). More precisely, dimension of space $M^0(\Omega; \emptyset, \Gamma)$ is exactly equal to the number of connected components of the boundary, minus one, and space $M^0(\Omega; \Gamma, \emptyset)$ parameterizes the second cohomology group of the open set Ω related to the number of holes in the domain. The main difficulty of our generalization is that the mixed boundary condition proposed in (4.42), *i.e.* formally $\varphi \bullet n = 0$ on Γ_1 and $\varphi \times n = 0$ on Γ_2 is the possible presence of bidimensional singularities if $n = 2$ (see Grisvard [Gr85]) or tridimensionnal if $n = 3$ (see Dauge [Da88]).

Theorem 2. Representation of vector fields.

Let Ω be a connected bounded domain of \mathbb{R}^N ($N = 2$ or 3) and (Γ_1, Γ_2) be a partition of its boundary Γ satisfying general hypotheses done at the beginning of section 4. Let $u \in L^2(\Omega)^N$ be a vector field. Then there exists two potentials φ and ψ satisfying the condition

$$(4.44) \quad \begin{cases} \varphi \in H^1(\Omega; \Gamma_1) \\ \psi \in H_0^1(\Omega; \Gamma_1, \Gamma_2), \end{cases}$$

uniquely and continuously defined if we impose the supplementary following conditions to vector potential ψ when $n = 3$:

$$(4.45) \quad \operatorname{div} \psi = 0 \text{ in } \Omega, \quad \Pi_{\Gamma_1, \Gamma_2}^1 \psi = 0,$$

$$(4.46) \quad \exists C > 0, \quad \|\varphi\|_{1, \Omega} \leq C \|u\|_{0, \Omega}, \quad \|\psi\|_{1, \Omega} \leq C \|u\|_{0, \Omega}$$

and chosen in such a way that u admits the following orthogonal decomposition in space $L^2(\Omega)^N$:

$$(4.47) \quad u = \nabla \varphi + \operatorname{curl} \psi + \Pi_{\Gamma_2, \Gamma_1}^0 u.$$

- In order to simplify the notations, we introduce the space $H_{00}^1(\Omega; \Gamma_1, \Gamma_2)$ of vector fields lying in space $H_0^1(\Omega; \Gamma_1, \Gamma_2)$ and satisfying moreover the two conditions (4.45) :

$$(4.48) \quad \begin{cases} H_{00}^1(\Omega; \Gamma_1, \Gamma_2) = \left\{ \varphi \in (H^1(\Omega))^N, \quad \widetilde{\gamma} \bullet \varphi = 0 \text{ on } \Gamma_1, \right. \\ \left. \widetilde{\gamma} \times \varphi = 0 \text{ on } \Gamma_2, \quad \operatorname{div} \varphi = 0, \quad \Pi_{\Gamma_1, \Gamma_2}^1 \varphi = 0 \right\}. \end{cases}$$

Proof of Theorem 2.

- We first remark that conditions (4.44) and (4.45) have been chosen in order to assume that decomposition (4.47) is orthogonal in space $L^2(\Omega)^N$. If we use generically letter γ for traces, we first have

$$\begin{aligned} (\nabla \varphi, \operatorname{curl} \psi) &= \langle \gamma \varphi, \gamma (\operatorname{curl} \psi \bullet n) \rangle_{H_{00}^{1/2}(\Gamma_2), (H_{00}^{1/2}(\Gamma_2))'} \\ &= \langle \gamma \varphi, \operatorname{div}_\Gamma (\widetilde{\gamma} \times \psi) \rangle_{H_{00}^{1/2}(\Gamma_2), (H_{00}^{1/2}(\Gamma_2))'} \end{aligned}$$

due to the classical relation $\operatorname{curl} \varphi \bullet n = \operatorname{div}_\Gamma (\varphi \times n)$ valid for regular fields (see e.g. [Du90]) and easily extended by duality. We observe now that $\widetilde{\gamma} \times \psi = 0$ in $TH_{00}^{1/2}(\Gamma_2)$ if $\psi \in H_0^1(\Omega; \Gamma_1, \Gamma_2)$ then $\operatorname{div}_\Gamma (\widetilde{\gamma} \times \psi)$ is null in space $(H_{00}^{1/2}(\Gamma_2))'$ (see Abboud-Starling [AS91] or Terrasse [Te93] concerning the fact that if vector

field ψ belongs to space $H(\text{curl}, \Omega)$, tangent field $\psi \times n$ belongs to $TH^{-1/2}(\Gamma)$ and moreover scalar boundary field $\text{div}_\Gamma(\psi \times n)$ belongs to space $H^{-1/2}(\Gamma)$ and we have

$$(4.49) \quad (\nabla\varphi, \text{curl}\psi) = 0.$$

Note also that relation (4.47) can also be obtained by integrating by parts on the other side. We have

$$\begin{aligned} (\nabla\varphi, \text{curl}\psi) &= \langle \gamma(\nabla\varphi) \times n, \gamma\psi \rangle_{(TH_{00}^{1/2}(\Gamma_2))', TH_{00}^{1/2}(\Gamma_2)} \\ &= \langle \widetilde{\gamma} \times (\nabla\varphi), n \times \gamma\psi \times n \rangle_{(TH_{00}^{1/2}(\Gamma_2))', TH_{00}^{1/2}(\Gamma_2)} = 0 \end{aligned}$$

because tangential gradient of φ is identically null on Γ_1 and tangential component of vector ψ is null on Γ_2 . Global coherence between spaces in duality show that formal way of making the calculus is justified.

• We have in an analogous way

$$(\nabla\varphi, \Pi_{\Gamma_2, \Gamma_1}^0 u) = \langle \gamma\varphi, \widetilde{\gamma} \bullet \left(\Pi_{\Gamma_2, \Gamma_1}^0 u \right) \rangle_{H_{00}^{1/2}(\Gamma_2), (H_{00}^{1/2}(\Gamma_2))'}$$

and $\widetilde{\gamma} \bullet \xi \equiv 0$ in dual space $(H_{00}^{1/2}(\Gamma_2))'$ if $\xi \in M^0(\Omega; \Gamma_2, \Gamma_1)$. Then we deduce

$$(4.50) \quad (\nabla\varphi, \Pi_{\Gamma_2, \Gamma_1}^0 u) = 0.$$

Finally we have

$$(\text{curl}\psi, \Pi_{\Gamma_2, \Gamma_1}^0 u) = - \langle \widetilde{\gamma} \times \psi, \gamma \left(\Pi_{\Gamma_2, \Gamma_1}^0 u \right) \rangle_{TH_{00}^{1/2}(\Gamma_1), (TH_{00}^{1/2}(\Gamma_1))'}$$

and we decompose the trace of vector $\xi \in M^0(\Omega; \Gamma_2, \Gamma_1)$ into tangential and normal components :

$$(4.51) \quad \gamma\xi = n \times (\widetilde{\gamma} \times \xi) + (\widetilde{\gamma} \bullet \xi) n,$$

the first one is null on Γ_1 if $\xi \in M^0(\Omega; \Gamma_2, \Gamma_1)$ because $\widetilde{\gamma} \times \xi = 0$ in space $(TH_{00}^{1/2}(\Gamma_1))'$ and the second one is normal to the boundary. Then we have

$$(4.52) \quad (\text{curl}\psi, \Pi_{\Gamma_2, \Gamma_1}^0 u) = 0.$$

• Taking into account orthogonality relations (4.49) and (4.50) developed at the previous point and representation (4.47), scalar potential φ is necessarily solution of the following variational problem

$$(4.53) \quad \begin{cases} \varphi \in H^1(\Omega; \Gamma_1) \\ (\nabla\varphi, \nabla\zeta) = (u, \nabla\zeta), \quad \forall \zeta \in H^1(\Omega; \Gamma_1). \end{cases}$$

Definition (4.3) of space $H^1(\Omega; \Gamma_1)$ and Rellich theorem show classically that the H^1 semi-norm is elliptic on space $H^1(\Omega; \Gamma_1)$:

$$(4.54) \quad \exists \alpha > 0, \quad \forall \zeta \in H^1(\Omega; \Gamma_1), \quad \|\nabla\zeta\|_{0,\Omega}^2 \geq \alpha \|\zeta\|_{1,\Omega}^2.$$

Then Lax-Milgram theorem [LM54] shows the existence and uniqueness of scalar potential φ satisfying (4.53), and in consequence the first parts of conditions (4.44) and (4.46).

- In an analogous way, Proposition 7 and in particular relation (4.37) show that the bilinear form $(H^1(\Omega))^{2N-3} \times (H^1(\Omega))^{2N-3} \ni (\varphi, \eta) \longmapsto (\text{curl } \varphi, \text{curl } \eta) \in \mathbb{R}$ is elliptic on space $H_{00}^1(\Omega; \Gamma_1, \Gamma_2)$:

$$(4.55) \quad \exists \beta > 0, \quad \forall \eta \in H_{00}^1(\Omega; \Gamma_1, \Gamma_2), \quad \|\text{curl } \eta\|_{0,\Omega}^2 \geq \beta \|\zeta\|_{1,\Omega}^2.$$

Independently, orthogonal decomposition (4.47), orthogonalities (4.49) and (4.52) and gauge conditions (4.45) show that vector potential ψ is necessarily solution of the problem

$$(4.56) \quad \begin{cases} \psi \in H_{00}^1(\Omega; \Gamma_1, \Gamma_2) \\ (\text{curl } \psi, \text{curl } \eta) = (u, \text{curl } \eta), \quad \forall \eta \in H_{00}^1(\Omega; \Gamma_1, \Gamma_2). \end{cases}$$

Due to ellipticity (4.55) and Lax-Milgram theorem, vector potential ψ is uniquely defined by problem (4.56) and continuously depends on datum u . Then inequalities (4.46) are completely established.

- We have now to characterize the residual vector field ξ defined by the relation

$$(4.57) \quad \xi = u - \nabla\varphi - \text{curl } \psi$$

that clearly belongs to space $L^2(\Omega)^N$, admits a divergence and a curl equal to zero and is such that the right hand sides of relation (4.52) and (4.56) associated to this field are equal to zero :

$$(4.58) \quad \begin{cases} (\xi, \nabla\zeta) = 0, \quad \forall \zeta \in H_0^1(\Omega; \Gamma_1) \\ (\xi, \text{curl } \eta) = 0, \quad \forall \eta \in H_{00}^1(\Omega; \Gamma_1, \Gamma_2). \end{cases}$$

Let $g \in H_{00}^{1/2}(\Gamma_2)$ be an arbitrary scalar field and $\zeta \in H_0^1(\Omega; \Gamma_1)$ be the variational solution of the problem

$$(4.59) \quad \begin{cases} \Delta\zeta = 0 & \text{in } \Omega \\ \zeta = 0 & \text{on } \Gamma_1 \\ \zeta = g & \text{on } \Gamma_2. \end{cases}$$

After integrating by parts the expression $(\operatorname{div} \xi, \zeta)$, first relation in (4.58) shows that

$$(4.60) \quad \langle \widetilde{\gamma} \bullet \xi, g \rangle_{(H_{00}^{1/2}(\Gamma_2))', H_{00}^{1/2}(\Gamma_2)} = 0$$

i.e. in a common way of speaking, $\xi \bullet n = 0$ on Γ_2 .

• In an analogous way, let $\mu \in TH_{00}^{1/2}(\Gamma_1)$ be a given tangential vector field and $\eta \in TH_{00}^1(\Omega; \Gamma_1, \Gamma_2)$ be the variational solution of the problem

$$(4.61) \quad \begin{cases} -\Delta \eta & = 0 & \text{in } \Omega \\ \operatorname{div} \eta & = 0 & \text{on } \partial\Omega \\ n \times \eta \times n & = \mu & \text{on } \Gamma_1 \\ \eta \times n & = 0 & \text{on } \Gamma_2 \\ \Pi_{\Gamma_1, \Gamma_2}^1 \eta & = 0 \end{cases}$$

that can also be written under the form

$$(4.62) \quad \begin{cases} \eta \in H_{00}^1(\Omega; \Gamma_1, \Gamma_2) \\ (\operatorname{curl} \eta, \operatorname{curl} \varphi) = - \langle \mu, \widetilde{\gamma} \times \varphi \rangle, \quad \forall \varphi \in H_{00}^1(\Omega; \Gamma_1, \Gamma_2). \end{cases}$$

Relation (4.58), interpretation (4.61) of variational formulation (4.62) and Green formula $(\operatorname{curl} \xi, \eta) = (\xi, \operatorname{curl} \eta) + \langle n \times \xi, \eta \rangle$ show that

$$(4.63) \quad \langle \widetilde{\gamma} \times \xi, \mu \rangle_{(TH_{00}^{1/2}(\Gamma_1))', TH_{00}^{1/2}(\Gamma_1)} = 0$$

which means that $\xi \times n = 0$ on Γ_1 . We have established that $\xi \in M^0(\Omega; \Gamma_2, \Gamma_1)$. Moreover, taking into account (4.57), we have, due to orthogonality relations (4.50) and (4.52),

$$(4.64) \quad (u - \xi, \theta) = 0, \quad \forall \theta \in M^0(\Omega; \Gamma_2, \Gamma_1).$$

In consequence, the relation $\xi = \Pi_{\Gamma_2, \Gamma_1}^0 u$ is completely established and Theorem 2 is proven. \square

5) A FIRST EXISTENCE AND UNIQUENESS RESULT.

• We prove in this section that the abstract result that mathematically modelizes the Stokes problem under vorticity-velocity-pressure formulation can effectively be used. The letter Ω still represents a connected bounded domain of \mathbb{R}^2 or \mathbb{R}^3 with a smooth boundary $\Gamma \equiv \partial\Omega$ which is, as in relations (2.25) and (2.26), supposed to have been partitionated in two ways, (Γ_m, Γ_p) on one hand and $(\Gamma_\theta, \Gamma_t)$ on

the other hand. We suppose in this section that we have the particular condition that $\Gamma_\theta \equiv \Gamma_m$ and $\Gamma_t \equiv \Gamma_p$.

$$(5.1) \quad \partial\Omega = \overline{\Gamma_m} \cup \overline{\Gamma_p}, \quad \Gamma_m \cap \Gamma_p = \emptyset$$

$$(5.2) \quad \Gamma_\theta = \Gamma_m \quad \text{and} \quad \Gamma_t = \Gamma_p.$$

- We suppose moreover the following technical hypothesis, which is quite strong and would be precized in a geometrical point of view in the future.

Hypothesis 1. No special functions between Γ_m and Γ_p .

$$(5.3) \quad M^0(\Omega; \Gamma_m, \Gamma_p) = \{0\}.$$

- We introduce also the three Hilbert spaces W , X and Y for vorticity, velocity and pressure respectively defined at relations (2.33) to (2.35), *i.e.*

$$(5.4) \quad W = \left\{ \begin{array}{l} \varphi \in H(\text{curl}, \Omega), \quad \varphi \times n = 0 \text{ on } \Gamma_\theta, \\ \text{i.e. } \widetilde{\gamma} \times \varphi = 0 \text{ in } (TH_{00}^{1/2}(\Gamma_\theta))' \end{array} \right\}$$

$$(5.5) \quad X = \{v \in H(\text{div}, \Omega), v \bullet n = 0 \text{ on } \Gamma_m, \text{ i.e. } \widetilde{\gamma} \bullet v = 0 \text{ in } (H_{00}^{1/2}(\Gamma_m))'\}$$

$$(5.6) \quad Y = \begin{cases} L^2(\Omega) & \text{if } \text{meas}(\Gamma_p) \neq 0 \\ \{q \in L^2(\Omega), (q, 1) = 0\} & \text{if } \text{meas}(\Gamma_p) = 0 \end{cases}$$

and datum ζ according to :

$$(5.7) \quad \left\{ \begin{array}{l} \zeta = (\Pi_0, \sigma_0, f) \in H_{00}^{1/2}(\Gamma_p) \times TH_{00}^{1/2}(\Gamma_t) \times (L^2(\Omega))^N \\ \|\zeta\|_{\text{data}} \equiv \|\Pi_0\|_{1/2, \Gamma_p} + \|\sigma_0\|_{1/2, \Gamma_t} + \|f\|_{0, \Omega} \end{array} \right.$$

- We can set and prove the following theorem with homogeneous Dirichlet boundary conditions and nonhomogeneous Neumann boundary conditions when formulated with help of the three fields of vorticity, velocity and pressure.

Theorem 3. A particular Stokes problem.

Let Ω be a connected bounded domain satisfying hypotheses recalled in the beginning of this section, a partition (Γ_m, Γ_p) of its boundary $\partial\Omega$ satisfying relation (5.1) and a second partition of the boundary $(\Gamma_\theta, \Gamma_t)$ chosen according to (5.2). We suppose moreover that the pair (Γ_m, Γ_p) satisfies Hypothesis 1. We consider datum ζ defined at relation (5.7). The Stokes problem under vorticity-velocity-pressure formulation

$$(5.8) \quad \begin{cases} \omega - \operatorname{curl} u &= 0 & \text{in } \Omega \\ \operatorname{curl} \omega + \nabla p &= f & \text{in } \Omega \\ \operatorname{div} u &= 0 & \text{in } \Omega \\ u \bullet n &= 0 & \text{on } \Gamma_m \\ \omega \times n &= 0 & \text{on } \Gamma_m \\ p &= \Pi_0 & \text{on } \Gamma_p \\ n \times u \times n &= \sigma_0 & \text{on } \Gamma_p \end{cases}$$

admits the following variational formulation

$$(5.9) \quad \omega \in W, \quad u \in X, \quad p \in Y$$

$$(5.10) \quad (\omega, \varphi) - (u, \operatorname{curl} \varphi) = \langle \sigma_0, \widetilde{\gamma} \times \varphi \rangle, \quad \forall \varphi \in W$$

$$(5.11) \quad (\operatorname{curl} \omega, v) - (p, \operatorname{div} v) = (f, v) - \langle \Pi_0, \widetilde{\gamma} \bullet v \rangle, \quad \forall v \in X$$

$$(5.12) \quad (\operatorname{div} u, q) = 0, \quad \forall q \in Y.$$

Problem (5.9)-(5.12) admits a unique solution $(\omega(\zeta), u(\zeta), p(\zeta)) \in W \times X \times Y$ that continuously depends on datum ζ defined in (5.7) :

$$(5.13) \quad \exists C > 0, \quad \|\omega(\zeta)\|_W + \|u(\zeta)\|_X + \|p(\zeta)\|_Y \leq C \|\zeta\|_{\text{data}}.$$

Proof of Theorem 3.

- Taking into account all the work done in Section 2 to obtain variational formulation (2.36)-(2.39) here written under the form (5.9)-(5.12), we just have to apply Theorem 1, *i.e.* verify that the four hypotheses (3.10)-(3.13) of this abstract result are true. The letters W , X and Y represent relatively to Theorem 1 the objects with the names introduced in (5.4)-(5.6). We set also $Z = L^2(\Omega)^N$, $Dv = \operatorname{div} v$ for all $v \in X$, $R\varphi = -\operatorname{curl} \varphi$ for all $\varphi \in W$. Then it is clear from definitions (2.3) and (2.4) of $H(\operatorname{div}, \Omega)$ and $H(\operatorname{curl}, \Omega)$ norms that properties (3.1) and (3.2) hold.

- We establish now the equality

$$(5.14) \quad \ker D = \operatorname{Im} R$$

that clearly implies relation (3.10). We first have the inclusion $\operatorname{Im} R \subset \ker D$. If $\varphi \in W$, we have $\varphi \times n = 0$ on $\Gamma_m \equiv \Gamma_\theta$, *i.e.* $\widetilde{\gamma} \times \varphi = 0$ in $(TH_{00}^{1/2}(\Gamma_m))'$ and in consequence $(\operatorname{curl} \varphi) \bullet n \equiv \operatorname{div}_\Gamma(\varphi \times n) = 0$ on Γ_m . Then first inclusion $\operatorname{Im} R \subset \ker D$ is established.

On the other hand and according to Theorem 2, let $v \in L^2(\Omega)^N$ be decomposed under the form

$$(5.15) \quad v = \nabla \rho + \operatorname{curl} \psi + \zeta$$

with scalar potential ρ , vector potential ψ and special vector field ζ chosen according to :

$$(5.16) \quad \begin{cases} \rho \in H_0^1(\Omega; \Gamma_p) \\ \psi \in H_{00}^1(\Omega; \Gamma_p, \Gamma_m) \\ \zeta \in M^0(\Omega; \Gamma_m, \Gamma_p) \end{cases} .$$

From relation (4.53), we know that scalar potential ρ is solution of the following problem :

$$(5.17) \quad \begin{cases} \rho \in H^1(\Omega; \Gamma_p) \\ (\nabla \rho, \nabla \mu) = (v, \nabla \mu), \quad \forall \mu \in H^1(\Omega; \Gamma_p). \end{cases}$$

But we have also the fact that $\mu \in H^1(\Omega; \Gamma_p)$ implies naturally that the trace $\gamma\mu$ belongs to space $H_{00}^{1/2}(\Gamma_m)$. In consequence, due to definition (5.5) of space X and if moreover $v \in \ker D$, we have :

$$(v, \nabla \mu) = -(\operatorname{div} v, \mu) + \langle \tilde{\gamma} \bullet v, \gamma\mu \rangle_{(H_{00}^{1/2}(\Gamma_m))', H_{00}^{1/2}(\Gamma_m)} = 0.$$

Then left hand side of (5.17) is identically null and ρ is identically equal to zero. In consequence v belongs to $\operatorname{Im} R$ because relation (5.3) of Hypothesis 1 implies $\zeta \equiv 0$.

- We consider now second hypothesis (3.11) of Theorem 1 :

$$(5.18) \quad \exists a > 0, \quad \inf_{q \in Y, q \neq 0} \sup_{v \in X, v \neq 0} \frac{(q, \operatorname{div} v)}{\|q\|_Y \|v\|_X} \geq a.$$

For doing this, we consider the following auxiliary variational problem :

$$(5.19) \quad \begin{cases} \eta \in H^1(\Omega; \Gamma_m) \\ (\nabla \eta, \nabla \mu) = -(q, \mu), \quad \forall \mu \in H^1(\Omega; \Gamma_m). \end{cases}$$

and even if $\operatorname{meas}(\Gamma_m) = 0$, problem (5.19) has a unique solution in space $H^1(\Omega; \Gamma_m)$ satisfying the continuity relatively to datum q :

$$(5.20) \quad \|\eta\|_{1,\Omega} \leq C \|q\|_{0,\Omega}$$

where $C > 0$ only depends on domain Ω and on sub-boundary Γ_m . We set

$$(5.21) \quad v = \nabla \eta$$

and this field satisfies relation

$$(5.22) \quad \operatorname{div} v = \Delta \eta = q \quad \in L^2(\Omega)$$

then v belongs to space $H(\operatorname{div}, \Omega)$. Moreover taking into account variational formulation (5.19), we deduce

$$(5.23) \quad \tilde{\gamma} \bullet v = \frac{\partial \eta}{\partial n} = 0 \quad \text{in } (H_{00}^{1/2}(\Gamma_m))'.$$

We deduce from (5.22) and (5.23) that vector field v belongs to space X . We have also, due to inequality (5.20) and characterization (5.22),

$$\|v\|_{\text{div}, \Omega}^2 \leq \|\eta\|_{1, \Omega}^2 + \|q\|_{0, \Omega}^2 \leq (1+C^2) \|q\|_{0, \Omega}^2 = (1+C^2) \left(\frac{(q, \text{div } v)}{\|q\|_{0, \Omega}} \right)^2$$

that establishes inf-sup condition (5.18) with the particular choice for constant a :

$$(5.24) \quad a = \frac{1}{\sqrt{1+C^2}}.$$

- Second inf-sup condition (3.12) is written with the following form :

$$(5.25) \quad \exists b > 0, \quad \inf_{v \in \ker D, v \neq 0} \sup_{\varphi \in W, \varphi \neq 0} \frac{(v, \text{curl } \varphi)}{\|v\|_X \|\varphi\|_W} \geq b.$$

We take here into account the fact that $\ker D = \text{Im } R$ (relation (5.14)). Then if $v \in \ker D$ is given, there exists $\varphi \in H_{00}^1(\Omega; \Gamma_p, \Gamma_m) = H_{00}^1(\Omega; \Gamma_t, \Gamma_\theta)$ such that

$$(5.26) \quad v = \text{curl } \varphi.$$

Moreover, due to (4.46), there exists some constant C_1 which is only a function of Ω and Γ_m such that

$$(5.27) \quad \|\varphi\|_{1, \Omega} \leq C_1 \|v\|_{0, \Omega}.$$

We deduce from (5.26) and (5.27) the following set of inequalities :

$$\|\varphi\|_{\text{curl}, \Omega} \leq \|\varphi\|_{1, \Omega} \leq C_1 \|v\|_{0, \Omega} = C_1 \frac{(v, \text{curl } \varphi)}{\|v\|_{0, \Omega}} = C_1 \frac{(v, \text{curl } \varphi)}{\|v\|_{\text{div}, \Omega}}$$

which establishes (5.25) with

$$(5.28) \quad b = \frac{1}{C_1}.$$

- The last hypothesis (3.13) of Theorem 1, *i.e.*

$$(5.28) \quad \exists d > 0, \quad \forall \varphi \in \ker R, \quad (\varphi, \varphi) \geq d \|\varphi\|_W^2$$

is straightforward to deduce from algebraic relation (2.4) that defines the norm in space $H(\text{curl}, \Omega)$. Then Theorem 3 is established. \square

6) CONCLUSION AND ACKNOWLEDGMENTS.

- We have proposed in this work a new formulation of the Stokes problem for mechanics of incompressible fluids. The key-point of this formulation is to consider a velocity field that belongs to Hilbert space $H(\text{div}, \Omega)$ of vector fields.

We have explored the natural choice for vorticity, *i.e.* the hypothesis that vorticity belongs to space $H(\text{curl}, \Omega)$ and this Hilbert space coincides with Sobolev space $H^1(\Omega)$ when domain Ω is included in \mathbb{R}^2 . We have developed an abstract result for triple-mixed formulation, a new theorem for the representation of squarely integrable vector fields and have finally obtained a positive result that states that the Stokes problem is well posed in a particular case of boundary conditions. In the general case, vorticity does not belong to space $H(\text{curl}, \Omega)$ (see Bernardi, Girault and Maday [BGM92] for two-dimensional domains) and we propose in [DSS01] a weaker form of our vorticity-velocity-pressure formulation.

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