Constructive aspects of functional analysis for the treatment of frictional contact

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CONSTRUCTIVE ASPECTS OF FUNCTIONAL ANALYSIS FOR THE TREATMENT OF FRICTIONAL CONTACT

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Abstract

This paper is a survey of our recent results concerning frictional unilateral contact problems. Mathematical and numerical results obtained over the past few years for both static and quasistatic problems are summarized. Undergoing developments concerning viscoelastic quasistatic problems are outlined.

1. INTRODUCTION

As most authors in this field, we focused to begin with on the following static formulation of friction expressed in terms of the displacements and not velocities : no displacement occurs if the tangential force does not reach the sliding limit, whereas a natural velocity formulation of friction would imply in such a case that the velocity is zero and therefore that the displacement can be non zero. Such a static formulation does not have a real mechanical meaning except for very specific cases such as for example when the loading is proportional and monotone (i.e. $F(t)=\lambda(t)F_0$ with λ monotone). Nevertheless, most of the mathematical results and of the first numerical methods have been developped in terms of this static formulation. In order to avoid the lack of regularity of the contact forces, a regularized problem corresponding to a nonlocal friction law was introduced by Duvaut [1]. In section 2 we comment on the existence and uniqueness results obtained for this problem under the condition that the friction coefficient is small (Cocu [2]). Some regularity results have also been established for this problem (see Cocu-Radoslovescu [3]). Rather than regularizing the friction by a nonlocal law, an alternative was to exploit the

regularization obtained through the finite element discretization. In Licht et al. [4], existence and conditional uniqueness (small friction) of the solution can be found for the discretized problem with P_1 elements, and the convergence of the algorithm used for the numerical computation is proved. Various numerical methods used in the group for Mechanics and Modelling of Contact in the LMA are presented.

Section 3 concerns recent mathematical and numerical results on evolution quasistatic problems in the presence of general loadings. Friction is now formulated in terms of velocity. A difficulty arises when writing the variational formulation which has to deal simultaneously with a displacement condition (unilateral condition) and a velocity relationship (friction). If a displacement or a velocity formulation is chosen, one of the two conditions has to be approximated. This difficulty is overcome by maintaining both variables, displacement and velocity and thus setting the problem as two coupled variational inequalities (see Cocu et al. [5]). One is an implicit variational inequality dealing essentially with friction and the other one is a variational inequality characterizing the unilateral condition ; they are strongly coupled. An incremental formulation ensues and, if the friction is sufficiently small, uniqueness of the solution of the incremental problem is established and appropriate estimates are given. Existence of a solution to the quasistatic problem is then obtained through the incremental solution. Two numerical methods are derived: one of them is written in terms of the velocities (increments of displacement), the other one is written in terms of the displacements. In the incremental formulation, a sequence of problems, very similar to the static one (but including extra terms), has to be solved : the numerical methods previously developed are then very useful. Finally an extension to elastoviscoplastic and viscoelastic materials is presented; existence and uniqueness results for a quasistatic contact problem with given friction are established.

2. ELASTOSTATIC CONTACT PROBLEMS

2.1. Unilateral contact with nonlocal friction

We consider an elastic body occupying a C^1-smooth domain Ω of $\mathbf{R}^d$, d = 2, 3, with boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, where Γ_1, Γ_2 and Γ_3 are open and disjoint parts of Γ. The body is fixed along Γ_1 with $\mathrm{mes}(\Gamma_1) > 0$ and a force density φ_2 is applied on Γ_2 . The solid is in contact with a rigid fixed support on Γ_3, the displacements on this part of the boundary being restricted by a unilateral condition of non penetration into the support (Signorini's conditions), and submitted to nonlocal friction when contact occurs. On Ω a volume force density φ_1 is applied. We suppose that we may neglect the inertial terms, that φ_1 belongs to $(L^2(\Omega))^d$ and that φ_2 belongs to $(L^2(\Gamma_2))^d$. We denote by $\sigma = (\sigma_{ij})$ the stress tensor, $e = (e_{ij})$ the strain tensor, $u = (u_i)$ the displacement field and $E = (a_{ijkl})$ the elasticity tensor of the material with the usual properties of symmetry and ellipticity. On Γ_3 we use

the classical notations for the normal and tangential components of the displacement and stress vectors.

Problem P_1 : Find u such that

$$\operatorname{div}\sigma = -\varphi_1 \text{ in } \Omega, \tag{2.1}$$

$$\sigma = E\,e \text{ in } \Omega, \tag{2.2}$$

$$e = \operatorname{grad}_s u \text{ in } \Omega, \tag{2.3}$$

$$u = 0 \text{ on } \Gamma_1, \tag{2.4}$$

$$\sigma.n = \varphi_2 \text{ on } \Gamma_2, \tag{2.5}$$

$$u_N \le 0,\ \sigma_N \ge 0,\ u_N \sigma_N = 0 \text{ on } \Gamma_3, \tag{2.6}$$

$$|\sigma_T| \le \mu|R\sigma_N| \quad \text{and} \quad \begin{cases} |\sigma_T| < \mu|R\sigma_N| & \Rightarrow u_T = 0 \\ |\sigma_T| = \mu|R\sigma_N| & \Rightarrow \exists \lambda \ge 0 \quad u_T = -\lambda\sigma_T \end{cases}, \quad \text{on } \Gamma_3, \tag{2.7}$$

where μ is the friction coefficient and $R\sigma_N$ is a regularization of the normal contact force see Duvaut [1] or Demkowicz-Oden [6].

In order to give a variational formulation we shall adopt the following notations:

$$V = \{ v \in (H^1(\Omega))^d ;\ v = 0 \text{ a.e. on } \Gamma_1\}, \tag{2.8}$$

$\|\ \|$ shall denote the norm induced by $(\ ,\ )$, the scalar product on V,

$$K = \{ v \in V;\ v_N \ge 0 \text{ a.e. on } \Gamma_3 \}, \tag{2.9}$$

$$a(u,v) = \int_\Omega \sigma_{ij}(u)e_{ij}(v), \tag{2.10}$$

$$j(u,v) = \int_{\Gamma_3} \mu|R\sigma_N(u)|\,|v_T|, \tag{2.11}$$

$$(f,v) = \int_\Omega \varphi_1 v + \int_{\Gamma_2} \varphi_2 v. \tag{2.12}$$

We suppose that $a_{ijkl} \in L^\infty(\Omega)$ $1 \le i,j,k,l \le d$, $\mu \in L^\infty(\Gamma_3)$ with $\mu \ge 0$ a.e. on Γ_3 and that $R: H^{-1/2}(\Gamma_3) \to L^2(\Gamma_3)$ is a compact linear operator, where (see [5]) $H^{-1/2}(\Gamma_3)$ is the dual space of

$$H^{1/2}(\Gamma_3) = \left\{ w|_{\Gamma_3} ;\ w \in H^{1/2}(\Gamma),\ w = 0 \text{ on } \Gamma_1 \right\}.$$

The following properties on a(.,.) and j(.,.) hold (see e.g. [7], [2]) :

$$\exists m > 0 \text{ such that } \forall u \in V \quad a(u,u) \ge m\|u\|^2, \tag{2.13}$$

$$\exists M > 0 \text{ such that } \forall u \in V \ \forall v \in V \quad |a(u,v)| \le M\|u\|\,\|v\|, \tag{2.14}$$

$$\exists C > 0 \quad \forall\ (u_1,u_2,v_1,v_2) \in V^4$$

$$|j(u_1,v_1) - j(u_1,v_2) - j(u_2,v_1) + j(u_2,v_2)| \le \hat{\mu}C\|u_1 - u_2\|\|v_1 - v_2\|, \tag{2.15}$$

where $\hat{\mu} = \|\mu\|_{L^\infty(\Gamma_3)}$.

We may now present the following weak formulation of problem P_1.

Problem P_2 : Find $u \in K$ such that

$$a(u, v-u) + j(u,v) - j(u,u) \geq (f, v-u) \quad \forall\, v \in K. \tag{2.16}$$

The proof of the formal equivalence, i.e. if u is sufficiently regular, is straightforward (see e.g. Duvaut - Lions[7] or Demkowicz - Oden[6]).

The main mathematical difficulty when considering Coulomb's friction law arises from the fact that, in the previous functional framework, the contact forces belong to $H^{-1/2}(\Gamma_3)$, so that the compactness condition, which is used in the general theory of variational methods, does not hold. The nonlocal friction law (2.7) enables us to overcome this difficulty; some physical and theoretical arguments in favour of nonlocal friction laws can be found in Oden-Pires [8].

The following result on this nonlocal formulation of a static frictional contact problem holds (see e.g.Cocu [2]).

Theorem 2.1. The problem P_2 has at least one solution. If μ satisfies the inequality $\hat{\mu} < m/C$, where m and C have been defined in (2.13) and (2.15), then problem P_2 has a unique solution.

The solution of P_2 is also locally regular. The proof of the following regularity result is based on the method of translation (see Cocu - Radoslovescu [3]).

Theorem 2.2. If Ω is C^3- smooth in all $x \in \Gamma_3$, $a_{ijkl} \in C^1(\overline{\Omega})$ $1 \leq i,j,k,l \leq d$, $\mu \in C^1(\overline{\Gamma}_3)$ and the regularized stress $R\sigma_N(u) \in C^1(\overline{\Gamma}_3)$ with $R\sigma_N(u) \geq 0$, then for every open set U such that $\overline{U} \subset \Omega \cup \Gamma_3$ we have $u \in [H^2(U)]^d$.

This regularity property enables us to consider pointwise unilateral contact conditions on Γ_3.

2.2. Regularization due to space discretization

Here the terms due to friction are not regularized by a nonlocal friction law as was the case in the preceeding section. The variational formulation of the problem is as follows.

Problem D_1 : Find $u \in K$ such that

$$a(u, v-u) - \langle \mu\sigma_N(u), |v_T| - |u_T| \rangle \geq (f, v-u) \qquad \forall v \in K, \tag{2.17}$$

where $\langle .,. \rangle$ denotes the duality pairing on $H^{1/2}(\Gamma_3) \times H^{-1/2}(\Gamma_3)$.

Not much can be said about problem D_1 from a mathematical point of view without adding extra assumptions, however if we discretize problem D_1 using a finite element method, the discretized problem can be shown to possess interesting properties (in fact the discretization has a similar regularizing effect on the friction terms as the nonlocal law).

Let us introduce the following notations:

- $\boldsymbol{T^h}$ is a regular triangulation of the domain Ω and $\overline{\Omega} = \bigcup_{T \in \boldsymbol{T^h}} T$,
- $X^h = \left\{ w \in C^0(\overline{\Omega}) \ ; \ w_{|T \in \boldsymbol{T^h}} \in P_1 \right\}$ where P_1 is the set of polynomials of degree 1,
- $V^h = \left\{ v \in (X^h)^d \ ; \ v = 0 \quad \text{on } \Gamma_1 \right\}$,
- $K^h = \left\{ v \in V^h \ ; \ v_N \le 0 \quad \text{on } \Gamma_3 \right\}$,
- $\tilde{X}^h$ is the space of traces of X^h on Γ_3,
- $\tilde{V}^h$ is the space of traces of V^h on Γ_3,
- Π^h is the restriction to Γ_3 of the interpolation operator associated with X^h. Note that Π^h possesses the two following properties which are due to the use of P_1 finite elements:

(a) $\Pi^h(v)$ is positive for all positive continuous v on Γ_3,

(b) $\exists c(h) \quad \left| \Pi^h(|v^h|) \right|_{L^2(\Gamma_3)} \le c(h) \left| v^h \right| \quad \forall v^h \in \tilde{V}^h$.

The following problem is then a discrete formulation of problem D_1 in V^h.

Problem D_2 : Find $u^h \in K^h$ such that $\forall v \in K^h$

$$a(u^h, v - u^h) - \left\langle \mu F_N^h(u^h), \Pi^h(|v_T| - |u_T^h|) \right\rangle \ge (f, v - u^h) \tag{2.18}$$

The extra projection $\Pi^h(|v|)$ shall prove to be very constructive for writing the finite dimensional problem presented in the next section. In order to simplify the notations we have supposed here that the friction coefficient μ is constant, the general case can be obtained by setting $\Pi^h \mu$ instead of μ.

Existence and uniqueness results are obtained by expressing problem D_2 as a fixed point problem on the sliding limit g. First if B is the set of positive linear mappings on $\tilde{X}^h$, for all g$\in$B the following problem,

Problem D_3 : Find $u_g^h \in K^h$ such that $\forall v \in K^h$

$$a(u_g^h, v - u_g^h) + \left\langle g, \Pi^h(|v_T| - |u_{gT}^h|) \right\rangle \ge (f, v - u_g^h) \tag{2.19}$$

has a unique solution. This is because the use of P_1 finite elements implies that the mapping $v \to \left\langle g, \left| \Pi^h (|v_T|) \right| \right\rangle$ is convex and therefore problem D_3 is a classical minimization problem of a convex functional with quadratic growth.

Let us now define the mapping T by $T(g) = -\mu F_N^h(u_g^h) \qquad \text{for } g \in B$.

T(g) is a positive linear mapping on $\tilde{X}^h$ and the existence of a fixed point of T is equivalent to the existence of a solution u^h of problem D_2. T can fairly easily be shown to possess the following two properties :

(i) $\exists$ c(h), a positive constant depending on h, such that $\forall g_1, g_2 \in B$,

$$|T(g_2) - T(g_1)|_* \le \mu\, c(h)\, |g_2 - g_1|_*$$

$$\text{where } |g|_* = \sup_{v^h \in \tilde{V}^h} \frac{\left| \langle g, v^h \rangle \right|}{|v^h|_{H^{1/2}(\Gamma_3)}}$$

(ii) $\exists$ M >0 such that $|T(g)|_* \le M \quad \forall g \in B$.

Theorem 2.3. There exists a solution to problem D_2 and this solution is unique if $\mu < \dfrac{1}{c(h)}$.

The existence of a solution to problem D_2 is obtained by applying Brouwer's theorem to the mapping T which is continuous because of (i) on the convex compact H, intersection of B with the ball of centre 0 and radius M (T(H) $\wp$ H because of (ii)).

The uniqueness of the solution to problem D_2 is obtained, under the condition that $\mu < \dfrac{1}{c(h)}$, by applying Banach's theorem to the mapping T which is then a contraction because of (i) and the fixed point of T is obtained as the limit of the fixed point iterates $g^{n+1}=T(g^n)$ with a given g^0.

2.3. Numerical treatment

The previous results are of course constructive and enable us to deduce a converging algorithm for the computation of the solution of (2.18). Using the fixed point method on the sliding limit g to determine the solution $\tilde{g} = \mu \left| F_N \right|$, we have to solve a sequence of variational inequalities given by (2.19). The properties of the bilinear form a imply that these problems are equivalent to minimizations of a non-differentiable functional under the constraint induced by the non-penetration condition. The finite dimensional problem is set as follows.

$$u \in K_h = \prod_{j=1}^{M} K^j$$

Problem N_1 : For a given G, find such that :

$$J(u) \leq J(v) \quad \forall v \in K_h \tag{2.20}$$

with $J(v) = \frac{1}{2} v^T A v - F^T v + G^T \left| v_T \right|$,

where : - $A_{ij} = a(w_i, w_j)$,

- $F_i = (f, w_i)$,

- $G_i = \int_{\Gamma_C} g\, w_i \, dx$,

- w_i is a basis function of the finite dimensional set associated with the finite element discretization,

- if I_c is the set of the subscripts related to a normal component of a displacement of a contact node, K^j is defined by :

$$K^j = \begin{cases} \mathbf{R} & \text{if } j \notin I_c \\ \mathbf{R}^- & \text{if } j \in I_c \end{cases}$$

A is a n∞n symmetric and positive definite. matrix, v and F are two n-dimensional vectors and v_T and G are two m-dimensional vectors, where n is the number of degrees of freedom (twice the number of nodes) and m is the number of tangential components of the contact displacement (number of contact nodes).

For solving problem N_1, various numerical methods can be used (see Raous et al [9], Lebon-Raous[10], Barbarin-Raous [11]):

- Successive Over-Relaxation with Projection,
- Aitken acceleration of Gauss-Seidel,
- Projected Conjugate Gradient with preconditioning.

Remark . Other numerical methods have been developed in our group "Mechanics and Modelling of Contact" at the LMA. They are based on other formulations of the original problem :

- mathematical programmming method (Lemke method) based on complementary formulation,
- Lagrangian and augmented Lagrangian formulations and associated solvers.

Some of these methods are compared in another paper of this special issue (Chabrand et al [12]).

3. QUASISTATIC CONTACT PROBLEMS

3.1. A unilateral contact problem with a nonlocal friction law in elasticity

We still consider a unilateral contact problem with a nonlocal friction law as in 2.1 but now the loading is supposed to be time dependent and the evolution to be quasistatic. For such evolution problems we can no longer adopt the static formulation of friction considered in section 2 so that the velocity field is present in the friction law. Two coupled variational inequalities constitute the variational formulation of this quasistatic problem. In this section we shall outline the proof of the existence of a solution to this quasivariational problem that can be found in [5], [13].

We use the same notations as in section 2.1. We suppose that the given forces φ_1 and φ_2 are sufficiently smooth in space and time, i.e. $\varphi_1 \in W^{1,2}(0,T;(L^2(\Omega))^d)$ and $\varphi_2 \in W^{1,2}(0,T;(L^2(\Gamma_2))^d)$ (see e.g.Cazenave - Haraux [14]) and that we may neglect the inertial terms. The quasistatic problem can be set as follows.

Problem Q_1 : Find u such that for all t□[0,T] equations (2.1) - (2.3) and conditions (2.4) - (2.6) hold together with the initial condition

$$u(0) = u_0 \quad \text{in } \Omega, \tag{3.1}$$

and with the following friction conditions for all t□[0,T]

$$|\sigma_T| \le \mu |R\sigma_N| \quad \text{and} \quad \begin{cases} \text{if } |\sigma_T| < \mu |R\sigma_N| \Rightarrow \dot{u}_T = 0 & \text{on } \Gamma_3, \\ \text{if } |\sigma_T| = \mu |R\sigma_N| \Rightarrow \exists \lambda \ge 0 \ \dot{u}_T = -\lambda \sigma_T & \text{on } \Gamma_3. \end{cases} \tag{3.2}$$

We denote by W the space $W^{1,2}(0,T;V)$, where V is defined by (2.8), and by $\|\cdot\|_W$ the norm on W and ((.,.)) the scalar product on $L^2(0,T;V)$. We suppose that μ□C(0,T;$L_+(\Gamma_3)$), that ∀t□[0,T] $\mu(t) \ge 0$ a.e. on Γ_3 and that the initial condition u_0 belongs to K and satisfies the following compatibility condition :

$$a(u_0, w-u_0) + j(u_0, w-u_0) \ge (f(0), w-u_0) \qquad \forall w \in K. \tag{3.3}$$

We adopt the following weak formulation of problem Q_1 (see [5], [13]).

Problem Q_2 : Find u□$W^{1,2}(0,T;V)$ such that $u(0)=u_0$ and for almost all t□[0,T], u(t)□K and satisfies

$$a(u(t), v-\dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \ge (f(t), v-\dot{u}(t)) + \langle \sigma_N(u(t)), v_N - \dot{u}_N(t) \rangle \quad \forall v \in V \tag{3.4}$$

$$\langle \sigma_N(u(t)), z_N - u_N(t) \rangle \ge 0 \quad \forall z \in K. \tag{3.5}$$

If a solution u of problem Q_2 is sufficiently regular it also solves problem Q_1.

The mapping j(.,.) involved in this variational formulation has the following property:

$$\exists C > 0 \quad \forall\ (u_1, u_2, v_1, v_2) \in V^4$$
$$|j(u_1, v_1) - j(u_1, v_2) - j(u_2, v_1) + j(u_2, v_2)| \le \bar{\mu} C |u_1 - u_2| |v_1 - v_2|$$

with $\bar{\mu} = \sup_{t \in [0,T]} \|\mu(t)\|_{L^\infty(\Gamma_3)}$.

An incremental formulation is obtained by operating a time discretization of problem Q_2, taking $n \in \mathbf{N}^*$ and setting $\Delta t = T/n$, $t_i = i.\Delta t$ and $f^i = f(t_i)$ for i = 0,...,n. We use an implicit scheme and obtain the following sequence $(Q_i^n)_{i=0,...,n-1}$ of variational inequalities defined for a given $u^0 \in K$ by

Problem Q_i^n : Find $u^{i+1} \in K$ such that

$$a(u^{i+1}, v - \frac{u^{i+1} - u^i}{\Delta t}) + j(u^{i+1}, v) - j(u^{i+1}, \frac{u^{i+1} - u^i}{\Delta t}) \ge (f^{i+1}, v - \frac{u^{i+1} - u^i}{\Delta t})$$
$$+ \left\langle \sigma_N(u^{i+1}), v_N - \frac{u_N^{i+1} - u_N^i}{\Delta t} \right\rangle \quad \forall v \in V, \qquad (3.6)$$

$$\left\langle \sigma_N(u^{i+1}), z_N - u_N^{i+1} \right\rangle \ge 0 \quad \forall z \in K. \qquad (3.7)$$

It can be shown by using its strong formulaton (see [5]) that problem (Q_i^n) is equivalent to the following problem:

Problem S_i^n : Find $u^{i+1} \in K$ such that

$$a(u^{i+1}, w - u^{i+1}) + j(u^{i+1}, w - u^i) - j(u^{i+1}, u^{i+1} - u^i) \ge (f^{i+1}, w - u^{i+1}) \quad \forall w \in K .$$

Inequalities such as S_i^n , for a given i, are of the same type as (2.16) and thus possess a unique solution if the friction coefficient is sufficiently small. To be more precise, the friction coefficient must satisfy the following inequality $\bar{\mu} < m / C$ where m and C have been defined in (2.13) and (2.15) . We shall from now on suppose that the friction coefficient is small enough to ensure that the above inequality is true. We suppose that u^0 satisfies the same condition (3.3) that we have imposed on u_0 , i.e. that u^0 satisfies :

$$a(u^0, w - u^0) + j(u^0, w - u^0) \ge (f^0, w - u^0) \qquad \forall w \in K . \qquad (3.8)$$

The following bounds for the incremental solution u^i and also for the difference $\Delta u^i = u^{i+1} - u^i$ can easily be obtained if $\bar{\mu} C < m$:

$$\left\|u^i\right\| \le \frac{\left\|f^i\right\|}{m-\bar{\mu}C} \qquad \forall i = 0,\dots,n \quad , \tag{3.9}$$

$$\left\|\Delta u^i\right\| \le \frac{C'\left\|\Delta f^i\right\|}{m-\bar{\mu}C} \qquad \forall i = 0,\dots,n-1 \quad . \tag{3.10}$$

We define, for all $n \in \mathbf{N}^*$, a function f_n: [0,T]$\varnothing$V by $f_n(0) = f(0)$ and $f_n(t) = f(t_{i+1})$ for $t \in]t_i,t_{i+1}]$ where $t_i = i.\text{Æ}t$ and $\text{Æ}t = T/n$.

We then consider inequalities S_i^n with $f^i = f(t_i)$ for i = 0,...,n and we define the function $u_n \in L^2(0,T;V)$ by $u_n(0) = u^0 = u_0$ where $u_0 \in K$ is the initial condition of problem Q_2 and for $t \in]t_i,t_{i+1}]$ by $u_n(t) = u^{i+1}$ solution of S_i^n i = 0,...,n-1.

Lemma 3.1. There exists a subsequence $(u_{n_p})_p$ of (u_n) such that for all $t \in [0,T]$ $(u_{n_p}(t))_p$ converges to u(t) weakly in V.

For the proof, see [5]. We shall omit the subscript p from now on for simplicity.

Remark. For all t belonging to [0,T] and for all $n \in \mathbf{N}^*$ $u_n(t)$ belongs to K so that the weak limit u(t) also belongs to K.

In order to show that u is sufficiently regular we shall now define the following sequence of functions :

$$\begin{cases} \tilde{u}_n(t) = u^i + \dfrac{(t-t_i)}{\Delta t}(u^{i+1}-u^i) & t \in]t_i,t_{i+1}] \\ \tilde{u}_n(0) = u^0 \end{cases}$$

where n belongs to the subset of **N** corresponding to the subsequence of (u_n) one has exhibited in the lemma 3.1. The mapping $\tilde{u}_n$ is obviously an element of $W^{1,2}(0,T;V)$. We have the following result.

Lemma 3.2. There exists a subsequence of $(\tilde{u}_n)$ that converges weakly in $W^{1,2}(0,T;V)$ and the weak limit of this subsequence is equal to u.

The functions u_n and $\tilde{u}_n$ are defined by using u^{i+1}, solution of S_i^n, but also solution of Q_i^n , for i=0,...,n-1. A solution of Q_i^n is such that $\langle \sigma_N(u^{i+1}), u_N^{i+1} \rangle = 0$ and as $\sigma_N(u^{i+1}) \le 0$ and $u_N^i \le 0$ we have $\langle \sigma_N(u^{i+1}), u_N^i \rangle \ge 0$ so that

$$\left\langle \sigma_N(u^{i+1}), v_N - \frac{u_N^{i+1} - u_N^i}{\Delta t} \right\rangle \geq \left\langle \sigma_N(u^{i+1}), v_N \right\rangle$$ and by integration on [0,T] we obtain for all $v \in L^2(0,T;V)$:

$$\int_0^T (a(u_n(t), v(t) - \frac{d}{dt}\tilde{u}_n(t)) + j(u_n(t), v(t)) - j(u_n(t), \frac{d}{dt}\tilde{u}_n(t)))dt \geq$$
$$\int_0^T (f_n(t), v(t) - \frac{d}{dt}\tilde{u}_n(t))dt + \int_0^T \langle \sigma_N(u_n(t)), v_N(t) \rangle dt \,. \qquad (3.11)$$

Lemma 3.3. For all $v \in L^2(0,T;V)$ we have

$$\lim_{n\to\infty} \int_0^T (f_n(t), v(t) - \frac{d}{dt}\tilde{u}_n(t))dt = \int_0^T (f(t), v(t) - \dot{u}(t))dt,$$
$$\lim_{n\to\infty} \int_0^T j(u_n(t), v(t))dt = \int_0^T j(u(t), v(t))dt,$$
$$\lim_{n\to\infty} \int_0^T a(u_n(t), v(t))dt = \int_0^T a(u(t), v(t))dt.$$

The proof is straightforward as (u_n) converges weakly towards u, $(\frac{d}{dt}\tilde{u}_n)$ converges weakly towards $\dot{u}$ in $L^2(0,T;V)$, (f_n) converges strongly towards f, and R is a compact operator.

Lemma 3.4.([5]) We have

$$\lim_{n\to\infty} \inf \int_0^T a(u_n(t), \frac{d}{dt}\tilde{u}_n(t))dt \geq \int_0^T a(u(t), \dot{u}(t))dt,$$
$$\lim_{n\to\infty} \inf \int_0^T j(u_n(t), \frac{d}{dt}\tilde{u}_n(t))dt \geq \int_0^T j(u(t), \dot{u}(t))dt.$$

By passing to the limit in inequality (3.11), using lemmas 3.3, 3.4 and the continuity property of σ_N the following result can be proved.

Theorem 3.1.([5]) The weak limit u of (u_n) satisfies :

$$\int_0^T (a(u(t), v(t) - \dot{u}(t)) + j(u(t), v(t)) - j(u(t), \dot{u}(t)))dt$$
$$\geq \int_0^T (f(t), v(t) - \dot{u}(t))dt + \int_0^T \langle \sigma_N(u(t)), v_N(t) - \dot{u}_N(t) \rangle dt \qquad (3.12)$$

$\forall v \in L^2(0,T;V)$, and the unilateral condition

$$\forall t \in [0,T] \quad \forall v \in K \qquad \langle \sigma_N(u(t)), v_N - u_N(t) \rangle \geq 0 . \tag{3.13}$$

Equation (3.12) is an integrated form of equation (3.4).

3.2. A contact problem with friction for a class of inelastic bodies

The purpose of this section is to study the existence and uniqueness of solutions of quasistatic contact problems with given friction (or Tresca's friction law), involving the time derivative of u for a general model of elastoviscoplastic materials with internal state variables ([15], [16]). This model includes some viscoelastic ones as, for example, the Maxwell's model (see [17], [18]).

The quasistatic evolution contact problem with given friction for a general model of elastoviscoplastic material with internal state variables has the following classical formulation.

Problem Q_3 : Find the fields of displacements u, inelastic strain tensor $e^i = (e^i_{ij})$ and internal variables $\alpha = (\alpha_1,..., \alpha_\nu)$ such that for all $t \in [0,T]$ (2.1), (2.3), (2.4), (2.5) and (3.1) hold together with the following equations and conditions:

$$\sigma = E\, e^e \ \text{ in } \Omega, \tag{3.14}$$

$$\dot{e}^i = \Phi_1(\sigma, \alpha) \ \text{ in } \Omega, \tag{3.15}$$

$$\dot{\alpha} = \Phi_2(\sigma, \alpha) \ \text{ in } \Omega, \tag{3.16}$$

$$e^i(x, 0) = e^i_0(x) \ \text{ in } \Omega, \tag{3.17}$$

$$\alpha(x, 0) = \alpha_0(x) \ \text{ in } \Omega, \tag{3.18}$$

$$\sigma_N = F_N \ \text{ on } \Gamma_3, \tag{3.19}$$

$$|\sigma_T| \leq g \quad \text{and} \quad \begin{cases} \text{if } |\sigma_T| < g \ \Rightarrow \dot{u}_T = 0 & \text{on } \Gamma_3, \\ \text{if } |\sigma_T| = g \ \Rightarrow \exists \lambda \geq 0 \ \dot{u}_T = -\lambda \sigma_T & \text{on } \Gamma_3 . \end{cases} \tag{3.20}$$

with $e(u) = e^e + e^i$, where $e = (e_{ij})$ and $e^e = (e^e_{ij})$ are respectively the total strain and the elastic strain, and $e^i_0 \in S$ with $S = \{\tau = (\tau_{ij}) ; \tau_{ij} \in L^2(\Omega) , \tau_{ij} = \tau_{ji} , 1 \leq i,j \leq d\}$ and $\alpha_0 \in [L^2(\Omega)]^\nu$.

We suppose that $\Phi_1 : R_s^{dxd} \times R^\nu \to R_s^{dxd}$ with $R_s^{dxd} = \{b = (b_{ij}) \in R^{dxd} ; b_{ij} = b_{ji} \ 1 \leq i, j \leq d\}$ and $\Phi_2 : R_s^{dxd} \times R^\nu \to R^\nu$ satisfy the following Lipschitz conditions:

$$\exists \gamma_1 > 0 \text{ such that } \forall b , c \in R_s^{dxd}, \ \forall \alpha , \beta \in R^\nu,$$
$$|\Phi_1(b, \alpha) - \Phi_1(c, \beta)|_{R^{dxd}} \leq \gamma_1 (|b - c|_{R^{dxd}} + |\alpha - \beta|_{R^\nu}). \tag{3.21}$$

$$\exists \gamma_2 > 0 \text{ such that } \forall b , c \in R_s^{dxd}, \ \forall \alpha, \beta \in R^\nu ,$$
$$|\Phi_2(b, \alpha) - \Phi_2(c, \beta)|_{R^\nu} \leq \gamma_2 (|b - c|_{R^{dxd}} + |\alpha - \beta|_{R^\nu}). \tag{3.22}$$

We suppose also that $\varphi_1 \in W^{1,2}(0,T;(L^2(\Omega))^d)$, $\varphi_2 \in W^{1,2}(0,T;(L^2(\Gamma_2))^d)$, $F_N \in W^{1,2}(0,T;L^2(\Gamma_3))$ and that, for simplicity, $g \in L^\infty(\Gamma_3)$ is independent of time.

In order to set the classical problem in variational form let us define:

$$j(v) = \int_{\Gamma_3} g\,|v_T| \qquad \forall\, v \in V, \tag{3.23}$$

$$(f,v) = \int_\Omega \varphi_1 v + \int_{\Gamma_2} \varphi_2 v + \int_{\Gamma_3} F_N v_N \qquad \forall\, v \in V, \tag{3.24}$$

$$b(\omega,\tau) = \int_\Omega a_{ijkl}\omega_{ij}\tau_{kl} \qquad \forall\, \omega, \tau \in S, \tag{3.25}$$

where V is defined by (2.8).

We suppose that the initial conditions u_0 and e^i_0 satisfy the following compatibility condition:

$$a(u_0,w-u_0) + j(w-u_0) \geq (f(0),w-u_0) + b(e^i_0, e(w-u_0)) \quad \forall w \in V, \tag{3.26}$$

where a is defined by (2.10).

If U is a Banach space we denote by $L^\infty_\lambda(0,T;U)$, $\lambda \geq 0$, the space $L^\infty(0,T;U)$ with the norm $\|y\|_{U,\lambda} = \text{ess sup}\{e^{-\lambda t}\|y(t)\|_U\ ;\ t \in [0,T]\}$.

For any $\lambda', \lambda'' \geq 0$ the norms $\|\ \|_{U,\lambda'}$ and $\|\ \|_{U,\lambda''}$ are equivalent but suitable choices for λ will be useful in proving the existence and uniqueness result. We set $\Lambda = [L^2(\Omega)]^\nu$.

If one knows the evolution of e^i on [0,T], then, using Green's formula, the solution u satisfies the following variational inequality for all $t \in [0,T]$

$a(u,v-\dot{u}) + j(v) - j(\dot{u}) \geq (f,v-\dot{u}) + b(e^i, e(v-\dot{u})) \quad \forall\, v \in V.$

We can now state the following mixed formulation of problem Q_3.

Problem Q_4 : Find $u \in W$, $e^i \in W^{1,\infty}(0,T;S)$ and $\alpha \in W^{1,\infty}(0,T;\Lambda)$ satisfying (3.14) to (3.16) and

$$a(u(t),v-\dot{u}(t)) + j(v) - j(\dot{u}(t)) \geq (f(t),v-\dot{u}(t)) + b(e^i(t), e(v-\dot{u}(t))) \quad \forall\, v \in V, \tag{3.27}$$

for almost all $t \in [0,T]$, with the initial conditions (3.1), (3.17) and (3.18), where W is defined by (3.3).

Let us state the following standard result.

Lemma 3.5. For each $\omega \in L^\infty(0,T;S)$ there exist a unique $\alpha(\omega) \in W^{1,\infty}(0,T;\Lambda)$ and a unique $e^i(\omega) \in W^{1,\infty}(0,T;S)$ which for almost all $t \in [0,T]$ satisfy the following equations:

$$\dot{e}^i = \Phi_1(\omega,\alpha) \text{ in } \Omega, \tag{3.28}$$

$$\dot{\alpha} = \Phi_2(\omega,\alpha) \text{ in } \Omega, \tag{3.29}$$

with the initial conditions (3.17) and (3.18).

The differential equation (3.29) satisfies the Cauchy - Lipschitz theorem for any given $\omega \in L^\infty(0,T;S)$ so that the proof of this lemma is straightforward.

Lemma 3.6. For each $\omega \in L^\infty(0,T;S)$ there exists a unique $u(\omega) \in W$ which satisfies the variational inequality (3.27), for almost all $t \in [0,T]$, where $e^i = e^i(\omega)$ is given by lemma 3.5.

For the proof, we observe that for each $\omega \in L^\infty(0,T;S)$ we have $e^i(\omega) \in W^{1,\infty}(0,T;S)$ so that (3.27) is a variational inequality for which, by using similar arguments as for the Problem Q_2, we obtain the existence of a solution $u(\omega) \in W$. The uniqueness readily follows.

Theorem 3.2. Let the above conditions on Φ_1, Φ_2, u_0, e^i_0, α_0, φ_1, φ_2, F_N and g hold. Then the problem (3.14) to (3.16) and (3.27) with the initial conditions (3.1), (3.17) and (3.18) has a unique solution

$(u, e^i, \alpha) \in W^{1,2}(0,T;V) \times W^{1,\infty}(0,T;S) \times W^{1,\infty}(0,T;\Lambda)$.

Proof.. For each $\omega \in L^\infty(0,T;S)$ let $s(\omega) = (s_{ij}(\omega))$ be given by

$$s_{ij}(\omega) = a_{ijkl}[e_{kl}(u(\omega)) - e^i_{kl}(\omega)] \quad 1 \le i, j, k, l \le N, \tag{3.30}$$

where $u(\omega)$ and $e^i(\omega)$ are defined as above. Thus we define a new map

$s: L^\infty(0,T;S) \to L^\infty(0,T;S)$.

We will show that $s : L^\infty_\lambda(0,T;S) \to L^\infty_\lambda(0,T;S)$ is a contraction mapping for a suitable choice of λ. Then s has a unique fixed point σ and $(u(\sigma), e^i(\sigma), \alpha(\sigma))$ is the unique solution of the problem.

Let $\omega, \tau \in L^\infty_\lambda(0,T;S)$. Then $z = u(\omega)$, $w = u(\tau) \in W^{1,2}(0,T;V)$ are the solutions of the following variational inequalities for almost all $t \in [0,T]$:

$$a(z, v-\dot{z}) + j(v) - j(\dot{z}) \ge (f, v-\dot{z}) + b(e^i(\omega), e(v-\dot{z})) \quad \forall\, v \in V, \tag{3.31}$$

$$a(w, v-\dot{w}) + j(v) - j(\dot{w}) \ge (f, v-\dot{w}) + b(e^i(\tau), e(v-\dot{w})) \quad \forall\, v \in V. \tag{3.32}$$

Choosing v = $\dot{w}$ in (3.31) and v = $\dot{z}$ in (3.32), integrating both sides of these inequalities and adding the resulting inequalities, we obtain, using the initial conditions,

$$\|z - w\|_{V,\lambda} \le C_1 \left\| e^i(\omega) - e^i(\tau) \right\|_{S,\lambda} + (C_1/\lambda) \left\| \dot{e}^i(\omega) - \dot{e}^i(\tau) \right\|_{S,\lambda} \quad \text{for all } \lambda \ge 0. \tag{3.33}$$

Now, by (3.30)

$$\|s(\omega) - s(\tau)\|_{S,\lambda} \le C_2 (\|z - w\|_{V,\lambda} + \left\| e^i(\omega) - e^i(\tau) \right\|_{S,\lambda}) \quad \text{for all } \lambda \ge 0, \tag{3.34}$$

and, using (3.28), (3.29) and the properties of Φ_1, Φ_2, one can infer (see [19]) the existence of $C_3, \lambda' > 0$ such that , for all $\lambda \ge \lambda'$

$$\left\| e^i(\omega) - e^i(\tau) \right\|_{S,\lambda} \le C_3 (\lambda - C_3)^{-1} \|\omega - \tau\|_{S,\lambda} . \tag{3.35}$$

Finally, from (3.34), (3.33), (3.35) and (3.28) it results that there exists $\lambda'' > 0$ such that if $\lambda \ge \lambda''$ then $s : L^\infty_\lambda(0,T;S) \to L^\infty_\lambda(0,T;S)$ is a contraction and the theorem is proved.

3.3. Numerical treatment

For solving the elastic quasistatic problem of section 3.1, the fixed point method on the sliding limit (introduced in section 2 with the problem D_3) is now associated to the incremental formulation of problem S^n_i. Then a problem similar to the static discrete problem N_1 is set for each t_{i+1}. It includes an extra term u^i_T which characterizes the memory of the system induced by friction.

Problem N_2 : Find $\bar{u}^{i+1}_G \in \prod_{j=1}^{M} K^j$ such that

$$J_1(\bar{u}^{i+1}_G) \le J_1(\bar{v}) \qquad \forall \bar{v} \in \prod_{j=1}^{M} K^j \tag{3.36}$$

with

$$J_1(\bar{v}) = \frac{1}{2} \bar{v}^T A \bar{v} - (\bar{F}^{i+1})^T \bar{v} + (\bar{\bar{G}}^{i+1})^T \left| \bar{\bar{v}}_T - \bar{\bar{u}}^i_T \right| . \tag{3.37}$$

The same methods as the ones of section 2.3 can be used with a slight modification for the treatment of the friction term. In problem N_2, it has to be emphasised that the convex set remains the same for every t_{i+1} because the formulation is written with respect to the displacements. As presented in Cocu et al. [5] an alternative resolution can be deduced from

the incremental formulation : it is set in terms of the velocities (increments of displacement), the friction term remains the same as the one of the static case but the constraint convex set changes at each step. The problem can be expressed in the following way.

Problem N_3 : Find $\Delta\bar{u}_G^{i+1} \in \prod_{j=1}^{M} K_j^i$ such that

$$J_2(\Delta\bar{u}_G^{i+1}) \le J_2(\bar{v}) \qquad \forall \bar{v} \in \prod_{j=1}^{M} K_j^i \tag{3.38}$$

with $$J_2(\bar{v}) = \frac{1}{2}\bar{v}^T A\bar{v} - (\bar{R}^i + \Delta\bar{F}^{i+1})^T\bar{v} + (\bar{\bar{G}}^{i+1})^T\left|\bar{\bar{v}}_T\right|, \tag{3.39}$$

where the convex set depends on the solution obtained at the previous step and is given by

$$K_j^i = \begin{cases} \mathbf{R} \text{ if } j \notin I_c \\ \{\alpha \ ; \ (\alpha + v_j^i) \le 0\} \text{ if } j \in I_c \end{cases}$$

$\bar{R}^i = \bar{F}^i - A\bar{u}^i$ being the previous residue of equilibrium (equal to the contact forces).

So, the various algorithms used for solving the static problem can be extended to the treatment of the quasistatic problem thanks to the simple modifications which give the dependance on the loading history.

In viscoelasticity or viscoplasticity, as shown in Raous [17][18][20], the problem can be set as a quasistatic elastic problem coupled with a differential equation by a viscous strain term. Various numerical methods can be used to integrate the differential equation, explicit Runge-Kutta methods such as Euler or Heun, or implicit ones such as θ−methods. In linear viscoelasticity, either explicit or implicit methods are easy to implement (see Raous [18] and Chabrand et al. [21]). A generalization to viscoplasticity is given in Raous [20] but explicit methods are prefered in that case.

4. CONCLUSION

Mathematical tools of functional analysis play a major part in this paper as the static and quasistatic unilateral contact problems with friction are written as implicit variational inequalities, of which the dual form would be quasivariational inequalities.

We have chosen to work directly with the initial formulation of the problem in terms of inequalities rather than adopting other formulations such as the compliance model. The results concerning the quasistatic problems constitute a new advance in the field and may help us in the formulation of dynamic frictional contact problems on which we are actually working.

REFERENCES

[1] G. DUVAUT, Equilibre d'un solide élastique avec contact unilatéral et frottement de Coulomb, *C. R. Acad. Sci. Paris*, 290, série A, 1980, 263 - 265.

[2] M. COCU, Existence of solutions of Signorini problems with friction, *Int. J. Engng. Sci.*, 22, 1984, 567 - 575.

[3] M. COCU and A. RADOSLOVESCU, Regularity properties for the solutions of a class of variational inequalities, *Nonlinear Analysis Theory Meth. Applic.,* Vol 11, 2, 1987, 221-230.

[4] C. LICHT, E. PRATT and M. RAOUS, Remarks on a numerical method for unilateral contact including friction, *International Series of Numerical Mathematics*, Vol. 101, Birkhäuser Verlag, Basel, 1991, 129 - 143.

[5] M. COCU, E. PRATT and M. RAOUS, Formulation and approximation of quasistatic frictional contact, *Int. J. Engng. Sci.*, to appear.

[6] L. DEMKOWICZ and J.T. ODEN, On some existence and uniqueness results in contact problems with nonlocal friction, *Nonlinear Analysis Theory Meth. Applic.,* 6, 1982, 1075 - 1093.

[7] G. DUVAUT and J.-L. LIONS, *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, 1972.

[8] J.T. ODEN and E. PIRES, Nonlocal and nonlinear friction laws and variational principles for contact problems in elasticity, *J. Appl. Mech.,* 50, 1983, 67 - 76.

[9] M. RAOUS, P. CHABRAND and F. LEBON, Numerical methods for frictional contact problems and applications, *J.ournal de Mécanique Théorique et Appliquée.*, Special issue, sup. 1 to vol. 7, 1988, 111 - 128.

[10] F. LEBON and M. RAOUS, Multibody contact problem including friction in structure assembly, *Comp. Struct.*, 43, 1992, 925 - 934.

[11] M. RAOUS and S. BARBARIN, Preconditioned conjugate gradient method for a unilateral problem with friction, in *"Contact Mechanics"*, Ed. A.Curnier, Presses Polytechniques et Universitaires Romandes, 1992, 423 - 432.

[12] P. CHABRAND, F. DUBOIS and M. RAOUS, Comparison of various numerical methods for solving unilateral contact problems with friction, in this issue.

[13] M. COCU, E. PRATT and M. RAOUS, Existence d'une solution du problème quasi-statique de contact unilatéral avec frottement non local, *C. R. Acad. Sci. Paris série I*, 320, 1995, 1413 -1417.

[14] T. CAZENAVE and A. HARAUX, *Introduction aux Problèmes d'Evolution Semi - linéaires*, Mathématiques et Applications, Ellipses, Paris, 1990.

[15] J. NECAS and I. HLAVÁCEK, *Mathematical theory of elastic and elasto plastic bodies* , Elsevier Scientific Publishing Company, Amsterdam, 1981.

[16] P. LABORDE, Approximation en viscoplasticité, *Journal de Mécanique Théorique et Appliquée* , vol 1, 1982, 419 - 428.

[17] M. RAOUS, Contacts unilatéraux avec frottement en viscoélasticité , in *"Unilateral Problems in Structural Analysis* , Eds. G. Del Piero, F. Maceri, CISM - Courses and lectures N° 288, 1985, 269 - 297.

[18] M. RAOUS, A periodic problem in viscoelasticity with variable coefficients, *Int. J. Engng. Sci.,* 19, 1981, 1145 - 1168.

[19] M. COCU, Unilateral contact problems with friction for an elastoviscoplastic material with internal state variables, in *"Contact Mechanics"*, Ed. A.Curnier, Presses Polytechniques et Universitaires Romandes, 1992, 207 - 215.

[20] M. RAOUS, *Fissuration sous contraintes alternnées en viscoélasticité et viscoplasticité,* Thèse de Doctorat d'Etat, Université de Provence, 1980.

[21] P. CHABRAND, C. LICHT, O. MAISONNEUVE, M. RAOUS, Residual thermal tempering stresses, *Comp. Struct.*, 31, 1989, 1003 - 1011.