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ABSTRACT: Various formulations are reviewed in the case of linear vibration analysis of bounded fluid-structure systems for low modal density situations. Compressibility effects in the fluid for interior structural-acoustic problems, and free surface gravity effects, as well as for hydroelastic-sloshing interaction problems in the case of incompressible liquids, are examined. Those matrix models leading to symmetric matrix systems are then described using static well-posed behaviour of the irrotational fluid. In this respect, the fluid-structure boundary value problem, expressed in terms of fluid scalar field variables for the fluid (and displacement variables for the structure) is regularized for zero-frequency limit. Reduced order models are then investigated.

KEY WORDS: Structural vibration; Fluid-structure intercation; reduced order model.

1 INTRODUCTION

We review in this paper various variational formulations for modal analysis of elastic structures containing an inviscid fluid (gas or liquid). Those methods allow us to construct *reduced models* expressed in terms of physical displacement vector field u in the structure, and generalized displacement vector $\mathbf{r}$ describing the behaviour of the fluid. Those models then lead to symmetric generalized eigenvalue matrix system ([1]) involving a reduced number of degrees of freedom for the fluid.

Two cases are analyzed. On one hand, we consider linear vibrations of an elastic structure completely filled with a compressible gas or liquid and on the other hand, we consider the case of an elastic structure containing an incompressible liquid with free surface effects due to gravity ([2], [3], [4], [5]) neglecting surface tension effects ([2], [6]) as well as liquid compressibility ([7], [8]).

The first case is a *structural acoustic* problem. In the case of a structure containing a gas, we consider a modal interaction between *structural modes in vacuo* and *acoustic modes in rigid motionless cavity*. For a structure containing a compressible liquid, we consider a modal interaction between *hydroelastic modes including "static" inertial and potential compressibility effects* and *acoustic modes in rigid motionless cavity*.

The second case is a *hydroelastic-sloshing* problem with a modal interaction between *incompressible hydroelastic structural modes* with *incompressible liquid sloshing modes* in rigid motionless cavity, involving an *elastogravity operator* related to the wall normal displacement of the fluid-structure interface, recently deeply analyzed theoretically and numerically in ([3], [4]).

For the construction of reduced models, the static behavior at zero frequency play an important role. Therefore, we review "regularized" variational formulations of the problem, in the sense that the static behaviour must also be in taken into account in the boundary value problem. Those "quasi-static" potential and inertial contributions plays a fundamental role in the Ritz-Galerkin procedure (error truncation).

2 SRUCTURAL-ACOUSTIC EQUATIONS

2.1 Structure subjected to a fluid pressure loading

We consider an elastic structure occupying the domain Ω_S at equilibrium. The interior fluid domain is denoted Ω_F and the fluid-structure interface is denoted Σ (see Figure 1).

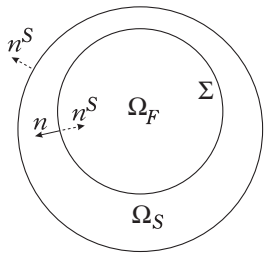

Figure 1. Elastic structure containing a gas.

The angular frequency of vibration is denoted as ω. The chosen unknown field in the structure domain Ω_S is the displacement field u. The linearized strain tensor is denoted as $\varepsilon_{ij}(u)$ and the corresponding stress tensor is denoted as $\sigma_{ij}(u)$. We denote by ρ_S the constant mass density at equilibrium and by n the unit normal, external to the structure domain Ω_S. Let δu be the test function, associated to u, belonging to the admissible space $\mathscr{C}_u$.

The weak variational formulation describing the response of the structure Ω_S to fluid pressure field p acting on the internal fluid-structure interface Σ is written as follows.

For all $\delta u \in \mathscr{C}_u$, find ω^2 and $u \in \mathscr{C}_u$ such that

$$\widetilde{k}(u,\delta u) - \omega^2 \int_{\Omega_S} \rho_S u.\delta u dx = \int_\Sigma p n.\delta u d\sigma \qquad (1)$$

in which

$$\widetilde{k}(u,\delta u) = k(u,\delta u) + k_G(u,\delta u) + k_{P_0}(u,\delta u) \qquad (2)$$

and where $k(u,\delta u)$ is the mechanical elastic stiffness such that

$$k(u,\delta u) = \int_{\Omega_S} \sigma_{ij}(u)\,\varepsilon_{ij}(\delta u)\,dx \tag{3}$$

and where $k_G(u,\delta u)$ is the geometric classical stiffness operator and where $k_{P_0}(u,\delta u)$ is such that

$$k_{P_0} = \int_\Sigma P_0 n_1(u).\delta u\,d\sigma \tag{4}$$

In equation (4), $k_{P_0}(u,\delta u)$ represents an additional load stiffness due to rotation of normal n, in which P_0 denotes the initial pressure existing in the reference equilibrium configuration. Finally, $n_1(u)$ represents the variation of normal n between the reference configuration and the actual configuration.

2.2 Fluid subjected to a wall normal displacement

Since the fluid is inviscid, instead of describing the small motion of the fluid by a fluid displacement vector field u_F which requires an appropriate discretization of the fluid irrotationality constraint $curl\,u_F = 0$ (see for instance [9]), we will use the pressure scalar field p.

Let us denote by c the (constant) sound speed in the fluid, and by ρ_F, the (constant) mass density of the fluid at rest ($c^2 = \beta/\rho_F$, where β denotes the bulk modulus). We denote as Ω_F the domain occupied by the fluid at rest (which is taken as the equilibrium state). The local equations describing the harmonic response of the fluid to a prescribed arbitrary normal displacement $u.n$ of the fluid-structure interface Σ are such that

$$\nabla p - \rho_F\,\omega^2 u_F = 0 \;|_{\Omega_F} \tag{5}$$

$$p = -\rho_F\,c^2\,\nabla.u_F \;|_{\Omega_F} \tag{6}$$

$$u_F.n = u.n \;|_\Sigma \tag{7}$$

$$\mathrm{curl}\,u_F = 0 \;|_{\Omega_F} \tag{8}$$

Equation (5) corresponds to the linearized Euler equation in the fluid. Equation (6) corresponds to the constitutive equation of the fluid (we consider here a barotropic fluid which means that p is only a function of ρ_F). Equation (7) corresponds to the wall slipping condition. Equation (8) corresponds to the irrotationality condition, only necessary in order to ensure that when $\omega \to 0$, u_F tends to static irrotational motion, which corresponds to the hypothesis that for $\omega = 0$, we only consider irrotational motions (for simply connected fluid domain).

A displacement potential φ defined up to an additive constant chosen for instance as follows $\int_{\Omega_F} \varphi\,dx = 0$ can be therefore introduced in order to recast the system defined by equations (5-8) into a scalar one. These aspects will be discussed below.

For $\omega = 0$, equations (6) and (7) lead to a constant static pressure field p^s which is related to the normal wall displacement by the relation

$$p^s = -\frac{\rho_F\,c^2}{|\Omega_F|}\int_\Sigma u.n\,d\sigma \tag{9}$$

in which $|\Omega_F|$ denotes the measure of Ω_F ([10]).

2.3 Equations in terms of p or φ and $u.n$

The elimination of u_F between equations (5), (6), (7) and (8) leads to

$$\nabla^2 p + \frac{\omega^2}{c^2}p = 0 \;|_{\Omega_F} \tag{10}$$

$$\frac{\partial p}{\partial n} = \rho_F\,\omega^2 u.n \;|_\Sigma \tag{11}$$

with the constraint

$$\frac{1}{\rho_F c^2}\int_{\Omega_F} p\,dx + \int_\Sigma u.n\,d\sigma = 0 \tag{12}$$

Equation (10) is the classical Helmholtz equation expressed in terms of p. Equation (11) corresponds to the kinematic condition defined by equation (7) ($\partial p/\partial n = -\rho_F \ddot{u}_F.n = \rho_F\,\omega^2 u.n\;|_\Sigma$). The linear constraint defined by equation (12) corresponds to the global mass conservation which ensures that the boundary problem defined by equations (10) to (11) is equivalent to the problem defined by equations (5) to (8). In the absence of the condition defined by equation (12), we would obtain a boundary value problem in terms of p which is not valid for $\omega = 0$ and which does not allow us to retrieve the value of p^s given by equation (9).

Using equations (8-9), the boundary value problem defined by equations (10-12) can be recasted into the following equivalent one using the displacement potential field φ introduced above such that $p = \rho_F\,\omega^2\varphi + p^s(u.n)$ with $\int_{\Omega_F}\varphi\,dx = 0$

$$\nabla^2\varphi + \frac{\omega^2}{c^2}\varphi - \frac{1}{|\Omega_F|}\int_\Sigma u.n\,d\sigma = 0 \;|_{\Omega_F} \tag{13}$$

$$\frac{\partial\varphi}{\partial n} = u.n \;|_\Sigma \tag{14}$$

with the constraint

$$\int_{\Omega_F}\varphi\,dx = 0 \tag{15}$$

The two boundary value problems expressed in terms of p or in terms of φ are well-posed in the static case ($\omega = 0$). They have been used, with further transformation, leading to appropriate so-called (u,p,φ) symmetric formulations with mass coupling (leading to a final (u,φ) formulation as described by [1, Chapter 8] or with stiffness coupling ([11]; [1, Chapter 8]).

2.4 Variational formulation in terms of (u,p)

Let δp be the test function, associated to p, belonging to the admissible space $\mathscr{C}_p$. The weak variational formulation corresponding to equations (10) to (12) is obtained by the usual test-function method using Green's formula. The weak variational formulation corresponding to the structural acoustic problem is then stated as follows. Find ω^2, $u \in \mathscr{C}_u$ and $p \in \mathscr{C}_p$, such that for all $\delta u \in \mathscr{C}_u$ and $\delta p \in \mathscr{C}_p$, we have

$$\widetilde{k}(u,\delta u) - \omega^2\int_{\Omega_S}\rho_S u.\delta u\,dx - \int_\Sigma p n.\delta u\,d\sigma = 0 \tag{16}$$

$$\frac{1}{\rho_F}\int_{\Omega_F}\nabla p.\nabla\delta p\,dx - \frac{\omega^2}{\rho_F\,c^2}\int_{\Omega_F} p\delta p\,dx - \omega^2\int_\Sigma u.n\delta p\,d\sigma = 0 \tag{17}$$

with the constraint

$$\frac{1}{\rho_F c^2}\int_{\Omega_F} p\,dx + \int_\Sigma u.n\,d\sigma = 0 \tag{18}$$

The variational formulation defined by equations (16), (17) and (18), due to the presence of the constraint defined by equation (18) which regularizes the (u,p) formulation, is therefore valid in the static case. In effect, usually, only equations (16) and (17) are written, and as pointed out above, are not valid for $\omega = 0$. In the case of a finite element discretization of equations (16), (17) and (18), we obtain a matrix system of the type $\mathbf{A}\,\mathbf{Y} - \omega^2\,\mathbf{B}\,\mathbf{Y} = \mathbf{0}$, in which $\mathbf{A}$ and $\mathbf{B}$ are not symmetric. As explained above, that is why various symmetric formulations using for the fluid pressure field p and displacement potential φ, defined up to an additive constant and such that $u_F = \nabla\varphi$, have been derived. The resulting *symmetric* formulations are then obtained by elimination of p or φ. In the present case, we are not considering a direct finite element approach of the variational formulation defined by equations (16), (17) and (18).

2.5 Symmetric reduced model

We will consider hereafter a dynamic substructuring approach through an appropriate decomposition of the admissible class into direct sum of admissible vector spaces (see Figure 2).

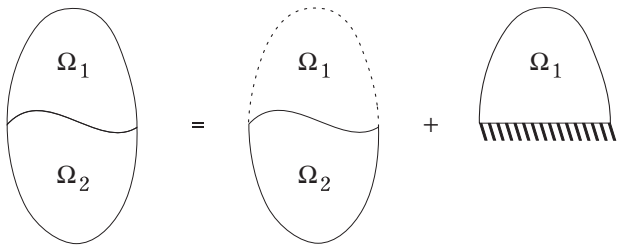

Figure 2. Dynamic fluid-structure substructuring scheme.

Let us consider the following two basic problems ([12], [13]). The first one corresponds to the acoustic modes in rigid motionless cavity and is obtained by setting $u = 0$ into equations (17) and (18). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace $\mathscr{C}_p^*$ of $\mathscr{C}_p$

$$\mathscr{C}_p^* = \left\{ p \in \mathscr{C}_p ; \int_{\Omega_F} p\,dx = 0 \right\} \tag{19}$$

the variational formulation of acoustic modes is stated as follows: find $\omega^2 > 0$ and $p \in \mathscr{C}_p^*$ such that, for all $\delta p \in \mathscr{C}_p^*$, we have

$$\frac{1}{\rho_F}\int_{\Omega_F} \nabla p.\nabla\delta p\,dx = \omega^2\,\frac{1}{\rho_F c^2}\int_{\Omega_F} p\,\delta p\,dx \tag{20}$$

with the constraint

$$\int_{\Omega_F} p\,dx = 0 \tag{21}$$

It should be noted that, in practice, we proceed as follows: the constraint condition (21) is "omitted" which means that that we only modify the initial acoustic problem by adding a first *non physical* zero frequency constant pressure mode, the other modes corresponding to $\omega \neq 0$ remaining the same as those defined by equations (20) and (21). In this new acoustic problem without equation (21), it can be easily seen that the condition defined by equation (21) can be considered as an orthogonality condition between all the modes and the first constant non physical mode corresponding to $\omega = 0$. This zero frequency mode must not be retained in any Ritz-Galerkin projection analysis. In addition, we have the following orthogonality conditions

$$\begin{aligned} &\frac{1}{\rho_F c^2}\int_{\Omega_F} p_\alpha p_\beta\,dx = \mu_\alpha\delta_{\alpha\beta} \\ &\frac{1}{\rho_F}\int_{\Omega_F} \nabla p_\alpha.\nabla p_\beta\,dx = \mu_\alpha\omega_\alpha^2\delta_{\alpha\beta} \end{aligned} \tag{22}$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement $u.n$. The solution, denoted as $p^s(u.n)$, is given by equation (9). For any deformation $u.n$ of the fluid-structure interface, $p^s(u.n)$ belongs to a subset of $\mathscr{C}_p$, denoted as $\mathscr{C}^{u.n}$

$$\mathscr{C}^{u.n} = \left\{ p^s \in \mathscr{C}_p\ ;\ p^s = -\frac{\rho_F c^2}{|\Omega_F|}\int_\Sigma u.n\,d\sigma \right\} \tag{23}$$

In the variational formulation defined by equations (16), (17) and (18), p is searched under the form

$$p = p^s(u.n) + \sum_{\alpha=1}^{N_p} r_\alpha p_\alpha \tag{24}$$

in which N_p denotes the number of retained acoustic modes. The decomposition (24) is unique. In addition, it should be noted that, since each eigenvector p_α corresponding to $\omega_\alpha \neq 0$, verifies the constraint defined by equation (21), then, using equation (9), we deduce that p and $u.n$ satisfy the constraint defined by equation (18). The decomposition defined by equation (24) corresponds to a decomposition of the admissible class $\mathscr{C}_p$ into the direct sum of the admissible classes defined respectively by equations (23) and (19)

$$\mathscr{C}_p = \mathscr{C}^{u.n} \oplus \mathscr{C}_p^* \tag{25}$$

Following equation (24), the test function δp is then searched under the following form

$$\delta p = p^s(\delta u.n) + \sum_{\alpha=1}^{N_p} \delta r_\alpha p_\alpha \tag{26}$$

Variational formulation in δu defined by equation (16) and corresponding to the eigenvalue problem defined by equations (16), (17), (18) becomes

$$\begin{aligned} &\widetilde{k}(u,\delta u) + k^s(u,\delta u) - \sum_{\alpha=1}^{N_p} r_\alpha \int_\Sigma p_\alpha\,n.\delta u\,d\sigma \\ &\qquad = \omega^2\int_{\Omega_S} \rho_S\,u.\delta u\,dx \end{aligned} \tag{27}$$

in which $\widetilde{k}(u,\delta u)$ is defined by equation (2) and $k^s(u,\delta u)$ is such that

$$k^s(u,\delta u) = \frac{\rho_F c^2}{|\Omega_F|}\left(\int_\Sigma u.n\,d\sigma\right)\left(\int_\Sigma \delta u.n\,d\sigma\right) \tag{28}$$

If we consider a finite element discretization of the structure, the corresponding discretized form of equation (27) can be written as

$$[\widetilde{\mathbf{K}}+\mathbf{K}^s]\,\mathbf{U}-\omega^2\,\mathbf{M}\,\mathbf{U}-\sum_{\alpha=1}^{n}\mathbf{C}_\alpha\, r_\alpha\,=\,\mathbf{0} \tag{29}$$

in which symmetric matrices $\widetilde{\mathbf{K}}$ and $\mathbf{K}^s$ correspond to finite element discretization of stiffness symmetric bilinear forms defined by equations (2), (3), (4) and (28) respectively. In equation (29), $\mathbf{M}$ denotes the structural symmetric mass matrix and rectangular coupling matrix $\mathbf{C}_\alpha$ corresponds to the discretization of the coupling fluid-structure contribution $\int_\sigma p\,\delta u.n\,d\sigma$. The discretized form of equation (17) in δp can then be written in generalized (acoustic) coordinates as

$$\omega_\alpha^2\mu_\alpha r_\alpha-\omega^2\mu_\alpha r_\alpha-\omega^2{\mathbf{C}_\alpha}^T\mathbf{U}=0 \tag{30}$$

From equations (29) and (30), we obtain the following symmetric reduced stiffness and mass matrices

$$\left[\begin{array}{cc}\mathbf{K}^{\rm tot} & \mathbf{0}\\ \mathbf{0} & \mathbf{Diag}\,\mu_\alpha\end{array}\right],\left[\begin{array}{cc}\mathbf{M}^{\rm tot} & \mathbf{D}\\ \mathbf{D}^T & \mathbf{Diag}\,(\frac{\mu_\alpha}{\omega_\alpha^2})\end{array}\right] \tag{31}$$

in which $\mathbf{r}$ denotes the vector of N generalized coordinates r_α, with $1\leq\alpha\leq N_p$, and

$$\mathbf{K}^{\rm tot}=\widetilde{\mathbf{K}}+\mathbf{K}^s \tag{32}$$

$$\mathbf{M}^{\rm tot}=\mathbf{M}+\sum_{\alpha=1}^{N_p}\frac{1}{\omega_\alpha^2\,\mu_\alpha}\mathbf{C}_\alpha\mathbf{C}_\alpha^T \tag{33}$$

$$\mathbf{D}_\alpha=\sum_{\alpha=1}^{N_p}\frac{1}{\omega_\alpha^2}\mathbf{C}_\alpha \tag{34}$$

Further diagonalization of equation (31) implies a projection of $\mathbf{U}$ on the solutions of the following eigenvalue problem

$$\mathbf{K}^{\rm tot}\,\mathbf{U}_\beta=\lambda_\beta\mathbf{M}^{\rm tot}\,\mathbf{U}_\beta \tag{35}$$

Setting

$$\mathbf{U}=\sum_{\beta=1}^{N_u}q_\beta\,\mathbf{U}_\beta \tag{36}$$

in which q_β are the generalized coordinates describing the structure. Using the orthogonality conditions associated with the solutions of equation (35), the set of matrices (31) becomes

$$\left[\begin{array}{cc}{\rm Diag}\lambda_\beta & 0\\ 0 & {\rm Diag}\omega_\alpha^2\end{array}\right],\left[\begin{array}{cc}I_{N_u} & \left[C_{\beta\alpha}\right]\\ \left[C_{\beta\alpha}\right]^T & I_{N_p}\end{array}\right] \tag{37}$$

It should be noted that two different situations are treated here.

For a heavy liquid filling the enclosure, one must mandatory use the eigenmodes defined by equation (35), i.e. *hydroelastic modes including "static" inertial and potential compressibility effects* .

For a light fluid such as a gas filling the enclosure, one may use instead *in vacuo structural modes* but the resulting matrix system would not be diagonal with respect to $\mathbf{U}$. In effect, looking at the eigenvalue problem corresponding to equation (30), the diagonalization is obtained by solving the 'structural' problem involving additional stiffness and mass due to static effects of the internal fluid. The *in vacuo* structural modes are orthogonal with respect to $\mathbf{K}$ and $\mathbf{M}$ but not with respect to $\mathbf{K}^{\rm tot}$ and $\mathbf{M}^{\rm tot}$.

Wall impedance condition corresponds to a particular fluid-structure interface modeling. This interface is considered as a third medium with infinitesimal thickness, without mass, and with the following constitutive equation

$$p=j\omega Z(\omega)(u.n-u_F.n) \tag{38}$$

in which $Z(\omega)$ denotes a complex impedance. Equations (7) and (11) must be replaced by equation (38), using $\partial p/\partial n=\rho_F\,\omega^2\,u_F.n$.

Let us consider a liquid with a free surface at rest denoted as Γ, If we neglect gravity effects, the boundary condition on Γ is such that

$$p=0\,|_\Gamma \tag{39}$$

In this case, constraint condition (12) (or (18)) is replaced by equation (39). Equation (9) is replaced by $p^s=0$. Admissible space involved in equation (19) becomes $\mathscr{C}_p^*=\{p\in\mathscr{C}_p\,;\,p=0\}$.

In this case, the static problem defined in Section 2.2, leads to a zero pressure field.

Let us remark that in this case, the "structural" modal basis may be constituted by the hydroelastic incompressible modes using the classical *added mass* operator ([1, Chapter 5]).

The reduced modal matrix models has been extended to the dissipative case using a wall local homogeneous impedance condition ([14]). or introducing a dissipative internal fluid with nonhomogeneous local impedance wall condition ([15]).

3 HYDROELASTIC-SLOSHING VIBRATION

We consider the linear vibrations of an elastic structure partially filled with an homogeneous, inviscid and incompressible liquid, taking into account gravity effects on the free surface Γ.

3.1 Structure subjected to a fluid pressure loading

The notations are those of Figure 3.

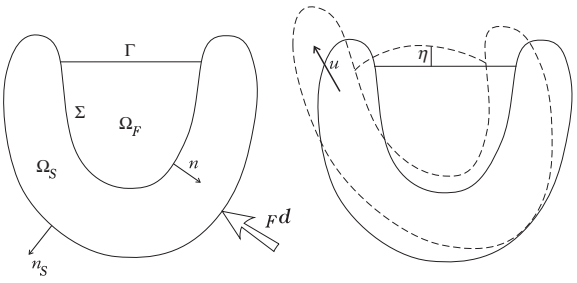

Figure 3. Structure containing a liquid with a free surface.

The weak variational formulation describing the response of the structure Ω_S to fluid pressure field p acting on the internal fluid-structure interface Σ is written as follows.

For all $\delta u\in\mathscr{C}_u$, find ω^2 and $u\in\mathscr{C}_u$ such that

$$\widehat{k}(u,\delta u)-\omega^2\int_{\Omega_S}\rho_S u.\delta u\,dx=\int_\Sigma p n.\delta u\,d\sigma \tag{40}$$

in which

$$\widehat{k} = \widetilde{k} + k_\Sigma \tag{41}$$

In equation (41), $\widetilde{k}(u,\delta u)$ is defined by equation (2), and k_Σ is the *elastogravity* symmetric operator such that ([1, Chapter 6]; [3], [4], [5]).

$$k_\Sigma(u,\delta u) = -\frac{1}{2}\rho_F g\{\int_\Sigma [zn_1(u).\delta u + u_z \delta u.n] d\sigma + \int_\Sigma [zn_1(\delta u).u + \delta u_z u.n]\, d\sigma\} \tag{42}$$

3.2 Fluid subjected to a wall normal displacement

We assume that the liquid is homogeneous, inviscid and incompressible. Free surface Γ is horizontal at equilibrium. We denote by z the external unit normal to Γ, and by g the gravity. The local equations describing the response of the fluid to a prescribed arbitrary normal displacement $u.n$ of the fluid-structure interface Σ are such that

$$\nabla p - \rho_F \omega^2 u_F = 0 \mid_{\Omega_F} \tag{43}$$
$$\nabla . u_F = 0 \mid_{\Omega_F} \tag{44}$$
$$u_F . n = u.n \mid_\Sigma \tag{45}$$
$$p = \rho_F g u_F . n \mid_\Gamma \tag{46}$$
$$\mathrm{curl}\, u_F = 0 \mid_{\Omega_F} \tag{47}$$

Equation (44) corresponds to the incompressibility condition. Equation (46) is the constitutive equation on the free surface Γ due to gravity effects.

A displacement potential φ defined up to an additive constant chosen for instance as follows $\int_\Gamma \varphi\, dx = 0$ can be therefore introduced in order to recast the system defined by equations (49-52) into a scalar one. These aspects will be discussed below.

For $\omega = 0$, equations (44), (45) and (46) lead to the constant static pressure field which is related to the normal wall displacement by the relation

$$p^s = -\frac{\rho_F g}{|\Gamma|}\int_\Sigma u.n\, d\sigma \tag{48}$$

in which $|\Gamma|$ denotes the measure of the area of free surface Γ ([10]).

3.3 Equations in terms of p or φ and u.n

The elimination of u_F between equations (43) to (47) leads to

$$\nabla^2 p = 0 \mid_{\Omega_F} \tag{49}$$
$$\frac{\partial p}{\partial n} = \rho_F \omega^2 u.n \mid_\Sigma \tag{50}$$
$$\frac{\partial p}{\partial z} = \frac{\omega^2}{g} p \mid_\Gamma \tag{51}$$

with the constraint

$$\frac{1}{\rho_F g}\int_\Gamma p\, d\sigma + \int_\Sigma u.n\, d\sigma = 0 \tag{52}$$

The linear constraint defined by equation (52) ensures that the boundary problem defined by equations (49) to (51) is equivalent to the problem defined by equations (43) to (47). This condition is usually omitted in literature.

Using equations (47) and (48), the boundary value problem defined by equations (49-52) can be recasted into the following equivalent one using the displacement potential field φ introduced above such that $p = \rho_F \omega^2 \varphi + p^s(u.n)$ with $\int_\Gamma \varphi\, dx = 0$

$$\nabla^2 \varphi = 0 \mid_{\Omega_F} \tag{53}$$
$$\frac{\partial \varphi}{\partial n} = u.n \mid_\Sigma \tag{54}$$
$$\frac{\partial \varphi}{\partial z} = \frac{\omega^2}{g}\varphi - -\frac{1}{|\Gamma|}\int_\Sigma u.n\, d\sigma \mid_\Gamma \tag{55}$$

with the constraint

$$\int_\Gamma \varphi\, d\sigma = 0 \tag{56}$$

The two boundary value problems expressed in terms of p or in terms of φ are well-posed in the static case ($\omega = 0$). The equations (53, 54, 55) have been used, using a different constraint relationship for φ, after the introduction of the elevation η of the free surface, to appropriate so-called (u,φ,η) symmetric formulations with mass coupling, leading to a (u,η) formulation ([1, Chapter 6]; [3], [4], [5], [6]).

3.4 Variational formulation in terms of (u,p)

Let δp be the test function, associated to p, belonging to the admissible space $\mathscr{C}_p$. Recalling equation (40), the variational formulation of the *hydroelastic-sloshing* problem is then stated as follows. Find ω^2, $u \in \mathscr{C}_u$ and $p \in \mathscr{C}_p$, such that for all $\delta u \in \mathscr{C}_u$ and $\delta p \in \mathscr{C}_p$, we have

$$\widehat{k}(u,\delta u) - \omega^2 \int_{\Omega_S} \rho_S u.\delta u\, dx = \int_\Sigma p n.\delta u\, d\sigma \tag{57}$$

$$\frac{1}{\rho_F}\int_{\Omega_F} \nabla p.\nabla \delta p\, dx = \frac{\omega^2}{\rho_F g}\int_\Gamma p\delta p\, dx + \omega^2 \int_\Sigma u.n\delta p\, d\sigma \tag{58}$$

with the constraint

$$\frac{1}{\rho_F g}\int_\Gamma p\, d\sigma + \int_\Sigma u.n\, d\sigma = 0 \tag{59}$$

3.5 Symmetric Reduced Matrix Model

Let us consider the following two basic problems ([12], [13]). The first one corresponds to the sloshing modes in rigid motionless cavity and is obtained by setting $u = 0$ into equations (50) and (52). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace $\mathscr{C}_p^*$ of $\mathscr{C}_p$

$$\mathscr{C}_p^* = \left\{ p \in \mathscr{C}_p ; \int_\Gamma p\, d\sigma = 0 \right\} \tag{60}$$

the variational formulation of acoustic modes is stated as follows: find $\omega^2 > 0$ and $p \in \mathscr{C}_p^*$ such that, for all $\delta p \in \mathscr{C}_p^*$,

we have

$$\frac{1}{\rho_F}\int_{\Omega_F}\nabla p.\nabla\delta p\,dx=\omega^2\frac{1}{\rho_F g}\int_\Gamma p\delta p\,d\sigma \qquad (61)$$

with the constraint

$$\int_\Gamma p\,d\sigma=0 \qquad (62)$$

It should be noted that, in practice, if the constraint condition (62) is "omitted", we only add a first *non physical* zero frequency constant pressure mode, the other modes corresponding to $\omega\neq 0$ remaining the same as those defined by equations (61) and (62). This zero frequency mode must not be retained in any Ritz-Galerkin projection analysis. In addition, we have orthogonality conditions similarly to equations (22).

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement $u.n$. The solution, denoted as $p^s(u.n)$, is given by equation (48). For any deformation $u.n$ of the fluid-structure interface, $p^s(u.n)$ belongs to a subset of $\mathscr{C}_p$, denoted as $\mathscr{C}^{u.n}$

$$\mathscr{C}^{u.n}=\left\{p^s\in\mathscr{C}_p\;;\;p^s=-\frac{\rho_F g}{|\Gamma|}\int_\Sigma u.n\,d\sigma\right\} \qquad (63)$$

In the variational formulation defined by equations (57) to (59), p is searched under the form

$$p=p^s(u.n)+\sum_{\alpha=1}^{N_p}r_\alpha p_\alpha \qquad (64)$$

in which N_p denotes the number of retained sloshing modes. The decomposition (64) is unique. In addition, it should be noted that, since each eigenvector p_α corresponding to $\omega_\alpha\neq 0$, verifies the constraint defined by equation (62), then, using equation (48), we deduce that p and $u.n$ satisfy the constraint defined by equation (59). The decomposition defined by equation (64) corresponds to a decomposition of the admissible class $\mathscr{C}_p$ into the direct sum of the admissible classes defined respectively by equations (60) and (63), $\mathscr{C}_p=\mathscr{C}^{u.n}\oplus\mathscr{C}_p^*$.

The variational formulation defined by equation (57) becomes

$$\widehat{k}(u,\delta u)+k^s(u,\delta u)-\sum_{\alpha=1}^{N_p}r_\alpha\int_\Sigma p_\alpha n.\delta u\,d\sigma = \omega^2\int_{\Omega_S}\rho_S u.\delta u\,dx \qquad (65)$$

in which $\widehat{k}(u,\delta u)$ is defined by equation (41) and $k^s(u,\delta u)$ is such that

$$k^s(u,\delta u)=\frac{\rho_F g}{|\Gamma|}\left(\int_\Sigma u.n\,d\sigma\right)\left(\int_\Sigma\delta u.n\,d\sigma\right) \qquad (66)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of equation (65) can be written as

$$[\widehat{\mathbf{K}}+\mathbf{K}^s]\,\mathbf{U}-\sum_{\alpha=1}^{n}\mathbf{C}_\alpha r_\alpha-\omega^2\mathbf{M}\mathbf{U}=\mathbf{0} \qquad (67)$$

in which symmetric matrices $\widehat{\mathbf{K}}$ and $\mathbf{K}^s$ correspond to finite element discretization of symmetric bilinear forms defined by equations (41) and (66) respectively. The discretized form of equation (58) in δp can then be written as

$$\omega_\alpha^2\mu_\alpha r_\alpha=\omega^2\mu_\alpha r_\alpha+\omega^2{\mathbf{C}_\alpha}^T\mathbf{U} \qquad (68)$$

From equations (67) and (68), we obtain a *symmetric matrix reduced model* whose expression is similar to the one given by expression (31).

Similarly to Section 2.5, further diagonalization can be obtained by using the eigenmodes of an eigenvalue problem similar to the one described by equation (35). We then obtain a similar matrix system than the one described by equation (37).

It should be noted that we can also use the *incompressible hydroelastic modes*, i.e. the modes of the coupled system constituted by the elastic structure containing an incompressible liquid, with $p=0$ on Γ (through an *added mass operator*). In this case, the resulting matrix system is not completely diagonal with respect to $\mathbf{U}$ variables.

4 CONCLUSION

We have reviewed various formulations for low modal density frequency computations of the eigenmodes of elastic structures containing linear inviscid homogeneous fluids for structural-acoustics problems, using *structural modes in vacuo* for structure containing a gas or *hydroelastic modes including "static" inertial and potential compressibility effects* for structure containing liquids, with *acoustic modes in rigid motionless cavity*, and incompressible hydroelastic-sloshing problems. Those formulations, using modal interaction schemes, with dynamic substructuring techniques lead to symmetric reduced matrix systems expressed in terms of generalized coordinates for the fluid-structure interior system.

REFERENCES

[1] H. Morand and R. Ohayon, *Fluid-Structure Interaction*, Wiley: Chichester, 1995.

[2] R. Ohayon and H. Morand, "Mechanical and numerical modelling of fluid-structure vibration instabilities of liquid propelled launch vehicle," *Chaos, Solitons and Fractals*, vol. 5, no. 9, pp. 1705–1724, 1995.

[3] J.S. Schotté and R. Ohayon, "Effect of gravity on a free-free elastic tank partially filled with incompressible liquid," *Int. J. of Fluids and Structures*, vol. 18, no. 2, pp. 215–226, 2003.

[4] J.S. Schotté and R. Ohayon, "Incompressible hydroelastic vibrations. finite element modelling of the elastogravity operator," *Computers and Structures*, vol. 83, no. 2-3, pp. 209–219, 2005.

[5] J.S. Schotté and R. Ohayon, "Various modelling levels to represent internal liquid behaviour in the vibration analysis of complex structures," *Comput. Methods Appl. Mech. Engrg*, vol. 198, no. 21-26, pp. 1913–1925, 2009.

[6] Schotté J.S. El-Kamali, M. and R. Ohayon, "Computation of the equilibium position of a liquid with surface tension inside a tank of complex geometry and extension to sloshing dynamic cases," *Computational Mechanics*, vol. 46, no. 1, pp. 169–184, 2010.

[7] O. Andrianarison and R. Ohayon, "Compressibility and gravity effects in internal fluid-structure vibrations: Basic equations and appropriate variational formulations," *Comput. Methods Appl. Mech. Engrg*, vol. 195, no. 17-18, pp. 1958–1972, 2006.

[8] O. Andrianarison and R. Ohayon, "Reduced models for modal analysis of fluid-structure systems taking into account compressibility and gravity effects," *Comput. Methods Appl. Mech. Engrg*, vol. 195, no. 41-43, pp. 5656–5672, 2006.

[9] A. Bermudez and R. Rodriguez, "Finite element computation of the vibration modes of a fluid-solid system," *Comp. Meth. Appl. Mech. Eng.*, vol. 119, pp. 355–370, 1994.

[10] R. Ohayon and C.A. Felippa, "The effect of wall motion on the governing equations of structures containing fluids," *Journal of Applied Mechanics*, vol. 57, pp. 783–785, 1990.

[11] G. Sandberg and P. Goransson, "A symmetric finite element formulation for acoustic fluid-structure interaction analysis," *J. Sound Vib.*, vol. 123, no. 3, pp. 507–515, 1988.
[12] R. Ohayon, "Reduced symmetric models for modal analysis of internal structural-acoustic and hydroelastic-sloshing systems," *Comput. Methods Appl. Mech. Engrg.*, vol. 190, pp. 3009–3019, 2001.
[13] R. Ohayon, "Fluid-structure interaction problems," *Encyclopedia of Computational Mechanics*, vol. 2, no. 21, pp. 683–693, 2004.
[14] V. Kehr-Candille and R. Ohayon, "Elasto-acoustic damped vibrations. finite element and modal reduction methods," in *New Advances in Comp. Struc. Mech., Elsevier, Amsterdam*, O.C. Zienkiewicz and P. Ladevèze, Eds., 1992, pp. 321–334.
[15] R. Ohayon and C. Soize, *Structural Acoustics and Vibration*, Academic Press, 1998.