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A consistent model coupling adhesion, friction, and unilateral contact

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Abstract

A model considering both unilateral contact, Coulomb friction, and adhesion is presented. In the framework of continuum thermodynamics, the contact zone is considered as a material boundary and the local constitutive laws are derived by choosing two specific surface potentials : the free energy and the dissipation potential. Because of the non regular properties of these potentials, convex analysis is used to derive the local behavior laws from the state and the complementary laws. The adhesion is characterized by an internal variable β , introduced by Frémond, which represents the intensity of adhesion. The continuous transition from a total adhesive condition to a possible pure frictional one is enforced by using elasticity coupled with damage for the interface. Non penetration conditions and Coulomb law are strictly imposed without using any penalty. The variational formulation for quasistatic problems is written as the coupling between an implicit variational inequality, a variational inequality, and a differential equation. An incremental formulation is proposed. An existence result under a condition on the friction coefficient is given. A numerical method is derived from the incremental formulation and various algorithms are implemented : they solve a sequence of minimization problems under constraints. The model is used to simulate a micro-indentation experiment conducted to characterize the behavior of fiber/matrix interface in a ceramic composite. Identification of the constitutive parameters is discussed.

1 Introduction

This paper is concerned with the coupling of unilateral contact, friction, and adhesion. The present model considers the interface as a material surface. The unilateral condition and the Coulomb friction law are strictly imposed without using any penalty. Adhesion and friction are strongly coupled through a specific compliance with damage that acts only in traction or shear and that disappears when the contact displacements increase. This insures a continuous transition between total adhesive and pure frictional states [3,37]. Adhesion is characterized by an intensity of adhesion β , previously introduced by Frémond [15]. In the present model, evolution of this intensity of adhesion β includes a viscosity effect. The thermodynamics basis of the model are given in Section 2 and a variational formulation is considered in Section 3. It is written under the form of the coupling between two variational inequalities (of which one is implicit) and a differential equation. Introducing an incremental formulation allows us to give an existence theorem when the friction coefficient is small. This incremental formulation suggests two numerical schemes to solve the problem. Several algorithms are presented in Section 4. In Section 5, the model is used to simulate the micro-indentation of a fiber of a composite material. Identification of the model parameters is presented, and ability of the model to describe the behavior of the fiber/matrix interface is discussed.

The present work is based on Frémond's work on adhesion [15] and on previous works on quasistatic problems with friction [8,11,37,38]. Most of the works on interface modelling have been conducted on fiber/matrix interface or on delamination. Some of them are analytical [17,28,36,41]. They give a global analysis of the decohesion, and friction modelling (when it is considered) is elementary (Tresca's law). Lemaitre [23] has given a formulation of interface damage based on a viscoplastic analysis restricted to a surface. Needleman [33,34] has proposed a decohesion model without friction, based on atomistic considerations. An exponential representation is used for the interface potential. A few years later, Tvergaard [45] introduced a non dimensional damage parameter to include a non reversible evolution of decohesion. The same author added the coupling with a frictional behavior, which is active only when adhesion is totally broken. This induces a non smooth evolution of the contact forces, which turn back to zero before friction starts. Recently Chaboche et al [5] have introduced a way, slightly different of ours [4], to couple the two phenomena ; it insures the continuity of the stress and displacement evolutions. Stupkiewicz [43] adds the influence of micro-asperities in a strip model [32] governed by energy or stress criteria. In these models, the contact conditions (both the non penetration condition and the friction law) are regularized and viscosity is not considered for the evolution of adhesion. For homogenization problems, decohesion between inclusions and matrix has been considered in [33] and in [31], where Michel et al use a Griffith criterion derived from interface damage model with elastic domain. For delamination problems, adhesion

has been widely studied by Allix and Ladeveze [1,2]. Contrary to the previous progressive decohesion models, several approaches have considered brittle break down for the interface adhesion (either analytical or numerical treatment) [17,22,29,30]. On the basis of material boundary hypothesis Klarbring [18] proposed a thermodynamic analysis of frictional contact and Strömberg et al [42] gave an extension to take wear into account.

Some works have been done considering an interphase model with the presence of a third body [6]. Although a third body approach would be attractive for the sake of simplicity of the concept, numerical simulations are difficult for two reasons : the parameters and the real geometry of the third body are very hard to determine, and the numerical treatment (mesh,...) is difficult because the interphase domain is one hundred or one thousand times smaller than the other domains under consideration. On the contrary, asymptotic methods and theoretical studies on the equivalent behavior of the third body when the thickness decreases to zero are very constructive [13,16,19,25,26,40].

2 A thermodynamic formulation for the coupling of adhesion and friction

2.1 Hypotheses and notations

Let Ω^1 and Ω^2 be two domains of R^d ($d = 2, 3$) occupied by two continuous bodies, with a boundary separated in three disjoint parts for each domain :

$$\partial\Omega^\alpha = \Gamma_U^\alpha \cup \Gamma_\phi^\alpha \cup \Gamma_C^\alpha \quad \alpha = 1, 2.$$

Let $\Gamma_\phi^1, \Gamma_\phi^2$ denote the parts of the boundary where external forces ϕ^1, ϕ^2 are

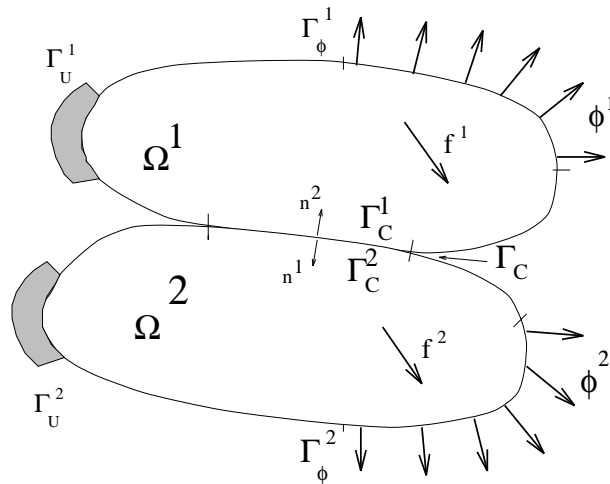


Fig. 1. *Contact between two elastic bodies*

respectively applied (see Fig. 1). In the same manner, Γ_U^1 and Γ_U^2 are the parts of $\partial\Omega^1$, $\partial\Omega^2$ where the displacements are prescribed and f^1 and f^2 denote the imposed volume force densities. Let Γ_C^1 , Γ_C^2 denote the parts of the boundary where the two solids are initially in contact. Introducing a gap in the unilateral conditions would allow the treatment of a case where contact may occur on a part larger than the present Γ_C . Assuming small displacements hypothesis, we have $n^1 = -n^2$, where n^α denotes the outward normal unit vector to $\partial\Omega^\alpha$, $\alpha = 1, 2$. Thus each particle of Γ_C^1 is coupled with a particle of Γ_C^2 in a single valued correspondence. Consequently, the two material boundaries $\Gamma_C^\alpha \subset \partial\Omega^\alpha$ ($\alpha = 1, 2$) define at the beginning a common zone of contact denoted Γ_C : $\Gamma_C^1 \simeq \Gamma_C^2 \simeq \Gamma_C$. The relative displacement between the two bodies is defined on Γ_C by $[u] = u^2 - u^1$ where u^1 and u^2 measure the displacements of two corresponding points. For the normal/tangential decomposition, the vector n^1 is chosen ($n = n^1 = -n^2$) : $[u] = U_N n + U_T$ with $U_N = [u] \cdot n$. According to this choice, the non penetration condition will be written $U_N \geq 0$ (convenient for a gap condition). When the second solid is rigid ($u^2 = 0$), the usual Signorini condition $U_N = u^1 \cdot n = u^1 \cdot n^1 \leq 0$ is obtained.

The internal force on Γ_C is denoted R . By using the principle of virtual power (see [12]), we obtain : $R = \sigma^1 n^1 = -\sigma^2 n^2$, according to the previous choice for $[u]$.

2.2 The general framework

Considering the contact area Γ_C as a material boundary [12,15,18,42], we introduce a surface density of internal energy E and a density of entropy S associated to the pseudo domain Γ_C . Then the Helmholtz free energies can be written $\Psi = E - ST$ on Γ_C and $\psi = e - sT$ in $\Omega^1 \cup \Omega^2$ (where e is the specific internal energy, s the specific entropy and T the absolute temperature). According to Frémond [14,15], we introduce the internal state variable β , which represents the intensity of adhesion ($\beta = 1$ means that the adhesion is total, $\beta = 0$ means that there is no adhesion and $0 < \beta < 1$ is the case of partial adhesion). If adhesion is described with a damage parameter D , we have $\beta = 1 - D$. As Michel et al [31] and Needlman [33], let us introduce the normalized relative displacements $u_N = U_N/l$ and $u_T = U_T/l$ where l is a characteristic length associated to the geometry such that $u_N < 1$ and $u_T < 1$. Numerous choices are possible : in the example of Section 6, we take l equal to the radius of the fiber. The following state variables are chosen : $\epsilon = (\epsilon_{ij})$, the strain tensor in $\Omega^1 \cup \Omega^2$, the normalized relative displacement u_T and u_N and the intensity of adhesion β on Γ_C . Classically, let σ^r denote the thermodynamic force associated to ϵ , where the superscript $(.)^r$ means reversible part. In the same manner, we take R_N^r and R_T^r as the thermodynamic forces associated to u_N and u_T . The internal variable β is then related to the thermodynamic force of decohesion denoted by G_β (see Table 1).

Table 1

The thermodynamic variables.

State variables	ϵ	u_N	u_T	β
Thermodynamic forces	σ^r	R_N^r	R_T^r	G_β

2.3 Free energies and state laws

Into the domain $\Omega^1 \cup \Omega^2$, the free energy ψ is the potential of elasticity (1), and the classical state law (2) is written as follows.

$$\psi^\alpha(\epsilon) = \frac{1}{2\rho} A_{ijkl}^\alpha \epsilon_{ij} \epsilon_{kl} \quad (1)$$

$$(\sigma^r)^\alpha = \rho \frac{\partial \psi^\alpha(\epsilon)}{\partial \epsilon} \quad (2)$$

In the following, we will consider only what concerns the contact boundary Γ_C , the part concerning $\Omega^1 \cup \Omega^2$ being a classical elasticity problem. Onto Γ_C , the surface density of free energy Ψ is chosen as follows :

$$\Psi(u_N, u_T, \beta) = \frac{1}{l} \left(\frac{\tilde{C}_N}{2} U_N^2 \beta^2 + \frac{\tilde{C}_T}{2} \|U_T\|^2 \beta^2 \right) - w h(\beta) + I_{\tilde{K}}(U_N) + I_P(\beta) \quad (3)$$

where $\tilde{K} = \{v / v \geq 0\}$ and $P = \{\gamma / 0 \leq \gamma \leq 1\}$.

Introducing the new variables $u_N = U_N/l$ and $u_T = U_T/l$, the potential (3) can be written as :

$$\Psi(u_N, u_T, \beta) = \frac{C_N}{2} u_N^2 \beta^2 + \frac{C_T}{2} \|u_T\|^2 \beta^2 - w h(\beta) + I_{\tilde{K}}(u_N) + I_P(\beta) \quad (4)$$

where $C_N = l \tilde{C}_N$ and $C_T = l \tilde{C}_T$.

The introduction of the indicator functions $I_{\tilde{K}}$ and I_P imposes the unilateral condition $u_N \geq 0$ and the condition $\beta \in [0, 1]$. The interfacial forces induced by the adhesion are introduced under the form of a compliance law depending on the current state of adhesion β and characterized by the initial stiffness C_N and C_T . In the work of Truong Dinh Tien [44], concerning a frictionless problem with adhesion, the penalty coefficients introduced for algorithmic reasons can be considered as large limit values of C_N and C_T . The term $w h(\beta)$ is a general form of the energy of decohesion which is supposed to be differentiable. With $h(\beta) = \beta$, w could be considered as Dupré's energy. Other forms for $h(\beta)$ could be considered.

The pseudo-potential Ψ can be separated into :

- a part (the first three terms) that is differentiable but not convex according to the couple (u, β) (it is only convex according to u or to β separately),
- a part (the last two terms) that is convex but not differentiable.

To write the state laws, the two difficulties (lack of convexity and lack of differentiability) are overcome by using local or partial subdifferentiation [3,12,15,37]. Then the state laws can be written as :

$$R_N^r \in \partial_{u_N} \Psi(u_N, u_T, \beta) \quad (5)$$

$$R_T^r \in \partial_{u_T} \Psi(u_N, u_T, \beta) \quad (6)$$

$$-G_\beta \in \partial_\beta \Psi(u_N, u_T, \beta) \quad (7)$$

where ∂_u and ∂_β denote respectively the subdifferential with respect to the variables u and β .

The following relation (8) is easily deduced from (6). By making explicit the subdifferentials ∂_{u_N} and ∂_β in (5) and (7), we obtain the following relations on the normal components (9) and on the thermodynamic force G_β (10).

$$R_T^r = C_T u_T \beta^2 \quad (8)$$

$$u_N \geq 0 \quad -R_N^r + C_N u_N \beta^2 \geq 0 \quad (-R_N^r + C_N u_N \beta^2) u_N = 0 \quad (9)$$

$$\begin{cases} G_\beta \geq w h'(\beta) & \text{if } \beta = 0 \\ G_\beta = w h'(\beta) - (C_N u_N^2 + C_T \|u_T\|^2) \beta & \text{if } \beta \in]0, 1[\\ G_\beta \leq w h'(\beta) - (C_N u_N^2 + C_T \|u_T\|^2) & \text{if } \beta = 1. \end{cases} \quad (10)$$

The state laws show that :

- the reversible (elastic) part of the tangential force depends on the square of the adhesion β ,
- the reversible parts of the normal components of R and u satisfy a generalized Signorini condition (unilateral contact),
- the thermodynamic force G_β , if $\beta \in]0, 1[$, is composed of the adhesive energy minus the elastic energy of the interface.

2.4 Dissipation and complementary laws

The two laws of thermodynamics give the following form of the Clausius-Duhem inequalities (in an isothermic evolution) :

$$\begin{cases} \rho \dot{\psi} \leq \sigma \cdot \dot{\epsilon} & \text{in } \Omega^1 \cup \Omega^2 \\ \dot{\Psi} \leq R_N \dot{u}_N + R_T \dot{u}_T & \text{on } \Gamma_C. \end{cases} \quad (11)$$

In Ω^α the behavior is totally reversible (elastic) and so $\sigma = \sigma^r$.

On Γ_C , using the state laws (5), (6), (7) to evaluate the time derivative of Ψ in the second Clausius Duhem inequality [3], the following expression for the dissipation D is obtained.

$$D = (R_N - R_N^r) \dot{u}_N + (R_T - R_T^r) \dot{u}_T + G_\beta \dot{\beta} \geq 0 \quad (12)$$

The irreversible part of the contact forces are $R_N^{ir} = R_N - R_N^r$ and $R_T^{ir} = R_T - R_T^r$. The only dissipative processes under consideration are friction and adhesion. To formulate the following complementary laws where the irreversible forces verify (12), we have to choose a pseudo-potential of dissipation $\Phi = \Phi(\dot{u}_T, \dot{\beta}; \chi_N)$ with $\chi_N = (R_N^r, u_N, \beta)$, positive, convex in $(\dot{u}_T, \dot{\beta})$, and zero in $(0, 0)$ (extension of standard generalized materials). We choose the following form for the pseudo-potential of dissipation :

$$\Phi(\dot{u}_T, \dot{\beta}; \chi_N) = \mu |R_N - C_N u_N \beta^2| \|\dot{u}_T\| + \frac{b}{p+1} |\dot{\beta}|^{p+1} + I_{C^-}(\dot{\beta}) \quad (13)$$

with $C^- = \{\gamma \in W / \gamma \leq 0\}$ and $p \leq 1$. A power law is considered for the evolution of the adhesion. The indicator function imposes that $\dot{\beta} \leq 0$: the adhesion is allowed only to decrease and cannot be regenerated (not reversible). This is in agreement with our application but other choices could be made for other situations. The parameter μ is the friction coefficient of the Coulomb law. The parameter b characterizes a time dependent evolution of the adhesion.

The complementary laws are then written :

$$R_N^{ir} = 0 \quad (14)$$

$$(R_T^{ir}, G_\beta) \in \partial \Phi(\dot{u}_T, \dot{\beta}; \chi_N) \quad (15)$$

and (15) can be written as :

$$R_T^{ir} \in \partial_{\dot{u}_T} \Phi(\dot{u}_T, \dot{\beta}; \chi_N) \quad (16)$$

$$G_\beta \in \partial_{\dot{\beta}} \Phi(\dot{u}_T, \dot{\beta}; \chi_N) \quad (17)$$

The normal behavior has been supposed to be totally elastic. Making explicit the subdifferentials in (16) and (17), we obtain on Γ_C :

$$\|R_T - C_T u_T \beta^2\| \leq \mu |R_N - C_N u_N \beta^2| \quad (18)$$

$$\begin{aligned} \text{with : } \quad & \|R_T - C_T u_T \beta^2\| < \mu |R_N - C_N u_N \beta^2| \Rightarrow \dot{u}_T = 0 \\ & \|R_T - C_T u_T \beta^2\| = \mu |R_N - C_N u_N \beta^2| \Rightarrow \exists \lambda \geq 0, \\ & \dot{u}_T = \lambda(R_T - C_T u_T \beta^2) \end{aligned}$$

$$\dot{\beta} = - \left(G_\beta^- / b \right)^{1/p}, \quad (19)$$

where G_β^- denotes the negative part of G_β .

2.5 The local model

Putting together the various relations obtained in the previous sections, the local model coupling adhesion, friction, and unilateral contact is written on Γ_C as follows.

Unilateral conditions with adhesion

$$-R_N + C_N u_N \beta^2 \geq 0, \quad u_N \geq 0, \quad (-R_N + C_N u_N \beta^2) u_N = 0 \quad (20)$$

Coulomb friction with adhesion

$$\begin{aligned} R_T^r &= C_T u_T \beta^2 \\ \|R_T - R_T^r\| &\leq \mu |R_N - C_N u_N \beta^2| \\ \|R_T - R_T^r\| &< \mu |R_N - C_N u_N \beta^2| \Rightarrow \dot{u}_T = 0 \\ \|R_T - R_T^r\| &= \mu |R_N - C_N u_N \beta^2| \Rightarrow \exists \lambda \geq 0, \quad \dot{u}_T = \lambda(R_T - R_T^r) \end{aligned} \quad (21)$$

Evolution of adhesion intensity

$$\begin{aligned} \dot{\beta} &= - \left[(1/b) (w h'(\beta) - (C_N u_N^2 + C_T \|u_T\|^2) \beta) \right]^{1/p} \text{ if } \beta \in [0, 1[\\ \dot{\beta} &\leq - \left[(1/b) (w h'(\beta) - (C_N u_N^2 + C_T \|u_T\|^2) \beta) \right]^{1/p} \text{ if } \beta = 1. \end{aligned} \quad (22)$$

The contact variables are $u_N, u_T, \beta, R_N, R_T$. The six parameters of the model are C_N, C_T (initial normal and tangential stiffness of the interface if adhesion

is complete), μ (the friction coefficient), b (viscosity of the adhesion evolution), w (limit of decohesion energy) and p (the power coefficient). In what follows, we take $h(\beta) = \beta$.

Let us now analyze the interface behavior for a 2D case. The initial conditions are supposed to be complete adhesion ($\beta = 1$) and zero displacement ($u_T = u_N = 0$). Considering first the normal behavior (see Fig. 2), under compressive action, the non penetration condition is strictly verified ($u_N = 0$).

Under traction ($u_N \geq 0$), an adhesive resistance ($R_N = C_N u_N \beta^2$) is active

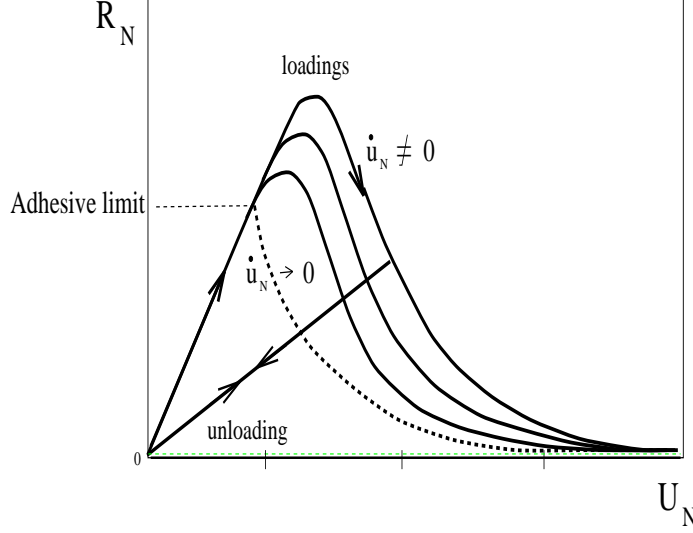


Fig. 2. Normal behavior (for $u_T = 0$ here)

(elasticity with damage). The intensity of adhesion starts to decrease when the displacement is sufficiently large such that the elastic energy becomes larger than the limit of adhesion energy w . Evolution of the adhesion is then governed by equation (22). When adhesion is totally broken, the classical Signorini problem is obtained.

Considering now the shear behavior (see Fig. 3), note first that friction acts only if a normal compression is applied ; if a normal traction is applied ($u_N > 0$), the sliding limit ($\mu |R_N - C_N u_N \beta^2|$) is zero because of (20) and the tangential behavior is elastic with damage ($R_T = C_T u_T \beta^2$). Under compression, the sliding limit is ($\mu |R_N|$) (because $u_N = 0$). As long as the norm of the tangential force $\|R_T\|$ is smaller than the sliding limit, sliding does not occur ($u_T = 0$ as initial condition and $\dot{u}_T = 0$ in relation (21)). When the sliding limit is reached, an elastic tangential displacement occurs. The adhesion begins to decrease when the adhesive limit is reached and evolution of β is then governed by (22). When adhesion is lost (β goes to zero), the usual Coulomb friction conditions are obtained.

If the loading remains constant (when the adhesion limit is overcome), the adhesion keeps decreasing (by relaxation). If the tangential loading is now

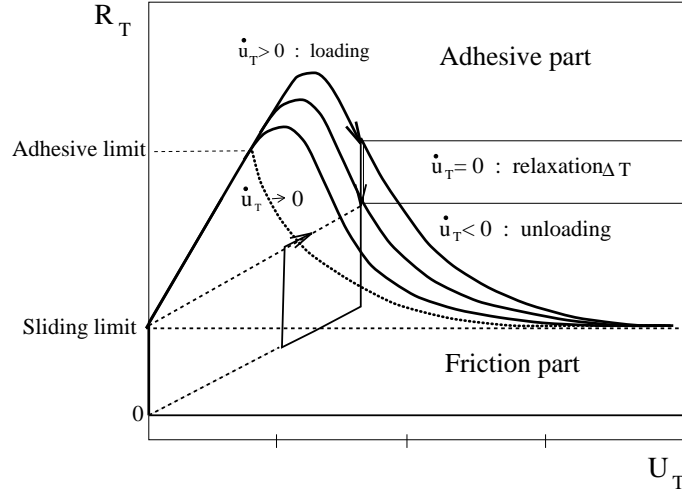


Fig. 3. *Tangential behavior (R_N is constant)*

backward, an opposite tangential displacement occurs only when the other side of the Coulomb cone is reached.

3 Variational formulation of the quasistatic problem

3.1 The quasistatic problem

Problem P1.

Find the displacements u^α , the stresses σ^α ($\alpha = 1, 2$), the strains ϵ , the contact force R such that :

$$\epsilon = \text{grad}_s u^\alpha \quad \text{in} \quad \Omega^\alpha \quad (23)$$

$$\text{div} \sigma^\alpha + f^\alpha = 0 \quad \text{in} \quad \Omega^\alpha \quad (24)$$

$$\sigma^\alpha n^\alpha = \phi^\alpha \quad \text{on} \quad \Gamma_\phi^\alpha \quad (25)$$

$$u^\alpha = 0 \quad \text{on} \quad \Gamma_U^\alpha \quad (26)$$

$$[u] = u_2 - u_1 = u_N n + u_T \quad \text{on} \quad \Gamma_C \quad (27)$$

$$\sigma^1 n^1 = -\sigma^2 n^2 = R_N n + R_T \quad \text{on} \quad \Gamma_C \quad (28)$$

$$\sigma^\alpha = K^\alpha : \epsilon \quad \text{in} \quad \Omega^\alpha \quad (29)$$

and on Γ_C :

$$-R_N + C_N u_N \beta^2 \geq 0, \quad u_N \geq 0, \quad (-R_N + C_N u_N \beta^2) u_N = 0 \quad (30)$$

$$R_T^r = C_T u_T \beta^2$$

$$\begin{aligned}
\|R_T - R_T^r\| &\leq \mu \left| R_N - C_N u_N \beta^2 \right| \\
\|R_T - R_T^r\| &< \mu \left| R_N - C_N u_N \beta^2 \right| \Rightarrow \dot{u}_T = 0 \\
\|R_T - R_T^r\| &= \mu \left| R_N - C_N u_N \beta^2 \right| \Rightarrow \exists \lambda \geq 0, \dot{u}_T = \lambda(R_T - R_T^r),
\end{aligned} \tag{31}$$

$$\begin{aligned}
\dot{\beta} &= - \left[(1/b) \left(w h'(\beta) - (C_N u_N^2 + C_T \|u_T\|^2) \beta \right) \right]^{1/p} \text{ if } \beta \in [0, 1[, \\
\dot{\beta} &\leq - \left[(1/b) \left(w h'(\beta) - (C_N u_N^2 + C_T \|u_T\|^2) \beta \right) \right]^{1/p} \text{ if } \beta = 1.
\end{aligned} \tag{32}$$

The time dependence of the loading is slow enough to make the inertial terms negligible.

3.2 The variational formulation

In what follows, we make the simplest hypothesis of a linear dissipation for the adhesion evolution (case $p = 1$).

We denote by V^α the following spaces :

$$V^\alpha = \left\{ v^\alpha \in [H^1(\Omega^\alpha)]^d; v^\alpha = 0 \text{ a.e. on } \Gamma_U^\alpha \right\}, \alpha = 1, 2,$$

and we set

$$V = V^1 \times V^2, \quad K = \{v = (v^1, v^2) \in V; v_N \geq 0 \text{ a.e. on } \Gamma_C\},$$

$$H = L^\infty(\Gamma_C).$$

For all $v \in V$ the norm on V is given by $\|v\|_V = \|v^1\|_{V^1} + \|v^2\|_{V^2}$ and $\langle \cdot, \cdot \rangle$ shall denote the duality pairing on $H^{\frac{1}{2}}(\Gamma_C) \times H^{-\frac{1}{2}}(\Gamma_C)$.

We suppose that :

$$\begin{aligned}
f &\in W^{1,2}(0, T; [L^2(\Omega^1)]^d \times [L^2(\Omega^2)]^d), \\
\phi &\in W^{1,2}(0, T; [L^2(\Gamma_\phi^1)]^d \times [L^2(\Gamma_\phi^2)]^d),
\end{aligned}$$

which imply that $F \in W^{1,2}(0, T; V)$, where

$$(F, v) = \sum_{\alpha=1,2} \left[\int_{\Omega^\alpha} f^\alpha \cdot v^\alpha dx + \int_{\Gamma_\phi^\alpha} \phi^\alpha \cdot v^\alpha ds \right] \quad \forall v \in V.$$

We shall adopt the following notations :

- $a : V \times V \longrightarrow \mathbb{R}$,

$$a(u, v) = a^1(u^1, v^1) + a^2(u^2, v^2) \quad \forall u = (u^1, u^2), v = (v^1, v^2) \in V,$$

$$\text{where } a^\alpha(u^\alpha, v^\alpha) = \int_{\Omega^\alpha} A_{ijkl}^\alpha \epsilon_{ij}(u^\alpha) \epsilon_{kl}(v^\alpha) dx, \quad \alpha = 1, 2.$$

- $j : H \times V \times V \longrightarrow \mathbb{R}$,

$$j(\beta, u, v) = \int_{\Gamma_C} \mu |\sigma_N^*(Pu^1) + C_N \beta^2 u_N| \|v_T\| ds,$$

where P is the projection of $W^{1,2}(0, T; V^1)$ on V_0^1 with

$$\begin{aligned} V_0^1 &= \left\{ w^1 \in W^{1,2}(0, T; V^1); \int_0^T a^1(w^1, \psi) dt \right. \\ &= \int_0^T (f^1, \psi)_{[L^2(\Omega^1)]^d} dt + \int_0^T (\phi^1, \psi)_{[L^2(\Gamma_\phi^1)]^d} dt, \\ &\quad \left. \forall \psi \in L^2(0, T; V^1), \psi = 0 \text{ a.e. on } \Gamma_C \times]0, T[\right\}, \end{aligned}$$

$$\mu \in L^\infty(\Gamma_C) \text{ and } \mu \geq 0,$$

$$(\cdot)^* : H^{-\frac{1}{2}}(\Gamma_C) \rightarrow L^2(\Gamma_C) \text{ is a linear and compact mapping.}$$

- $c_N, c_T : H \times V \times V \longrightarrow \mathbb{R}$,

$$c_N(\beta, u, v) = \int_{\Gamma_C} C_N \beta^2 u_N v_N ds \quad \text{and} \quad c_T(\beta, u, v) = \int_{\Gamma_C} C_T \beta^2 u_T \cdot v_T ds.$$

- $y(\beta, u) = -\frac{1}{b} \left[w - (C_N u_N^2 + C_T \|u_T\|^2) \beta \right]^-.$

Then the local problem **P1** admits the following variational formulation (see [3,11,37]).

Problem P2.

Find $(u, \beta) \in W^{1,2}(0, T; V) \times W^{1,2}(0, T; H)$ such that $u(0) = u_0$, $\beta(0) = \beta_0$ and for almost all $t \in [0, T]$, $u(t) \in K$ and

$$\begin{aligned} a(u, v - \dot{u}) + j(\beta, u, v) - j(\beta, u, \dot{u}) + c_T(\beta, u, v - \dot{u}) \geq \\ (F, v - \dot{u}) + \langle \sigma_N(u^1), v_N - \dot{u}_N \rangle \quad \forall v \in V \end{aligned} \quad (33)$$

$$\langle \sigma_N(u^1), z_N - u_N \rangle + c_N(\beta, u, z - u) \geq 0 \quad \forall z \in K, \quad (34)$$

$$\dot{\beta} = y(\beta, u) \quad \text{a.e. on } \Gamma_C, \quad (35)$$

where the initial conditions $u_0 \in K$, $\beta_0 \in H$, $\beta_0 \in [0, 1[$ a.e. on Γ_C and satisfy the following compatibility condition :

$$\begin{aligned} a(u_0, w - u_0) + j(\beta_0, u_0, w - u_0) + c_T(\beta_0, u_0, w - u_0) \geq \\ (F(0), w - u_0) \quad \forall w \in K. \end{aligned} \quad (36)$$

This problem is a generalization of the quasistatic unilateral contact problem with friction considered by Cocu et al in [11].

4 Incremental formulation and mathematical results

First, let us focus on the properties of the bilinear form a and the mapping j involved in the problem **P2** :

$a(., .)$ is continuous on $V \times V$ and coercive i.e. it satisfies

$$\exists M > 0 \quad \forall u \in V \quad \forall v \in V \quad |a(u, v)| \leq M \|u\|_V \|v\|_V, \quad (37)$$

$$\exists m > 0 \quad \forall u \in V \quad a(u, u) \geq m \|u\|_V^2, \quad (38)$$

and the mapping $j(., ., .)$ satisfies the following property :

$$\begin{aligned} \exists C > 0 \quad \forall u, \bar{u}, v, \bar{v} \in V_0^1 \\ |j(\beta, u, v) - j(\beta, u, \bar{v}) - j(\beta, \bar{u}, v) + j(\beta, \bar{u}, \bar{v})| \leq \bar{\mu} C \|u - \bar{u}\|_V \|v - \bar{v}\|_V, \end{aligned} \quad (39)$$

with $\bar{\mu} = |\mu|_{L^\infty(\Gamma_C)}$.

An incremental formulation is obtained by operating a time discretization of problem **P2**, taking $n \in \mathbb{N}^*$ and setting $\Delta t = T/n$, $t^i = i \Delta t$ and $F^i = F(t^i)$ for $i = 0, \dots, n$. Using an implicit scheme we obtain the following sequence of problems (P_i^n) , $i = 0, \dots, n-1$, defined for a given $(u^0, \beta^0) \in K \times H$.

Problem P_i^n .

Find $(u^{i+1}, \beta^{i+1}) \in K \times H$ such that :

$$\begin{aligned} a(u^{i+1}, v - u^{i+1}) + j(\beta^{i+1}, u^{i+1}, v - u^i) - j(\beta^{i+1}, u^{i+1}, u^{i+1} - u^i) \\ + c(\beta^{i+1}, u^{i+1}, v - u^{i+1}) \geq (F^{i+1}, v - u^{i+1}) \quad \forall v \in K \end{aligned} \quad (40)$$

$$\beta^{i+1} - \beta^i = \Delta t y(\beta^{i+1}, u^{i+1}) \quad \text{a.e. on } \Gamma_C, \quad (41)$$

where $c(\cdot) = c_N(\cdot) + c_T(\cdot)$.

The incremental problem P_i^n is solved using a fixed point method as follows.

For every $\bar{u} \in K$ let us denote by $s(\bar{u}) = \beta$ the solution of

$$\beta = \Delta t y(\beta, \bar{u}) + \beta^i. \quad (42)$$

For every $\beta \in H$ we denote by $u(\beta)$ the solution of

$$\begin{aligned} u \in K \quad & a(u, v - u) + j(\beta, u, v - u^i) - j(\beta, u, u - u^i) \\ & + c(\beta, u, v - u) \geq (F^{i+1}, v - u) \quad \forall v \in K. \end{aligned} \quad (43)$$

The existence of the solution for problem (42) is clear and inequalities such as (43) have a unique solution if the friction coefficient is sufficiently small i.e. $\bar{\mu} < \frac{m}{C}$ (see [9]). We suppose from now on that μ satisfies this condition.

We define the mapping $T : K \longrightarrow K$ by

$$\forall \bar{u} \in K \quad T(\bar{u}) = u(s(\bar{u})).$$

By a straightforward computation, we obtain that $\exists k_1 > 0$ such that for all $\bar{u}_1, \bar{u}_2 \in K$

$$|s(\bar{u}_1) - s(\bar{u}_2)| \leq k_1 \Delta t (\|\bar{u}_1\| + \|\bar{u}_2\|) \|\bar{u}_1 - \bar{u}_2\| \quad a.e. \text{ on } \Gamma_C. \quad (44)$$

Let us now set $u_1 = u(s(\bar{u}_1))$, $u_2 = u(s(\bar{u}_2))$. Adding the inequalities (43) with $u = u_1$, $v = u_2$ and $u = u_2$, $v = u_1$ and using the properties of a , j and b one can show that $\exists k_2 > 0$ such that for all $\bar{u}_1, \bar{u}_2 \in K$

$$\|u_1 - u_2\|^2 \leq k_2 \int_{\Gamma_C} |u_2| |u_1 - u_2| |s(\bar{u}_1) - s(\bar{u}_2)| ds. \quad (45)$$

From the inequalities (44), (45) and the relation $\|u\| \leq k_3$ (with k_3 independent of \bar{u}), satisfied by $u(s(\bar{u}))$, it follows that T is a contraction mapping for sufficiently small Δt . Then T has a unique fixed point u and $(u, s(u))$ is the solution of the incremental problem \mathbf{P}_i^n (see also [10]).

5 Numerical methods and algorithms

The previous sections showed that the two variational inequalities (one is implicit) coupled with the differential equation on β can be reduced for the

incremental formulation to the only one variational inequality (40). By extending the results of [11], it is evidenced that this variational inequality is very similar to the one obtained for the static problem of Coulomb friction with unilateral contact without adhesion. As shown in [11], two alternative discrete formulations (one set on the variables u^{i+1} and the other one set on the increments $\Delta u^{i+1} = u^{i+1} - u^i$) can be associated : they include extra terms related to the previous time steps (velocity formulation of the friction) and to the variable β (coupling with adhesion).

For the time discretization, an implicit Euler method was introduced in the previous section. The implicit or semi-implicit character of the integration is important because the intensity of adhesion β may decrease very fast during the loading. A θ -method is also used ; it improves the accuracy of the solution. For the sake of simplicity, only the Euler method is presented here.

A finite element approximation is used to solve the problem \mathbf{P}_i^n at each time step t^{i+1} . Only 2D problems are considered here. In Note 2, a convenient treatment for 3D problems is given.

The approximate problem \mathbf{P}_i^h associated to problem \mathbf{P}_i^n is written:

Problem \mathbf{P}_i^h .

For each time step t^{i+1} , find $\bar{\beta}^h$ fixed point of the application $s(\cdot)$:

$$\beta^h \longrightarrow s(u^h(\beta^h)) = \beta_h^i + \Delta y(\beta^h, u^h(\beta^h)) ,$$

where $u^h(\beta^h)$ is solution of Problem \mathbf{Q}_i^h .

Problem \mathbf{Q}_i^h .

Find $u^h(\beta^h) \in K^h$ such that $\forall w^h \in K^h$,

$$\begin{aligned} a(u^h, w^h - u^h) + \pi_h j_h(\beta^h, u^h, w^h - u_h^i) - \pi_h j_h(\beta^h, u^h, u^h - u_h^i) \\ + c(\beta^h, u^h, w^h - u^h) \geq (F_h^{i+1}, w^h - u^h), \end{aligned}$$

where :

- $K^h = \{v^h \in U^h / v^h \geq 0 \text{ on } \Gamma_C\}$ is the set of the admissible displacements with $(U^h)_h$ a family of finite dimensional spaces which constitutes a finite element approximation of V ,
- π_h denotes the projection on the finite element discretization; the choice of this extra projection for the approximate problem makes the approximation of the absolute value in $j(\cdot, \cdot, \cdot)$ much simpler (as shown in [27]),
- β_h^i, u_h^i are the approximate solutions computed for the previous step t^i ,
- $y(\beta^h, u^h(\beta^h)) = -\frac{1}{b} [w - (C_N u_N^h)^2 + C_T \|u_T^h\|^2] \beta^h$
- $c(\beta^h, u^h, w^h - u^h) = \int_{\Gamma_C} C_N \beta^{h^2} u_N^h (w_N^h - u_N^h) ds + \int_{\Gamma_C} C_T \beta^{h^2} u_T^h \cdot (w_T^h - u_T^h) ds$,
- $j_h(\beta^h, u^h, w^h - u_h^i) = \int_{\Gamma_C} \mu |R_N(u^h) - C_N \beta^{h^2} u_N^h| \|w_T^h - u_{hT}^i\| ds$.

For solving the implicit variational inequality in problem \mathbf{Q}_i^h , let us introduce a fixed point method on the sliding limit. As show in [21,27,38], problem \mathbf{Q}_i^h is then equivalent to the problem \mathbf{R}_i^h .

Problem \mathbf{R}_i^h .

Find \bar{g} fixed point of the application $t(\cdot)$:

$$g \longrightarrow t(u^h(g)) = \mu \left| R_N(u^h(g)) - C_N \beta^{h^2} u_N^h(g) \right|,$$

where $u^h(g)$ is solution of problem \mathbf{S}_i^h .

Problem \mathbf{S}_i^h .

Find $u^h(g) \in K^h$ such that $\forall w^h \in K^h$,

$$\begin{aligned} a(u^h(g), w^h - u^h) + \pi_h j^*(w^h - u_h^i) - \pi_h j^*(u^h(g) - u_h^i) \\ + c(\beta^h, u^h(g), w^h - u^h(g)) \geq (F_h^{i+1}, w^h - u^h(g)), \end{aligned}$$

$$\text{where } j^*(v) = \int_{\Gamma_C} g \|v_T\| ds.$$

Now, problem \mathbf{S}_i^h is a classical variational inequality problem associated to a Tresca problem with a given sliding limit g . A minimum principle can be associated, thus problem \mathbf{S}_i^h is equivalent to the following problem \mathbf{T}_i^h .

Problem \mathbf{T}_i^h .

Find u_g^h such that

$$J(u_g^h) \leq J(v) \quad \forall v \in K^h$$

with

$$J(v) = \frac{1}{2} a(v, v) + j^*(v - u_h^i) + \frac{1}{2} c(\beta^h, v, v) - (F_h^{i+1}, v).$$

This is a minimization problem under constraint ($v \in K^h$) of a quadratic functional including a non differentiable part ($j^*(v - u_h^i)$). The discrete problem is then written as follows :

Problem \mathbf{U}_i^h .

Find $\mathbf{u} \in \mathbf{IK}$ such that

$$J(\mathbf{u}) \leq J(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{IK}$$

with

$$J(\mathbf{v}) = \frac{1}{2} \mathbf{v}^T \mathbf{A} \mathbf{v} + G^T |\mathbf{v} - u_h^i| + \frac{1}{2} \mathbf{v}^T \mathbf{C}(\beta) \mathbf{v} - F_h^{i+1T} \mathbf{v},$$

where :

- $\mathbf{IK} = \{\Pi K_i \text{ with } K_i = R^+ \text{ if } i \in I_N^c \text{ and } K_i = R \text{ if not } \}$ (if the shape functions are strictly positive the approximation is internal : that is the case when linear elements (T3 or Q4) are used).
- I_N^c is the set of the number of degrees of freedom concerning the normal components of the contact nodes,
- A is the matrix of dimension $N = \dim(V) : A_{ij} = a(w_i, w_j)$,
- C is the diagonal matrix of dimension M (M is the number of contact nodes) : $C_{kl} = c(\beta^h, w_k, w_l)$,
- G is the vector of dimension $M : G_j = \int_{\Gamma_C} g w_j ds$.

Problem \mathbf{U}_i^h is very similar to the one solved in [11,21,38]. The main difference is that, at each step, the terms of A concerning the contact variables have to be modified by adding the term C_{kl} related to the current contact condition. The memory of the loading history (velocity formulation of the friction) is given by the term $\|w^h - u_h^i\|$ in the functional j^* of the variational inequality of problem \mathbf{S}_i^h , where u_h^i is the solution at the previous time step.

Various solvers of problem \mathbf{U}_i^h have been implemented :

- Successive Over-Relaxation with Projection (see [21,38]),
- Gauss Seidel with Aitken acceleration and Projection,
- Projected Conjugate Gradient with Preconditioning (see [39]).

These algorithms are robust, and the average number of iterations is smaller than the number of degrees of freedom. The convergence of the fixed point of the sliding limit g (Problem \mathbf{R}_i^h) is fast (generally 7 or 8 iterations), and a diagonal process (coarse resolution for the first values of g) improves the efficiency of the algorithm. Computational times are given in the next section.

Note 1 : An alternative choice of the unknowns.

As given in [11], problem \mathbf{S}_i^h can also be written under the form of an equivalent problem where the unknown is the increment of displacement $\Delta u^{i+1} = \frac{u^{i+1} - u^i}{\Delta t}$ (velocity formulation). In that case, the convex \mathbf{IK} changes at each step, and the memory of the loading appears in the contribution to the loading term of the contact forces at the previous time step.

Note 2: An alternative solver (see [7,8,20,35]).

The incremental problem \mathbf{P}_i^h can also be written as a complementarity problem. Two extra variables must be introduced (see [20]) : sliding is separated into right and left slidings. After condensation of the problem, which is a reduction of the problem to only the contact variables, various algorithms of mathematical programming can be used. We have implemented Lemke's method (see [7,8,35,38]). It is a powerful pivoting direct method. Comparison with other methods can be found in [8]. By making the Coulomb cone

polygonal, 3D problems can be treated (see [20]).

6 An application : modelling of the fiber/matrix interface of a composite material

We have used the present model to describe the behavior of the fiber/matrix interface of composite materials. The global study deals with the interaction of a crack in the matrix with a fiber/matrix interface. The final topic is to optimize the characteristics of the interface in order to enforce the resistance of the composite to macro- and micro-crack progression. The first step of this study consists of the validation of the model by considering simulations on micro-indentation experiments carried out at the ONERA (Office National d'Etudes et de Recherches Aérospatiales). This is widely discussed from a mechanical point of view in [24]. Some corrections of the experiment data have to be made to take into account the plastic zone under the indenter, and the global elasticity of the testing bench. Various geometries are considered, boundary conditions are discussed, and residual thermal stresses are taken into account. To complete the validation, different loadings are simulated : single loading, cycles, relaxation, different loading velocities, etc.

In the present paper, we present only the ability of the model to describe the complex behavior of a fiber/matrix interface during a micro-indentation experiment (push-in). A qualitative identification of the model parameters is conducted on a single experiment, and the numerical results are discussed in relation to the fundamental choices of the model : smooth evolution from adhesion to friction, strict unilateral conditions, etc.

6.1 Experiment and model

A push-in experiment carried out at the ONERA on a SIC/SIC composite is used to test the model. The parameters are identified on the plot of the evolution of the force on the indenter relatively to its prescribed displacement. Figure 4 shows the model geometry. The radius of the fiber is $R_f = 8.6\mu m$. By analyzing the volume density of fibers ($V_f = 40\%$), the radius $R_m = 12.6\mu m$ is chosen to define a zone of pure matrix. The extra zone is defined as an equivalent homogeneous material equivalent to the composite one by using a mixture rule. Various global dimensions were tested. Results are given for $L = 200\mu m$ (radius of the domain 23 times larger than the fiber one) and $h = 1600\mu m$. The boundary conditions are given on Fig. 4.

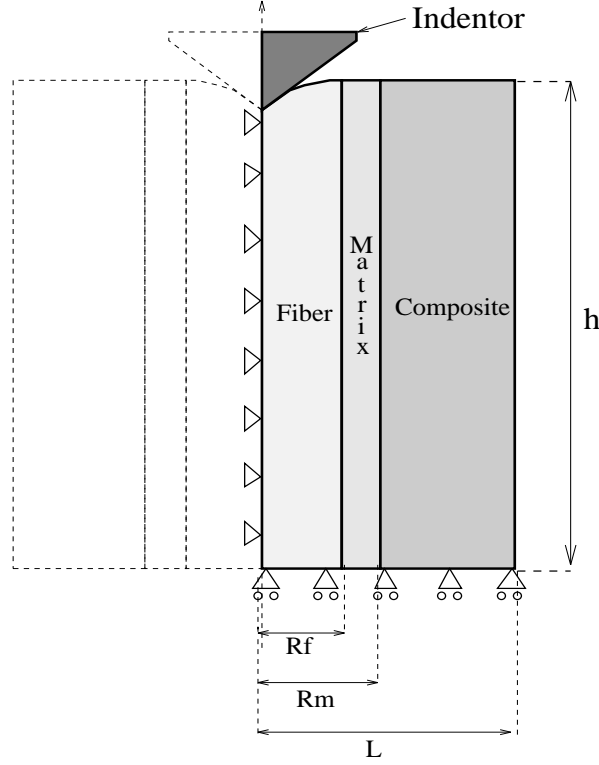


Fig. 4. *The micro-indentation test*

The elasticity coefficients of the different materials are given on Table 2. The characteristic length l is taken equal to the fiber radius : $l = 8.6\mu m$. A vertical prescribed displacement is applied on the indenter. A frictionless unilateral contact is considered between the indenter and the upper part of the fiber.

Table 2

Elasticity coefficients.

	E (Gpa)	ν
fiber	200	0.25
matrix	350	0.2
composite	290	0.22

6.2 Simulation and identification of the parameters

The finite element mesh is given on Fig. 5 and 6. Triangle T_3 are used. The mesh has 1419 nodes of which 300 are contact nodes of the interface.

Results of the identification of the model parameters are given on Table 3

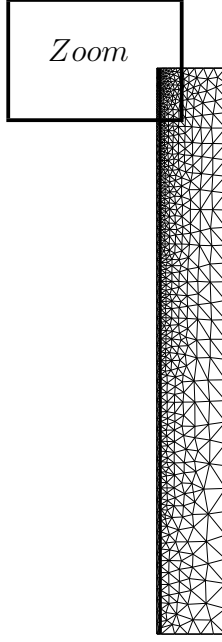


Fig. 5. *Initial Mesh.*

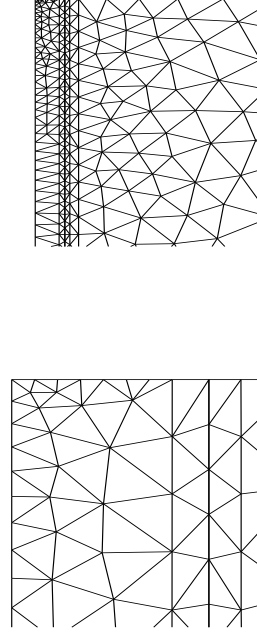


Fig. 6. *Zooms.*

and on Fig. 7. The same contact stiffness was chosen for the normal and the tangential interface behavior : $C_N = C_T = C$. Figure 7 shows the ability of the model to fit the force/displacement evolution of the push-in experiment. A more realistic identification is conducted in [24] on a collection of experiments. Mechanical considerations give a specified range of variations for each parameter. For the contact stiffness, the range is evaluated by considering the composition and the thickness of the layer of carbon and oxyde in the interface. On Fig. 7, we have also plotted the results of the simulation by making Table 3

First parameter identification.

parameters	values
Friction coefficient μ	0.075
Contact stiffness C	$0.007 \text{ N}/\mu\text{m}^2$
Energy of adhesion w	$1 \text{ J}/\text{m}^2$
Adhesion viscosity b	$25 \text{ N.s}/\text{m}$

$\mu = 0$ (no friction) and $w = 0$ (no adhesion) in the values of the parameters given on Table 3. This is helpful to better understand the experimental results

: the change of the slope is strongly related to the lost of adhesion and the final slope to the residual friction between the fiber and the matrix when adhesion is broken.

On Fig. 8, the contact condition is presented for a given step of the load-

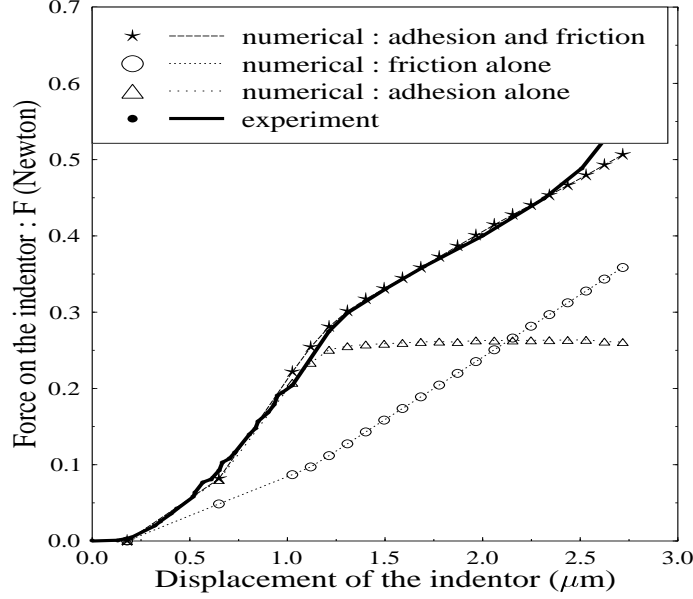


Fig. 7. *Simulation and experimental results.*

ing. The values of the ratio R_T/R_N (scale on the left), the adhesion intensity β , and the tangential sliding u_T (scales on the right) are plotted along the interface. Three zones are clearly characterized :

- zone 1 : close to the indenter, adhesion is totally broken ($\beta \simeq 0$) and only friction remains active ($R_T/R_N = \mu$),
- zone 2 : in this transition zone, adhesion is partial ($0 < \beta < 1$) and the elasticity acts ($R_T/R_N > \mu$),
- zone 3 : on this part, the interface is still weakly affected, the adhesion is total ($\beta = 1$).

The smooth evolution of the solution underlines the good property of the model, which gives a continuous transition from total adhesion to pure frictional contact.

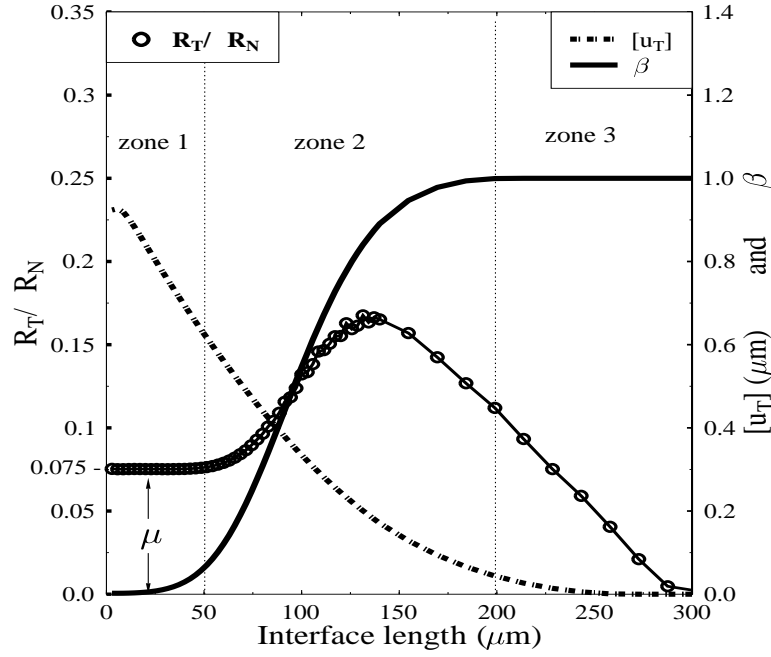


Fig. 8. Contact forces, adhesion intensity β and tangential displacement u_T along the interface for a given loading step.

6.3 Computational aspects

In that case, the Aitken - Gauss Seidel solver has been used. The number of iterations is quite the same as it is for the successive overrelaxation solver but the determination of the optimal relaxation parameter is not needed in that case. For the time discretization, the number of steps is 20. The time steps are chosen so as to correctly follow the changes in the condition of the contact: there is no convergence problem with large increments. For each step of the loading, the average number of iterations of the fixed point on the sliding limit g is 7 ; the one on the fixed point on the adhesion intensity β (treatment of the implicit integration) is 3. The average number of Aitken - Gauss Seidel solver is 600. This includes the fixed point iterations on g because a diagonal process is used : for the first values of g , a coarse resolution is conducted (the convergence test is coarse), and when more precise values of g are computed the accuracy of the resolution is enforced. The global CPU time, for the complete computation of the results given on Fig. 7 is 40 mn on a VAX 8400/bi-processor 350MHz.

7 Conclusion

A model coupling adhesion and friction that is based on the choice of convenient thermodynamics potentials has been elaborated. Damage, through traction and shear but not compression, that is introduced in the initial elasticity of the interface provides a smooth transition between the adhesive and frictional states. The unilateral contact conditions are strictly imposed without penalty regularization. It remains possible to add some elasticity in compression in case of mechanical necessity because the problem is then more regular. Viscosity of the evolution of the intensity of adhesion is considered. The limit case when viscosity may be neglected can be treated without numerical difficulties.

The variational formulation given as the coupling between an implicit variational inequality, a regular variational inequality, and a differential equation is constructive to establish an existence result and a condition of uniqueness, and to propose a discretization scheme.

The numerical methods based on optimization methods are robust. A direct mathematical programming method, presented in [7,8,35], may also be used.

This application to the simulation of fiber/matrix interface is only a test on the ability of the model to describe this kind of phenomena. A more complete analysis has been conducted in collaboration with ONERA [24]. An extended paper will be presented. The role of viscosity and of anisotropy in the behavior of such an interface should be investigated. Our undergoing research concerns the interaction of a crack with an interface so as to better understand crack progression in composite materials [30].

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