Implicit variational inequalities arising un frictional contact mechanics : analysis and numerical solutions for quasistatic problems

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IMPLICIT VARIATIONAL INEQUALITIES ARISING IN FRICTIONAL UNILATERAL CONTACT MECHANICS: ANALYSIS AND NUMERICAL SOLUTION OF QUASISTATIC PROBLEMS

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Abstract:

This paper is a survey on implicit variational inequalities arising in the study of unilateral contact problems with friction. Recent works on mathematical and numerical approaches of quasistatic problems are presented. The coupling of unilateral contact, friction, and adhesion is considered and previous results are generalized to this case.

1.1 INTRODUCTION

It can be said that the bases of the mathematical formulation of unilateral and frictional contact have been set by a French-Italian school. On the basis of Signorini's formulation (Signorini, 1959), the unilateral contact problems have been studied by variational methods and by an intensive development of the theory of variational inequalities by (Fichera, 1964; 1972), (Lions and Stampacchia, 1967; Duvaut and Lions, 1972; Mosco, 1975), etc.. The introduction of friction has complicated the mathematical problems, even in the static case, since the problems are then related to implicit variational inequalities or to quasi variational inequalities (in a dual formulation). The advances during the last twenty years are due to several groups: in Austin (Oden, Kikuchi), for static and dynamic formulations, in Linköping (Klarbring, Andersson), for modelling, mathematical programming and analysis, in Montpellier (Moreau, Jean), for convex analysis and dynamic problems, in Prague (Jarusek, Haslinger, Hlavaček, Nečas), for mathematical and numerical analysis, and in Thessaloniki (Panagiotopoulos), for extensions to nonsmooth mechanics. The names of Curnier (Lausanne), Marques, Martins and Pires (Lisbon), Mróz and Telega (Warsaw), Pfeiffer and Glocker (Munich), Wriggers (Darmstadt) have also to be mentionned. In this paper, we are concerned with a mathematical analysis of quasistatic unilateral contact problems with nonlocal friction law. A generalization to the coupling with adhesion is also considered and various numerical treatments are given.

1.2 QUASISTATIC CONTACT PROBLEMS WITH FRICTION

In this section we present the mathematical and numerical results which have been recently obtained for quasistatic unilateral problems with a nonlocal friction law (Cocu et al., 1996). Unilateral conditions are strictly imposed, we use neither penalization nor normal compliance. Existence results for quasistatic frictional contact problems with a normal compliance law have been proved by (Andersson, 1991; 1995) and, in the presence of a time regularization, by (Klarbring et al., 1991). We consider a variational formulation of these problems, combining an inequality describing the friction and an inequality representing the unilateral contact conditions. A time discretization is adopted and if the

friction coefficient is sufficiently small then the incremental formulation that can be derived from this discretization has a unique solution. By using appropriate estimates for the incremental solutions, we are able to prove the existence of a quasistatic solution.

1.2.1 The classical formulation of unilateral contact problems with friction

Consider a linear elastic body occupying a domain Ω of $\mathbf{R}^d$, $d = 2, 3$, with sufficiently regular boundary $\Gamma = \bar{\Gamma}_u \cup \bar{\Gamma}_\phi \cup \bar{\Gamma}_c$. We assume that on Γ_u (with $\mathrm{mes}(\Gamma_u) > 0$) the displacement vector equals zero. On Γ_ϕ a surface force density ϕ is prescribed and on Ω a volume force density f is applied. The solid is initially in contact with a rigid fixed support along Γ_c and the displacements on this part of the boundary satisfy the unilateral constraints of non penetration into the support (Signorini's conditions) and are submitted to nonlocal friction when contact occurs. We suppose that we may neglect the inertial terms. We denote by $u = (u_i)$ the displacement vector, $\sigma = (\sigma_{ij})$ the stress tensor, $\epsilon = (\epsilon_{ij})$ the strain tensor and by $A = (A_{ijkl})$ the elasticity tensor. We make the classical decomposition into normal and tangential components for the contact forces and the displacement vector on Γ_c :

$$u = u_N\, n + u_T\,,\ \sigma\, n = \sigma_N\, n + \sigma_T \quad \text{with} \quad u_N = u.n\,,\ \sigma_N = (\sigma\, n).n\,,$$

where n is the outward unit normal vector to Γ. The classical formulation of this problem is as follows.

$\mathbf{P}_1$: Find a displacement field $u = u(t, x)$ which satisfies, for all $t \in [0, T]$, the following equations and conditions:

$$div\,\sigma(u) + f = 0 \quad \text{in} \quad \Omega\,, \tag{1.1}$$

$$\sigma(u) = A : \epsilon(u) \quad \text{in} \quad \Omega\,, \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_u\,, \tag{1.3}$$

$$\sigma(u)n = \phi \quad \text{on} \quad \Gamma_\phi\,, \tag{1.4}$$

$$u_N \le 0, \quad \sigma_N(u) \le 0 \ \text{ and } \sigma_N(u).u_N = 0 \quad \text{on} \quad \Gamma_c\,, \tag{1.5}$$

$$\begin{array}{c} |\sigma_T(u)| \le \mu\,|\sigma_N^*(u)| \ \text{ and} \\ |\sigma_T(u)| < \mu\,|\sigma_N^*(u)| \ \Rightarrow\ \dot{u}_T = 0\,, \\ |\sigma_T(u)| = \mu\,|\sigma_N^*(u)| \ \Rightarrow \\ \exists \lambda \ge 0 \quad \dot{u}_T = -\lambda \sigma_T(u) \ \text{ on } \ \Gamma_c\,, \end{array} \tag{1.6}$$

where μ is the friction coefficient and σ_N^* is a regularization of the normal contact force (Duvaut, 1980; Demkowicz and Oden, 1982; Oden and Pires, 1983; Cocu et al., 1996).

1.2.2 Variational formulation

We define the space $V = \left\{ v \in \left[H^1(\Omega)\right]^d ; v = 0 \text{ a.e. on } \Gamma_u \right\}$ and we set $K = \{v \in V ; v_N \leq 0 \text{ a.e. on } \Gamma_c\}$. The norm on V is given by $||.||$ and $\langle .,. \rangle$ shall denote the duality pairing on $H^{\frac{1}{2}}(\Gamma_c) \times H^{-\frac{1}{2}}(\Gamma_c)$. We suppose that $f \in W^{1,2}(0,T;\left[L^2(\Omega)\right]^d), \phi \in W^{1,2}(0,T;\left[L^2(\Gamma_\phi)\right]^d)$, so that we have $F \in W^{1,2}(0,T;V)$, where

$$(F,v) = \int_\Omega f.v dx + \int_{\Gamma_\phi} \phi.v ds \quad \forall\, v \in V.$$

We shall adopt the following notations :

- $a : V \times V \longrightarrow I\!R$,

  $a(u,v) = \int_\Omega A_{ijkl}\epsilon_{ij}(u)\epsilon_{kl}(v)dx \;\; \forall\, u,v \in V,$ where $A_{ijkl} \in L^\infty(\Omega)$,

  $1 \leq i,j,k,l \leq d$, with the usual properties of ellipticity and symmetry.

- $j : V \times V \longrightarrow I\!R$,

  $j(u,v) = \int_{\Gamma_c} \mu|\sigma_N^*(Pu)||v_T|ds$, where

  P is the projection of $W^{1,2}(0,T;V)$ on V_0 with

  $$V_0 = \left\{ w \in W^{1,2}(0,T;V) ; \int_0^T a(w,\psi)\, dt = \int_0^T (f,\psi)_{[L^2(\Omega)]^d}\, dt + \int_0^T (\phi,\psi)_{[L^2(\Gamma_\phi)]^d}\, dt, \forall\, \psi \in L^2(0,T;V),\ \psi = 0 \text{ a.e. on } \Gamma_c \times]0,T[\right\},$$

  $\mu \in L^\infty(\Gamma_c)$ and $\mu \geq 0$ a.e. on Γ_c,

  $(.)^* : H^{-\frac{1}{2}}(\Gamma_c) \to L^2(\Gamma_c)$ is a linear and compact mapping.

Then the problem (1.1)-(1.4), (1.5), (1.6) admits the following variational formulation (Cocu et al., 1996).

$\mathbf{P}_2$: Find $u \in W^{1,2}(0,T;V)$ such that $u(0) = u_0$ and for almost all $t \in [0,T]$ $u(t) \in K$ and

$$a(u, v - \dot{u}) + j(u,v) - j(u,\dot{u}) \geq (F, v - \dot{u}) + \langle \sigma_N(u), v_N - \dot{u}_N \rangle \;\; \forall v \in V, \quad (1.7)$$

$$\langle \sigma_N(u), z_N - u_N \rangle \geq 0 \quad \forall\, z \in K, \quad (1.8)$$

where the initial displacement u_0 belongs to K and satisfies the following compatibility condition :

$$a(u_0, w - u_0) + j(u_0, w - u_0) \geq (F(0), w - u_0) \quad \forall\, w \in K.$$

A similar variational formulation was proposed in (Telega, 1991). The following well known property of the mapping j involved in the problem $\mathbf{P}_2$ holds with $\bar{\mu} = |\mu|_{L^\infty(\Gamma_c)}$:

$$\exists C > 0 \quad \forall\, t \in [0,T] \quad \text{and} \quad \forall\, u, \bar{u}, v, \bar{v} \in V_0$$
$$|j(u,v) - j(u,\bar{v}) - j(\bar{u},v) + j(\bar{u},\bar{v})| \leq \bar{\mu} C ||u - \bar{u}||\, ||v - \bar{v}||.$$

1.2.3 *Incremental formulations*

Incremental formulations can be obtained by operating a time discretization of problem $\mathbf{P}_2$, taking $n \in I\!N^*$, $t_i = i\,\Delta t$ and $F^i = F(t_i)$ for $i = 0, ..., n$. Using an implicit scheme we consider the following sequence of problems $\mathbf{P}_2^{n,i}$, $i = 0, ..., n-1$, defined for $u^0 = u_0$.

$\mathbf{P}_2^{n,i}$: Find $u^{i+1} \in K$ such that :

$$a(u^{i+1}, v - \frac{u^{i+1} - u^i}{\Delta t}) + j(u^{i+1}, v) - j(u^{i+1}, \frac{u^{i+1} - u^i}{\Delta t}) \geq (F^{i+1}, v - \frac{u^{i+1} - u^i}{\Delta t}) + \langle \sigma_N(u^{i+1}), v_N - \frac{u_N^{i+1} - u_N^i}{\Delta t} \rangle \quad \forall\, v \in V\,, \tag{1.9}$$

$$\langle \sigma_N(u^{i+1}), z_N - u_N^{i+1} \rangle \geq 0 \quad \forall\, z \in K\,. \tag{1.10}$$

Problems $\mathbf{P}_2^{n,i}$, $i = 0, ..., n-1$, are equivalent to the following sequence $\mathbf{S}_2^{n,i}$, $i = 0, ..., n-1$, where $\mathbf{S}_2^{n,i}$: Find $u^{i+1} \in K$ such that

$$a(u^{i+1}, w - u^{i+1}) + j(u^{i+1}, w - u^i) - j(u^{i+1}, u^{i+1} - u^i) \geq (F^{i+1}, w - u^{i+1}) \qquad \forall\, w \in K\,. \tag{1.11}$$

1.2.4 *Existence results*

Inequalities such as $\mathbf{S}_2^{n,i}$ are elliptic quasi variational inequalities so that have a unique solution if the friction coefficient is sufficiently small ($\bar{\mu} < m/C$, where m is the coerciveness constant of the bilinear form a) (Cocu, 1984; Capatina and Cocu, 1991). We suppose from now that μ satisfies this condition. In order to pass to the limit in the incremental formulations, the following bounds for the incremental solution u^i and also for the difference $\Delta u^i = u^{i+1} - u^i$ can be obtained (Cocu et al., 1996) :

$$||u^i|| \leq C_1\, ||F^i|| \;\; \forall i = 0, ..., n \quad \text{and} \quad ||\Delta u^i|| \leq C_1\, ||\Delta F^i|| \;\; \forall i = 0, ..., n-1\,.$$

We define the sequences $(u_n)_n$, $(\tilde{u}_n)_n$, such that $u_n \in L^2(0,T;V)$, $\tilde{u}_n \in W^{1,2}(0,T;V) \quad \forall n \in I\!N^*$, given by $u_n(0) = \tilde{u}_n(0) = u_0$ and

$$u_n(t) = u^{i+1} \quad \forall t \in]t_i, t_{i+1}]\,,$$

$$\tilde{u}_n(t) = u^i + \frac{(t - t_i)}{\Delta t}(u^{i+1} - u^i) \quad \forall t \in]t_i, t_{i+1}]\,,$$

where u^k is the solution of $\mathbf{S}_2^{n,k}$. Then the following existence and approximation result holds (Cocu et al., 1996).

Theorem 1 *There exists a subsequence* $(\tilde{u}_{n_p})_p$ *of* $(\tilde{u}_n)_n$ *that converges weakly in* $W^{1,2}(0,T;V)$ *to an element* u, *such that* $\forall t \in [0,T]$ $(u_{n_p}(t))_p$ *converges weakly to* $u(t)$ *in* V. *Every such weak limit* u *is a solution of* $\mathbf{P}_2$.

1.3 EXTENSION TO A MODEL COUPLING ADHESION AND FRICTION

This section is concerned with a model where unilateral contact conditions, adhesion and friction between two elastic bodies are strongly coupled. The interface is considered as a material surface and the derivation of this model follows from the principle of virtual power and the principles of thermodynamics (Cangémi, 1997), (Raous et al., 1998; 1997). Adhesion is characterized by a new variable, the intensity of adhesion β introduced by (Frémond, 1987). We propose a variational formulation which generalizes the one already given for the quasistatic friction problems in Section 2. The use of an implicit scheme leads us to incremental formulations for which an existence and uniqueness result is given if the friction coefficient is sufficiently small. Algorithms for computing the incremental solutions are presented.

1.3.1 The classical formulation of quasistatic adhesion problems

Let Ω^1 and Ω^2 be two disjoint domains of $\mathbf{R}^d$, $d = 2, 3$, occupied by two linear elastic bodies with sufficiently regular boundaries $\Gamma^\alpha = \bar{\Gamma}^\alpha_u \cup \bar{\Gamma}^\alpha_\phi \cup \bar{\Gamma}_c$, $\alpha = 1, 2$, where Γ_c is the contact boundary. Suppose that on $\Omega^1 \cup \Omega^2$ and on Γ^1_ϕ, Γ^2_ϕ the volume force densities f^1, f^2 and the surface force densities ϕ^1, ϕ^2 are respectively applied. On Γ^1_u and Γ^2_u (with $\mathrm{mes}(\Gamma^\alpha_u) > 0$, $\alpha = 1, 2$) the displacements equal zero. We denote by $[u] = u^1 - u^2$ the relative displacement on Γ_c and by n^α the outward unit normal vector to Γ^α, $\alpha = 1, 2$.

The equilibrium equations can be deduced from the principle of virtual power to the whole system $\Omega^1 \cup \Omega^2 \cup \Gamma_c$. The constitutive laws are obtained from the state laws , the complementary laws for the bodies and the contact boundary Γ_c which will be considered in what follows as a material boundary with a specific thermodynamic behaviour (Frémond, 1987; Klarbring, 1990). The state variables are: the infinitesimal strain tensor $\epsilon = (\epsilon_{ij})$, the jump [u] on Γ_c and the intensity of adhesion β ($\beta = 1$ means that the adhesion is total, $\beta = 0$ means that there is no adhesion and $0 < \beta < 1$ is the case of partial adhesion). For the interface, we choose the following free energy:

$$\Psi([u_N],[u_T],\beta) = \frac{C_N}{2}\,[u_N]^2\beta^2 + \frac{C_T}{2}\,[u_T]^2\beta^2 - w\,\beta + + I_{\bar{K}}([u]) + I_P(\beta)\,,$$

with $I_{\tilde{K}}, I_P$ the indicator functions of $\tilde{K} = \{v\,;\, v_N \leq 0\}$ and $P = \{\gamma\,;0 \leq \gamma \leq 1\}$. We introduce also a pseudo-potential of dissipation Φ, having the following form (Cocu et al., 1998; Raous et al., 1998):

$$\Phi\left([\dot{u}_T], \dot{\beta}; \chi_N\right) = \mu \left|\sigma_N^* + C_N\,[u_N]\beta^2\right|\,|[\dot{u}_T]| + \frac{b}{2}\dot{\beta}^2 + I_{C^-}(\dot{\beta}),$$

where $\chi_N = (\sigma_N, [u_N], \beta)$, μ is the friction coefficient, σ_N^* is a regularization of the normal contact force σ_N and $C^- = \{\gamma\,;\gamma \leq 0\}$. The adhesion is governed here by a compliance law, where C_N and C_T are material characteristic coefficients. As shown in (Cocu et al., 1998; Raous et al., 1998; 1997), the state laws and the complementary laws give the following constitutive laws for the interface, coupling unilateral contact, friction and adhesion:

- unilateral contact conditions

$$\sigma_N + C_N\,[u_N]\beta^2 \leq 0\,, \quad [u_N] \leq 0 \text{ and } \left(\sigma_N + C_N\,[u_N]\beta^2\right)\,[u_N] = 0\,, \tag{1.12}$$

- adhesion and friction law

$$\begin{aligned}
&\left|\sigma_T + C_T\,[u_T]\beta^2\right| \leq \mu\left|\sigma_N^* + C_N\,[u_N]\beta^2\right|\,,\\
&\left|\sigma_T + C_T\,[u_T]\beta^2\right| < \mu\left|\sigma_N^* + C_N\,[u_N]\beta^2\right| \;\Rightarrow\; [\dot{u}_T] = 0\,,\\
&\left|\sigma_T + C_T\,[u_T]\beta^2\right| = \mu\left|\sigma_N^* + C_N\,[u_N]\beta^2\right| \;\Rightarrow\\
&\qquad\qquad \exists\lambda \geq 0 \quad \sigma_T + C_T\,[u_T]\beta^2 = -\lambda\,[\dot{u}_T]\,,
\end{aligned} \tag{1.13}$$

- damage evolution equation

$$b\dot{\beta} = -\left[w - (C_N\,[u_N]^2 + C_T\,[u_T]^2)\beta\right]^-\,. \tag{1.14}$$

The conditions (1.12) represent a contact law where no interpenetration between the two bodies occurs. The friction-adhesion law (1.13) expresses the coupling between a (regularized) Coulomb force and an extra compliance force in the tangential direction. When total decohesion occurs ($\beta = 0$), the relations (1.12), (1.13) are the classical unilateral contact (or Signorini's) conditions and friction conditions for the two elastic bodies.

1.3.2 Variational and incremental formulations of adhesion problems

We denote by V^α, $\alpha = 1, 2$, the spaces
$V^\alpha = \left\{v^\alpha \in \left[H^1(\Omega^\alpha)\right]^d\,; v^\alpha = 0 \text{ a.e. on } \Gamma_u^\alpha\right\}$ and we set

$\mathbf{V} = V^1 \times V^2$, $\mathbf{K} = \left\{ v = (v^1, v^2) \in \mathbf{V}; [v_N] \leq 0 \text{ a.e. on } \Gamma_c \right\}$, $H = L^\infty(\Gamma_c)$.
The norm on $\mathbf{V}$ is given by $||v|| = ||v^1||_{V^1} + ||v^2||_{V^2}$ for all $v \in \mathbf{V}$.

We suppose that :

$(f^1, f^2) \in W^{1,2}(0,T; \left[L^2(\Omega^1)\right]^d \times \left[L^2(\Omega^2)\right]^d)$,

$(\phi^1, \phi^2) \in W^{1,2}(0,T; \left[L^2(\Gamma_\phi^1)\right]^d \times \left[L^2(\Gamma_\phi^2)\right]^d)$,

which imply $\mathbf{F} \in W^{1,2}(0,T;\mathbf{V})$, where

$$(\mathbf{F}, v) = \sum_{\alpha=1,2} \left[\int_{\Omega^\alpha} f^\alpha . v^\alpha dx + \int_{\Gamma_\phi^\alpha} \phi^\alpha . v^\alpha ds \right] \quad \forall\, v = (v^1, v^2) \in \mathbf{V}.$$

We shall adopt the following notations :

- $\mathbf{a} : \mathbf{V} \times \mathbf{V} \longrightarrow I\!R$,
  $\mathbf{a}(u,v) = a^1(u^1, v^1) + a^2(u^2, v^2) \quad \forall\, u = (u^1, u^2), v = (v^1, v^2) \in \mathbf{V}$,
  where $\quad a^\alpha(u^\alpha, v^\alpha) = \displaystyle\int_{\Omega^\alpha} A^\alpha_{ijkl} \epsilon_{ij}(u^\alpha) \epsilon_{kl}(v^\alpha) dx, \ \alpha = 1,2,$

- $\mathbf{j} : H \times \mathbf{V} \times \mathbf{V} \longrightarrow I\!R$,
  $\mathbf{j}(\beta, u, v) = \displaystyle\int_{\Gamma_c} \mu |\sigma_N^*(P^1 u^1) + C_N \beta^2 [u_N]| |[v_T]| ds, \quad$ where
  P^1 is the projection of V^1 on V_0^1 with
  $$V_0^1 = \left\{ w^1 \in W^{1,2}(0,T;V^1) ; \int_0^T a^1(w^1, \psi^1)\, dt = \int_0^T (f^1, \psi^1)_{[L^2(\Omega^1)]^d}\, dt \right.$$
  $$\left. + \int_0^T (\phi^1, \psi^1)_{\left[L^2(\Gamma_\phi^1)\right]^d} dt, \forall\, \psi^1 \in L^2(0,T;V^1), \ \psi^1 = 0 \text{ a.e. on } \Gamma_c \times]0,T[\right\},$$
  $\mu \in L^\infty(\Gamma_c)$ and $\mu \geq 0$ a.e. on Γ_c ,
  $(.)^* : H^{-\frac{1}{2}}(\Gamma_c) \to L^2(\Gamma_c)$ is a linear and compact mapping ,

- $b_N, b_T : H \times \mathbf{V} \times \mathbf{V} \longrightarrow I\!R$,
  $b_N(\beta, u, v) = \displaystyle\int_{\Gamma_c} C_N \beta^2 [u_N][v_N] ds$ and $b_T(\beta, u, v) = \displaystyle\int_{\Gamma_c} C_T \beta^2 [u_T][v_T] ds,$

- $g(\beta, u) = -\dfrac{1}{b} \left[w - (C_N [u_N]^2 + C_T [u_T]^2)\beta \right]^-$.

Then the local problem (1.12)-(1.14) admits the following variational formulation.

$\mathbf{P}_3$: Find $(u, \beta) \in W^{1,2}(0,T;\mathbf{V}) \times W^{1,2}(0,T;H)$ such that $u(0) = u_0$, $\beta(0) = \beta_0$ and for almost all $t \in [0,T]$, $u(t) \in \mathbf{K}$ and

$$\begin{aligned} \mathbf{a}(u, v - \dot{u}) + \mathbf{j}(\beta, u, v) - \mathbf{j}(\beta, u, \dot{u}) + b_T(\beta, u, v - \dot{u}) \geq \\ (\mathbf{F}, v - \dot{u}) + \langle \sigma_N(u^1), [v_N] - [\dot{u}_N] \rangle \quad \forall\, v \in \mathbf{V}, \end{aligned} \tag{1.15}$$

$$\langle \sigma_N(u^1), [z_N] - [u_N] \rangle + b_N(\beta, u, z-u) \geq 0 \quad \forall\, z \in \mathbf{K}\,, \tag{1.16}$$

$$\dot{\beta} = g(\beta, u) \quad \text{a.e. on} \quad \Gamma_c\,, \tag{1.17}$$

where the initial conditions $u_0 \in \mathbf{K}$, $\beta_0 \in H$ with $\beta_0 \in [0,1[$ a.e. on Γ_c, satisfy the following compatibility condition: for all w $\in \mathbf{K}$

$\mathbf{a}(u_0, w-u_0) + \mathbf{j}(\beta_0, u_0, w-u_0) + \mathbf{b}(\beta_0, u_0, w-u_0) \geq (\mathbf{F}(0), w-u_0)$,
where $\mathbf{b} = b_N + b_T$. The mapping $\mathbf{j}(.,.,.)$ satisfies the following property:

$\exists \bar{C} > 0 \quad \forall\, \beta \in H$ with $\beta \in [0,1[$ a.e. on Γ_c, $\quad \forall\, u, \bar{u}, v, \bar{v} \in \mathbf{V}_0^1$
$|\mathbf{j}(\beta, u, v) - \mathbf{j}(\beta, u, \bar{v}) - \mathbf{j}(\beta, \bar{u}, v) + \mathbf{j}(\beta, \bar{u}, \bar{v})| \leq \bar{\mu}\bar{C}||u-\bar{u}||\,||v-\bar{v}||$.

In the same way as in Section 2, the following sequence of incremental problems $\mathbf{P}_3^{n,i}$, $i = 0, ..., n-1$, can be obtained.

$\mathbf{P}_3^{n,i}$: Find $(u^{i+1},\, \beta^{i+1}) \in \mathbf{K} \times H$ such that

$$\begin{aligned} \mathbf{a}(u^{i+1}, v-u^{i+1}) + \mathbf{j}(\beta^{i+1}, u^{i+1}, v-u^i) - \mathbf{j}(\beta^{i+1}, u^{i+1}, u^{i+1}-u^i) \\ + \mathbf{b}(\beta^{i+1}, u^{i+1}, v-u^{i+1}) \;\geq\; (\mathbf{F}^{i+1}, v-u^{i+1}) \quad \forall\, v \in \mathbf{K}\,, \end{aligned} \tag{1.18}$$

$$\beta^{i+1} - \beta^i = \Delta t\, g(\beta^{i+1}, u^{i+1}) \quad \text{a.e. on } \Gamma_c\,. \tag{1.19}$$

1.3.3 Mathematical results

We may solve the incremental problem $\mathbf{P}_3^{n,i}$ without additional regularity conditions by a fixed point method (Cangémi, 1997; Cocu et al., 1998). For every $\bar{u} \in \mathbf{K}$ let us denote by $s(\bar{u}) = \beta$ the solution of

$$\beta = \Delta t\, g(\beta, \bar{u}) + \beta^i \tag{1.20}$$

and for every $\beta \in H$ we denote by $u = u(\beta) \in \mathbf{K}$ the solution of

$$\begin{aligned} \mathbf{a}(u, v-u) + \mathbf{j}(\beta, u, v-u^i) - \mathbf{j}(\beta, u, u-u^i) \\ + \mathbf{b}(\beta, u, v-u) \;\geq\; (\mathbf{F}^{i+1}, v-u) \quad \forall\, v \in \mathbf{K}\,. \end{aligned} \tag{1.21}$$

The existence of the solution $s(\bar{u})$ for the problem (1.20) is clear, while inequalities such as (1.21) have a unique solution $u(\beta)$ if the friction coefficient is sufficiently small i.e. $\bar{\mu} < \bar{m}/\bar{C}$, where $\bar{m}$ is the coerciveness constant of the bilinear form $\mathbf{a}$ (Cocu, 1984; Capatina and Cocu, 1991). In the following we shall suppose that μ satisfies this condition.

We define the mapping $\mathbf{T}: \mathbf{K} \longrightarrow \mathbf{K}$ by: $\forall\, \bar{u} \in \mathbf{K} \quad \mathbf{T}(\bar{u}) = u(s(\bar{u}))$. By a straightforward computation, we can establish that $\exists\, k_1 > 0$ such that for all $\bar{u}_1, \bar{u}_2 \in \mathbf{K}$

$$|s(\bar{u}_1) - s(\bar{u}_2)| \leq k_1 \Delta t\, ||\bar{u}_1| + |\bar{u}_2||\; |[\bar{u}_1 - \bar{u}_2]| \quad \text{a.e. on} \quad \Gamma_c\,. \tag{1.22}$$

Let us now set $u_1 = u(s(\bar{u}_1))$, $u_2 = u(s(\bar{u}_2))$. Adding the inequalities (1.21) with $u = u_1$, $v = u_2$ and $u = u_2$, $v = u_1$, and using the properties of $\mathbf{a}$, $\mathbf{j}$ and $\mathbf{b}$, one can show that $\exists\, k_2 > 0$ such that for all $\bar{u}_1, \bar{u}_2 \in \mathbf{K}$

$$||u_1 - u_2||^2 \leq k_2 \int_{\Gamma_c} |[u_2]|\, |[u_1 - u_2]|\, |s(\bar{u}_1) - s(\bar{u}_2)| ds\,. \tag{1.23}$$

From the inequalities (1.22), (1.23) and the relation $||u|| \leq k_3$ (with k_3 independant of $\bar{u}$), satisfied by $u(s(\bar{u}))$, it follows that $\mathbf{T}$ is a contraction mapping for sufficiently small Δt. In this way the following existence and uniqueness result holds.

Theorem 2 *For a sufficiently small Δt the mapping $\mathbf{T}$ has a unique fixed point u and $(u, s(u))$ is the solution of the incremental problem $\mathbf{P}_3^{n,i}$.*

1.4 NUMERICAL METHODS

For dealing with unilateral contact with friction for evolution problems, either quasistatic or dynamic, the key point is to solve a "static" problem. This will be present in the subsection 4.1. By using the incremental formulations, algorithms adapted to quasistatic problems and to the coupling with adhesion are given in subsections 4.2 and 4.3. An extension to dynamic problems is considered in subsection 4.4.

1.4.1 Various solvers for the static problem

The static contact problem has been widely studied (Fichera, 1964; 1972; Duvaut and Lions, 1972; Duvaut, 1980; Demkowicz and Oden, 1982; Glowinski et al., 1976 ; etc.) and leads to variational or implicit variational inequalities. For a unilateral contact with Coulomb's friction law, the following problem is obtained.

$\mathbf{P}_4$: Find $u \in K$ such that

$$a(u, v - u) + j(u, v) - j(u, u) \geq (F, v - u) \quad \forall\, v \in K\,. \tag{1.24}$$

We focus in the present paper on methods treating strictly the unilateral and the friction conditions without using a regularization of the unilateral condition. Penalty and augmented lagrangian methods, which are widely used in large finite element codes, are approximations of the contact conditions and need the choice of some computational parameters. When precise analysis is conducted, such as the analysis of instabilities due to friction, see (Martins et al., 1998), or the coupling with adhesion (Raous et al., 1998; 1997), the contact solution has to be independent of computational parameters. Among the

various numerical methods which can be used, we can distinguish two classes : projection techniques associated to optimization and mathematical programming. By introducing a fixed point on the sliding limit, the problem (1.24) can be written as a sequence of classical variational inequalities : find $\tilde{p}$, fixed point of the application $p \longrightarrow S(p)$ with $S(p) = \mu\,|\sigma_N(u_p)|$, where u_p is the solution of the following Tresca friction problem (i.e. when the sliding limit is given) .

$\mathbf{P}_5$: Find $u_p \in K$ such that

$$a(u_p, v - u_p) + \int_{\Gamma_c} p(|v_T| - |u_{Tp}|)ds \geq (F, v - u_p) \quad \forall\, v \in K\,. \tag{1.25}$$

Elimination of the implicit character gives the convexity of the problem (1.25) . The following minimization problem under constraints is equivalent to $\mathbf{P}_5$.

$\mathbf{P}_6$: Find $u_p \in K$ such that $G(u) \leq G(v) \quad \forall\, v \in K$, where

$$G(v) = \frac{1}{2}\,a(v, v) - (F, v) + \int_{\Gamma_c} p|v_T|ds \quad \forall\, v \in K\,.$$

The finite element approximation can be easily written and the discrete problem consists in solving a minimization problem under constraints ($v \in K$) for a functional including a non differentiable term due to friction . As presented in (Licht et al., 1991), an extra projection on the finite element basis of the friction term gives a simple discrete form (for the absolute value). Then various algorithms have been implemented to solve problem $\mathbf{P}_6$, see (Raous et al., 1988; Lebon and Raous, 1992; Raous and Barbarin, 1992) or in (Panagiotopoulos et al., 1998) :

- Over-relaxation with projection (very easy to implement, simple treatment of the absolute value and of the unilateral condition),
- Gauss Seidel algorithm with Aitken acceleration (the convergence is equivalent to the previous method but no optimal relaxation parameter is needed),
- Projected conjugate gradient with preconditioning (a regularization of the absolute value is needed and a convenient projection is used to preserve the conjugacy property) .

Convergence of the fixed point method is fast (a diagonal process is used to save the CPU time) . The two algorithms based on Gauss Seidel methods are robust as they still work for large friction coefficients. A preliminary condensation of the problem to the only contact variables reduces the number of iterations (this is efficient when several loading cases are treated with the same elasticity matrix) .

An alternative to these methods is given by mathematical programming techniques applied to the problem written as a linear complementarity problem.

By introducing an extra variable (right and left slidings are treated separately), see (Klarbring, 1986), it has been shown that the problem can be written as follows after condensation:

$$\begin{aligned} w - A^* u &= f \\ u &\geq 0 \\ w &\geq 0 \\ u.w &= 0\,, \end{aligned}$$

where A^* is a non symmetric matrix because of the friction.

Various algorithms can be implemented, see (Cottle et al., 1979). Our Lemke's algorithm implementation can be found in (Raous et al., 1988; Chabrand et al., 1998) and in (Panagiotopoulos et al., 1998). This direct method (pivoting technique) is very fast but needs more programming work. Generalization to 3 D problems is possible by using a Coulomb's cone with facets, see (Klarbring, 1986). All these solvers have been implemented in our finite element code GYPTIS. Some extensions for finite deformations and plasticity problems have been implemented in SIMEM 3.

1.4.2 The quasistatic problem

It has been shown in Section 2 that the incremental formulation of a quasistatic contact problem with friction can be written under the form of the implicit variational inequality (1.11). At each time step, by using the same fixed point process on the sliding limit, as presented in subsection 4.1, the following problem has to be solved.

$\mathbf{P}_7$: Find $u^{i+1} \in K$ such that $J(u^{i+1}) \leq J(v) \quad \forall\, v \in K$, where

$$J(v) = \frac{1}{2}\, a(v,v) - (F^{i+1}, v) + \int_{\Gamma_c} p|v_T - u_T^i| ds\,.$$

This problem is very similar to problem $\mathbf{P}_6$. The extra term in the part related to the friction characterizes the memory due to the velocity formulation of the friction. Algorithms of subsection 4.1 can be used. An alternative solving based on the increments of displacements is given in (Cocu et al., 1996).

1.4.3 Coupling of adhesion and friction

Once the problems has been written under the incremental form $\mathbf{P}_3^{n,i}$, it is easy to show that, by using the fixed point argument, we have to solve for each subiteration the following problem.

$\mathbf{P}_8$: Find $(u^{i+1},\, \beta^{i+1}) \in \mathbf{K} \times H$ such that

$$\mathbf{J}(u^{i+1}, \beta^{i+1}) \leq \mathbf{J}(v, \beta^{i+1}) \quad \forall\, v \in \mathbf{K}\,, \tag{1.26}$$

$$\beta^{i+1} - \beta^i = \Delta t\, g(\beta^{i+1}, u^{i+1}) \quad \text{a.e. on } \Gamma_c \text{ with} \tag{1.27}$$

$$\mathbf{J}(v, \beta^{i+1}) = \frac{1}{2}\,\mathbf{a}(v,v) - (\mathbf{F}^{i+1}, v) + \int_{\Gamma_c} p|v_T - u_T^i|ds \; + \frac{1}{2}\,\mathbf{b}(\beta^{i+1}, v, v)\,,$$

where p is now the fixed point of the application $p \longrightarrow \bar{S}(p)$ and $\bar{S}(p) = \mu|\sigma_N^*(u_p) + C_N\beta^2[u_{Np}]|$. Problem $\mathbf{P}_8$ is very similar to problem $\mathbf{P}_7$; the stiffness matrix will be modified by the extra term $\mathbf{b}(\beta^{i+1}, v, v)$ related to the stiffness of the contact. The integration of the differential equation on β is here presented with an implicit Euler method, but a θ-method have been implemented. The implicit character of the integration needs a few loops on β^{i+1} because of the nonlinearity. So, for each time t^{i+1} and for each value of β^{i+1}, the problem is solved by one of the algorithms presented in subsection 4.1. Numerical results and application to a fiber/matrix interface can be found in (Raous et al., 1998). Simulation of a micro-indentation experiment is carried out and an excellent agreement between numerical and experimental results has been observed.

1.4.4 An extension to dynamic problems

The LCP algorithms can be extended to the treatment of dynamic problems. LCP formulation of frictional and unilateral contact for a viscoelastic dynamic problem has been given in (Vola et al., 1998). The differential equation is written in terms of differential measures and the integration is conducted by combining two variants of θ-methods. A correct modelling of shocks and impacts is obtained.

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