



HAL
open science

Time-domain analysis of viscoelastic systems

Lucie Rouleau, Jean-François Deü

► **To cite this version:**

Lucie Rouleau, Jean-François Deü. Time-domain analysis of viscoelastic systems. X International Conference on Structural Dynamics, EURODYN 2017, Sep 2017, Rome, Italy. pp.384-390, 10.1016/j.proeng.2017.09.057 . hal-03179086

HAL Id: hal-03179086

<https://hal.science/hal-03179086v1>

Submitted on 21 Jul 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



X International Conference on Structural Dynamics, EURODYN 2017

Time-domain analysis of viscoelastic systems

Lucie Rouleau^{a,*}, Jean-François Deü^a

^aStructural Mechanics and Coupled Systems Laboratory Conservatoire national des arts et métiers (Cnam), 292 Rue Saint-Martin, F-75141 Paris cedex 03, France

Abstract

The use of constrained viscoelastic materials has been regarded as a convenient strategy to reduce noise and vibrations in many types of industrial applications. The presence of local nonlinearities in the system or the implementation of a hyper-visco-elastic behaviour law, cannot be appropriately dealt with in the frequency domain and require the analysis to be performed in the time domain. This work aims at presenting a general framework for the computation of time responses of viscoelastically damped systems, by using two-step recurrence formulas involving internal variables into an unconditionally stable Newmark time discretization scheme. After validating the implementation of the methods on a sandwich beam, a numerical analysis of the modified Newmark algorithms is carried out. The methodology is then applied to a structure with more complex geometry to assess memory and CPU requirements.

© 2017 The Authors. Published by Elsevier Ltd.

Peer-review under responsibility of the organizing committee of EURODYN 2017.

Keywords: Viscoelastic damping ; Finite element method ; Transient dynamic analysis

1. Introduction

The use of constrained viscoelastic materials has been regarded as a convenient strategy to reduce noise and vibrations in many types of industrial applications. Many modeling approaches have been developed to account for the frequency- and temperature-dependency of viscoelastic properties. While the use of these viscoelastic models is quite straightforward in the frequency domain, some difficulties arise from their application in the time domain. The presence of local nonlinearities in the system or the implementation of a hyper-visco-elastic behaviour law, cannot be appropriately dealt with in the frequency domain and require the analysis to be performed in the time domain.

The purpose of this work is to present a general framework for the computation of time responses of viscoelastically damped systems, by using two-step recurrence formulas involving internal variables into an unconditionally stable Newmark time discretization scheme. Three of the most common viscoelastic models are under study: the generalized Maxwell model, the Golla-Hughes-McTavish model and the fractional derivative model. After presenting the Newmark schemes adapted to each representation of the behaviour law, the proposed approach is applied to the computation of the time response of a structure treated with constrained viscoelastic layer for validation and qualita-

* Corresponding author. Tel.: +33-1-40-27-24-46 ; fax: +33-1-40-27-27-19.

E-mail address: lucie.rouleau@lecnam.fr

tive comparison. Since the accuracy of the time response is highly dependent on the quality of the curve fitting of the master curves (which is driven by the number of parameters in the viscoelastic model), the parameters of each model are identified so that they fit experimental master curves with a similar degree of precision. The implementation is validated on a simple model by comparing the time response computed for each model with a small time step. The stability, the order of convergence of the modified Newmark algorithms are then studied. Finally, these algorithms are applied to a more complex structure modelled by finite element to assess the memory and CPU requirements for each model so as to highlight the drawbacks and advantages of choosing a viscoelastic over another.

2. Numerical integration of viscoelasticity

2.1. Viscoelastic models

In the context of linear viscoelasticity, the 1D constitutive law which links the stress σ to the strain ε can be written in its convolution integral form [1]:

$$\sigma(t) = \int_{-\infty}^t E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \quad (1)$$

where $E(t)$ is the relaxation modulus. If the initial strain is $\varepsilon(t) = 0$ for $t > 0$, and the modulus is written as:

$$E(t) = E(t \rightarrow 0) - h(t), \quad (2)$$

where $h(t)$ is the memory function, the stress-strain relation becomes:

$$\sigma(t) = E(t \rightarrow 0)\varepsilon(t) - \int_{-\infty}^t h(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \quad (3)$$

Various expressions of the relaxation modulus exist in the literature, each one leading to a specific viscoelastic damping model. The relaxation modulus is commonly defined in the frequency/Laplace domain, since its expression in the time-domain is not available for some models:

$$\sigma(s) = E(s)\varepsilon(s) \quad \text{or} \quad \sigma^*(\omega) = E^*(\omega)\varepsilon^*(\omega) \quad (4)$$

where s is the complex frequency variable and ω is the natural pulsation. The mechanical properties of a viscoelastic material are usually defined through the components of the complex modulus E^* :

$$E^*(\omega) = E'(\omega) + iE''(\omega) = E'(\omega)(1 + i\eta(\omega)) \quad (5)$$

where $E'(\omega)$ is called the storage modulus, $E''(\omega)$ is the loss modulus and $\eta(\omega)$ is the loss factor.

Table 1 presents the three viscoelastic models under study in this work: the generalized Maxwell model, the Golla-Hughes-McTavish model and the fractional derivative model. In the models, E_0 is the relaxed modulus ($E_0 = E(\omega \rightarrow 0) = E(t \rightarrow \infty)$) and E_∞ is the unrelaxed modulus ($E_\infty = E(\omega \rightarrow \infty) = E(t \rightarrow 0)$). τ and τ_k are relaxation times, E_k and $E_0\alpha_k$ are moduli, ω_k and ζ_k are respectively the natural pulsation and the damping ratio of mini-oscillators, and α is a fractional coefficient comprised between 0 and 1.

To implement viscoelasticity in the finite element formulation of the system, dissipation variables associated to the memory function $h(t)$ from Eq.(3) are introduced. The internal variables $\bar{\varepsilon}_k$ are defined as a strain function so that the stress-strain relation becomes:

$$\sigma(t) = E_\infty\varepsilon(t) - \sum_{k=1}^N p_k \bar{\varepsilon}_k(t) \quad (6)$$

for viscoelastic models with series representation, and:

$$\sigma(t) = E_\infty\varepsilon(t) - p\bar{\varepsilon}(t) \quad (7)$$

for the fractional derivative model. The coefficients p_k and p are to be identified from the model.

This change of variables implies that Eq.(3) does not explicitly contain a convolution term any more. A recurrence formula can be obtained after time discretisation for the computation of the internal dissipation variables at each time step. More details on the determination of the internal variables and the obtaining of the recurrence formula for each models are given in [2].

Table 1. Representation of relaxation modulus in the Laplace domain for various viscoelastic models.

Viscoelastic model	Relaxation modulus representation
Generalised Maxwell model (GM)	$E(s) = E_0 + \sum_{k=1}^N E_k \frac{\tau_k s}{1 + \tau_k s}$
Golla-Hughes-McTavish model (GHM)	$E(s) = E_0 + \sum_{k=1}^N E_0 \alpha_k \frac{s^2 + 2\zeta_k \omega_k s}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$
Fractional Derivative model (FD)	$E(s) = E_0 + (E_\infty - E_0) \frac{(\tau s)^\alpha}{1 + (\tau s)^\alpha}$

2.2. Finite element formulation

The equation of motion for a structure with viscoelastic materials after time and space discretisation can be written as:

$$\mathbf{K}(t_n)\mathbf{q}(t_n) + \mathbf{M}\ddot{\mathbf{q}}(t_n) = \mathbf{F}(t_n) \tag{8}$$

where \mathbf{M} is the mass matrix of the system, \mathbf{q} is the vector of degrees of freedom and \mathbf{F} is the loading vector. The time-dependent stiffness matrix of the system \mathbf{K} is obtained by assembly of element stiffness matrices \mathbf{K}^e which are computed from:

$$\mathbf{K}^e(t_n)\mathbf{q}(t_n) = \int_{\Omega_e} \mathbf{B}^T \boldsymbol{\sigma}(t_n) d\Omega_e \tag{9}$$

where \mathbf{B} is the gradient matrix operator and Ω_e is an element volume. For element volume of elastic material:

$$\int_{\Omega_e} \mathbf{B}^T \boldsymbol{\sigma}(t_n) d\Omega_e = \mathbf{K}_e^e \mathbf{q}(t_n) \tag{10}$$

where \mathbf{K}_e^e is the element stiffness matrix of the elastic part of the structure. Assuming a constant Poisson ratio for the viscoelastic media, the integral term in Equation (9) becomes:

$$\int_{\Omega_e} \mathbf{B}^T \boldsymbol{\sigma}(t_n) d\Omega_e = \overline{\mathbf{K}}_v^e \mathbf{q}(t_n) + \overline{\mathbf{C}}_v^e \dot{\mathbf{q}}(t_n) - \overline{\mathbf{F}}_k^e(t_n) \tag{11}$$

where $\overline{\mathbf{K}}_v^e$ is an element constant stiffness matrix associated with the viscoelastic part of the structure, $\overline{\mathbf{C}}_v^e$ is a constant matrix proportional to $\overline{\mathbf{K}}_v^e$ and $\overline{\mathbf{F}}_k^e$ is a vector which contains all the terms depending on the internal degrees of freedom $\overline{\mathbf{q}}_k$ at various timesteps, associated with the internal dissipation variables $\overline{\varepsilon}_k$ introduced in the previous section. In the following, the term $\overline{\mathbf{F}}^e$ will be referred to as a modified loading vector. It should be noted that the expression of the matrices $\overline{\mathbf{K}}_v^e$ and $\overline{\mathbf{C}}_v^e$ and the modified loading vector $\overline{\mathbf{F}}_k^e$ depend on the considered viscoelastic model:

- for the generalised Maxwell model: $\overline{\mathbf{F}}_k^e(t_{n+1}) = \sum_{k=1}^N -\frac{E_k}{E_0} \exp\left(-\frac{\Delta t}{\tau_k}\right) \mathbf{K}_c^e \overline{\mathbf{q}}_k(t_n)$
- for the fractional derivative model: $\overline{\mathbf{F}}_k^e(t_{n+1}) = -c \frac{E_\infty - E_0}{E_0} \sum_{j=1}^{N_i} A_{j+1} \overline{\mathbf{q}}_{n+1-j}(t_{n+1-j})$
- for the Golla-Hughes-McTavish model: $\overline{\mathbf{F}}_k^e(t_{n+1}) = \sum_{k=1}^N \alpha_k \theta_k \left(\left(\frac{1}{\Delta t^2} + \frac{2\zeta_k \omega_k}{\Delta t} \right) \overline{\mathbf{q}}_k(t_n) - \frac{1}{2\Delta t^2} \overline{\mathbf{q}}_k(t_{n-1}) \right)$

In order to solve Equation (8), a inconditionnally stable Newmark scheme ($\beta = 1/4$ and $\gamma = 1/2$) is considered. This algorithm has been applied to compute the time response of viscoelastically damped structures in [3] and provided good results. The modified Newmark algorithm implemented in this work is given in Algorithm 1. It is based on a two-step recurrence formula: the first recurrence formula related the modified loading vector $\overline{\mathbf{F}}_k$ to $\overline{\mathbf{q}}_k$ at various timesteps (line 6 in Algorithm 1), and the second recurrence formula allows the calculation of the internal degrees of freedom $\overline{\mathbf{q}}_k$ from $\overline{\mathbf{q}}_k$, \mathbf{q} and $\dot{\mathbf{q}}$ at previous timesteps (line 10 in Algorithm 1).

Algorithm 1 Modified Newmark algorithm.

1. Enter data (integration parameters, model parameters, assembled matrices)
2. Initialisation of $\mathbf{q}(0)$, $\dot{\mathbf{q}}(0)$, $\bar{\mathbf{q}}_k(0)$ and

$$\ddot{\mathbf{q}}(0) = \mathbf{M}^{-1} (\mathbf{F}(0) - (\mathbf{K}_e + \bar{\mathbf{K}}_v)\mathbf{q}(0))$$

3. Evaluation of the matrix \mathbf{S}

$$\mathbf{S} = \mathbf{M} + \gamma\Delta t\bar{\mathbf{C}}_v + \beta\Delta t^2(\mathbf{K}_e + \bar{\mathbf{K}}_v)$$

4. **for** $n = 1$ **to** N_{dt} **do**

5. Prediction of the displacement and the velocity

$$\mathbf{q}^{pr}(t_{n+1}) = \mathbf{q}(t_n) + \Delta t\dot{\mathbf{q}}(t_n) + (0.5 - \beta)\Delta t^2\ddot{\mathbf{q}}(t_n)$$

$$\dot{\mathbf{q}}^{pr}(t_{n+1}) = \dot{\mathbf{q}}(t_n) + (1 - \gamma)\Delta t\ddot{\mathbf{q}}(t_n)$$

6. Calculation of the modified loading $\bar{\mathbf{F}}_k(t_{n+1})$.

7. Computation of the residual

$$\mathbf{R}(t_{n+1}) = \mathbf{F}(t_{n+1}) + \bar{\mathbf{F}}(t_{n+1}) - (\mathbf{K}_e + \bar{\mathbf{K}}_v)\mathbf{q}^{pr}(t_{n+1})$$

8. Evaluation of the acceleration

$$\ddot{\mathbf{q}}(t_{n+1}) = \mathbf{S}^{-1}\mathbf{R}(t_{n+1})$$

9. Correction of the displacement and the velocity

$$\mathbf{q}(t_{n+1}) = \mathbf{q}^{pr}(t_{n+1}) + \beta\Delta t^2\ddot{\mathbf{q}}(t_{n+1})$$

$$\dot{\mathbf{q}}(t_{n+1}) = \dot{\mathbf{q}}^{pr}(t_{n+1}) + \gamma\Delta t\ddot{\mathbf{q}}(t_{n+1})$$

10. Evaluation of the internal degrees of freedom $\bar{\mathbf{q}}_k(t_{n+1})$

11. **end for**

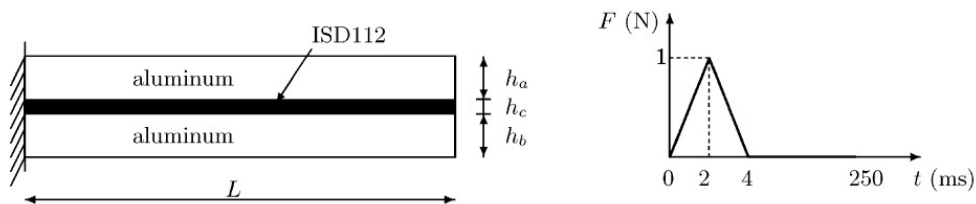


Fig. 1. Sandwich beam studied and description of the triangular impulse imposed at the free end of the beam (from [3]).

3. Results and analysis

For validation purposes, the implementation of each viscoelastic model is tested on the example of a cantilever sandwich beam with viscoelastic core and symmetrical faces, subjected to a transverse triangular impulse at its free end, presented in [3] (see Figure 1).

The geometry of the structure is: length $L = 200$ mm, width $b = 10$ mm, thickness of top and bottom faces $h_a = h_b = 1$ mm and thickness of the viscoelastic core layer $h_c = 0.2$ mm. A shear coefficient value of $k = 5/6$ is considered. The elastic faces are made of aluminium ($E = 70.3 \cdot 10^3$ Pa, $\nu = 0.3$, $\rho = 2690$ kg/m³) and the viscoelastic core is made of ISD112 at 27°C ($\nu = 0.495$, $\rho = 1600$ kg/m³). The frequency-dependency of the Young modulus is modelled by one of the four viscoelastic models investigated in this work. The parameters of the models have been

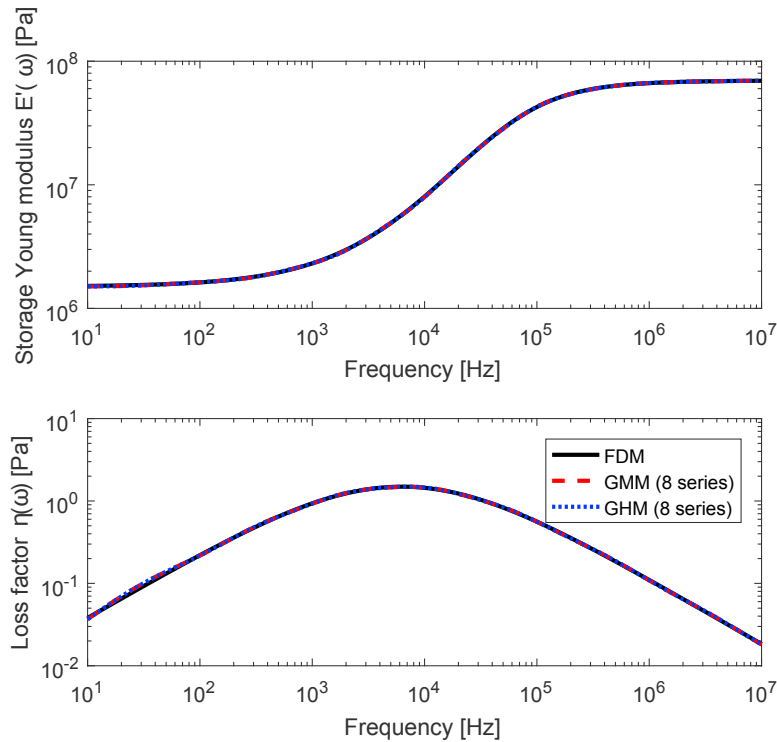


Fig. 2. Master curves of ISD112 at 27°C, modelled by three viscoelastic models.

identified so that the corresponding master curves are similar. The results of the identification are given in Figure 2. For the models with series representation, 8 series were considered, which leads to a relative error inferior to 1% between each model.

As in [3], the structure is modelled using sandwich finite elements [4]. The time response of the sandwich beam to the transverse load described in Figure 1 is computed with a very fine time step of $50\mu\text{s}$ for each viscoelastic model implemented. The results are presented in Figure 3. A good correlation is observed between the three models, which validates the implementation of the models in the time-domain.

For each model, the time response computed with a very fine time step ($\Delta t = 10^{-5}\text{s}$) is used as a reference solution to estimate the order of convergence of the modified Newmark algorithms (which are related to the approximation scheme used to update the internal variables in the 10th step of Algorithm 1). A displacement error is considered in this work:

$$\epsilon_U = \frac{\|\mathbf{U}_{ref} - \mathbf{U}\|}{\|\mathbf{U}_{ref}\|} \quad (12)$$

Figure 4 shows that all three models gives similar results in terms of errors and that the tested methods are first-order accurate.

4. Conclusion

The goal of this work was to present a general framework for the computation of time responses of viscoelastically damped systems, by using two-step recurrence formulas involving internal variables in the time discretisation scheme. The generalised Maxwell model, the Golla-Hughes-McTavish model and a four-parameter fractional derivative model were implemented within a modified Newmark scheme to compute the time response of a sandwich beam subjected to a transverse triangular load. It is shown that by identifying the models' parameters such that the produced master curves are almost identical, the tip displacements of the sandwich beam computed with each models are similar.

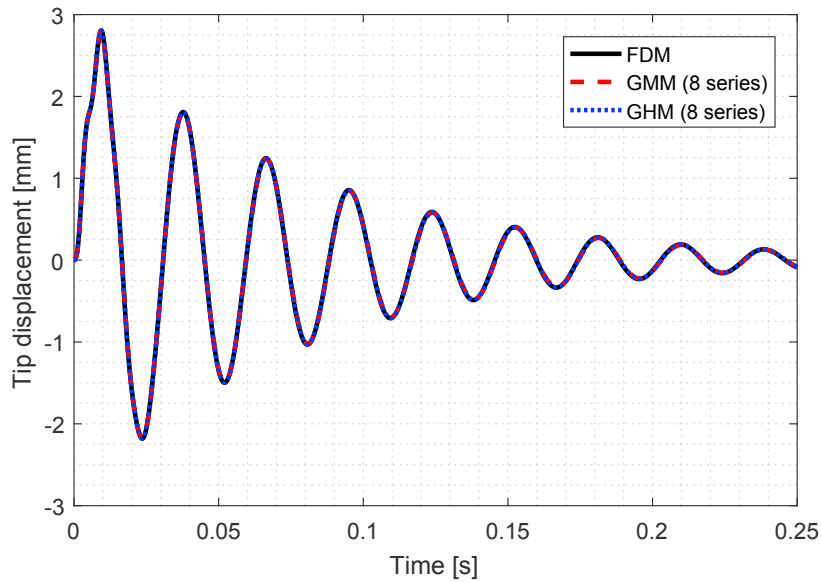


Fig. 3. Time response of the sandwich beam to the transverse load for each viscoelastic model implemented.

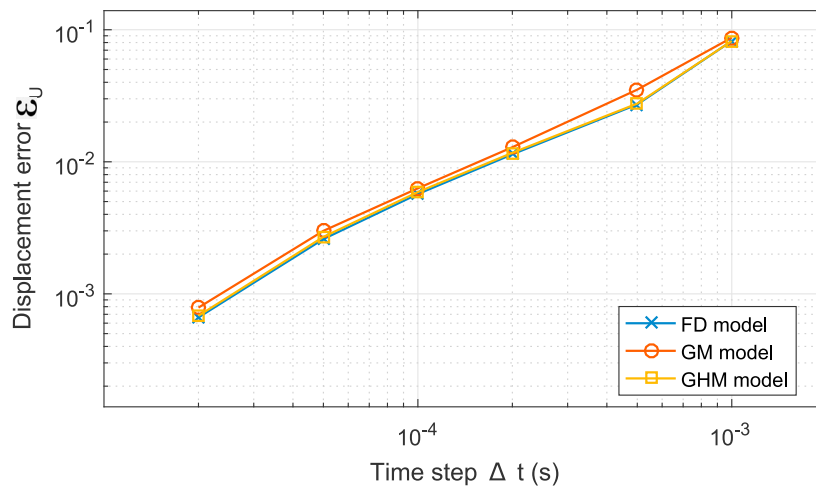


Fig. 4. Displacement error as a function of the time step.

An analysis of the displacement error shows that the numerical methods tested are first-order accurate and give similar results. However, the memory and CPU requirements for each model will differ as the number of variables to store at each time step for the GM model and the GHM model is directly related to the number of series considered in the model, while in the case of the FD model, it is related to the number of Grünwald coefficients taken in the approximation of the fractional derivative [3].

In order to assess the memory and CPU requirements for each model, the time integration scheme described in 1 is applied to compute the time response of a turbine nozzle with constrained viscoelastic layer, represented in Figure 5, when subjected to a triangular point load $F(t)$.

This study will help identifying the drawbacks and advantages of the viscoelastic models under study for a transient dynamic analysis.

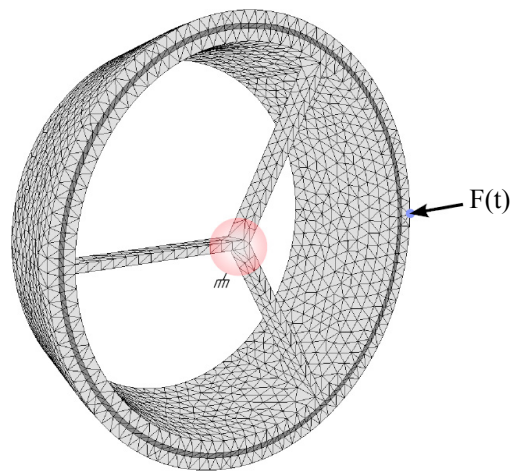


Fig. 5. Finite element modeling of a turbine nozzle with constrained viscoelastic damping.

References

- [1] R.M. Christensen *Theory of viscoelasticity*, Dover Publications (1982).
- [2] L. Rouleau, J.-F. Deü *Time domain analysis of viscoelastic models*, Proc. of ISMA 2016, 19-21 September 2016, Leuven, Belgium.
- [3] A.C. Galucio, J.-F. Deü, R. Ohayon. *Finite element formulation of viscoelastic sandwich beams using fractional derivative operators*, Comp. Mech., Vol. 33, (2004), pp. 282-291.
- [4] M.A. Trindade, A. Benjeddou, R. Ohayon *Finite element modelling of hybrid active-passive vibration damping of multilayer piezoelectric sandwich beams - part I: formulation; part II: System analysis*, Int. J. of Num. Meth. in Eng., Vol. 51, (2001), pp. 835-864.