PERIODIC POINTS AND SHADOWING PROPERTY FOR GENERIC LEBESGUE MEASURE PRESERVING INTERVAL MAPS

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Abstract. In this article we study dynamical behaviour of generic Lebesgue measure-preserving interval maps. We show that for each $k \geq 1$ the set of periodic points of period at least k is a Cantor set of Hausdorff dimension zero and of upper box dimension one. Moreover, we obtain analogous results also in the context of generic Lebesgue measure-preserving circle maps. Furthermore, building on the former results, we show that there is a dense collection of transitive Lebesgue measure-preserving interval maps whose periodic points have full Lebesgue measure and whose periodic points of period k have positive measure for each $k \geq 1$. Finally, we show that the generic continuous maps of the interval which preserve the Lebesgue measure satisfy the shadowing and periodic shadowing property.

1. Introduction

In what follows a *residual* set denotes a dense G_δ set and we call a property *generic* if it is satisfied on at least a residual set of the underlying Baire space. The roots of studying generic properties in dynamical systems can be found in the article by Oxtoby and Ulam from 1941 [28] in which they showed that for a finite-dimensional compact manifold with a non-atomic measure which is positive on open sets, the set of ergodic measure-preserving homeomorphisms is generic in the strong topology. Subsequently, Halmos in 1944 [17],[18] introduced approximation techniques to a purely metric situation: he studied interval maps which are invertible almost everywhere and preserve the Lebesgue measure and showed that the generic invertible map is weakly mixing, i.e., has continuous spectrum. Then, Rohlin in 1948 [33] showed that the set of (strongly) mixing measure-preserving invertible maps is of the first category in the same space. Two decades later, Katok and Stepin in 1967 [20] introduced the notion of a speed of approximations. One of the notable applications of their method is the genericity of ergodicity and weak mixing for certain classes of interval exchange transformations. One of the most outstanding results using approximation theory is the Kerckhoff, Masur, Smillie theorem on the existence of polygons for which the billiard flow is ergodic [21], as well as its quantitative version by Vorobets [38]. Many more details on the history of approximation theory can be found in the surveys [5], [14], [37].

In what follows we denote by I the *unit interval* $[0, 1]$, $\mathbb{S}^1$ the *unit circle* and λ the *Lebesgue measure* on an underlying manifold. Our present study focuses on generic topological properties in the spaces of Lebesgue measure-preserving (not necessarily invertible) continuous maps on the interval $C_\lambda(I)$ and the circle $C_\lambda(\mathbb{S}^1)$. For the rest of the article we equip the two spaces with the uniform metric, which makes the spaces complete. The study of generic properties on $C_\lambda(I)$ was initiated in [7] and continued recently in [8]. It is well known that every such map has

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a dense set of periodic points (see for example [8]). Furthermore, except for the two exceptional maps id and $1 - \mathrm{id}$, every such map has positive metric entropy. Recently, basic topological and measure-theoretical properties of generic maps from $C_\lambda(I)$ were studied in [8]. We say that an interval map f is *locally eventually onto (leo)* if for every open interval $J \subset I$ there exists a non-negative integer n so that $f^n(J) = I$. This property is also sometimes referred in the literature as *topological exactness*. The $C_\lambda(I)$-generic function

(a) is weakly mixing with respect to λ [8, Th. 15],
(b) is leo [8, Th. 9],
(c) satisfies the periodic specification property [8, Cor. 10],
(d) has a knot point at λ almost every point [7],
(e) maps a set of Lebesgue measure zero onto $[0, 1]$ [8, Cor. 22],
(f) has infinite topological entropy [8, Prop. 26],
(g) has Hausdorff dimension = lower Box dimension $= 1 <$ upper Box dimension $= 2$ [34].

It was furthermore shown that the set of mixing maps in $C_\lambda(I)$ is dense [8, Cor. 14] and in analogy to Rohlin's result [33] that this set is of the first category [8, Th. 20].

In this paper we study periodic structure of generic Lebesgue measure-preserving maps on manifolds of dimension 1. Our choice of $C_\lambda(I)$ is motivated by the fact that they are one-dimensional versions of volume-preserving maps, or more broadly, conservative dynamical systems; ergodic maps preserving Lebesgue measure are the most fundamental examples of maps having a unique physical measure. Since generic maps in $C_\lambda(I)$ are weakly mixing [8], the Ergodic theorem implies that for a generic map in $C_\lambda(I)$ the closure of a typical trajectory has full Lebesgue measure, thus the statistical properties of typical trajectories can be revealed by physical observations.

The class $C_\lambda(I)$ contains very large spectrum of maps; on one hand nowhere differentiable ones or even without finite or infinite one-sided derivative [7] and on the other hand piecewise monotone or even piecewise smooth maps. One can construct many interesting examples using Lemma 11 from [9] and the fact that $C_\lambda(I)$ is closed.

On the other hand, they represent a variety of possible one-dimensional dynamics as highlighted in the following remark.

Remark. *Let f be an interval map. The following conditions are equivalent.*

(i) *f has a dense set of periodic points, i.e., $\overline{\mathrm{Per}(f)} = I$.*
(ii) *f preserves a nonatomic probability measure μ with $\operatorname{supp} \mu = I$.*
(iii) *There exists a homeomorphism h of I such that $h \circ f \circ h^{-1} \in C_\lambda(I)$.*

To see the above equivalence it is enough to combine a few facts from the literature. The starting point is [3], where the dynamics of interval maps with dense set of periodic points has been described; while this article is purely topological it easily implies that such maps must have non-atomic invariant measures with full support. The Poincaré Recurrence Theorem and the fact that in dynamical systems given by an interval map the closures of recurrent points and periodic points coincide [12] provides connection between maps preserving a probability measure with full support and dense set of periodic points. Finally, for μ a non-atomic probability measure with full support the map $h\colon I \to I$ defined as $h(x) = \mu([0, x])$ is a homeomorphism of I; moreover, if f preserves μ then $h \circ f \circ h^{-1} \in C_\lambda(I)$ (see the proof of Theorem 2 for more detail on this construction). Therefore, the topological properties that are proven in [8] and later in this paper are generic also for interval maps preserving measure μ.

A basic tool to understand the dynamics of interval maps is to understand the structure, dimension and Lebesgue measure of the set of its periodic points. For what follows let $f \in C_\lambda(I)$. Since generic maps from $C_\lambda(I)$ are weakly mixing with respect to λ it follows that the Lebesgue measure of the set of periodic points is 0. However, it is still natural to ask:

Question A. *What is the cardinality, structure and dimension of periodic points for generic maps in $C_\lambda(I)$?*

Akin et. al. proved in [1, Theorems 9.1 and 9.2(a)] that the set of periodic points of generic homeomorphisms of $\mathbb{S}^1$ is a Cantor set. In an unpublished sketch, Guihéneuf showed that the set of periodic points of a generic volume preserving homeomorphism f of a manifold of dimension at least two (or more generally preserving a good measure in the sense of Oxtoby and Ulam [28]) is a dense set of measure zero and for any $\ell \geq 1$ the set of fixed points of f^ℓ is either empty or a perfect set [15]. On the other hand, Carvalho et. al. have shown that the upper box dimension of the set of periodic points is full for generic homeomorphisms on compact manifolds of dimension at least one [11][1].

In the above context we provide the general answer about the cardinality and structure of periodic points of period k for f (denoted by $\mathrm{Per}(f,k)$), of fixed points of f^k (denoted by $\mathrm{Fix}(f,k)$) and of the union of all periodic points of f (denoted by $\mathrm{Per}(f)$) and its respective lower box, upper box and Hausdorff dimensions. Namely, we prove:

Theorem 1. *For a generic map $f \in C_\lambda(I)$, for each $k \geq 1$:*

(1) the set $\mathrm{Fix}(f,k)$ is a Cantor set,
(2) $\mathrm{Per}(f,k)$ is a relatively open dense subset of $\mathrm{Fix}(f,k)$,
(3) the set $\mathrm{Fix}(f,k)$ has Hausdorff dimension and lower box dimension zero. In particular, $\mathrm{Per}(f,k)$ has Hausdorff dimension and lower box dimension zero. As a consequence, the Hausdorff dimension of $\mathrm{Per}(f)$ is also zero.
(4) the set $\mathrm{Per}(f,k)$ has upper box dimension one. Therefore, $\mathrm{Fix}(f,k)$ has upper box dimension one as well.

The proof of the above theorem works also for the generic continuous maps which we believe is not known yet. Furthermore, we can also address the setting of $C_\lambda(\mathbb{S}^1)$, however, due to the presence of rotations, we need to treat degree 1 maps separately (for the related statement of the degree one case we refer the reader to Theorem 17).

Related to the study above, there is an interesting question about the possible Lebesgue measure on the set of periodic points for maps from $C_\lambda(I)$.

Question B. *Does there exist a transitive (or even leo) map in $C_\lambda(I)$ with positive Lebesgue measure on the set of periodic points?*

As mentioned already above, generic maps from $C_\lambda(I)$ will have Lebesgue measure 0 since λ is weakly mixing. Therefore, the previous question asks about the complement of generic maps from $C_\lambda(I)$ and requires on the first glance contradicting properties. The often noted discrepancy between the topological and measure theoretic notions of density is again displayed here and we obtain the following result. We answer Question B and even prove a stronger statement.

Theorem 2. *The set of leo maps in $C_\lambda(I)$ whose periodic points have full Lebesgue measure and whose periodic points of period k have positive measure for each $k \geq 1$ is dense in $C_\lambda(I)$.*

[1]this statement only appears in the published version of [11].

Another motivation for the study in this article was the following natural question.

Question C. *Is the shadowing property generic in $C_\lambda(I)$?*

Shadowing is a classical notion in topological dynamics and it serves as a tool to determine whether any hypothetical orbit is actually close to some real orbit of a topological dynamical system; this is of great importance in systems with sensitive dependence on initial conditions, where small errors may potentially result in a large divergence of orbits. The dynamics of maps that satisfy the shadowing property can be physically observed, in particular, through computer simulations. Pilyugin and Plamenevskaya introduced in [31] a nice technique to prove that shadowing is generic for homeomorphisms on any smooth compact manifold without a boundary. This led to several subsequent results that shadowing is generic in topology of uniform convergence, also in dimension one (see [27, 22] for recent results of this type). On the other hand, there are many cases known, when shadowing is not present in an open set in C^1 topology (see survey paper by Pilyugin [30] and [29, 32] for the general overview on the recent progress related with shadowing).

For continuous maps on manifolds of dimension one, Mizera proved that shadowing is indeed a generic property [26]. In the context of volume preserving homeomorphisms on manifolds of dimension at least two (with or without boundary), the question above was solved recently in the affirmative by Guihéneuf and Lefeuvre [16].

Our last main theorem provides the affirmative answer on Question C. Let us mention at this point why this setting is difficult, compared to expanding maps on surfaces. For interval maps, we lose expanding at some points due to the existence of critical points that necessarily appear. This may in turn disable shadowing, even in very regular settings. For example, it was proved in [13] that in the tent map family there is a dense set of parameters where there is shadowing, however there its dense complement has no shadowing property. All depends on the trajectory of the unique critical point, whose dynamics can change dramatically with a slight change of the slope. On the other hand, for many maps in the core tent map family we have strong mixing properties, like topological mixing, locally eventually onto, or even periodic specification property [6] (see also [10]).

Theorem 3. *Shadowing and periodic shadowing are generic properties for maps from $C_\lambda(I)$.*

Let us briefly describe the structure of the paper. In Preliminaries we give general definitions that we will need in the rest of the paper. In particular, our main tool throughout the most of the paper will be the controlled use of the approximation techniques which we introduce in the end of Section 2. In Section 3 we turn our attention to the study of periodic points and prove Theorem 1. The proof relies on a precise control of perturbations introduced in Section 2 which turns out to be particularly delicate. With some additional work we consequently obtain Theorem 2. We conclude the section with the study of periodic points for maps from $C_\lambda(\mathbb{S}^1)$ in Subsection 3.1. In Section 4 we provide a proof of Theorem 3. Similarly as in [22] we use covering relations, however the main obstacle is the preservation of Lebesgue measure which makes obtaining such coverings a more challenging task. We conclude the paper with Subsection 4.2 where we address a notion stronger than shadowing called the s-limit shadowing (see Definition 20) in the contexts of $C_\lambda(I)$. We prove that s-limit shadowing is dense in the respective environments. The approach resembles the one taken in [25], however due to our more restrictive setting our proof requires better control of perturbations. This result, in particular, implies that limit shadowing is dense in the respective environments as well.

2. Preliminaries

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let λ denote the Lebesgue measure on the unit interval $I := [0, 1]$. We denote by $C_\lambda(I) \subset C(I)$ the family of all continuous Lebesgue measure-preserving functions of I being a proper subset of the family of all continuous interval maps equipped with the *uniform metric* ρ:

$$\rho(f, g) := \sup_{x \in I} |f(x) - g(x)|.$$

It follows from the following proposition that $(C_\lambda(I), \rho)$ is a complete metric space as well.

Proposition 4. *$(C_\lambda(I), \rho)$ is closed in $(C(I), \rho)$; in particular it is a complete metric space.*

Proof. Fix a sequence $(f_n)_n$ of maps from $C_\lambda(I)$ which converge to a map f. Then f is continuous and we need to show that $f \in C_\lambda(I)$. Fix an open interval $J \subset [0, 1]$. The set $f^{-1}(J)$ is a union of open intervals, so for a given $\varepsilon > 0$ we can find finitely many connected components (open intervals) $J_1, \ldots J_m$ of $f^{-1}(J)$ such that

$$\lambda(\bigcup_{i=1}^{m} J_i) > \lambda(f^{-1}(J)) - \varepsilon; \tag{2.1}$$

since $f_n \xrightarrow{n} f$ we have $\lim_{n\to\infty} \lambda(f_n^{-1}(J) \cap J_i) = \lambda(J_i)$ and

$$\lambda(J) \geq \lim_{n\to\infty} \lambda(f_n^{-1}(J) \cap \bigcup_{i=1}^{m} J_i) = \lambda(\bigcup_{i=1}^{m} J_i).$$

Combining this with (2.1) we conclude $\lambda(J) \geq \lambda(f^{-1}(J)) - \varepsilon$. Taking into account that ε was chosen arbitrarily we get

$$\lambda(J) \geq \lambda(f^{-1}(J)) \tag{2.2}$$

and thus

$$\lambda(f^{-1}(x)) = 0 \text{ for each } x \in [0, 1]. \tag{2.3}$$

Applying (2.2) we obtain: $\frac{1}{2} \geq \lambda(f^{-1}((0, \frac{1}{2})))$ and $\frac{1}{2} \geq \lambda(f^{-1}((\frac{1}{2}, 1)))$. Applying (2.3) we obtain: $0 = \lambda(f^{-1}(0)) = \lambda(f^{-1}(\frac{1}{2})) = \lambda(f^{-1}(1))$. Together these imply

$$\tfrac{1}{2} = \lambda(f^{-1}((0, \tfrac{1}{2}))) = \lambda\left(f^{-1}\left((\tfrac{1}{2}, 1)\right)\right).$$

We proceed similarly for the dyadic intervals for each $j \in \{0, \ldots, 2^k - 1\}$ to conclude

$$\tfrac{1}{2^k} = \lambda\left(f^{-1}\left((\tfrac{j}{2^k}, \tfrac{j+1}{2^k})\right)\right). \tag{2.4}$$

Since λ is a *regular measure* (i.e., every measurable set can be approximated from above by open sets and from below by compact sets), (2.4) implies $f \in C_\lambda(I)$. □

A *critical point* of f is a point $x \in I$ such that there exists no neighborhood of x on which f is strictly monotone. Denote by $\mathrm{Crit}(f)$ the set of all critical points of f. A point x is called *periodic of period* $N \in \mathbb{N}$, if $f^N(x) = x$ and $f^i(x) \neq x$ for $1 \leq i < N$. Let us denote by $\Xi(f)$ the set of points from I for which no neighborhood has a constant slope under f. Obviously, $\mathrm{Crit}(f) \subset \Xi(f)$. Let $\mathrm{PA}(I) \subset C(I)$ denote the set of *piecewise affine* functions; i.e., functions that are affine on every interval of monotonicity and have finitely many points in the set $\Xi(f)$. Let $\mathrm{PA}_\lambda(I) \subset C_\lambda(I)$ denote the set of piecewise affine functions that preserve Lebesgue measure and $\mathrm{PA}_{\lambda(\mathrm{leo})}(I) \subset \mathrm{PA}_\lambda(I)$ such functions that are additionally *locally eventually onto* (i.e., the image under sufficiently large iterations of nonempty open sub-intervals cover I).

For a metric space (X,d) we shall use $B(x,\xi)$ for the open ball of radius ξ centered at $x \in X$ and for a set $U \subset X$ we shall denote

$$B(U,\xi) := \bigcup_{x\in U} B(x,\xi).$$

In the rest of the paper we use letter d to denote the Euclidean distance on I and $\mathbb{S}^1$.

2.1. **Window perturbations in Lebesgue measure-preserving setting.** In this subsection we briefly discuss the setting of Lebesgue measure-preserving interval maps and introduce the techniques that we will apply in the rest of the paper.

Definition 5. *We say that continuous maps $f,g:[a,b]\subset I\to I$ are* λ-equivalent *if for each Borel set $A\in\mathcal{B}$,*

$$\lambda(f^{-1}(A)) = \lambda(g^{-1}(A)).$$

For $f\in C_\lambda(I)$ and $[a,b]\subset I$ we denote by $C(f;[a,b])$ the set of all continuous maps λ-equivalent to $f|[a,b]$. We define

$$C_*(f;[a,b]) := \{h\in C(f;[a,b]) : h(a)=f(a),\ h(b)=f(b)\}.$$

Remark 6. *It follows from Definition 5 that if we take any function $f:[a,b]\to I$ and any function $q\in C_\lambda([a,b])$, then f and $f\circ q$ are λ-equivalent.*

The following definition is illustrated by Figure 1.

Definition 7. *Let f be from $C_\lambda(I)$ and $[a,b]\subset I$. For any fixed $m\in\mathbb{N}$, let us define the map $h=h\langle f;[a,b],m\rangle\colon [a,b]\to I$ for $j\in\{0,\dots,m-1\}$ by:*

$$h(a+x) := \begin{cases} f\left(a+m\left(x-\frac{j(b-a)}{m}\right)\right) & \text{if } x\in\left[\frac{j(b-a)}{m},\frac{(j+1)(b-a)}{m}\right],\ j \text{ even},\\ f\left(a+m\left(\frac{(j+1)(b-a)}{m}-x\right)\right) & \text{if } x\in\left[\frac{j(b-a)}{m},\frac{(j+1)(b-a)}{m}\right],\ j \text{ odd}. \end{cases}$$

Then $h\langle f;[a,b],m\rangle\in C(f;[a,b])$ for each m and $h\langle f;[a,b],m\rangle\in C_*(f;[a,b])$ for each m odd.

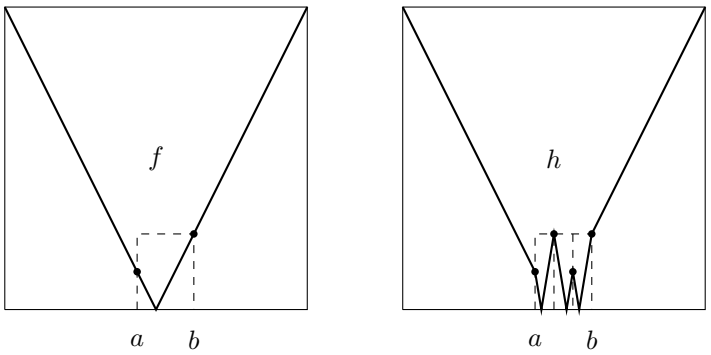

FIGURE 1. For $f\in C_\lambda(I)$ shown on the left, we show on the right the regular 3-fold piecewise affine window perturbation of f by $h=h\langle f;[a,b],3\rangle\in C_*(f;[a,b])$.

Definition 8. *For a fixed $h\in C_*(f;[a,b])$, the map $g=g\langle f,h\rangle\in C_\lambda(I)$ defined by*

$$g(x) := \begin{cases} f(x) & \text{if } x\notin[a,b],\\ h(x) & \text{if } x\in[a,b] \end{cases}$$

will be called the window perturbation *of f (by h on $[a,b]$). In particular, if $h = h\langle f;[a,b],m\rangle$, m odd, (resp. h is piecewise affine), we will speak of* regular m-fold (resp. piecewise affine) window perturbation *g of f (on $[a,b]$).*

Remark 9. *We will repeatedly use the following facts about regular m-fold piecewise affine window perturbations which either are easy to verify or follow directly from the definition. First, for every $f \in C_\lambda(I)$ Lemma 5 from* [8] *gives that for each $\varepsilon > 0$ there is a positive integer n_0 such that for each $n > n_0$, if $I_j = [j/n, (j+1)/n]$ and*

$$g|_{I_j} = h\langle f; I_j, m(j)\rangle$$

with odd numbers $m(j)$ for every $j \in \{0,\dots,n-1\}$, then independently of numbers $m(j)$ it holds $\rho(f,g) < \varepsilon$. Second, we will also often use the fact that if $f \in \mathrm{PA}_\lambda(I)$ then any regular m-fold piecewise affine window perturbation $g\langle f,h\rangle \in \mathrm{PA}_\lambda(I)$.

3. Cardinality and dimension of periodic points for generic Lebesgue measure-preserving interval and circle maps

Since generic maps from $C_\lambda(I)$ are weakly mixing (Theorem 15 from [8]) it follows that the Lebesgue measure of the periodic points of generic maps from $C_\lambda(I)$ is 0. The main result of this section is Theorem 1 which describes the structure, cardinality and dimensions of this set.

Let

$$\mathrm{Fix}(f,k) := \{x : f^k(x) = x\}$$
$$\mathrm{Per}(f,k) := \{x : f^k(x) = x \text{ and } f^i(x) \neq x \text{ for all } 1 \le i < k\}$$
$$k(x) := k \text{ for } x \in \mathrm{Per}(f,k)$$

and

$$\mathrm{Per}(f) := \bigcup_{k\ge 1} \mathrm{Per}(f,k) = \bigcup_{k\ge 1} \mathrm{Fix}(f,k).$$

Definition 10. *A periodic point $p \in \mathrm{Per}(f,k)$ is called* transverse *if there exist three adjacent intervals $A = (a_1,a_2), B = [a_2,c_1], C = (c_1,c_2)$, with $p \in B$, B possibly reduced to a point, such that (1) $f^k(x) = x$ for all $x \in B$ and either (2.a) $f^k(x) > x$ for all $x \in A$ and $f^k(x) < x$ for all $x \in C$ or (2.b) $f^k(x) < x$ for all $x \in A$ and $f^k(x) > x$ for all $x \in C$.*

Remark 11. *In our constructions we will apply the above definition only for the case when B degenerates to a point.*

To prove Theorem 1 we will use the following lemma.

Lemma 12. *For each $k \ge 1$ there is a dense set $\{g_i\}_{i\ge 1}$ of maps in $C_\lambda(I)$ such that $g_i \in \mathrm{PA}_\lambda(I)$, $\mathrm{Per}(g_i,k) \neq \emptyset$, and for each i all points in $\mathrm{Fix}(g_i,k)$ are transverse.*

Proof. The set $\mathrm{PA}_\lambda(I)$ is dense in $C_\lambda(I)$ ([7], see also Proposition 8 in [8]). Each $f \in \mathrm{PA}_\lambda(I)$ (in fact each $f \in C_\lambda(I)$) has a fixed point, so using a 3-fold window perturbation around the fixed point we can approximate f arbitrarily well by a map $f_1 \in \mathrm{PA}_\lambda(I)$ with $\mathrm{Per}(f_1,k) \neq \emptyset$.

Fix $f \in \mathrm{PA}_\lambda(I)$ with $\mathrm{Per}(f,k) \neq \emptyset$. We claim that by an arbitrarily small perturbation of f we can construct a map $g \in \mathrm{PA}_\lambda(I)$ such that

(3.1) $\mathrm{Per}(g,k) \neq \emptyset$ and all points in $\mathrm{Fix}(g,k)$ are transverse.

We do this in several steps. The first step is to perturb f to g in such a way that the points 0 and 1 are not in $\mathrm{Fix}(g,k)$; we will treat only the point 0, the arguments for the point 1 are analogous. If 0 is a fixed point we can make an arbitrarily small window perturbation as in Figure 2 so that this is no longer the case.

Now consider the case when $f^j(0) = 0$, where $j > 1$ is the period of the point 0, and $j|k$. We assume that a is so small that $f^i([0,a]) \cap [0,a] = \emptyset$ for $i = 1,2,\dots,j-1$

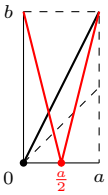

FIGURE 2. Small perturbations of a map $f \in \mathrm{PA}_\lambda(I)$ near 0.

and choose a so that $f^j(a) \neq 0$. Let g be the map resulting from a regular 2-fold window perturbation of f on the interval $[0, a]$ (see Figure 3). Thus $g(0) = f(a)$ and $g^{j-1} = f^{j-1}$ on the interval $[f(0), f(a)]$, and so $g^j(0) = g^{j-1} \circ f(a) = f^j(a) \neq 0$.

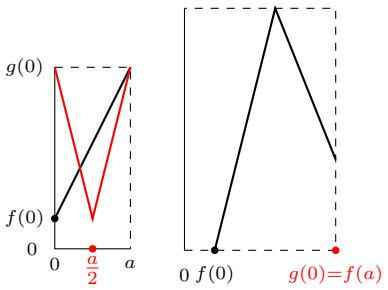

FIGURE 3. Left: small perturbations of a map $f \in \mathrm{PA}_\lambda(I)$ near 0; Right: $g^{j-1} = f^{j-1}$ on $[f(0), f(a)]$.

Thus we can choose a dense set of $f \in \mathrm{PA}_\lambda(I)$ with $\mathrm{Per}(f, k) \neq \emptyset$, and $\mathrm{Fix}(f, k) \cap \{0, 1\} = \emptyset$. Fix such a map f. We claim the following.

Claim 13. *By an arbitrarily small perturbation of f we can construct a $g \in \mathrm{PA}_\lambda(I)$ with $\mathrm{Per}(g, k) \neq \emptyset$ and $\mathrm{Fix}(g, k) \cap \{0, 1\} = \emptyset$ such that for every $c \in \mathrm{Crit}(g)$ we have $g^i(c) \notin \mathrm{Crit}(g)$ for all $1 \leq i \leq k$.*

Proof of Claim 13. Suppose that for some $c_1, c_2 \in \mathrm{Crit}(f)$ we have an $\ell \geq 1$ such that $f^\ell(c_1) = c_2$ and $f^i(c_1) \notin \mathrm{Crit}(f)$ for $1 \leq i \leq \ell - 1$. We call this orbit a *critical connection of length ℓ*. Choose c_1, c_2 with the minimal such ℓ, if there are several choices fix one of them. We will perturb f to a map g for which this critical connection is destroyed, so g has one less critical connection of length ℓ. Since there are finitely many critical connections of a given length, a finite number of such perturbations will remove all of them, and a countable sequence of perturbations will finish the proof of the claim.

If $c_1 \neq c_2$ it suffices to use a small window perturbation around c_2 as in Figure 4; if the window perturbation is disjoint from the orbit segment $f^i(c_1)$ for $i \in \{0, 1, \ldots, \ell - 1\}$ then for the resulting map g we have $g^\ell(c_1) = c_2 < \bar{c}_2$ and $g^i(c_1) \notin \mathrm{Crit}(g) = \{\bar{c}_2\} \cup \mathrm{Crit}(f) \setminus \{c_2\}$ for $i = 1, 2, \ldots, \ell - 1$, thus we have destroyed the critical connection. Let $Q = \{f^i(c) : 0 \leq i < \ell \text{ and } c \in \mathrm{Crit}(f)\} \setminus \{c_2\}$. Notice that $g(\bar{c}_2) = f(c_2)$, thus taking the neighborhood for the perturbations sufficiently small to be disjoint from Q guaranties that we did not create new critical connections of length ℓ or shorter.

Now consider the case $c_1 = c_2$. Suppose that c_1 is a local minimum of f; the other cases are similar. If $\ell = 1$ then we again move the peak using the window perturbation as in Figure 4 to destroy the connection. If $\ell > 1$ then by assumption the map $f^{\ell-1}$ in a neighborhood $U = (a, b)$ of the point $f(c_1)$ is strictly monotone.

FIGURE 4. Small perturbations of a map $f \in \mathrm{PA}_\lambda(I)$.

Using a window perturbation around c_1 as in Figure 5, yields a map $g \in C_\lambda(I)$ with $\mathrm{Per}(g,k) \neq \emptyset$ such that

$$\mathrm{Crit}(g) = \{\bar{c}_1\} \cup \mathrm{Crit}(f) \setminus \{c_1\}.$$

The critical set $\mathrm{Crit}(f)$ is finite since f is piecewise affine. If this perturbation is

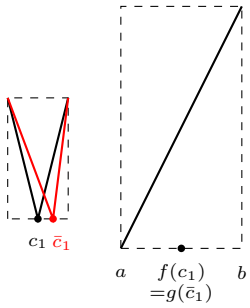

FIGURE 5. Left: small perturbations of a map $f \in \mathrm{PA}_\lambda(I)$; Right: $g^{\ell-1} = f^{\ell-1}$.

small enough to be disjoint from the set $\mathcal{U} := \cup_{i=1}^{\ell-1} f^i(U)$ then the resulting map $g|\mathcal{U} = f|\mathcal{U}$, and so $g^{\ell-1}|U = f^{\ell-1}|U$. Furthermore, if the perturbation is small enough so that $g(\bar{c}_1) \in U$ then

$$g^{\ell-1} \circ g(\bar{c}_1) = f^{\ell-1} \circ g(\bar{c}_1) = f^{\ell-1} \circ f(c_1) = c_1.$$

Moreover, we can choose the perturbation so small that these two points are arbitrarily close, i.e., $g^\ell(\bar{c}_1) \in (c_1 - \varepsilon, c_1)$, for any fixed $\varepsilon > 0$.

If ε is small enough then there are no critical points of f in the interval $(c_1 - \varepsilon, c_1)$. Furthermore, since $\bar{c}_1 > c_1$, then if the perturbation and ε are small enough it holds that $\bar{c}_1$ is also not in this interval. This procedure possibly creates new critical connections but of length at least $\ell + 1$; but inductive application of both cases will eventually get rid of the critical connections of length at most k.

If we choose the interval of perturbation small enough then no new critical connections of length at most k can be created, thus the proof of the claim is finished. □

Claim 13 implies that no critical point nor endpoint is in $\mathrm{Fix}(g,k)$ for the constructed perturbation g of f. But the map g is piecewise affine with absolute value of slope larger than 1. Therefore all points in $\mathrm{Fix}(g,k)$ must be transverse periodic points.

By Proposition 8 from [8] there is a sequence of maps $\{f_i\}_{i\geq 1} \subset \mathrm{PA}_\lambda(I)$ which is dense in $C_\lambda(I)$. For each i and n use (3.1) and define maps $\{g_{i,n}\}_{i\geq 1} \subset C_\lambda(I)$ with $\rho(f_i, g_{i,n}) < 1/n$. The sequence $\{g_{i,n}\}_{i\geq 1}$ is by definition dense in $C_\lambda(I)$ but now each $g_{i,n}$ satisfies $\mathrm{Per}(g_{i,n},k) \neq \emptyset$ and all points in $\mathrm{Fix}(g_{i,n},k)$ are transverse. This completes the proof, after renumbering the sequence $\{g_{i,n}\}_{i\geq 1}$. □

Now we have prepared all the tools to give the proof of Theorem 1. In what follows, by $\underline{\dim}_{Box}$, $\overline{\dim}_{Box}$ and $\dim_H$ we denote the lower box dimension, the upper box dimension and the Hausdorff dimension of the underlying sets respectively.

Proof of Theorem 1. First note that the last part of (3) follows from the first part of (3) since

$$\dim_H(\mathrm{Per}(f)) \leq \sup_{k\geq 1} \dim_H(\mathrm{Fix}(f,k)) \leq \sup_{k\geq 1} \underline{\dim}_{Box}(\mathrm{Fix}(f,k)) = 0.$$

For the proofs of 1), 2), 3) and 4) we fix $k \in \mathbb{N}$.

Thus we can choose a countable set $\{g_i\}_{i\geq 1} \subset \mathrm{PA}_\lambda(I)$, such that no g_i has slope ± 1 on any interval, which is dense in $C_\lambda(I)$ with each g_i satisfying the conclusion of Lemma 12, i.e., (3.1). The advantage of such g_i is that for each point in $\mathrm{Fix}(g_i,k)$, there is at least one corresponding periodic point in $\mathrm{Fix}(g,k)$ if the perturbed map g is sufficiently close to g_i.

Consider the shortest length

$$\gamma_i := \min\{|c-c'| : c,c' \in \mathrm{Crit}(g_i) \cup \{0,1\} \text{ and } c \neq c'\}$$

of the intervals of monotonicity of g_i, note that $\gamma_i > 0$ since $g_i \in \mathrm{PA}_\lambda(I)$.

Since $g_i \in \mathrm{PA}_\lambda(I)$ do not have slope ± 1 the set $\mathrm{Fix}(g_i,k)$ is finite, suppose it consists of ℓ_i disjoint orbits and the set $\mathrm{Per}(g_i,k)$ consists of $\bar{\ell}_i \leq \ell_i$ distinct orbits. In particular

$$\ell_i \leq \#\mathrm{Fix}(g_i,k) \leq k\ell_i \text{ and } \mathrm{Per}(g_i,k) = k\bar{\ell}_i. \tag{3.2}$$

Choosing one point from each of the orbits in $\mathrm{Fix}(g_i,k)$ defines the set $\{x_{l,i} : 1 \leq l \leq \ell_i\} \subset \mathrm{Fix}(g_i,k)$. Let $k(x_{l,i})$ denote the minimal period of $x_{l,i}$.

By the definition of g_i, the minimal distance

$$\eta_i := \min\{|g_i^m(x_{l,i}) - c| : 0 \leq m \leq k(x_{l,i})-1,\ \ 1 \leq l \leq \ell_i,\ c \in \mathrm{Crit}(g_i) \cup \{0,1\}\}$$

of the periodic orbits to the set $\mathrm{Crit}(g_i) \cup \{0,1\}$ is strictly positive.

If $k = \ell_i = 1$ let $\beta_i := 1$, otherwise we consider the minimal distance

$$\beta_i := \frac{1}{2}\min\{|x-x'| : x \neq x' \in \mathrm{Fix}(g_i,k)\}.$$

Let τ_i be a positive real number such that the slope of every $|(g_i^k)'(x)| < \tau_i$ for every point x where g_i^k is differentiable.

The construction in the proof depends on integers $n_i \geq 1$ which will be defined in the proof, for most of the estimates it suffices to have $n_i = 1$, but for the upper box dimension estimates we will need n_i growing sufficiently quickly. We define a new map $h_i \in \mathrm{PA}_\lambda(I)$ by applying a regular $2n_i+1$-fold window perturbation of g_i of diameter

$$a_i \leq \frac{1}{2\tau_i}\min(\frac{1}{ik\ell_i}, \eta_i, \gamma_i, \beta_i, (k\ell_i)^{-i})$$

around each of the points $x_{l,i}$ keeping the map g_i unchanged elsewhere, in particular it is unchanged around the other points in $\mathrm{Fix}(g_i,k)$. The perturbations are disjoint from one another (perturbation around $x_{l,i}$ and $x_{l',i}$) by the definition of a_i. The bound on a_i guarantees that these maps satisfy the following properties:

i) The collection $\{h_i\}_{i\geq 1}$ is dense in $C_\lambda(I)$. Namely, the slope of g_i is bounded by τ_i and h_i can differ from g_i on an interval of length at most $k\ell_i a_i$ where we perform the window perturbation. If a, b are points from that window, then

$$|g_i(a) - g_i(b)| \leq \tau_i k\ell_i a_i \leq \frac{1}{i} \to 0.$$

and therefore $\rho(h_i, g_i) \to 0$.

ii) Suppose $x_{l,i} \in \mathrm{Fix}(g_i,k)$.

a) The map $h_i^{k(x_{l,i})}$ has exactly $2n_i+1$ fixed points in the interval $I_{l,i} := [x_{l,i}-a_i, x_{l,i}+a_i]$,
b) The map h_i^k has $(2n_i+1)^{k/k(x_{l,i})}$ fixed points in this interval.
c) The full branches of $h_i^{k/k(x_{l,i})}$ have length $a_i/(2n_i+1)^{k/k(x_{l,i})}$, thus each subinterval of $I_{l,i}$ of length $2a_i/(2n_i+1)^{k/k(x_{l,i})}$ contains at least one full branch and at most parts of three branches, and thus at least one fixed point and at most 3 fixed points of h_i^k.
d) The map h_i has a point of period k in each interval $I_{l,i}$.

iii) The total number $N_{l,i}$ of fixed points of h_i^k arising from the orbit of $x_{l,i}$ satisfies

$$N_{l,i} = (2n_i+1)^{k/k(x_{l,i})}k(x_{l,i}).$$

Summing over the points $x_{l,i}$ and using $1 \le k(x_{l,i}) \le k$ yields

$$\max((2n_i+1)\ell_i, (2n_i+1)^k) \le \#\mathrm{Fix}(h_i,k) = \sum_{l=1}^{\ell_i} N_{l,i} \le (2n_i+1)^k k\ell_i.$$

Note that the last lower bound follows from the fact that we have ℓ_i different points $x_{l,i}$ and among them there is a fixed point of g_i, so there is at least one l with $k(x_{l,i})=1$.

iv) If $x_{l,i} \in \mathrm{Per}(g_i,k)$ (i.e., $k(x_{l,i})=k$) then the $N_{l,i}=(2n_i+1)k$ points are not only in $\mathrm{Fix}(h_i,k)$ but also in $\mathrm{Per}(h_i,k)$; thus $\#\mathrm{Per}(h_i,k) \ge (2n_i+1)k\bar{\ell}_i$. The reason is that we make perturbation close to point $x_{l,i}$ of period k so new periodic points obtained by perturbation have to visit all k small disjoint intervals defined by the orbit of that point (we make window perturbation on a small interval around considered point $x_{l,i}$).

v) Any interval of length $a_i/(2n_i+1)^k$ covers at most two points of $\mathrm{Fix}(h_i,k)$ (since h_i restricted to an interval of length $a_i/(2n_i+1)$ has at most one critical point).

Consider $\delta_i > 0$ and

$$\mathcal{G} := \bigcap_{j\in\mathbb{N}} \bigcup_{i\ge j} B(h_i,\delta_i).$$

The set $\mathcal{G}$ is a dense G_δ set.

(1) We claim that if $\delta_i>0$ goes to zero sufficiently quickly then $\mathrm{Fix}(f,k)$ is a Cantor set for each $f\in\mathcal{G}$. The set $\mathrm{Fix}(h_i,k)$ is finite, choose ζ_i so small that the balls of radius ζ_i around distinct points of $\mathrm{Fix}(h_i,k)$ are disjoint and such that $\zeta_i \to 0$. We can choose δ_i so small that if $f\in B(h_i,\delta_i)$ then $\mathrm{Fix}(f,k)\subset B(\mathrm{Fix}(h_i,k),\zeta_i)$; in particular the set $\mathrm{Fix}(f,k)$ can not contain an interval whose length is longer than $2\zeta_i$.

Consider the open cover of $\mathrm{Fix}(h_i,k)$ by pairwise disjoint intervals of length $2a_i$, by (ii) these intervals contain $(2n_i+1)^{k/k(x_{l,i})}$ fixed points of h_i for some $k(x_{l,i})$. Fix these covering intervals and choose δ_i sufficiently small so that all fixed points of f^k of any $f\in B(h_i,\delta_i)$ are contained in the covering intervals and so that there are at least $\#\mathrm{Fix}(h_i,k)\ge(2n_i+1)\ell_i$ such points (iii).

Fix $f\in\mathcal{G}$, thus $f\in B(h_{i_j},\delta_{i_j})$ for some subsequence i_j. Since $\zeta_{i_j}\to 0$ the set $\mathrm{Fix}(f,k)$ can not contain an interval. By its definition the set $\mathrm{Fix}(f,k)$ is closed. By the above discussion on $\#\mathrm{Fix}(h_i,k)$, there are no isolated points in $\mathrm{Fix}(f,k)$ since for any periodic point of period k we can find another point from $\mathrm{Fix}(f,k)$ arbitrary close. This completes the proof of (1).

To prove (2) we additionally assume that ζ_i converges to zero sufficiently fast, to ensure $\mathrm{Per}(f,k)\subset B(\mathrm{Per}(h_i,k),\zeta_i)$ for each $f\in B(h_i,\delta_i)$. We first claim that $x_{l,i}\in\mathrm{Per}(g_i,k)$ has period k then the $(2n_i+1)$ fixed points of h_i in the interval $I_{l,i}$ all have period k. Suppose now $x_{l,i}\in\mathrm{Fix}(g_i,k)$ has period $k(x_{l,i})$ (a strict

divisor of k), then the corresponding $(2n_i+1)$ points in $\mathrm{Fix}(h_i,k)\cap I_{l,i}$ have period $k(x_{l,i})$. If we consider the map $h_i^{k(x_{l,i})}$ restricted to $I_{l,i}$ then it is a $(2n_i+1)$-fold tent map, thus it has periodic orbits of all periods. In particular, periodic points of $h_i^{k(x_{l,i})}$ with period $k/k(x_{l,i})$ belongs to the set $\mathrm{Per}(h_i,k)$. The number of such periodic points is strictly larger than 1, and the claim follows. Fix any $f\in\mathcal{G}$, any $x\in\mathrm{Fix}(f,k)$ and any $\alpha>0$. Take i such that $\zeta_i<\alpha/3$, $2a_i<\alpha/3$ and $f\in B(h_i,\delta_i)$. Then

- $\mathrm{Fix}(f,k)\subset B(\mathrm{Fix}(h_i,k),\zeta_i)$,
- $\mathrm{Per}(f,k)\subset B(\mathrm{Per}(h_i,k),\zeta_i)$,
- $\mathrm{Fix}(h_i,k)\subset B(\mathrm{Per}(h_i,k),2a_i)$ (see ii) d) above).

Therefore $\mathrm{Per}(f,k)\cap(x-\alpha,x+\alpha)\neq\emptyset$. Thus $\mathrm{Per}(f,k)$ is dense in $\mathrm{Fix}(f,k)$.

Finally notice that $\mathrm{Per}(f,k)=\mathrm{Fix}(f,k)\setminus\cup_{1\le\ell<k:\ell|k}\mathrm{Fix}(f,\ell)$. But this finite union is closed, thus $\mathrm{Per}(f,k)$ is relatively open subset of $\mathrm{Fix}(f,k)$. This completes the proof of (2).

(3) Since $\mathrm{Per}(f,k)\subset\mathrm{Fix}(f,k)$ it suffices to prove the statement for $\mathrm{Fix}(f,k)$. Remember that the number $k\ge1$ and the sequence ℓ_i are fixed. We claim that if $\delta_i>0$ goes to zero sufficiently quickly then the lower box dimension of $\mathrm{Fix}(f,k)$ is zero for any $f\in\mathcal{G}$.

Consider the open cover of $\mathrm{Fix}(h_i,k)$ by intervals of length a_i guaranteed by (ii). Fix these covering intervals and choose δ_i sufficiently small so that all points of $\mathrm{Fix}(f,k)$ of any $f\in B(h_i,\delta_i)$ are contained in the covering intervals.

To prove the claim fix $f\in\mathcal{G}$, thus $f\in B(h_{i_j},\delta_{i_j})$ for some subsequence i_j. Let $N(\varepsilon)$ denote the number of intervals of length $\varepsilon>0$ needed to cover $\mathrm{Fix}(h_{i_j},k)$. By the choice of δ_{i_j}, these intervals of length a_{i_j} also cover $\mathrm{Fix}(f,k)$. Equation (3.2) combined with (ii) implies that $\ell_{i_j}\le N(a_{i_j})\le k\ell_{i_j}$. Combining this with the fact that $a_{i_j}\le(k\ell_{i_j})^{-i_j}$ yields

$$\frac{\log(N(a_{i_j}))}{\log(1/a_{i_j})}\le\frac{\log(k\ell_{i_j})}{\log(1/a_{i_j})}\le\frac{1}{i_j}$$

and thus the lower box dimension of $\mathrm{Fix}(f,k)$ defined as

$$\liminf_{\varepsilon\to0}\frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}$$

is 0.

(4) We begin by calculating the upper box dimension of $\mathrm{Fix}(f,k)$ of $f\in\mathcal{G}$. Here we will need to choose the sequence n_i growing sufficiently quickly. Instead of covering $\mathrm{Fix}(h_i,k)$ by intervals of length a_i we cover it by intervals of length $b_i:=2a_i/(2n_i+1)^k$. By (iic) each such interval covers at most three points of $\mathrm{Fix}(h_i,k)$. Thus we need at least $(\#\mathrm{Fix}(h_i,k))/3$ such intervals to cover $\mathrm{Fix}(h_i,k)$; so by (iii) we need at least $(2n_i+1)^k/3$ such intervals to cover $\mathrm{Fix}(h_i,k)$. Fix such a covering and choose δ_i sufficiently small so that all periodic points of period k of any $f\in B(h_i,\delta_i)$ are contained in the covering intervals.

Thus

$$\frac{\log(N(b_i))}{\log(1/b_i)}\ge\frac{\log((2n_i+1)^k/3)}{\log(1/b_i)}=\frac{\log((2n_i+1)^k)-\log(3)}{\log((2n_i+1)^k)-\log(2a_i)}. \tag{3.3}$$

The sequence a_i has been fixed above, if n_i grows sufficiently quickly the last term in (3.3) approaches one. We can not cover $\mathrm{Fix}(h_i,k)$ by fewer intervals, and thus we can not cover $\mathrm{Fix}(f,k)$ by fewer intervals for any $f\in B(h_i,\delta_i)$.

By the above discussion, if we fix any $f \in \mathcal{G}$ then $f \in B(h_{i_j}, \delta_{i_j})$ for some subsequence i_j and therefore, the upper box dimension of $\mathrm{Fix}(f,k)$ defined as

$$\limsup_{\varepsilon\to 0} \frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}$$

is 1.

Next we modify the above proof to calculate the upper box dimension of $\mathrm{Per}(f,k)$ for $k \geq 2$. Instead of $\mathrm{Fix}(h_i,k)$ we consider $\mathrm{Per}(h_i,k)$ in the calculations. As before, an interval of length b_i covers at most three points of this set, thus we need at least $\#(\mathrm{Per}(h_i,k))/3$ such intervals to cover it. Let $x_{l,i}$ be a fixed point and $I_{l,i}$ the associated interval, then

$$\begin{aligned}
\#(\mathrm{Per}(h_i,k)) &\geq \#(\mathrm{Fix}(h_i,k)\cap I_{l,i}) - \sum_{\ell|k,1\leq\ell<k} \#(\mathrm{Fix}(h_i,\ell)\cap I_{l,i}) \\
&= (2n_i+1)^k - \sum_{\ell|k,1\leq\ell<k} (2n_i+1)^\ell \\
&\geq (2n_i+1)^k \left(1-\sqrt{k}(2n_i+1)^{\sqrt{k}-k}\right).
\end{aligned}$$

Here the last inequality holds since the largest divisor of k is at most $\sqrt{k}$ and there are at most $\sqrt{k}$ positive divisors of k.

If we additionally suppose that $n_i \geq 2$ then for any $k \geq 2$ we have

$$\left(1-\sqrt{k}(2n_i+1)^{\sqrt{k}-k}\right) \geq 1-\sqrt{2}5^{\sqrt{2}-2} > \frac{2}{5}.$$

Thus the estimate (3.3) becomes

$$\begin{aligned}
\frac{\log(N(b_i))}{\log(1/b_i)} &> \frac{\log(\frac{2}{5}(2n_i+1)^k/3)}{\log(1/b_i)} \\
&= \frac{\log((2n_i+1)^k)-\log(2/15)}{\log((2n_i+1)^k)-\log(2a_i)}
\end{aligned}$$

and the rest of the proof follows in a similar manner. □

Remark 14. *The proof of Theorem 1 can easily be adapted to show that generic maps in $C(I)$ have the same properties, this does not seem to be known in our setting. Related results have been proven for homeomorphisms on manifolds of dimension at least two in* [15] *(unpublished sketch) and* [11].

While positive Lebesgue measure of periodic points can not be realized for ergodic maps, it turns out it can be seen in many leo Lebesgue measure-preserving maps. To this end let us first introduce some needed definitions.

Let $M_f(I)$ be the space of invariant Borel probability measures on I equipped with the *Prohorov metric* D defined by

$$D(\mu,\nu) = \inf\left\{\varepsilon\colon \begin{array}{c}\mu(A)\leq\nu(B(A,\varepsilon))+\varepsilon \text{ and } \nu(A)\leq\mu(B(A,\varepsilon))+\varepsilon \\ \text{for any Borel subset } A\subset I\end{array}\right\}$$

for $\mu,\nu \in M_f(I)$. The following (asymmetric) formula

$$D(\mu,\nu) = \inf\{\varepsilon\colon \mu(A)\leq\nu(B(A,\varepsilon))+\varepsilon \text{ for all Borel subsets } A\subset I\}$$

is equivalent to original definition, which means we need to check only one of the inequalities. It is also well known, that the topology induced by D coincides with the *weak*-topology* for measures, in particular $(M_f(I),D)$ is a compact metric space (for more detail on Prohorov metric and weak*-topology the reader is referred to [19]).

Lemma 15. *Fix $k \geq 1$. Assume that $\mathrm{Fix}(f,k)$ is a Cantor set and $\mathrm{Per}(f,k)$ is non-empty. For any open set $U \subset I$ such that $\mathrm{Per}(f,k) \cap U \neq \emptyset$ the set $\mathrm{Per}(f,k) \cap U$ contains a Cantor set C. Fix such a Cantor set in U, $x \in C$ and $\varepsilon > 0$. Let μ_x be the unique f-invariant Borel probability measure supported on the orbit of x. Then there is a non-atomic measure ν supported on $C \subset \mathrm{Per}(f,k)$ such that $D(\mu_x, \nu) < \varepsilon$.*

Proof. For any open set $U' \subset \overline{U'} \subset U$ if $C := \mathrm{Per}(f,k) \cap \overline{U'} = \mathrm{Fix}(f,k) \cap \overline{U'}$ is non-empty, then since $\mathrm{Fix}(f,k)$ is a Cantor set, C is a Cantor set as well. Assume U' additionally satisfies that the sets $f^i(C)$ are pairwise disjoint with $\mathrm{diam}(f^i(C)) < \varepsilon$ for $i = 0, \ldots, k-1$. Let $\hat{\nu}$ be any non-atomic probability measure on C and put $\nu = \frac{1}{k} \sum_{i=0}^{k-1} \hat{\nu} \circ f^i$. Clearly ν is f-invariant. Note that $\nu(B(f^i(x), \varepsilon)) \geq \nu(f^i(C)) = 1/k$, which yields that $D(\mu_x, \nu) < \varepsilon$. □

Proof of Theorem 2. Using Theorem 1 and the results of [8] we can choose a map $f \in C_\lambda(I)$ that is leo, ergodic and $\mathrm{Per}(f,k) \cap U$ contains a Cantor set for each k and any open set U such that $U \cap \mathrm{Per}(f,k) \neq \emptyset$. By result of Blokh, every topologically mixing interval map has the periodic specification property [6] (see also [8], Corollary 10). By a well known result of Sigmund [35, 36], so called CO-measures, i.e., ergodic measures supported on periodic orbits, are dense in the space of invariant probability measures for maps with periodic specification property. In our context it means that Lebesgue measure can be approximated arbitrarily well by a CO-measure supported on a periodic orbit. As a consequence, Lemma 15 implies that there exists a sequence μ_k of non-atomic measures supported on a subset of $\mathrm{Per}(f)$ such that $\lim_{k\to\infty} D(\mu_k, \lambda) = 0$.

Let us fix any $\varepsilon > 0$ and without loss of generality assume that $D(\mu_k, \lambda) < \varepsilon$ for every k.

Consider the measure

$$\nu := \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \mu_k.$$

By definition ν is an f-invariant Borel probability measure, so f preserves both measures λ and ν. As a combination of non-atomic measures, ν is non-atomic, and since $\lim_{k\to\infty} D(\mu_k, \lambda) = 0$, ν has full support, i.e., $\mathrm{supp}\, \nu = I$.

We define a map $h\colon I \to I$ by $h(x) := \nu([0,x])$, since ν has full support and is non-atomic, the map h is a homeomorphism. Note that by the definition of the metric D we have

$$\nu([0,x]) \leq \lambda([0, x+\varepsilon]) + \varepsilon = x + 2\varepsilon$$

and

$$x - \varepsilon = \lambda([0, x-\varepsilon]) \leq \nu([0,x]) + \varepsilon$$

hence $|x - h(x)| < 2\varepsilon$.

For each Borel set A in I we can equivalently write

$$\lambda(h(A)) = \nu(A) \text{ or } \lambda(A) = \nu(h^{-1}(A)). \tag{3.4}$$

We claim that $g := h \circ f \circ h^{-1} \in C_\lambda(I)$. Using (3.4) for any Borel set A in I we have

$$\lambda(A) = \nu(h^{-1}(A)) = \nu(f^{-1}(h^{-1}(A))) = \lambda(h(f^{-1}(h^{-1}(A)))) = \lambda(g^{-1}(A)).$$

Moreover, the maps g and f are topologically conjugated, so the map g is also leo and $h(\mathrm{Per}(f)) = \mathrm{Per}(g)$. But by (3.4) again

$$\lambda(\mathrm{Per}(g)) = \lambda(h(\mathrm{Per}(f))) = \nu(\mathrm{Per}(f)) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \mu_k(\mathrm{Per}(f)) = 1.$$

In the above construction, we may take ε arbitrarily small, therefore g can be arbitrarily small perturbation of f.

Now assume that we are given a leo map $f \in C_\lambda(I)$ for which $\lambda(\mathrm{Per}(f)) = 1$. Consider the measure

$$\eta := \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \eta_k,$$

where each η_k is obtained by application of Lemma 15 to a point $x \in \mathrm{Per}(f,k)$. Then η is non-atomic and $\eta(\mathrm{Per}(f,k)) > 0$ for every k.

We repeat the above proof (construction of map g) using measures $\nu_{\varepsilon_i} := \varepsilon_i \cdot \eta + (1-\varepsilon_i) \cdot \lambda$ where the sequence $0 < \varepsilon_i < 1$ decreases to 0. The resulting maps $\{g_{\varepsilon_i}\}$ satisfy $\rho(g_{\varepsilon_i}, f) \to 0$, thus completing the proof. □

3.1. **Periodic points for generic circle maps.** Let $C_{\lambda,d}(\mathbb{S}^1)$ denote the set of degree d maps in $C_\lambda(\mathbb{S}^1)$. The proof of Theorem 1 immediately shows:

Theorem 16. *Theorem 1 holds for generic maps in $C_{\lambda,d}(\mathbb{S}^1)$ for each $d \in \mathbb{Z} \setminus \{1\}$.*

For $C_{\lambda,1}(\mathbb{S}^1)$ the situation is more complicated, consider the open set

$$C_p := \{f \in C_{\lambda,1}(\mathbb{S}^1) : f \text{ has a transverse periodic point of period } p\}.$$

In this setting the proof of Theorem 1 yields a similar result,

Theorem 17. *For any f in a dense G_δ subset of $\overline{C}_p$ we have that for each $k \in \mathbb{N}$*

1. *the set $\mathrm{Fix}(f,kp)$ is a Cantor set,*
2. *the set $\mathrm{Per}(f,kp)$ is a relatively open dense subset of $\mathrm{Fix}(f,kp)$,*
3. *the set $\mathrm{Fix}(f,kp)$ has Hausdorff dimension and lower box dimension zero. In particular, $\mathrm{Per}(f,kp)$ has Hausdorff dimension and lower box dimension zero. As a consequence, the Hausdorff dimension of $\mathrm{Per}(f)$ is also zero.*
4. *the set $\mathrm{Per}(f,kp)$ has upper box dimension one. Therefore, $\mathrm{Fix}(f,kp)$ and $\mathrm{Per}(f)$ have upper box dimension one as well.*

Remark 18. *As in the interval case, the proof of the previous two results can easily be adapted to show that the generic degree d map in $C(\mathbb{S}^1)$ has the same properties, again this does not seem to be known in our setting.*

To interpret this result we investigate the set $C_\infty := C_{\lambda,1}(\mathbb{S}^1) \setminus \cup_{p\geq 1}\overline{C_p}$. As we already saw in the proof of Theorem 1, a periodic point can be transformed to a transverse periodic point by an arbitrarily small perturbation of the map, thus the set C_∞ consists of maps without periodic points. Using the same argument we see that $\cup_{p\geq 1}\overline{C_p}$ contains an open dense set. Therefore, C_∞ is nowhere dense in $C_{\lambda,1}(\mathbb{S}^1)$.

Proposition 19. *The set C_∞ consists of irrational circle rotations.*

Proof. Clearly C_∞ contains all irrational circle rotations. Each rational rotation is in $\overline{C_p}$ for some $p \in \mathbb{N}$.

We claim that any $f \in C_\infty$ must be invertible. For each point z denote by J_z the largest interval containing z such that $f^n(z) \notin J_z$ for all $n > 0$. Suppose that $f(x) = f(y)$ for some $x \neq y$. By definition each $f \in C_\infty$ does not have periodic point, so by [2, Theorem 1] we obtain that $J_x = J_y$, in particular both are nondegenerate intervals. By the same result intervals $f^n(J_x)$ are pairwise disjoint for all $n \geq 0$. The Poincaré recurrence theorem states that almost every point is recurrent, which is a contradiction since interior of J_x consists of non-recurrent points. Indeed f is invertible, and so all elements of C_∞ are homeomorphisms.

But each homeomorphism in C_∞ preserves Lebesgue measure by definition, thus it is an isometry. This means that elements of C_∞ are rotations, and so they are irrational rotations by the previous discussion. □

4. Shadowing is generic for Lebesgue measure-preserving interval and circle maps

First we recall the definition of shadowing and its related extensions that we will work with in the rest of the paper. For $\delta > 0$, a sequence $(x_n)_{n\in\mathbb{N}_0} \subset I$ is called a *δ-pseudo orbit* of $f \in C(I)$ if $d(f(x_n), x_{n+1}) < \delta$ for every $n \in \mathbb{N}_0$. A *periodic δ-pseudo orbit* is a δ-pseudo orbit for which there exists $N \in \mathbb{N}_0$ such that $x_{n+N} = x_n$, for all $n \in \mathbb{N}_0$. We say that the sequence $(x_n)_{n\in\mathbb{N}_0}$ is an *asymptotic pseudo orbit* if $\lim_{n\to\infty} d(f(x_n), x_{n+1}) = 0$. If a sequence $(x_n)_{n\in\mathbb{N}_0}$ is a δ-pseudo orbit and an asymptotic pseudo orbit then we simply say that it is an asymptotic δ-pseudo orbit.

Definition 20. *We say that a map $f \in C(I)$ has the:*

- shadowing property *if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following condition: given a δ-pseudo orbit $\mathbf{y} := (y_n)_{n\in\mathbb{N}_0}$ we can find a corresponding point $x \in I$ which ε-traces $\mathbf{y}$, i.e.,*
$$d(f^n(x), y_n) < \varepsilon \text{ for every } n \in \mathbb{N}_0.$$
- periodic shadowing property *if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following condition: given a periodic δ-pseudo orbit $\mathbf{y} := (y_n)_{n\in\mathbb{N}_0}$ we can find a corresponding periodic point $x \in I$, which ε-traces $\mathbf{y}$.*
- limit shadowing *if for every sequence $(x_n)_{n\in\mathbb{N}_0} \subset I$ so that*
$$d(f(x_n), x_{n+1}) \to 0 \text{ when } n \to \infty$$
there exists $p \in I$ such that
$$d(f^n(p), x_n) \to 0 \text{ as } n \to \infty.$$
- s-limit shadowing *if for every $\varepsilon > 0$ there exists $\delta > 0$ so that*
  1. *for every δ-pseudo orbit $\mathbf{y} := (y_n)_{n\in\mathbb{N}_0}$ we can find a corresponding point $x \in I$ which ε-traces $\mathbf{y}$,*
  2. *for every asymptotic δ-pseudo orbit $\mathbf{y} := (y_n)_{n\in\mathbb{N}_0}$ of f, there is $x \in I$ which ε-traces $\mathbf{y}$ and*
$$\lim_{n\to\infty} d(y_n, f^n(x)) = 0.$$

The notions of shadowing and periodic shadowing are classical but let us comment less classical notions of limit and s-limit shadowing. While limit shadowing seems completely different than shadowing, it was proved in [23] that transitive maps with limit shadowing also have the shadowing property. In general it can happen that for an asymptotic pseudo orbit which is also a δ-pseudo orbit, the point which ε-traces it and the point which traces it in the limit are different [4]. This shows that possessing a common point for such a tracing is a stronger property than the shadowing and limit shadowing properties together and this property introduced in [24] is called the s-limit shadowing.

Observation 21. *S-limit shadowing implies both classical and limit shadowing.*

4.1. Proof of genericity of shadowing.

The main step in the proof of genericity of the shadowing property in the context of maps from $C_\lambda(I)$ is the following lemma.

Lemma 22. *For every $\varepsilon > 0$ and every map $f \in C_\lambda(I)$ there are $\delta < \frac{\varepsilon}{2}$ and $F \in C_\lambda(I)$ such that:*

1. *F is piecewise affine and $\rho(f, F) < \frac{\varepsilon}{2}$,*
2. *if $g \in C_\lambda(I)$ and $\rho(F, g) < \delta$ then every δ-pseudo orbit $\mathbf{x} := \{x_i\}_{i=0}^{\infty}$ for g is ε-traced by a point $z \in I$. Furthermore, if $\mathbf{x}$ is a periodic sequence of period n, then z can be chosen to be a periodic point of period at most n.*

Proof. **Step 1. Partition.** First, let $0 < \gamma < \varepsilon/2$ be such that, if $|a-b| < \gamma$ then $|f(a) - f(b)| < \varepsilon/2$.

Let us assume that f is piecewise affine with the absolute value of the slope at least 4 on every piece of monotonicity. Indeed, we can assume that f is piecewise affine due to Proposition 8 from [8]. Furthermore, we can also assume that the absolute value of the slope of f is at least 4 on every piece of monotonicity by using regular window perturbations from Definition 8 and thus we can approximate arbitrarily well any piecewise affine map from $C_\lambda(I)$ by a piecewise affine map from $C_\lambda(I)$ having absolute value of the slope at least 4 on every piece of monotonicity. We set γ to be smaller than the length of the shortest piece of monotonicity of f. Since f preserves the Lebesgue measure it must have non-zero slope on every interval of monotonicity. Thus we can assume we have a partition $\mathcal{P} := \{[a_i, a_{i+1}] : i \in \{0, \dots n+1\}\}$ where $0 = a_0 < a_1 < \ldots < a_n < a_{n+1} = 1$ such that:

(i) $\gamma/2 \le a_{i+1} - a_i \le \gamma$ for $i = 0, \dots, n$,
(ii) if $f(a_i) \notin \{0, 1\}$ then $f(a_i) \neq a_j$ for every j.
(iii) if $f(a_i) \notin \{0, 1\}$ then $a_i \notin f(\mathrm{Crit}(f))$.

Thus $\operatorname{diam} f([a_i, a_{i+1}]) \ge \gamma$ and so $f([a_i, a_{i+1}])$ intersects interiors of at least two consecutive intervals of the partition.

Step 2. Perturbation. By the definition of the partition, there is $\delta > 0$ such that for each $j = 0, \dots, n$ we have

$$\{i : f([a_j, a_{j+1}]) \cap (a_i, a_{i+1}) \neq \emptyset\} = \{i : B(f([a_j, a_{j+1}]), 3\delta) \cap (a_i, a_{i+1}) \neq \emptyset\}. \tag{4.1}$$

Using (i) and the fact that the slope of f is at least 4 combined with assuming that δ is sufficiently small we may require that if $f([a_j, a_{j+1}]) \cap [a_i, a_{i+1}] \neq \emptyset$ then

$$f([a_j, a_{j+1}]) \supset [a_i, a_i + 2\delta] \quad \text{or} \quad f([a_j, a_{j+1}]) \supset [a_{i+1} - 2\delta, a_{i+1}]. \tag{4.2}$$

Now, repeating the construction behind Proposition 8 of [8] we construct a map F by replacing each $f|[a_i, a_{i+1}]$ by its regular m-fold window perturbation (see Definition 8 and Figure 1), with odd m large enough to satisfy $1/m < \delta$. This way F is still piecewise affine and its minimal slope is larger than the maximal slope of f and such that

$$F([a_i, a_i + \delta]) = F([a_{i+1} - \delta, a_{i+1}]) = F([a_i, a_{i+1}]) = f([a_i, a_{i+1}]). \tag{4.3}$$

Since $C_\lambda(I)$ is invariant under window perturbations we conclude $F \in C_\lambda(I)$.

Step 3. ε-shadowing. For any $x \in I$ and any $J \subset I$ in what follows denote $\operatorname{dist}(x, J) := \inf\{d(x, y) : y \in J\}$. Also, for an interval $J \subset I$ let $\operatorname{diam}(J) := \sup\{d(x, y) : x, y \in J\}$. Take any $g \in C_\lambda(I)$ such that $\rho(F, g) < \delta$ and let $\mathbf{x} := \{x_i\}_{i=0}^{\infty}$ be a δ-pseudo orbit for g. We claim that there is a sequence of closed intervals J_i such that

(1) $\operatorname{diam} J_i \le \gamma$ and if $i > 0$ then $J_i \subset g(J_{i-1})$,
(2) $\operatorname{dist}(x_i, J_i) < \gamma$,
(3) for every i there is p such that $F(J_i) = F([a_p, a_{p+1}])$ and $x_i \in [a_p, a_{p+1}]$.

Take $p \ge 0$ such that $[a_p, a_{p+1}] \ni x_0$ and put $J_0 = [a_p, a_{p+1}]$. Then conditions (1)–(3) are satisfied for $i = 0$.

Next assume that for $i = 0, \dots, m$ there are intervals J_i such that conditions (1)–(3) are satisfied. We will show how to construct J_{m+1}. Denote $F(J_m) =: [a, b]$. By (3) and the definition of F, namely (4.2) and (4.3), there are nonnegative integers $\hat{i}, \hat{j}$, $\hat{j} - \hat{i} \ge 2$ such that

$$[a_{\hat{i}+1} - 2\delta, a_{\hat{j}-1} + 2\delta] \subset [a, b] \subset [a_{\hat{i}}, a_{\hat{j}}]. \tag{4.4}$$

Furthermore, if $a_{\hat{i}} \neq 0$ then $a > a_{\hat{i}} + 2\delta$ and if $b_{\hat{j}} < 1$ then $b < a_{\hat{j}} - 2\delta$. From this it follows that $B([a, b], 2\delta) \subset [a_{\hat{i}}, a_{\hat{j}}]$. To see this, note that $[a, b] = F(J_m) =$

$f([a_p, a_{p+1}])$ and so $b-a \geq \gamma$. So there is i such that $[a_i, a_i+2\delta] \subset [a,b]$ (or symmetrically $[a_i - 2\delta, a_i] \subset [a,b]$). Now condition (4.1) implies that $[a,b] \cap (a_{i-1}, a_i) \neq \emptyset$ and therefore, by the choice of δ, we get $[a_i - 2\delta, a_i] \subset [a,b]$. This way we have candidates for $\hat{i}, \hat{j}$ which could be $i-1$ and $i+1$. But clearly it may happen that $[a,b] \setminus [a_{i-1}, a_{i+1}] \neq \emptyset$. Then we decrease or increase these indices, to find the smallest possible interval such that $[a,b] \subset [a_{\hat{i}}, a_{\hat{j}}]$ and then (4.4) follows.

Since $\rho(F,g) < \delta$ it holds

$$[a_{\hat{i}+1} - \delta, a_{\hat{j}-1} + \delta] \subset g(J_m) \subset [a_{\hat{i}}, a_{\hat{j}}] \tag{4.5}$$

and

$$g(x_m) \in B(F(x_m), \delta) \subset B([a,b], \delta)$$

and therefore

$$x_{m+1} \in B([a,b], 2\delta) \subset [a_{\hat{i}}, a_{\hat{j}}].$$

Then there is $\hat{i} \leq q < \hat{j}$ such that $x_{m+1} \in [a_q, a_{q+1}]$ and if we put $L := [a_q, a_q + \delta]$ and $R := [a_{q+1} - \delta, a_{q+1}]$ then by (4.5) it follows $L \subset g(J_m)$ or $R \subset g(J_m)$. Now, we put $J_{m+1} = L$ or $J_{m+1} = R$ depending on the situation, obtaining that $J_{m+1} \subset g(J_m)$. Additionally, $\operatorname{dist}(x_{m+1}, J_{m+1}) < \gamma$ since both L and R are contained in $[a_q, a_{q+1}]$ and by the definition of F it follows from (4.3) that

$$F(J_{m+1}) = F(L) = F(R) = F([a_q, a_{q+1}]).$$

This finishes the inductive construction.

Observe that if $\mathbf{x}$ is periodic of period n then by (3) J_n satisfies $F(J_n) = F([a_p, a_{p+1}])$ and $x_0 = x_n \in [a_p, a_{p+1}]$ so $F(J_n) = F(J_0)$ by the definition. Therefore, the choice of $\hat{i}, \hat{j}$ in the construction for J_1 and J_{n+1} is the same, and since the choice of J_1, J_{n+1} is determined only by $\hat{i}, \hat{j}$, we obtain $J_1 = J_{n+1}$. Thus, replacing J_0 by J_n will give a periodic sequence. In other words, we may require that $J_{jn+k} = J_k$ for each $j, k \geq 0$ in the case of periodic $\mathbf{x}$ of period n.

By (1) there is a point $z \in I$ such that $z \in \bigcap_{i=0}^{\infty} g^{-i}(J_i)$. Then $g^i(z) \in J_i$ for every $i \geq 0$ and so by (1) and (2) we obtain that

$$d(g^i(z), x_i) \leq \operatorname{diam} J_i + \operatorname{dist}(x_i, J_i) < 2\gamma < \varepsilon.$$

We have just proved that the pseudo orbit $\mathbf{x}$ is ε-traced by the point z.

To finish the proof, let us assume that $\mathbf{x}$ is additionally periodic. Since the sequence J_i is periodic of period n, covering relation implies that we may select z being a periodic point of period at most n. □

Proof of Theorem 3. Fix $\{\varepsilon_n\}_{n\in\mathbb{N}}$, where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Let us also fix a dense collection of maps $\{f_k\}_{k\in\mathbb{N}} \subset C_\lambda(I)$. Define the set

$$A_n := \{f \in C_\lambda(I) : \exists \delta > 0 \text{ so that every } \delta\text{-pseudo orbit is } \varepsilon_n\text{-traced}\}.$$

Let us fix $k, n \in \mathbb{N}$. By Lemma 22 it holds that for every $f \in C_\lambda(I)$ and for all integers $s > 1/\varepsilon_n$ there exist $F_{k,s} \in C_\lambda(I)$ and $\xi_{k,s} > 0$ so that $\rho(F_{k,s}, f_k) < 1/s$ and $B(F_{k,s}, \xi_{k,s}) \subset A_n$. Define

$$Q_n := \bigcup_{s > \frac{1}{\varepsilon_n}} \bigcup_{k=1}^{\infty} B(F_{k,s}, \xi_{k,s}) \subset A_n.$$

Observe that since f_k is in the closure of Q_n for all $k \in \mathbb{N}$ it follows that Q_n is dense in $C_\lambda(I)$. Also $B(F_{k,s}, \xi_{k,s})$ is an open set and thus Q_n is open in $C_\lambda(I)$ as well. Now, taking the intersection of the collection $\{Q_n\}_{n\in\mathbb{N}}$ we thus get a dense G_δ set $Q \subset C_\lambda(I)$. Clearly, if $f \in Q$ then for every $\varepsilon > 0$ there is $\delta > 0$ so that every δ-pseudo orbit is ε-traced by some trajectory of f and if δ-pseudo orbit is periodic then such trajectory of f can be required to be periodic as well. □

4.2. **S-limit shadowing for Lebesgue measure-preserving interval and circle maps.** In this subsection we address the level of occurrence of the strongest of the above presented notions related with shadowing. Let us note that in the context of Lebesgue measure-preserving circle maps we proved that s-limit shadowing is generic [9]. In the context of $C_\lambda(I)$, however, the methods used in [9] do not apply and Proposition 23 is the best we are able to prove about the s-limit shadowing (see the final section in [9] for an extended discussion on this topic).

Let us put $\mathrm{LS}_\lambda(I) := \{f \in C_\lambda(I)\colon f \text{ has the s-limit shadowing property}\}$.

Proposition 23. *The set* $\mathrm{LS}_\lambda(I)$ *is dense in* $C_\lambda(I)$.

Proof. Choose $\varepsilon > 0$. Let $g_0 \in \mathrm{PA}_{\lambda(\mathrm{leo})}(I)$. We will show how to perturb g_0 to obtain a map $g \in C_\lambda(I)$ close to g_0 - it will be specified later - which has the limit shadowing property. We will proceed analogously as in the proof of Lemma 22. In that proof for a given $\varepsilon > 0$ a perturbation of f defining F assumes a special finite partition $\mathcal{P}$ and related positive parameters γ, δ, m. We will call the whole procedure (such a map F) $(\varepsilon, \mathcal{P}, \gamma, \delta, m)$-perturbation of f.

Fix a decreasing sequence $(\varepsilon_n)_{n\geq 1}$ of positive numbers such that

$$\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon \tag{4.6}$$

Step 1. We put $f = g_0$ and consider

$$F = g_1 \text{ as } (\varepsilon_1, \mathcal{P}_1, \gamma_1, \delta_1, m_1) - \text{perturbation of } f.$$

We assume that $g_0|P_i^1$ is monotone for each $P_i^1 \in \mathcal{P}_1$. From Lemma 22 (1) it follows that $\rho(g_0, g_1) < \varepsilon_1/2$ and Lemma 22 (2) implies that for each $g \in B(g_1, \delta_1)$ (hence also for g_1 itself) every δ_1-pseudo orbit is ε_1-traced. In addition we can require $\delta_1 < \varepsilon/2$.

Step 2. We put $f = g_1$ and consider

$$F = g_2 \text{ as } (\varepsilon_2, \mathcal{P}_2, \gamma_2, \delta_2, m_2) - \text{perturbation of } f.$$

We assume that $g_1|P_i^2$ is monotone for each $P_i^2 \in \mathcal{P}_2$. Moreover, we choose $\mathcal{P}_2$ to be a refinement of $\mathcal{P}_1$, i.e., each element of $\mathcal{P}_1$ is a union of some elements of $\mathcal{P}_2$. We consider γ_2 and δ_2 so small that

$$B(g_1, \delta_1) \supset \overline{B(g_2, \delta_2)};$$

Lemma 22 implies that for each $g \in B(g_2, \delta_2)$ (hence also for g_2 itself) every δ_2-pseudo orbit is ε_2-traced.

Step n. We put $f = g_{n-1}$ and consider

$$F = g_n \text{ as } (\varepsilon_n, \mathcal{P}_n, \gamma_n, \delta_n, m_n\}) - \text{perturbation of } f.$$

We assume that $g_{n-1}|P_i^n$ is monotone for each $P_i^n \in \mathcal{P}_n$ and choose $\mathcal{P}_n$ to be a refinement of $\mathcal{P}_{n-1}$. We consider γ_n and δ_n so small that

$$B(g_{n-1}, \delta_{n-1}) \supset \overline{B(g_n, \delta_n)}; \tag{4.7}$$

Lemma 22(2) implies that for each $g \in B(g_n, \delta_n)$ (hence also for g_n itself) every δ_n-pseudo orbit is ε_n-traced.

The proof of Lemma 22 shows that for a fixed map $g \in B(g_n, \delta_n)$, for every δ_n-pseudo orbit $(x_i)_{i\geq 0}$, if $x_i \in [a^n_{q(i)}, a^n_{q(i)+1}] \in \mathcal{P}_n$ for each $i \geq 0$, there exists a sequence of intervals

$$J_i^n \in \{[a^n_{q(i)}, a^n_{q(i)} + \delta_n], [a^n_{q(i)+1} - \delta_n, a^n_{q(i)+1}]\} \tag{4.8}$$

such that

$$g(J_{i-1}^n) \supset J_i^n \tag{4.9}$$

and a point $z \in \bigcap_{i=0}^{\infty} g^{-i}(J_i^n)$ satisfies

$$|g^i(z) - x_i| < \varepsilon_n \tag{4.10}$$

for each $i \geq 0$.

By our construction, the convergence of the sequence $(g_n)_{n\geq 0}$ is uniform in $C_\lambda(I)$ hence $\lim_{n\to\infty} g_n = G \in C_\lambda(I)$. Moreover, since by (4.7),

$$G \in \bigcap_n B(g_n, \delta_n),$$

and by the previous the map G has the shadowing property, i.e., for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit is ε-traced.

Let us show that the map G has the s-limit shadowing property. Due to the definition of s-limit shadowing let us assume that a sequence $(x_i)_{i\geq 0}$ is satisfying

$$|G(x_i) - x_{i+1}| \to 0,\ i \to \infty.$$

Obviously there is an increasing sequence $(\ell(n))_{n\geq 1}$ of nonnegative integers (w.l.o.g. we assume that $\ell(1) = 0$, i.e., $(x_i)_{i\geq 0}$ is an asymptotic δ_1-pseudo orbit) such that

$$|G(x_i) - x_{i+1}| < \delta_n,\ i \geq \ell(n),$$

i.e., each sequence $(x_i)_{i\geq \ell(n)}$ is a δ_n-pseudo orbit. Now we repeatedly use the procedure describe after the equation (4.7) and containing the equations (4.8)-(4.10). By that procedure, for each $n \in \mathbb{N}$ we can find sequences $(J_i^n)_{i\geq \ell(n)}$ (to simplify our notation on the nth level we index J_i^n from $\ell(n)$) such that for each

$$z \in \bigcap_{i=\ell(n)}^{\infty} G^{-i}(J_i^n) \quad G^{\ell(n)}(z) \quad \varepsilon_n - \text{traces } (x_i)_{i\geq \ell(n)} \text{ for } G. \tag{4.11}$$

But by (4.3) and (4.1) of Step 3 in the proof of Lemma 22, we have $G(J_i^n) = g_n(J_i^n)$ for each n and i and the sequence $(\mathcal{P}_n)_{n\geq 1}$ is nested, so by (4.1) of Step 3 in the proof of Lemma 22, (4.8) and (4.9) for each n we get

$$G(J^n_{\ell(n+1)-1}) \supset J^{n+1}_{\ell(n+1)}. \tag{4.12}$$

If we define a new sequence $(K_i)_{i\geq 0}$ of subintervals of I by

$$K_i = J_i^n,\ \ell(n) \leq i \leq \ell(n+1) - 1,$$

then by (4.12) the intersection

$$K = \bigcap_{i=0}^{\infty} G^{-i}(K_i)$$

is nonempty. It follows from (4.11) and (4.6) that for each $z \in K$, $|G^i(z) - x_i| \to 0$, $i \to \infty$. If asymptotic pseudo orbit was δ-pseudo orbit at start, then the choice of intervals J_i^1 in the first step ensures ε-tracing.

In order to finish the proof let us recall that we have chosen $\varepsilon_1 < \varepsilon$ and $\delta_1 < \varepsilon/2$ hence

$$\rho(g_0, G) < \rho(g_0, g_1) + \rho(g_1, G) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since the set $\mathrm{PA}_{\lambda(\mathrm{leo})}(I)$ is dense in $C_\lambda(I)$, the conclusion of our theorem follows. □

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