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# Reduced symmetric models for modal analysis of internal structural-acoustic and hydroelastic-sloshing systems

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It is proposed to investigate several variational formulations for modal analysis of bounded fluid–structure systems, taking into account in particular compressibility effects in the fluid for internal vibroacoustic problems, either gravity effects, in the context of an incompressible fluid, for hydroelastic-sloshing interaction problems. Appropriate reduced symmetric matrix models are constructed using zero-frequency static solutions of the fluid. In this respect, attention is paid to the behaviour of the fluid–structure equations with regard to their validity for zero-frequency static limit.

## 1. Introduction

This paper deals with appropriate computational methods for modal analysis of elastic structures containing an inviscid fluid (gas or liquid). These methods, based on Ritz–Galerkin projection using appropriate functional basis, allow us to construct *reduced models* expressed in terms of physical displacement vector field  $u$  in the structure, and generalized coordinates vector  $\mathbf{r}$  describing the behavior of the fluid. Those reduced models lead to symmetric generalized eigenvalue matrix system involving a reduced number of degrees of freedom for the fluid. More precisely, we construct symmetric matrix models of the fluid considered as a subsystem, by considering the response of the fluid to a prescribed normal displacement of the fluid–structure interface. Two distinct situations are analyzed, namely, linear vibrations of an elastic structure completely filled with a compressible gas or liquid and linear vibrations of an elastic structure containing an incompressible liquid with free surface effects due to gravity. The first case is a *structural-acoustic* problem with modal interaction between *structural modes in vacuo* for structure containing a gas or *incompressible hydroelastic modes* for structure containing liquids, with *acoustic modes in rigid motionless cavity*. Wall impedance can also be easily introduced in order to take into account fluid–structure interface dissipation, for further forced response studies. The second case is a *hydroelastic-sloshing* problem with modal interaction between *incompressible hydroelastic structural modes* with *incompressible liquid sloshing modes* in rigid motionless cavity, involving an elastogravity operator *elastogravity operator* related to the wall normal displacement of the fluid–structure interface. For the construction of reduced models, the static behavior at zero frequency plays an important role. This is why we start from “well-posed” variational formulations of the problem, in the sense that zero-frequency behavior must be well retrieved in the equations. It should be noted that the so-called “quasi-static correction” term plays a fundamental role in the Ritz–Galerkin procedure (error truncation). The general methodology corresponds to dynamic

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substructuring procedures adapted to fluid–structure modal analysis. For general presentations of computational methods using appropriate finite element and dynamic substructuring procedures applied to modal analysis of elastic structures containing inviscid fluids (sloshing, hydroelasticity and structural-acoustics), we refer the reader to [5], and for the case of various damping effects, we refer the reader to [7].

## 2. Structural acoustic problem

We consider here the harmonic vibrations of an elastic structure completely filled with a homogeneous, inviscid and compressible fluid considered as weightless (linear acoustic fluid). Among the applications, we find, for instance, the important problem of acoustic comfort of “passengers” related to the vibroacoustic environment such as in launcher payloads, aircraft, automobiles. The particular case of a compressible liquid with a free surface, neglecting gravity effects, is also considered (for applications involving compressibility effects in huge cryogenic tanks).

After an original derivation of the linearized equations of the fluid–structure coupled system, introducing an appropriate linear constraint in order to obtain a well-posed problem at zero frequency, we discuss the basic ideas leading to a reduced model of the “fluid subsystem”. For this purpose, acoustic modes in rigid motionless cavity are introduced as Ritz projection vector basis, including mandatory zero-frequency (static) solution of the coupled system. Let us recall that the internal fluid–structure system is characterized by a strong resonant behavior. Therefore, a direct finite element description can be costly. That is why, starting from one of the possible variational formulations of the problem, *reduced symmetric matrix models* are constructed.

## 3. Structural acoustic equations

### 3.1. Structure subjected to a fluid pressure loading

We consider an elastic structure occupying the domain  $\Omega_S$  at equilibrium. The internal fluid–structure interface is denoted as  $\Sigma$ . The angular frequency is denoted as  $\omega$ . The chosen unknown field in the structure domain  $\Omega_S$  is the displacement field  $u$ . The linearized deformation tensor is denoted as  $\epsilon_{ij}(u)$  and the corresponding stress tensor is denoted as  $\sigma_{ij}(u)$ . We denote by  $\rho_S$  the constant mass density at equilibrium and by  $n$  the unit normal, external to the structure domain  $\Omega_S$ . Let  $\delta u$  be the test function, associated to  $u$ , belonging to the admissible space  $\mathcal{C}_u$ .

The weak variational formulation describing the response of the structure  $\Omega_S$  to given harmonic forces of amplitude  $F^d$  on the external structure boundary  $\partial\Omega_S \setminus \Sigma$ , and to fluid pressure field  $p$  acting on the internal fluid–structure interface  $\Sigma$  is written as follows. For all real  $\omega$  and  $\forall \delta u \in \mathcal{C}_u$ , find  $u \in \mathcal{C}_u$  such that

$$\tilde{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \delta u \, dx - \int_{\Sigma} p n \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \delta u \, d\sigma, \quad (1)$$

in which

$$\tilde{k}(u, \delta u) = k(u, \delta u) + k_G(u, \delta u) + k_{P_0}(u, \delta u), \quad (2)$$

and where  $k(u, \delta u)$  is the mechanical elastic stiffness such that

$$k(u, \delta u) = \int_{\Omega_S} \sigma_{ij}(u) \epsilon_{ij}(\delta u) \, dx, \quad (3)$$

and where  $k_G(u, \delta u)$  and  $k_{P_0}(u, \delta u)$  are such that

$$k_G(u, \delta u) = \int_{\Omega_S} \sigma_{ij}^0 u_{1,i} \delta u_{1,j} \, dx, \quad k_{P_0} = \int_{\Sigma} P_0 n_1(u) \delta u \, d\sigma. \quad (4)$$

In Eqs. (4) and (5),  $k_G(u, \delta u)$  represents the classical structural prestress geometric symmetric bilinear form in which  $\sigma_{ij}^0$  denotes the prestress tensor, and  $k_{P_0}(u, \delta u)$  represents an additional prestress symmetric bilinear form due to rotation of normal  $n$ , in which  $P_0$  denotes the initial pressure existing in the reference equilibrium configuration. Finally,  $n_1(u)$  represents the variation of normal  $n$  between the reference configuration and the actual configuration.

### 3.2. Fluid subjected to a wall normal displacement

Since the fluid is inviscid, instead of describing the small motion of the fluid by a fluid displacement vector field  $u_F$  (see for instance [1]), which requires an appropriate discretization of the fluid irrotationality constraint  $\text{curl } u_F = 0$ , we will use the pressure scalar field  $p$ . It should be noted that the small movements corresponding to  $\omega \neq 0$  are obviously irrotational, but in the static limit case, i.e. at zero frequency, we consider only fluids which exhibit a physical irrotational behavior.

Let us denote by  $c$  the (constant) sound speed in the fluid, and by  $\rho_F$ , the (constant) mass density of the fluid at rest. We denote as  $\Omega_F$  the domain occupied by the fluid at rest (which is taken as the equilibrium state). The local equations describing the harmonic response of the fluid to a prescribed arbitrary normal displacement  $un$  of the fluid–structure interface  $\Sigma$  are such that

$$\nabla p - \rho_F \omega^2 u_F = 0|_{\Omega_F}, \quad (5)$$

$$p = -\rho_F c^2 \nabla u_f|_{\Omega_F}, \quad (6)$$

$$u_F n = un|_{\Sigma}, \quad (7)$$

$$\text{curl } u_F = 0|_{\Omega_F}. \quad (8)$$

Eq. (5) corresponds to the linearized Euler equation in the fluid. Eq. (6) corresponds to the constitutive equation of the barotropic fluid. Eq. (7) corresponds to the wall slipping condition. Eq. (8) corresponds to the irrotationality condition, only necessary in order to ensure that when  $\omega \rightarrow 0$ ,  $u_F$  tends to static irrotational motion, which corresponds to the hypothesis that for  $\omega = 0$ , we only consider irrotational motions (for simply connected fluid domain).

### 3.3. Relation between static pressure $p^s$ and $un$

For  $\omega = 0$ , Eqs. (6) and (7) lead to the constant pressure static field which is related to the normal wall displacement by the relation (see [5,6], and for the dissipative case [7])

$$p^s = -\frac{\rho_F c^2}{|\Omega_F|} \int_{\Sigma} un d\sigma, \quad (9)$$

in which  $|\Omega_F|$  denotes the measure of the volume occupied by domain  $\Omega_F$ .

### 3.4. Equations in terms of $p$ and $un$

The elimination of  $u_F$  between Eqs. (5)–(8) leads to

$$\Delta p + \frac{\omega^2}{c^2} p = 0|_{\Omega_F}, \quad (10)$$

$$\frac{\partial p}{\partial n} = \rho_F \omega^2 un|_{\Sigma}, \quad (11)$$

with the constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, dx + \int_{\Sigma} u n \, d\sigma = 0. \quad (12)$$

Eq. (10) is the classical Helmholtz equation expressed in terms of  $p$ . Eq. (11) corresponds to the kinematic condition defined by Eq. (7). The linear constraint defined by Eq. (12) corresponds to the global mass conservation which ensures that the boundary problem defined by Eqs. (10) and (11) is equivalent to the problem defined by Eqs. (5)–(8). In the absence of the condition defined by Eq. (12), we would obtain a boundary value problem in terms of  $p$  which is not valid for  $\omega = 0$  and which does not allow us to retrieve the value of  $p^s$  given by Eq. (9). This condition is usually omitted in the literature.

### 3.5. Variational formulation in terms of $(u, p)$

Let  $\delta p$  be the test function, associated to  $p$ , belonging to the admissible space  $\mathcal{C}_p$ . The weak variational formulation corresponding to Eqs. (10)–(12) is obtained by the usual test-function method using Green's formula. The weak variational formulation corresponding to the modal analysis of the structural-acoustic problem is then obtained by setting  $F^d = 0$  in Eq. (1). The eigenvalue structural-acoustic variational formulation is then stated as follows. Find  $\omega^2$  and  $u \in \mathcal{C}_u$  and  $p \in \mathcal{C}_p$ , such that for all  $\delta u \in \mathcal{C}_u$  and  $\delta p \in \mathcal{C}_p$ , we have

$$\tilde{k}(u, \delta u) - \int_{\Sigma} p n \, \delta u \, d\sigma = \omega^2 \int_{\Omega_S} \rho_S u \, \delta u \, dx, \quad (13)$$

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \nabla \delta p \, dx = \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} p \delta p \, dx + \omega^2 \int_{\Sigma} u n \delta p \, d\sigma, \quad (14)$$

with the constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, dx + \int_{\Sigma} u n \, d\sigma = 0. \quad (15)$$

The variational formulation defined by Eqs. (13)–(15) is original due to the presence of the constraint defined by Eq. (15) which regularizes the  $(u, p)$  formulation for  $\omega = 0$  (see also [8]). In effect, usually, only Eqs. (13) and (14) are written, and as pointed out above, are not valid for  $\omega = 0$ . In the case of a direct finite element discretization of Eqs. (13)–(15), we obtain a matrix eigenvalue system of the type  $\mathbf{A}\mathbf{Y} = \omega^2 \mathbf{B}\mathbf{Y}$ , in which  $\mathbf{A}$  and  $\mathbf{B}$  are not symmetric. That is why various symmetric formulations using for the fluid pressure field  $p$  and displacement potential  $\varphi$ , defined up to an additive constant and such that  $u_F = \nabla \varphi$ , have been derived. The resulting *symmetric* formulations are then obtained by elimination of  $p$  or  $\varphi$  (see [2,4]), or for detailed complete analysis, [5]. In the present case, we are not considering a direct finite element approach of the variational formulation defined by Eqs. (13)–(15).

### 3.6. Symmetric reduced model

Let us consider the following two basic problems. The first one corresponds to the acoustic modes in rigid motionless cavity and is obtained by setting  $u = 0$  into Eqs. (14) and (15). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace  $\mathcal{C}_p^*$  of  $\mathcal{C}_p$

$$\mathcal{C}_p^* = \left\{ p \in \mathcal{C}_p; \int_{\Omega_F} p \, dx = 0 \right\}, \quad (16)$$

the variational formulation of acoustic modes is stated as follows: find  $\omega^2 > 0$  and  $p \in \mathcal{C}_p^*$  such that,  $\forall \delta p \in \mathcal{C}_p^*$ , we have

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \nabla \delta p \, dx = \omega^2 \frac{1}{\rho_F c^2} \int_{\Omega_F} p \delta p \, dx, \quad (17)$$

with the constraint

$$\int_{\Omega_F} p \, dx = 0. \quad (18)$$

It should be noted that, in practice, if the constraint condition (18) is “omitted”, we only add a first *nonphysical* zero-frequency constant pressure mode, the other modes corresponding to  $\omega \neq 0$  remaining the same as those defined by Eqs. (17) and (18). This zero-frequency mode must not be retained in any Ritz–Galerkin projection analysis. In addition, we have the following orthogonality conditions

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p_\alpha p_\beta \, dx = \mu_\alpha \delta_{\alpha\beta}, \quad \frac{1}{\rho_F} \int_{\Omega_F} \nabla p_\alpha \nabla p_\beta \, dx = \mu_\alpha \omega_\alpha^2 \delta_{\alpha\beta}. \quad (19)$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement  $un$ . The solution, denoted as  $p^s(un)$ , is given by Eq. (9). For any deformation  $un$  of the fluid–structure interface,  $p^s(un)$  belongs to a subset of  $\mathcal{C}_p$ , denoted as  $\mathcal{C}^{un}$

$$\mathcal{C}^{un} = \left\{ p^s \in \mathcal{C}_p; p^s = -\frac{\rho_F c^2}{|\Omega_F|} \int_{\Sigma} un \, d\sigma \right\}. \quad (20)$$

In the variational formulation defined by Eqs. (13)–(15),  $p$  is searched under the form

$$p = p^s(un) + \sum_{\alpha=1}^{N_p} r_\alpha p_\alpha, \quad (21)$$

in which  $N_p$  denotes the number of retained acoustic modes. The decomposition (21) is unique. In addition, it should be noted that, since each eigenvector  $p_\alpha$  corresponding to  $\omega_\alpha \neq 0$ , verifies the constraint defined by Eq. (18), then, using Eq. (9), we deduce that  $p$  and  $un$  satisfy the constraint defined by Eq. (15). The decomposition defined by Eq. (21) corresponds to a decomposition of the admissible class  $\mathcal{C}_p$  into the direct sum of the admissible classes defined, respectively, by Eqs. (20) and (16)

$$\mathcal{C}_p = \mathcal{C}^{un} \oplus \mathcal{C}_p^*. \quad (22)$$

Following Eq. (21), the test function  $\delta p$  is then searched under the following form:

$$\delta p = p^s(\delta un) + \sum_{\alpha=1}^{N_p} \delta r_\alpha p_\alpha. \quad (23)$$

The variational formulation in  $\delta u$  defined by Eq. (13) becomes

$$\tilde{k}(u, \delta u) + k^s(u, \delta u) - \sum_{\alpha=1}^{N_p} r_\alpha \int_{\Sigma} p_\alpha n \, \delta u \, d\sigma = \omega^2 \int_{\Omega_S} \rho_S u \, \delta u \, dx, \quad (24)$$

in which  $\tilde{k}(u, \delta u)$  is defined by Eq. (2) and  $k^s(u, \delta u)$  is such that

$$k^s(u, \delta u) = \frac{\rho_F c^2}{|\Omega_F|} \left( \int_{\Sigma} un \, d\sigma \right) \left( \int_{\Sigma} \delta un \, d\sigma \right). \quad (25)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of Eq. (24) can be written as

$$[\tilde{\mathbf{K}} + \mathbf{K}^s]\mathbf{U} - \sum_{\alpha=1}^n \mathbf{C}_\alpha r_\alpha = \omega^2 \mathbf{M}\mathbf{U}, \quad (26)$$

in which symmetric matrices  $\tilde{\mathbf{K}}$  and  $\mathbf{K}^s$  correspond to finite element discretization of stiffness symmetric bilinear forms defined by Eqs. (2)–(4) and (25), respectively. In Eq. (26),  $\mathbf{M}$  denotes the structural-symmetric mass matrix and rectangular coupling matrix  $\mathbf{C}_\alpha$  corresponds to the discretization of the coupling fluid–structure contribution  $\int_\sigma p \delta u n d\sigma$ . The discretized form of Eq. (14) in  $\delta p$  can be rewritten as

$$\omega_\alpha^2 \mu_\alpha r_\alpha = \omega^2 \mu_\alpha r_\alpha + \omega^2 \mathbf{C}_\alpha^T \mathbf{U}. \quad (27)$$

From Eqs. (26) and (27), we obtain the following *symmetric matrix reduced model*

$$\begin{pmatrix} \mathbf{K}^{\text{tot}} & \mathbf{0} \\ \mathbf{0} & \text{Diag}(\mu_\alpha) \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{r} \end{pmatrix} = \omega^2 \begin{pmatrix} \mathbf{M}^{\text{tot}} & \mathbf{D} \\ \mathbf{D}^T & \text{Diag}(\mu_\alpha/\omega_\alpha^2) \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{r} \end{pmatrix}, \quad (28)$$

in which  $\mathbf{r}$  denotes the vector of  $N$  generalized coordinates  $r_\alpha$ , with  $1 \leq \alpha \leq N_p$ , and

$$\mathbf{K}^{\text{tot}} = \tilde{\mathbf{K}} + \mathbf{K}^s, \quad (29)$$

$$\mathbf{M}^{\text{tot}} = \mathbf{M} + \sum_{\alpha=1}^{N_p} \frac{1}{\omega_\alpha^2 \mu_\alpha} \mathbf{C}_\alpha \mathbf{C}_\alpha^T, \quad (30)$$

$$\mathbf{D}_\alpha = \sum_{\alpha=1}^{N_p} \frac{1}{\omega_\alpha^2} \mathbf{C}_\alpha. \quad (31)$$

Further diagonalization of Eq. (28) implies a projection of  $\mathbf{U}$  on the solutions of the following eigenvalue problem

$$\mathbf{K}^{\text{tot}} \mathbf{U}_\beta = \lambda_\beta \mathbf{M}^{\text{tot}} \mathbf{U}_\beta. \quad (32)$$

Setting

$$\mathbf{U} = \sum_{\beta=1}^{N_u} q_\beta \mathbf{U}_\beta, \quad (33)$$

in which  $q_\beta$  are the generalized coordinates describing the structure. Using the orthogonality conditions associated with the solutions of Eq. (32), i.e.  $\mathbf{U}_{\beta'}^T \mathbf{M}^{\text{tot}} \mathbf{U}_\beta = \mu_\beta^s \delta_{\beta\beta'}$  and  $\mathbf{U}_{\beta'}^T \mathbf{K}^{\text{tot}} \mathbf{U}_\beta = \mu_\beta^s \lambda_\beta \delta_{\beta\beta'}$ , Eq. (28) becomes

$$\begin{pmatrix} \text{Diag} \lambda_\beta & \mathbf{0} \\ \mathbf{0} & \text{Diag} \omega_\alpha^2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \omega^2 \begin{pmatrix} I_{N_u} & [\mathbf{C}_{\beta\alpha}] \\ [\mathbf{C}_{\beta\alpha}]^T & I_{N_p} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix}. \quad (34)$$

It should be noted that *in vacuo structural modes* can also be used (instead of those defined by Eq. (32)), but the resulting matrix system would not be diagonal with respect to  $\mathbf{U}$ .

### 3.7. Wall impedance condition

Referring to [3] or for more detailed modeling, to [7], wall impedance condition corresponds to a particular fluid–structure interface modeling. This interface is considered as a third medium with infinitesimal thickness, without mass, and with the following constitutive equation

$$p = j\omega Z(\omega)(un - u_F n), \quad (35)$$

in which  $Z(\omega)$  denotes a complex impedance. Eqs. (7) and (11) must be replaced by Eq. (35), using  $\partial p / \partial n = \rho_F \omega^2 u_F n$ .

### 3.8. Case of a liquid with a free surface

Let us consider a liquid with a free surface at rest denoted as  $\Gamma$ . If we neglect gravity effects, the boundary condition on  $\Gamma$  is such that

$$p = 0|_{\Gamma}. \quad (36)$$

In this case, constraint condition (12) or (15) is replaced by Eq. (36). Eq. (9) is replaced by  $p^s = 0$ . Admissible space defined by Eq. (16) becomes  $\mathcal{C}_p^* = \{p \in \mathcal{C}_p; p = 0\}$ .

In this case, the static problem defined in Section 3.3 leads to a zero-pressure field.

Let us remark that in this case, the ‘‘structural’’ modal basis may be constituted by the hydroelastic incompressible modes using the classical added mass operator.

The reduced modal matrix models has been extended to the dissipative case (dissipative internal fluid with nonhomogeneous local impedance wall condition) in [7].

## 4. Incompressible hydroelastic-sloshing problem

We consider the harmonic vibrations of an elastic structure partially filled with an homogeneous, inviscid and incompressible liquid, taking into account gravity effects on the free surface  $\Gamma$ . After an original derivation of the linearized equations of the fluid–structure coupled problem, introducing an appropriate linear constraint in order to obtain a well-posed problem at zero frequency, we discuss the main ideas leading to a reduced model of the ‘‘liquid subsystem’’. For this purpose, sloshing modes in rigid motionless cavity are introduced as Ritz projection vector basis, including mandatory zero-frequency (static) solution of the coupled system. Let us recall that the internal fluid–structure system is characterized by a strong resonant behavior. Therefore, a direct finite element description can be costly. That is why, starting from one of the possible variational formulations of the problem, reduced symmetric matrix models of this system are constructed.

## 5. Hydroelastic-sloshing equations

### 5.1. Structure subjected to a fluid pressure loading

The notations are the same that those defined in Section 3.1. The weak variational formulation describing the response of the structure  $\Omega_S$  to given variation  $F^d$  of the applied forces with respect to the equilibrium state on the external structure boundary  $\partial\Omega_S \setminus \Sigma$ , and to fluid pressure field  $p$  acting on the internal fluid–structure interface  $\Sigma$  is written as follows.

For all real  $\omega$  and  $\forall \delta u \in \mathcal{C}_u$ , find  $u \in \mathcal{C}_u$  such that

$$\widehat{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \delta u dx - \int_{\Sigma} p n \delta u d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \delta u d\sigma, \quad (37)$$

in which

$$\widehat{k} = \tilde{k} + k_{\Sigma}. \quad (38)$$

In Eq. (38),  $\tilde{k}(u, \delta u)$  is defined by Eq. (2), and  $k_{\Sigma}$  is the *elastogravity* symmetric bilinear form such that (see [5], chapter 6)

$$k_{\Sigma}(u, \delta u) = -\frac{1}{2} \rho_F g \left\{ \int_{\Sigma} [zn_1(u) \delta u + u_z \delta un] d\sigma + \int_{\Sigma} [zn_1(\delta u)u + \delta u_z un] d\sigma \right\}. \quad (39)$$



### 5.2. Fluid subjected to a wall normal displacement

We assume that the liquid is homogeneous, inviscid and incompressible. Free surface  $\Gamma$  is horizontal at equilibrium. We denote by  $z$  the external unit normal to  $\Gamma$ , and by  $g$  the gravity. The notations are similar to those of Section 3.2. The local equations describing the response of the fluid to a prescribed arbitrary normal displacement  $un$  of the fluid–structure interface  $\Sigma$  are such that

$$\nabla p - \rho_F \omega^2 u_F = 0|_{\Omega_F}, \quad (40)$$

$$\nabla u_f = 0|_{\Omega_F}, \quad (41)$$

$$u_F n = un|_{\Sigma}, \quad (42)$$

$$p = \rho_F g u_F n|_{\Gamma}, \quad (43)$$

$$\text{curl } u_F = 0|_{\Omega_F}. \quad (44)$$

Eq. (41) corresponds to the incompressibility condition. Eq. (43) is the constitutive equation on the free surface  $\Gamma$  due to gravity effects.

### 5.3. Relation between static pressure $p^s$ and $un$

For  $\omega = 0$ , Eqs. (41)–(43) lead to the constant pressure static field which is related to the normal wall displacement by the relation (see [5,6])

$$p^s = -\frac{\rho_F g}{|\Gamma|} \int_{\Sigma} un d\sigma, \quad (45)$$

in which  $|\Gamma|$  denotes the measure of the area of free surface  $\Gamma$ .

### 5.4. Equations in terms of $p$ and $un$

The elimination of  $u_F$  between Eqs. (40)–(44) leads to

$$\Delta p = 0|_{\Omega_F}, \quad (46)$$

$$\frac{\partial p}{\partial n} = \rho_F \omega^2 un|_{\Sigma}, \quad (47)$$

$$\frac{\partial p}{\partial z} = \frac{\omega^2}{g} p|_{\Gamma}, \quad (48)$$

with the constraint

$$\frac{1}{\rho_F g} \int_{\Gamma} p d\sigma + \int_{\Sigma} un d\sigma = 0. \quad (49)$$

The linear constraint defined by Eq. (49) ensures that the boundary problem defined by Eqs. (46)–(49) is equivalent to the problem defined by Eqs. (40)–(44). This condition is usually omitted in the literature.

### 5.5. Variational formulation in terms of $(u, p)$

Let  $\delta p$  be the test function, associated to  $p$ , belonging to the admissible space  $\mathcal{C}_p$ . The weak variational formulation corresponding to Eqs. (46)–(49) is obtained by the usual test-function method using Green's

formula. Recalling Eq. (37), the variational formulation of the *hydroelastic-sloshing* problem is then stated as follows. Find  $u \in \mathcal{C}_u$  and  $p \in \mathcal{C}_p$ , such that for all  $\delta u \in \mathcal{C}_u$  and  $\delta p \in \mathcal{C}_p$ , we have

$$\widehat{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \delta u \, dx - \int_{\Sigma} p n \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \delta u \, d\sigma, \quad (50)$$

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \nabla \delta p \, dx = \frac{\omega^2}{\rho_F g} \int_{\Gamma} p \delta p \, dx + \omega^2 \int_{\Sigma} u n \delta p \, d\sigma, \quad (51)$$

with the constraint

$$\frac{1}{\rho_F g} \int_{\Gamma} p \, d\sigma + \int_{\Sigma} u n \, d\sigma = 0. \quad (52)$$

The variational formulation defined by Eqs. (50)–(52) is original due to the presence of the constraint defined by Eq. (49) which regularizes the  $(u, p)$  formulation for  $\omega = 0$  [8].

### 5.6. Symmetric reduced matrix model

Let us consider the following two basic problems. The first one corresponds to the sloshing modes in rigid motionless cavity and is obtained by setting  $u = 0$  into Eqs. (47) and (49). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace  $\mathcal{C}_p^*$  of  $\mathcal{C}_p$

$$\mathcal{C}_p^* = \left\{ p \in \mathcal{C}_p; \int_{\Gamma} p \, d\sigma = 0 \right\}, \quad (53)$$

the variational formulation of acoustic modes is stated as follows: find  $\omega^2 > 0$  and  $p \in \mathcal{C}_p^*$  such that, for all  $\delta p \in \mathcal{C}_p^*$ , we have

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \nabla \delta p \, dx = \omega^2 \frac{1}{\rho_F g} \int_{\Gamma} p \delta p \, d\sigma, \quad (54)$$

with the constraint

$$\int_{\Gamma} p \, d\sigma = 0. \quad (55)$$

It should be noted that, in practice, if the constraint condition (55) is “omitted”, we only add a first *nonphysical* zero-frequency constant pressure mode, the other modes corresponding to  $\omega \neq 0$  remaining the same as those defined by Eqs. (54) and (55). This zero-frequency mode must not be retained in any Ritz–Galerkin projection analysis. In addition, we have the following orthogonality conditions

$$\frac{1}{\rho_F g} \int_{\Gamma} p_\alpha p_\beta \, d\sigma = \mu_\alpha \delta_{\alpha\beta}, \quad \frac{1}{\rho_F} \int_{\Omega_F} \nabla p_\alpha \nabla p_\beta \, dx = \mu_\alpha \omega_\alpha^2 \delta_{\alpha\beta}. \quad (56)$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement  $un$ . The solution, denoted as  $p^s(un)$ , is given by Eq. (45). For any deformation  $un$  of the fluid–structure interface,  $p^s(un)$  belongs to a subset of  $\mathcal{C}_p$ , denoted as  $\mathcal{C}^{un}$

$$\mathcal{C}^{un} = \left\{ p^s \in \mathcal{C}_p; p^s = -\frac{\rho_F g}{|\Gamma|} \int_{\Sigma} u n \, d\sigma \right\}. \quad (57)$$

In the variational formulation defined by Eqs. (50)–(52),  $p$  is searched under the form

$$p = p^s(un) + \sum_{\alpha=1}^{N_p} r_\alpha p_\alpha, \quad (58)$$

in which  $N_p$  denotes the number of retained sloshing modes. The decomposition (58) is unique. In addition, it should be noted that, since each eigenvector  $p_x$  corresponding to  $\omega_x \neq 0$ , verifies the constraint defined by Eq. (55), then, using Eq. (45), we deduce that  $p$  and  $un$  satisfy the constraint defined by Eq. (52). The decomposition defined by Eq. (58) corresponds to a decomposition of the admissible class  $\mathcal{C}_p$  into the direct sum of the admissible classes defined, respectively, by Eqs. (56) and (57),  $\mathcal{C}_p = \mathcal{C}^{un} \oplus \mathcal{C}_p^*$ .

The variational formulation in  $u$  defined by Eq. (50) becomes

$$\widehat{k}(u, \delta u) + k^s(u, \delta u) - \sum_{x=1}^{N_p} r_x \int_{\Sigma} p_x n \delta u \, d\sigma = \omega^2 \int_{\Omega_S} \rho_S u \delta u \, dx, \quad (59)$$

in which  $\widehat{k}(u, \delta u)$  is defined by Eq. (38) and  $k^s(u, \delta u)$  is such that

$$k^s(u, \delta u) = \frac{\rho_F g}{|\Gamma|} \left( \int_{\Sigma} u n \, d\sigma \right) \left( \int_{\Sigma} \delta u n \, d\sigma \right). \quad (60)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of Eq. (60) can be written as

$$[\widehat{\mathbf{K}} + \mathbf{K}^s] \mathbf{U} - \sum_{x=1}^n \mathbf{C}_x r_x - \omega^2 \mathbf{M} \mathbf{U} = \mathbf{F}^d, \quad (61)$$

in which symmetric matrices  $\widehat{\mathbf{K}}$  and  $\mathbf{K}^s$  correspond to finite element discretization of stiffness symmetric bilinear forms defined by Eqs. (38) and (60), respectively. The discretized form of Eq. (51) in  $\delta p$  can then be written as

$$\omega_x^2 \mu_x r_x = \omega^2 \mu_x r_x + \omega^2 \mathbf{C}_x^T \mathbf{U}. \quad (62)$$

From Eqs. (61) and (62), we obtain a *symmetric matrix reduced model* whose expression is similar to the one given by expression (28).

Similarly to Section 3.6, further diagonalization can be obtained by setting

$$\mathbf{U} = \sum_{\beta=1}^{N_u} q_{\beta} \mathbf{U}_{\beta}, \quad (63)$$

in which  $q_{\beta}$  are the generalized coordinates describing the structure and  $\mathbf{U}_{\beta}$  are the eigenmodes of an eigenvalue problem similar to the one described by Eq. (32). We then obtain a similar matrix system than the one described by Eq. (34)

$$\begin{pmatrix} \text{Diag } \lambda_{\beta} & 0 \\ 0 & \text{Diag } \omega_x^2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} - \omega^2 \begin{pmatrix} I_{N_u} & [C_{\beta x}] \\ [C_{\beta x}]^T & I_{N_p} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^d \\ \mathbf{0} \end{pmatrix}. \quad (64)$$

It should be noted that we can also use the *incompressible hydroelastic modes*, i.e. the modes of the coupled system constituted by the elastic structure containing an incompressible liquid, with  $p = 0$  on  $\Gamma$  (through an *added mass operator*). In this case, the resulting matrix system is not completely diagonal with respect to  $\mathbf{U}$  variables [5].

## 6. Conclusion

Using a new approach, we have presented convenient and versatile tools for low frequency modal analysis of elastic structures containing linear inviscid homogeneous fluids for *structural-acoustics* and *incompressible hydroelastic-sloshing* problems. Those formulations, together with dynamic substructuring techniques [9], lead to *symmetric reduced matrix systems* expressed in terms of generalized coordinates for the fluid (and if necessary for the structure).

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