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Incompressible hydroelastic vibrations: finite element modelling of the elastogravity operator

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This paper deals with the low frequency vibratory analysis of fluid–structure interactions in an elastic tank partially filled with an incompressible inviscid liquid. The originality of this work is to give an exact expression of the gravity interface operator whereas other standard hydroelastic formulations treat this effect through approximations. The properties of this so-called elastogravity operator will be studied here from theoretical and numerical point of view. Experimental measurements will validate the computational model and allow to quantify the effect of coupling between the liquid sloshing and the hydroelastic deformations of the structure.

Keywords: Fluid–structure interactions; Internal liquid; Sloshing; Hydroelasticity; Gravity; Hydrostatic pressure

1. Introduction

Many structures designed by mechanical engineers are intended to contain a more or less important quantity of liquid. Let us cite for example all the means of transport whose tanks are filled with liquid propellant (car, aircraft, space launcher, etc.), sea and road tankers, nuclear reactors or chemical industry containers. Many industrial domains are then interested in the problematic of the coupling between an elastic structure and an internal incompressible fluid, and a huge theoretical and numerical work has been done on this topic for the last decades, in particular in the aerospace domain (see for instance [1]).

We are interested here especially in the vibratory analysis of this coupled fluid–structure system and neglect all flow-induced vibration effects (see for instance [2] and also [3–5] for coupling with nonlinear structural displacements). To compute the eigenmodes of an elastic structure coupled with an internal inviscid and incompressible fluid, the standard hydroelastic model is generally used. However, this modelling assumes that the sloshing potential energy of the fluid, which is its gravity potential energy, can be neglected with respect to the deformation potential energy of the structure. Consequently, the gravity is completely omitted in this modelling. The fluid is then only represented by a kinetic energy and its contribution to the system is reduced to an added-mass effect [6]. The results obtained with this formulation are correct if the frequency domain of the coupled system eigenmodes is much higher than the frequency of the first sloshing eigenmodes. In the case of

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slender and flexible structures, such as space launchers or civil aircraft wings, the first structural eigenfrequencies are very low (between 1 and 2 Hz) and the decoupling between the liquid sloshing in tanks and the structural deformations is no longer a valid assumption. Some authors proposed to solve this problem by simply adding the gravity potential energy of the fluid to the Lagrangian of the system (see for instance [7,6]). In 1966, Tong had proposed a simplified expression for the fluid added-stiffness matrix due to gravity [8] which has been often used (see for instance [9]). However, in 1992, Morand and Ohayon proposed the first exact formulation of this operator [10] and the present study comes within the following of their works.

In this paper, we will first detail the derivation of the linear local equations of the problem when the effects of gravity are taken into account. The case of pressurized closed containers will also be considered. A symmetric variational formulation will then be deduced. The expression of the so-called *elastogravity operator* will be given and its properties will be discussed. The finite element discretization of this hydroelastic modelling with gravity will be carried out and a first experimental validation will be finally exposed.

2. Linearized local equations of the coupled system

Fig. 1 represents the generic internal fluid–structure system we consider here. A structure, denoted by Ω^S is partially filled by a fluid Ω^F . The fluid free surface and fluid–structure interface are respectively denoted by Γ and Σ_i . The tank ullage, which may be occupied by a pressurized gas, is denoted by Ω^G . This system is subjected to external surface loads f on Σ_f and to the action of gravity g . By definition, the liquid free surface is initially orthogonal to gravity which is chosen to be in the opposite direction of the z -axis. Furthermore, the displacement of the structure is supposed to be known (equal to 0 for the sake of convenience) on Σ_u . The par-

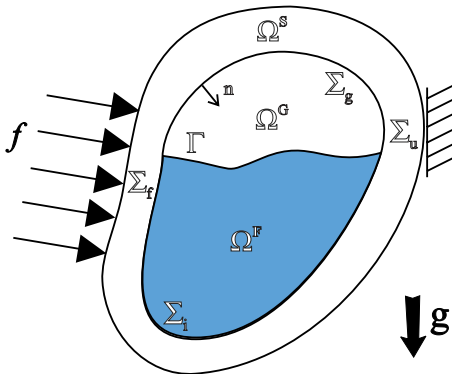


Fig. 1. Domain and external load definition.

ticular case of free fluid–structure systems has already been studied in previous works [11]. In this first section, we are going to write the coupled local equations of this system. The chosen variable for the structure is ordinarily its displacement field \mathbf{u}^S . The velocity field \mathbf{v}^F is the natural variable to describe the fluid state. It is particularly suitable for flowing fluid, but in our case, the fluid is enclosed and to study its movement, we prefer to chose the displacement field \mathbf{u}^F as variable.

2.1. Basic hypotheses and fluid model

As explained in the introduction, herein we are interested only in the linear vibrations of this system. We then suppose that external excitations are harmonic and seek stationary solutions of pulsation ω . Furthermore, we assume that the structure is elastic and the fluid is a Newtonian homogeneous liquid. Since we can consider the conservative associated system, we omit the damping aspect in this modelling and suppose that the liquid is inviscid (dynamic viscosity $\mu = 0$). In the frequency range of interest, the compressibility effects in the liquid can also be neglected which gives the relation

$$\nabla \cdot \mathbf{v}^F = 0 \quad (1)$$

With all these assumptions, the Navier–Stokes equation written as

$$\begin{aligned} \rho^F \frac{\partial \mathbf{v}^F}{\partial t} + \nabla \left(\frac{(\mathbf{v}^F)^2}{2} + (\nabla \wedge \mathbf{v}^F) \wedge \mathbf{v}^F \right) \\ = -\nabla P + \rho^F \mathbf{g} + \mu \nabla^2 \mathbf{v}^F + \frac{\mu}{3} \nabla (\nabla \cdot \mathbf{v}^F) \end{aligned} \quad (2)$$

can be simplified and linearized with respect to the variable \mathbf{v}^F to become the incompressible Euler equation

$$\rho^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = -\nabla P + \rho^F \mathbf{g} \quad (3)$$

where ρ^F is the density of the liquid and P is the pressure in the liquid.

By taking the rotational of Eq. (3) and using the homogeneous property ($\nabla \rho_F = 0$), we derive the following relation:

$$\rho^F \frac{\partial^2}{\partial t^2} (\nabla \wedge \mathbf{u}^F) = 0 \quad (4)$$

We then conclude that the irrotationality of the linear movements of such a liquid is a consequence of the previous assumptions (if we suppose that $\nabla \wedge \mathbf{v}^F_{(t=0)} = \nabla \wedge \mathbf{u}^F_{(t=0)} = 0$):

$$\nabla \wedge \mathbf{u}^F = 0 \quad (5)$$

2.2. Fluid local equations

Eq. (3) cannot be directly interpreted as the fluid vibratory equation, because, in this expression, \mathbf{u}^F is

not the displacement of the fluid with respect to the initial configuration around which the system vibrates, but with respect to a natural state of the fluid submitted to no external loads (not even the gravity field). When a hydrostatic pressure P_0 exists in the initial reference state, the initial configuration cannot be considered as a natural state. The hydrostatic pressure, and then the gravity, acts as a prestress for our fluid–structure system [12]. In this condition, we cannot use the classical simplifications due to "small perturbations", in particular the confusion of the Lagrangian and Eulerian definitions of variables [13]. For instance, it has been shown that the Lagrangian and Eulerian pressure fluctuations have different expressions [10]. If \mathbf{u}_0^F is the fluid displacement in the initial configuration and $\mathbf{U}^F = \mathbf{u}^F - \mathbf{u}_0^F$ is the fluid displacement with respect to the initial state, we note a difference between the Eulerian pressure fluctuation p_ε , defined at a geometrical fluid point M^F by

$$p_\varepsilon(M^F) = P(M^F) - P_0(M^F) \quad (6)$$

and the Lagrangian pressure fluctuation

$$p_\varphi(M^F) = P(M^F + \mathbf{U}^F) - P_0(M^F) \quad (7)$$

defined by following a fluid particle initially at M^F . After linearization, the difference can be expressed as

$$p_\varphi - p_\varepsilon = \nabla P_0 \cdot \mathbf{U}^F + o(\mathbf{U}^F) \quad (8)$$

This expression shows clearly that, in presence of a hydrostatic pressure, both definitions are not equivalent.

As the initial state of the fluid is static ($\frac{\partial^2 \mathbf{u}_0^F}{\partial t^2} = 0$), Eq. (3) gives the following relation for P_0 :

$$\nabla P_0 = \rho^F \mathbf{g} \quad (9)$$

Using this relation in the derivative of the Eulerian pressure fluctuation definition, we obtain

$$\nabla p_\varepsilon = \nabla P - \nabla P_0 = -\rho^F \frac{\partial^2 \mathbf{U}^F}{\partial t^2} \quad (10)$$

Since we are seeking stationary solutions \mathbf{U}^F of pulsation ω , the modified Euler equation for a "vibrating fluid" in presence of gravity is finally written as

$$-\rho^F \omega^2 \mathbf{U}^F = -\nabla p_\varepsilon \quad (11)$$

Paradoxically, gravity does not appear in this expression. However, if we rewrite it in function of the Lagrangian pressure fluctuation p_φ , using Eqs. (8), (9) and (5), a non-classical term depending on the gravity appears:

$$-\rho^F \omega^2 \mathbf{U}^F = -\nabla p_\varphi + \rho^F (\mathbf{g} \cdot \nabla) \mathbf{U}^F \quad (12)$$

We now need to express the incompressibility condition (1) in term of \mathbf{U}^F . In this aim, the mass conservation law in Lagrangian coordinates can be used:

$$\rho^F \det(J) = \rho_0^F \quad (13)$$

where ρ_0^F is the fluid density in the initial state and J is the Jacobian of the transformation associated with the displacement \mathbf{U}^F . Since the fluid is incompressible ($\rho^F = \rho_0^F$) and $\det(J) = 1 + \nabla \cdot \mathbf{U}^F + o(\mathbf{U}^F)$, the linearized relation expressing the incompressibility of the fluid is written as

$$\nabla \cdot \mathbf{U}^F = 0 \quad (14)$$

Some authors keep the fluid displacement as unknown of the problem, but because of the difficulty of imposing the irrotationality condition (5), spurious modes tend to appear in such formulation [9,14]. A good way to avoid this and also to reduce the size of the problem is to use this irrotationality property to represent the fluid displacement vector through a scalar displacement potential φ defined by

$$\mathbf{U}^F = \nabla \varphi \quad (15)$$

Since this definition gives a value of φ up to an additive constant, an unicity condition, denoted by $\ell(\varphi) = 0$, where ℓ is a linear form such as $\ell(1) \neq 0$, will be chosen later.

From (12), (14) and (15), we are now able to write the linearized local equations of the liquid in (φ, p_φ) :

$$\nabla p_\varphi = \rho^F \nabla^2 \varphi \mathbf{g} + \rho^F \omega^2 \nabla \varphi \quad \text{in } \Omega^F$$

$$\Delta \varphi = 0 \quad \text{in } \Omega^F$$

$$\ell(\varphi) = 0 \quad \text{with } \ell(1) \neq 0$$

The boundary conditions on $\partial\Omega^F = \Sigma_i \cup \Gamma$ must now be established. The coupling condition at fluid–structure interface Σ_i is given by the normal displacement continuity between the structure and the fluid, and will be expressed later (cf. Section 2.3). For the boundary condition on liquid free surface Γ , we use the property of pressure continuity at the interface between the fluid and the tank ullage gas above the fluid. In term of p_φ , we obtain

$$p_\varphi = P^G - P_0^G \quad \text{on } \Gamma \quad (16)$$

where P_0^G and P^G are the pressure in the tank ullage gas respectively in the initial and current states. For an open tank, P_0^G and P^G are equal to the atmospheric pressure and the relation (16) simply yields the following free surface condition:

$$p_\varphi = 0 \quad \text{on } \Gamma \quad (17)$$

Sometimes in the literature, the free surface boundary condition $p = -\rho^F \mathbf{g} \cdot \mathbf{U}^F$ is used instead of (17) when gravity effects are taken into account, but we stress that in this case, the pressure fluctuation considered is the Eulerian one.

For a closed tank, the gas pressure fluctuation can be estimated through a quasi-static effect (homogeneous pressure in all the gas domain) by using the isentropic law of transformation of a perfect gas ($PV^\gamma = \text{Cst}$).

The linearized fluctuation of the gas pressure is then given by

$$P^G - P_0^G = -P_0^G \gamma \frac{\delta V}{|\Omega_0^G|} \quad (18)$$

where $|\Omega_0^G|$ is the initial gas volume and δV is the linearized variation of the gas volume which is equal to the variation of the cavity volume because of the liquid incompressibility. Finally, the free surface boundary condition for a closed tank is written as

$$p_{\mathcal{G}} = -P_0^G \gamma \frac{\delta V}{|\Omega_0^G|} \quad \text{on } \Gamma \quad (19)$$

We note that for a large tank ullage or for small pressurization, the gas pressure fluctuation is small and this boundary condition can be approximated by (17).

2.3. Structure local equations

Let us now deal with the structure. We first remark that the structure, like the fluid, is prestressed in its initial configuration by the fluid hydrostatic pressure (and eventually by its own weight and external loads). In this case, the initial state, around which the structure vibrates, is not a natural state and the classical simplifications due to “small perturbations” cannot be used. In particular, the equilibrium equations of the structure cannot be simplified by substituting the Eulerian variables to the Lagrangian ones. However, thanks to the small displacement assumption, we show that geometries of the different states of the system are close enough to be still confused. The structural displacement \mathbf{u}^S must then satisfy the following equation:

$$\rho_0^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} = \text{Div} \mathbf{\Pi}(\mathbf{u}^S) + \rho_0^S \mathbf{g} \quad \text{in } \Omega^S \quad (20a)$$

$$\mathbf{S}(\mathbf{u}^S) = \mathbf{H}\mathbf{E}(\mathbf{u}^S) \quad \text{in } \Omega^S \quad (20b)$$

where $\mathbf{\Pi}$ is the first non-symmetric Piola–Kirchhoff stress tensor, \mathbf{S} is the second symmetric Piola–Kirchhoff stress tensor, \mathbf{E} is the Euler–Lagrange strain tensor and \mathbf{H} is Hooke’s elasticity tensor. The equation verified by the deformation of the structure in the initial configuration \mathbf{u}_0^S is deduced by considering that the initial state is a static equilibrium ($\frac{\partial^2 \mathbf{u}_0^S}{\partial t^2} = 0$):

$$\mathbf{0} = \text{Div} \mathbf{\Pi}(\mathbf{u}_0^S) + \rho_0^S \mathbf{g} \quad \text{in } \Omega^S \quad (21a)$$

$$\mathbf{S}(\mathbf{u}_0^S) = \mathbf{H}\mathbf{E}(\mathbf{u}_0^S) \quad \text{in } \Omega^S \quad (21b)$$

Using (20a) and (21a), we obtain the (nonlinear) local equation verified by the structure displacement with respect to the initial state, denoted by $\mathbf{U}^S = \mathbf{u}^S - \mathbf{u}_0^S$:

$$\rho_0^S \frac{\partial^2 \mathbf{U}^S}{\partial t^2} = \text{Div}[\mathbf{\Pi}(\mathbf{u}^S) - \mathbf{\Pi}(\mathbf{u}_0^S)] \quad \text{in } \Omega^S \quad (22)$$

We transform the right hand side member of this equation by using (20b) and (21b) and the relation between $\mathbf{\Pi}$ and \mathbf{S} :

$$\begin{aligned} \mathbf{\Pi}(\mathbf{u}^S) - \mathbf{\Pi}(\mathbf{u}_0^S) &= (\mathbf{I} + \nabla \mathbf{u}^S) \mathbf{S}(\mathbf{u}^S) - (\mathbf{I} + \nabla \mathbf{u}_0^S) \mathbf{S}(\mathbf{u}_0^S) \\ &= (\mathbf{I} + \nabla \mathbf{u}^S) \mathbf{H}[\mathbf{E}(\mathbf{u}^S) - \mathbf{E}(\mathbf{u}_0^S)] \\ &\quad + \nabla \mathbf{U}^S \mathbf{S}(\mathbf{u}_0^S) \end{aligned} \quad (23)$$

To linearize this expression with respect to displacements \mathbf{u}_0^S and \mathbf{U}^S , we distinguish the linear part and the quadratic part of \mathbf{E} , respectively denoted by $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^Q$:

$$\begin{aligned} \mathbf{\Pi}(\mathbf{u}^S) - \mathbf{\Pi}(\mathbf{u}_0^S) &= (\mathbf{I} + \nabla \mathbf{u}^S) \mathbf{H}[\boldsymbol{\varepsilon}(\mathbf{U}^S) + \boldsymbol{\varepsilon}^Q(\mathbf{u}^S, \mathbf{u}_0^S) \\ &\quad - \boldsymbol{\varepsilon}^Q(\mathbf{u}_0^S, \mathbf{u}_0^S)] + \nabla \mathbf{U}^S \mathbf{S}(\mathbf{u}_0^S) \\ &= \mathbf{H}\boldsymbol{\varepsilon}(\mathbf{U}^S) + \nabla \mathbf{U}^S \boldsymbol{\sigma}(\mathbf{u}_0^S) + o(\mathbf{u}_0^S, \mathbf{U}^S) \end{aligned} \quad (24)$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor. In this expression, the first term is the classical elastic deformation term, whereas the second term is associated with the presence of a prestress $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}(\mathbf{u}_0^S)$ in the initial state. Since we are seeking stationary solutions \mathbf{U}^S of pulsation ω , the linearized equation for a “prestressed vibrating structure” is finally written as

$$-\rho_0^S \omega^2 \mathbf{U}^S = \text{Div} \tilde{\boldsymbol{\sigma}} \quad \text{in } \Omega^S \quad (25)$$

where $\tilde{\boldsymbol{\sigma}}$ is the “prestressed stress tensor” defined by

$$\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S) = \mathbf{H}\boldsymbol{\varepsilon}(\mathbf{U}^S) + \nabla \mathbf{U}^S \boldsymbol{\sigma}_0 \quad (26)$$

Let us remark that gravity \mathbf{g} does not appear directly in Eq. (25) as an external load but through the prestress $\boldsymbol{\sigma}_0(\mathbf{g})$.

The boundary conditions must now be established on $\partial\Omega^S = \Sigma_i \cup \Sigma_g \cup \Sigma_f \cup \Sigma_{ii}$. The first one is obtained from the property of continuity of the normal velocities at interface Σ_i , which is the consequence of the frictionless contact between the inviscid fluid and the structure. It can be shown that, after linearization, this condition of continuity can be transferred on the normal displacements [10]:

$$\mathbf{u}^F \cdot \mathbf{n} = \mathbf{u}^S \cdot \mathbf{n} \quad \text{on } \Sigma_i \quad (27)$$

By subtracting the similar relation written for the initial state of the system, we deduce the boundary condition in term of \mathbf{U}^F and \mathbf{U}^S :

$$\mathbf{U}^F \cdot \mathbf{n} = \mathbf{U}^S \cdot \mathbf{n} \quad \text{on } \Sigma_i \quad (28)$$

The second boundary condition on Σ_i is deduced from the property of continuity of stresses at the fluid–structure interface. Since the simplifications due to “small perturbations” cannot be used, it is mandatory to write this relation in Lagrangian coordinates. However, the reader must remind that, thanks to the small displacement assumption, geometries of the different configurations can be confused. By using the Lagrangian transport of surface element, this boundary condition is then written [15]:

$$\mathbf{\Pi}(\mathbf{u}^S)\mathbf{n} = -P \det(\mathbf{I} + \nabla \mathbf{u}^S)^t (\mathbf{I} + \nabla \mathbf{u}^S)^{-1} \mathbf{n} \quad \text{on } \Sigma_i \quad (29)$$

With the aim of making displacement \mathbf{U}^S appear, we subtract from this equation a similar relation written on the initial configuration:

$$\begin{aligned} [\mathbf{\Pi}(\mathbf{u}^S) - \mathbf{\Pi}(\mathbf{u}_0^S)]\mathbf{n} &= -(P - P_0) \det(\mathbf{I} + \nabla \mathbf{u}^S)^t \\ &\quad \times (\mathbf{I} + \nabla \mathbf{u}^S)^{-1} \mathbf{n} - P_0 [\det(\mathbf{I} + \nabla \mathbf{u}^S)^t \\ &\quad \times (\mathbf{I} + \nabla \mathbf{u}^S)^{-1} - \det(\mathbf{I} + \nabla \mathbf{u}_0^S)^t \\ &\quad \times (\mathbf{I} + \nabla \mathbf{u}_0^S)^{-1}] \mathbf{n} \end{aligned} \quad (30)$$

To linearize this equation, we first need to express term $(P - P_0)$. The difficulty is to introduce in this expression the Lagrangian pressure fluctuation defined by (7) for every fluid particle M^F whereas the pressures involved here are pressures associated with structural points. Due to the relative displacement between liquid and structure, in Lagrangian coordinates, we need to distinguish pressures associated with structural points from pressures associated with fluid particles [10,12]:

$$\begin{aligned} P(M^S) &= P(M^F) + \nabla P(M^F) \cdot (\mathbf{U}^S - \mathbf{U}^F) + o(\mathbf{U}^S, \mathbf{U}^F) \\ &= P(M + \mathbf{U}^F) + \nabla P_0(M^F) \cdot (\mathbf{U}^S - \mathbf{U}^F) + o(\mathbf{U}^S, \mathbf{U}^F) \end{aligned} \quad (31)$$

Term $(P - P_0)$, which could be denoted by $p_{\mathcal{S}}^S$ and called the Lagrangian pressure fluctuation at structure points, is then linked to the Lagrangian pressure fluctuation of fluid particles through the following linearized relation, obtained by using the value of ∇P_0 (9):

$$P - P_0 = p_{\mathcal{S}}^S = p_{\mathcal{S}} + \rho^F \mathbf{g} \cdot (\mathbf{U}^S - \mathbf{U}^F) \quad (32)$$

Let us remark that the difference between these two Lagrangian pressure fluctuations vanished in absence of gravity. By using this relation, linearization of Eq. (30) finally gives a second linearized boundary condition on Σ_i :

$$\begin{aligned} \tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} &= -p_{\mathcal{S}}^S \mathbf{n} - P_0 [(1 + \nabla \cdot \mathbf{u}^S)(\mathbf{I} - {}^t \nabla \mathbf{u}^S) \\ &\quad - (1 + \nabla \cdot \mathbf{u}_0^S)(\mathbf{I} - {}^t \nabla \mathbf{u}_0^S)] \mathbf{n} \\ &= -[p_{\mathcal{S}} + \rho^F \mathbf{g} \cdot (\mathbf{U}^S - \mathbf{U}^F)] \mathbf{n} \\ &\quad - P_0 (\nabla \cdot \mathbf{U}^S \mathbf{I} - {}^t \nabla \mathbf{U}^S) \mathbf{n} \quad \text{on } \Sigma_i \end{aligned} \quad (33)$$

In the right hand side member of this expression, the first term represents the effect of pressure fluctuation $p_{\mathcal{S}}^S$ defined previously, whereas the second term can be interpreted as a follower force effect of fluid pressure and associated with the rotation of the normal \mathbf{n} at the fluid–structure interface that we note in the following

$$\boldsymbol{\tau}(\mathbf{U}^S) = (\nabla \cdot \mathbf{U}^S \mathbf{I} - {}^t \nabla \mathbf{U}^S) \mathbf{n} \quad (34)$$

The boundary condition on Σ_g is very similar to the previous one. The property of continuity of stresses at the gas–structure interface, written in Lagrangian coordinates, gives

$$\mathbf{\Pi}(\mathbf{u}^S)\mathbf{n} = -P^G \det(\mathbf{I} + \nabla \mathbf{u}^S)^t (\mathbf{I} + \nabla \mathbf{u}^S)^{-1} \mathbf{n} \quad \text{on } \Sigma_g \quad (35)$$

The same relation can be written on the initial configuration in term of \mathbf{u}_0^S and subtracted from that equation to yield a nonlinear expression of the boundary condition similar to (30). Using (18), we linearize this equation to obtain

$$\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} = P_0^G \gamma \frac{\delta V}{|\Omega_0^G|} \mathbf{n} - P_0^G (\nabla \cdot \mathbf{U}^S \mathbf{I} - {}^t \nabla \mathbf{U}^S) \mathbf{n} \quad \text{on } \Sigma_g \quad (36)$$

where the linearized variation of cavity volume δV depends on structure deformation \mathbf{U}^S by the following relation:

$$\delta V = \int_{\Omega^G} \nabla \cdot \mathbf{U}^S d\Omega = - \int_{\Sigma_i \cup \Sigma_g} \mathbf{U}^S \cdot \mathbf{n} d\Sigma \quad (37)$$

Since the case of follower external forces has been studied for free fluid–structure systems in [11], we suppose here that external forces \mathbf{f} are dead loads. It means that their direction of application is independent of the structure displacement. In this case, by neglecting the surface element dilatation, we can write a simplified condition of stress continuity on Σ_f :

$$\mathbf{\Pi}(\mathbf{u}^S)\mathbf{n} = \mathbf{f} \quad \text{on } \Sigma_f \quad (38)$$

After linearization of the equation obtained by subtracting the similar relation written for the initial state, we obtain the boundary condition on Σ_f of our problem:

$$\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} = \tilde{\mathbf{f}} \quad \text{on } \Sigma_f \quad (39)$$

where $\tilde{\mathbf{f}} = \mathbf{f} - \mathbf{f}_0$ is supposed to be known.

The last boundary condition on Σ_u is directly the consequence that, for all time t , $\mathbf{u}^S(t) = \mathbf{0}$ on Σ_u :

$$\mathbf{U}^S = \mathbf{0} \quad \text{on } \Sigma_u \quad (40)$$

2.4. Formulation in terms of $(\mathbf{U}^S, p_{\mathcal{S}}, \varphi)$

Finally the linearized local equations of a vibrating fluid–structure system in presence of gravity can be summarized here:

$$\nabla p_{\mathcal{S}} = \rho^F \nabla^2 \varphi \mathbf{g} + \rho^F \omega^2 \nabla \varphi \quad \text{in } \Omega^F \quad (41a)$$

$$\Delta \varphi = 0 \quad \text{in } \Omega^F \quad (41b)$$

$$\ell(\varphi) = 0 \quad \text{with } \ell(1) \neq 0 \quad (41c)$$

$$p_{\mathcal{S}} = \frac{P_0^G \gamma}{|\Omega_0^G|} \int_{\Sigma_i \cup \Sigma_g} \mathbf{U}^S \cdot \mathbf{n} d\Sigma \quad \text{on } \Gamma \quad (41d)$$

$$\nabla \varphi \cdot \mathbf{n} = \mathbf{U}^S \cdot \mathbf{n} \quad \text{on } \Sigma_i \quad (41e)$$

$$\text{Div } \tilde{\boldsymbol{\sigma}} = -\rho_0^S \omega^2 \mathbf{U}^S \quad \text{in } \Omega^S \quad (41f)$$

$$\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S) = \mathbf{H}\boldsymbol{\varepsilon}(\mathbf{U}^S) + \nabla \mathbf{U}^S \boldsymbol{\sigma}_0 \quad \text{in } \Omega^S \quad (41g)$$

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} &= -[p_\varphi + \rho^F \mathbf{g} \cdot (\mathbf{U}^S - \nabla\varphi)]\mathbf{n} \\ &\quad - P_0\boldsymbol{\tau}(\mathbf{U}^S) \quad \text{on } \Sigma_i\end{aligned}\quad (41h)$$

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} &= -\frac{P_0^G\gamma}{|\Omega_0^G|} \int_{\Sigma_i \cup \Sigma_g} \mathbf{U}^S \cdot \mathbf{n} d\Sigma \Big) \mathbf{n} \\ &\quad - P_0^G\boldsymbol{\tau}(\mathbf{U}^S) \quad \text{on } \Sigma_g\end{aligned}\quad (41i)$$

$$\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} = \tilde{\mathbf{f}} \quad \text{on } \Sigma_f \quad (41j)$$

$$\mathbf{U}^S = \mathbf{0} \quad \text{on } \Sigma_u \quad (41k)$$

We remark that, by integrating Eq. (41a), p_L can be expressed as a function of φ and eliminated from the formulation:

$$\frac{p_\varphi}{\rho^F} = \mathbf{g} \cdot \nabla\varphi + \omega^2\varphi + \pi \quad \text{in } \Omega^F \quad (42)$$

where π is a constant on Ω^F which is determined by choosing the unicity condition ℓ on φ . We then potentially obtain a formulation only in terms of (\mathbf{U}^S, φ) . However, we have shown in previous works that this formulation is non-symmetric [16]. To symmetrize the formulation, it is needed to keep π and to introduce a new unknown η representing the vertical displacement of the free surface:

$$\eta = \nabla\varphi \cdot \mathbf{i}_z \quad \text{on } \Gamma \quad (43)$$

where \mathbf{i}_z is the unit vector of the z -axis.

If we consider this new expression for δV , function of \mathbf{U}^S and η

$$\delta V = - \int_{\Sigma_g} \mathbf{U}^S \cdot \mathbf{n} d\Sigma - \int_{\Gamma} \eta d\Gamma \quad (44)$$

the local equations of the problem in $(\mathbf{U}^S, \varphi, \eta, \pi)$ are written:

$$\Delta\varphi = 0 \quad \text{in } \Omega^F \quad (45a)$$

$$\ell(\varphi) = 0 \quad \text{with } \ell(1) \neq 0 \quad (45b)$$

$$\begin{aligned}\rho^F \mathbf{g}\eta &= \rho^F \omega^2\varphi + \rho^F \pi \\ &\quad - \frac{P_0^G\gamma}{|\Omega_0^G|} \int_{\Sigma_g} \mathbf{U}^S \cdot \mathbf{n} d\Sigma + \int_{\Gamma} \eta d\Gamma \Big) \quad \text{on } \Gamma\end{aligned}\quad (45c)$$

$$\nabla\varphi \cdot \mathbf{i}_z = \eta \quad \text{on } \Gamma \quad (45d)$$

$$\nabla\varphi \cdot \mathbf{n} = \mathbf{U}^S \cdot \mathbf{n} \quad \text{on } \Sigma_i \quad (45e)$$

$$\text{Div } \tilde{\boldsymbol{\sigma}} = -\rho_0^S \omega^2 \mathbf{U}^S \quad \text{in } \Omega^S \quad (45f)$$

$$\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S) = \mathbf{H}\boldsymbol{\varepsilon}(\mathbf{U}^S) + \nabla\mathbf{U}^S\boldsymbol{\sigma}_0 \quad \text{in } \Omega^S \quad (45g)$$

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} &= -(\rho^F \omega^2\varphi + \rho^F \pi + \rho^F \mathbf{g} \cdot \mathbf{U}^S)\mathbf{n} \\ &\quad - P_0\boldsymbol{\tau}(\mathbf{U}^S) \quad \text{on } \Sigma_i\end{aligned}\quad (45h)$$

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} &= -\frac{P_0^G\gamma}{|\Omega_0^G|} \int_{\Sigma_g} \mathbf{U}^S \cdot \mathbf{n} d\Sigma + \int_{\Gamma} \eta d\Gamma \Big) \mathbf{n} \\ &\quad - P_0^G\boldsymbol{\tau}(\mathbf{U}^S) \quad \text{on } \Sigma_g\end{aligned}\quad (45i)$$

$$\tilde{\boldsymbol{\sigma}}(\mathbf{U}^S)\mathbf{n} = \tilde{\mathbf{f}} \quad \text{on } \Sigma_f \quad (45j)$$

$$\mathbf{U}^S = \mathbf{0} \quad \text{on } \Sigma_u \quad (45k)$$

3. Symmetric variational formulation in $(\mathbf{U}^S, \varphi, \eta, \pi)$

To discretize and numerically solve this problem, we write the associated weak formulation by applying the virtual work principle.

If the admissible space for φ is denoted by $\mathfrak{C}_\varphi = \mathcal{H}^1(\Omega^F)$ and \mathfrak{C}_φ^* is its restriction to solutions φ satisfying the condition $\ell(\varphi) = 0$, the variational formulation associated with (45a), (45d) and (45e) is written

$$\begin{aligned}\exists \varphi \in \mathfrak{C}_\varphi^*, \quad \forall \delta\varphi \in \mathfrak{C}_\varphi, \\ \mathcal{F}(\varphi, \delta\varphi) - \mathcal{B}(\eta, \delta\varphi) + \mathcal{C}(\mathbf{U}^S, \delta\varphi) = 0\end{aligned}\quad (46)$$

with

$$\mathcal{F}(\varphi, \delta\varphi) = \rho^F \int_{\Omega^F} \nabla\varphi \cdot \nabla\delta\varphi d\Omega \quad (47)$$

$$\mathcal{B}(\eta, \delta\varphi) = \rho^F \int_{\Gamma} \eta\delta\varphi d\Gamma \quad (48)$$

$$\mathcal{C}(\mathbf{U}^S, \delta\varphi) = \rho^F \int_{\Sigma_i} \mathbf{U}^S \cdot \mathbf{n}\delta\varphi d\Sigma \quad (49)$$

In this expression, \mathcal{F} is the symmetric positive bilinear form associated with the liquid kinetic energy and \mathcal{C} is the classical fluid–structure coupling operator. Let us remark that relation (46) is non-symmetric because φ and $\delta\varphi$ do not belong to the same admissible space. Since $\mathfrak{C}_\varphi = \mathfrak{C}_\varphi^* \oplus \mathbb{R}$, we replace (46) by two equations obtained for $\delta\varphi \in \mathfrak{C}_\varphi^*$ and $\delta\pi \in \mathbb{R}$:

$$\begin{aligned}\exists \varphi \in \mathfrak{C}_\varphi^*, \quad \forall \delta\varphi \in \mathfrak{C}_\varphi^*, \\ \mathcal{F}(\varphi, \delta\varphi) - \mathcal{B}(\eta, \delta\varphi) + \mathcal{C}(\mathbf{U}^S, \delta\varphi) = 0\end{aligned}\quad (50)$$

$$\forall \delta\pi \in \mathbb{R}, \quad -\mathcal{B}(\eta, 1)\delta\pi + \mathcal{C}(\mathbf{U}^S, 1)\delta\pi = 0 \quad (51)$$

Next, the variational formulation of Eq. (45c) is written for all $\delta\eta$ in $\mathfrak{C}_\eta = \mathcal{L}^2(\Gamma)$:

$$\begin{aligned}\exists \eta \in \mathfrak{C}_\eta, \quad \forall \delta\eta \in \mathfrak{C}_\eta, \\ \mathcal{I}(\eta, \delta\eta) + \mathcal{K}_\Gamma(\eta, \delta\eta) - \pi\mathcal{B}(1, \delta\eta) - \omega^2\mathcal{B}(\varphi, \delta\eta) \\ + \mathcal{D}(\mathbf{U}^S, \delta\eta) = 0\end{aligned}\quad (52)$$

with

$$\mathcal{I}(\eta, \delta\eta) = \rho^F \int_{\Gamma} \eta\delta\eta d\Gamma \quad (53)$$

$$\mathcal{K}_\Gamma(\eta, \delta\eta) = \frac{P_0^G \gamma}{|\Omega_0^G|} \left(\int_\Gamma \eta \, d\Gamma \right) \left(\int_\Gamma \delta\eta \, d\Gamma \right) \quad (54)$$

$$\mathcal{D}(\mathbf{U}^S, \delta\eta) = \frac{P_0^G \gamma}{|\Omega_0^G|} \left(\int_{\Sigma_g} \mathbf{U}^S \cdot \mathbf{n} \, d\Sigma \right) \left(\int_\Gamma \delta\eta \, d\Gamma \right) \quad (55)$$

We recognize here the bilinear form associated with the liquid sloshing potential energy \mathcal{S} which is symmetric positive and definite. Operator \mathcal{K}_Γ , which represents an added stiffness on the free surface due to the pressurized gas, is symmetric positive.

From local Eqs. (45f)–(45i), (45g), (45k), we derive the variational formulation for the structure for all $\delta\mathbf{U}$ in $\mathfrak{C}_u = \mathcal{H}^1(\Omega^S)$. We then introduce the expression of the initial fluid pressure P_0 in function of the coordinate z (defined with respect to the liquid free surface):

$$P_0 = P_0^G - \rho^F g z \quad (56)$$

After some simplifications, the variational formulation for the structure is given by:

$$\begin{aligned} \exists \mathbf{U}^S \in \mathfrak{C}_u, \quad \forall \delta\mathbf{U} \in \mathfrak{C}_u, \\ \mathcal{K}_E(\mathbf{U}^S, \delta\mathbf{U}) + \mathcal{K}_G(\mathbf{U}^S, \delta\mathbf{U}) \\ + \mathcal{K}_\Sigma(\mathbf{U}^S, \delta\mathbf{U}) + \mathcal{K}_P(\mathbf{U}^S, \delta\mathbf{U}) \\ + \mathcal{K}_C(\mathbf{U}^S, \delta\mathbf{U}) - \omega^2 \mathcal{M}(\mathbf{U}^S, \delta\mathbf{U}) \\ + \omega^2 \mathcal{C}(\varphi, \delta\mathbf{U}) + \pi \mathcal{C}(1, \delta\mathbf{U}) + \mathcal{D}(\eta, \delta\mathbf{U}) \\ = f(\delta\mathbf{U}) \end{aligned} \quad (57)$$

with

$$\mathcal{K}_E(\mathbf{U}^S, \delta\mathbf{U}) = \int_{\Omega^S} \text{Tr}[\mathbf{H}\boldsymbol{\varepsilon}(\mathbf{U}^S)\boldsymbol{\varepsilon}(\delta\mathbf{U})] \, d\Omega \quad (58)$$

$$\mathcal{K}_G(\mathbf{U}^S, \delta\mathbf{U}) = \int_{\Omega^S} \text{Tr}[\nabla\mathbf{U}^S \boldsymbol{\sigma}_0' \nabla\delta\mathbf{U}] \, d\Omega \quad (59)$$

$$\begin{aligned} \mathcal{K}_\Sigma(\mathbf{U}^S, \delta\mathbf{U}) = -\rho^F g \int_{\Sigma_i} (\mathbf{i}_z \cdot \mathbf{U}^S) (\delta\mathbf{U} \cdot \mathbf{n}) \\ + z \tau(\mathbf{U}^S) \cdot \delta\mathbf{U} \, d\Sigma \end{aligned} \quad (60)$$

$$\mathcal{K}_C(\mathbf{U}^S, \delta\mathbf{U}) = \frac{P_0^G \gamma}{|\Omega_0^G|} \left(\int_{\Sigma_g} \mathbf{U}^S \cdot \mathbf{n} \, d\Sigma \right) \left(\int_{\Sigma_g} \delta\mathbf{U} \cdot \mathbf{n} \, d\Sigma \right) \quad (61)$$

$$\mathcal{K}_P(\mathbf{U}^S, \delta\mathbf{U}) = P_0^G \int_{\Sigma_i \cup \Sigma_g} \tau(\mathbf{U}^S) \cdot \delta\mathbf{U} \, d\Sigma \quad (62)$$

$$\mathcal{M}(\mathbf{U}^S, \delta\mathbf{U}) = \int_{\Omega^S} \rho_0^S \mathbf{U}^S \cdot \delta\mathbf{U} \, d\Omega \quad (63)$$

$$f(\delta\mathbf{U}) = \int_{\Sigma_f} \tilde{f} \cdot \delta\mathbf{U} \, d\Sigma \quad (64)$$

One remark that, in this expression, four additional stiffness terms are added to the classical elastic stiffness of

the structure. \mathcal{K}_G is a symmetric bilinear form representing the geometric stiffness due to prestresses in the initial state and \mathcal{K}_Σ is related to the presence of a follower force (liquid pressure) on the fluid–structure interface. Both are directly dependent on gravity and, added to \mathcal{K}_E , they form the so-called *elastogravity operator*. \mathcal{K}_C and \mathcal{K}_P are proportional to tank pressurization and vanish if P_0^G is low. The first one is a symmetric positive bilinear form depending on the tank-ullage gas compressibility and the second one is due to the follower force effect of gas pressure on the structure and is similar to \mathcal{K}_Σ [17]. Let us show that \mathcal{K}_Σ and \mathcal{K}_P are both symmetric bilinear forms so that the global stiffness of the structure, denoted by \mathcal{K} , is also symmetric. In this aim, quantity $\mathcal{K}_\Sigma(\mathbf{U}, \mathbf{V}) - \mathcal{K}_\Sigma(\mathbf{V}, \mathbf{U})$ is evaluated. By using the following analytical formula

$$(\nabla \cdot \mathbf{U})\mathbf{V} - (\nabla\mathbf{U})\mathbf{V} - (\nabla \cdot \mathbf{V})\mathbf{U} - (\nabla\mathbf{V})\mathbf{U} = \nabla \wedge (\mathbf{V} \wedge \mathbf{U}) \quad (65)$$

the expression of $\mathcal{K}_\Sigma(\mathbf{U}, \mathbf{V}) - \mathcal{K}_\Sigma(\mathbf{V}, \mathbf{U})$ can be simplified:

$$\begin{aligned} \mathcal{K}_\Sigma(\mathbf{U}, \mathbf{V}) - \mathcal{K}_\Sigma(\mathbf{V}, \mathbf{U}) = -\rho^F g \int_{\Sigma_i} (\mathbf{i}_z \wedge (\mathbf{U} \wedge \mathbf{V})) \cdot \mathbf{n} \\ + z(\nabla \wedge (\mathbf{V} \wedge \mathbf{U})) \cdot \mathbf{n} \, d\Sigma \end{aligned} \quad (66)$$

The Stokes formula allows to transform the second term of the right hand side member:

$$\begin{aligned} \int_{\Sigma_i} z(\nabla \wedge (\mathbf{V} \wedge \mathbf{U})) \cdot \mathbf{n} \, d\Sigma \\ = \int_{\Sigma_i} \nabla \wedge (z\mathbf{V} \wedge \mathbf{U}) \cdot \mathbf{n} \, d\Sigma - \int_{\Sigma_i} (\nabla z \wedge (\mathbf{V} \wedge \mathbf{U})) \cdot \mathbf{n} \, d\Sigma \\ = \int_{\partial\Sigma_i} z(\mathbf{V} \wedge \mathbf{U}) \, d\mathbf{l} - \int_{\Sigma_i} (\mathbf{i}_z \wedge (\mathbf{V} \wedge \mathbf{U})) \cdot \mathbf{n} \, d\Sigma \end{aligned} \quad (67)$$

Since all points of $\partial\Sigma_i$ are on Γ , their coordinate z is equal to 0 and the previous term nullifies the expression (66) which demonstrates that \mathcal{K}_Σ is symmetric. The proof for \mathcal{K}_P can be easily deduced from this one and will not be presented here [12].

4. Finite element discretization

The weak formulation, made up of (57), (52), (50) and (51), is discretized by the Ritz–Galerkin finite-element method. The approximated solution (U, φ, η, π) satisfies the following coupled system of matricial equations:

$$\begin{aligned} (K_E + K_G + K_\Sigma + K_C + K_P)U - \omega^2 MU + \omega^2 C\varphi \\ + c\pi + D\eta = f \end{aligned} \quad (68)$$

$$(S + K_\Gamma)\eta - b\pi - \omega^2 B\varphi + {}^t DU = 0 \quad (69)$$

$$F\varphi - {}^t B\eta + {}^t CU = 0 \quad (70)$$

$$-{}^t b \eta + {}^t c U = 0 \quad (71)$$

with $b = B1_\varphi$ and $c = C1_\varphi$, where 1_φ is a φ -size vector of 1.

We notice that Eq. (70) should allow to express φ as a function of U and η . However, matrix F is singular (since $F1_\varphi = 0$). The unicity condition ℓ on φ has to be applied to remove the indetermination on φ . We choose here the first component of vector φ to be 0. Rows and columns corresponding to this d.o.f. are removed from the equations. The truncated matrices and vectors are then denoted by subscript 2. The unknown φ_2 is determined and eliminated from the equations:

$$\varphi = \begin{pmatrix} 0 \\ F_{22}^{-1t} B_2 \eta - F_{22}^{-1t} C_2 U \end{pmatrix} \quad (72)$$

$$\begin{pmatrix} \begin{bmatrix} K & D & c \\ {}^t D & S + K_\Gamma & -b \\ {}^t c & -{}^t b & 0 \end{bmatrix} \\ -\omega^2 \begin{bmatrix} M + C_2 F_{22}^{-1t} C_2 & -C_2 F_{22}^{-1t} B_2 & 0 \\ -B_2 F_{22}^{-1t} C_2 & B_2 F_{22}^{-1t} B_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{pmatrix} \begin{pmatrix} U \\ \eta \\ \pi \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} \quad (73)$$

with

$$K = K_E + K_G + K_\Sigma + K_C + K_P \quad (74)$$

We verify with this expression that the formulation in terms of (U, η, π) is symmetric. Furthermore, we note that π can be interpreted as a Lagrange multiplier for the constraint of fluid volume invariance (due to fluid incompressibility).

Nevertheless, this discretized form has a drawback: it involves several full matrices like K_C , D and K_Γ which can lead to prohibited computing times. To avoid this problem, we propose to introduce an auxiliary variable $\lambda = -\delta V$ with δV defined by relation (44). In this case, it can easily be shown that the following relation is equivalent to (73)

$$\begin{pmatrix} \begin{bmatrix} \tilde{K} & 0 & c & \beta d \\ 0 & S & -b & \beta b \\ {}^t c & -{}^t b & 0 & 0 \\ \beta' d & \beta' b & 0 & -\beta \end{bmatrix} \\ -\omega^2 \begin{bmatrix} M + C_2 F_{22}^{-1t} C_2 & -C_2 F_{22}^{-1t} B_2 & 0 & 0 \\ -B_2 F_{22}^{-1t} C_2 & B_2 F_{22}^{-1t} B_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{pmatrix} \begin{pmatrix} U \\ \eta \\ \pi \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (75)$$

with

$$\tilde{K} = K_E + K_G + K_\Sigma + K_P \quad (76)$$

$$\beta = \frac{P_0^G \gamma}{(\rho^F)^2 |\Omega_0^G|} \quad (77)$$

$$d(\pi, \delta U) = \rho^F \pi \int_{\Sigma_g} \delta U \cdot \mathbf{n} d\Sigma \quad (78)$$

5. Validations

This formulation has been implemented in MSC/NASTRAN through DMAP programming. To validate this model, we used among other things experimental results published by Chai and Genevoux [18,19]. They have measured the coupled eigenmodes of a container filled with water illustrated Fig. 2. The rectangular container is made of plexiglass and can be considered as rigid in the frequency domain of interest. Its dimensions are $L = 0.231$ m and $l = 0.1$ m. The deformable part of this structure is a cantilever plate, clamped to the bottom of the container and excited in bending by an electromagnet. The gap between the plate and the container wall is so small that any flow between both fluid domains can be neglected. The duralium plate characteristics (density $\rho^S = 7800$ kg/m³, Young's modulus $E = 2.1 \times 10^{11}$ N/m², thickness $e = 0.57$ mm and length $h_p = 0.266$ m) are such that its first dry bending frequency is close to the first sloshing modes of the fluid. It ensures a significant coupling between the liquid sloshing and the hydroelastic deformations of the plate. This configuration is then well suited to demonstrate the capacity of our model to predict these couplings.

The first bidimensional eigenmodes of this system were measured by Chai and Genevoux for various levels of water. Table 1 gives the first three eigenfrequencies for a water height h_w of 0.21 m. We realized an equivalent numerical model and computed these eigenmodes with our fluid-structure formulation (Figs. 3–5). The

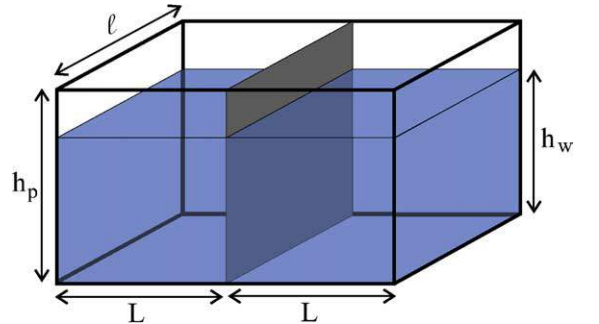


Fig. 2. Experimental configuration.

Table 1
Comparison between analytical, experimental and numerical eigenfrequencies

Theory (Hz)	Experiment (Hz)	Computation (Hz)	Error (%)
1.57	1.56	1.58	1.5
2.42	2.39	2.42	1
2.97	2.95	2.98	1

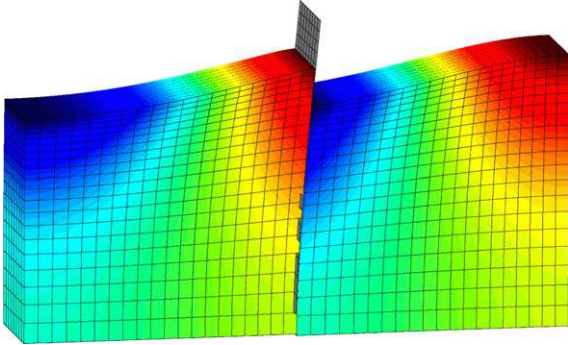


Fig. 3. First computed mode.

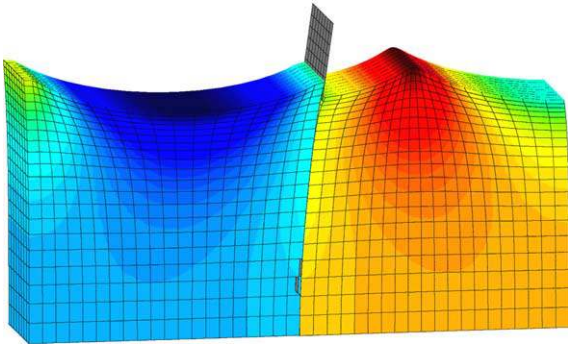


Fig. 4. Second computed mode.

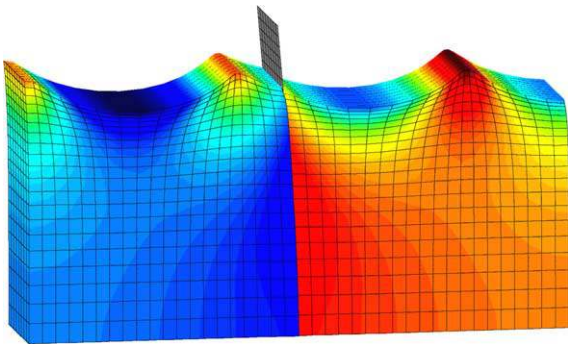


Fig. 5. Third computed mode.

comparison with experimental and analytical values shows a very good agreement (Table 1).

In order to illustrate the gravity effects on the prediction of these modes, we compare the eigenfrequencies we obtained with those given by a classical hydroelastic modelling (Figs. 6 and 7). For the first mode, the error

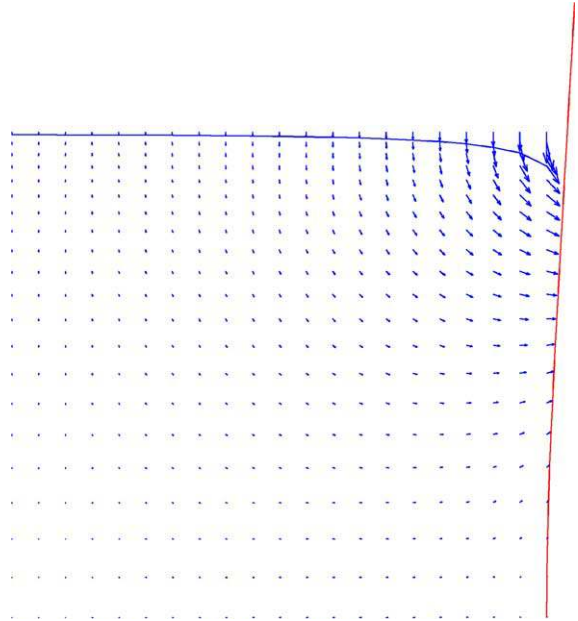


Fig. 6. First hydroelastic mode without gravity (2.19 Hz).

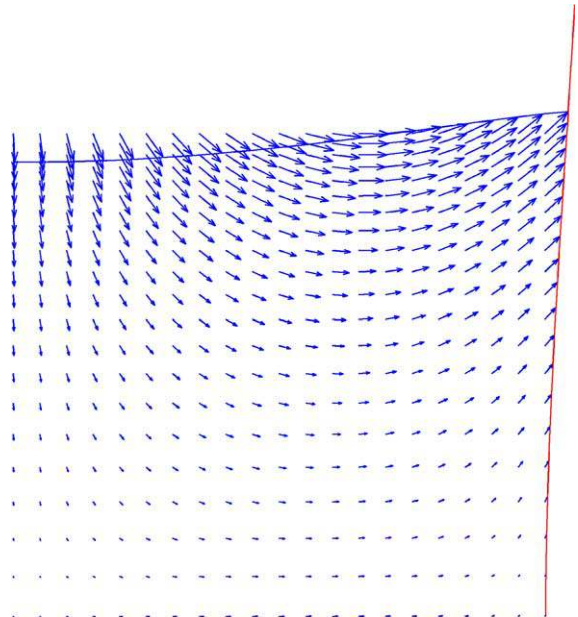


Fig. 7. First hydroelastic mode with gravity (1.58 Hz).

committed by neglecting gravity effects is more than 30 percent.

6. Conclusions and perspectives

For the vibratory analysis of slender structures containing incompressible liquids, we have presented a formulation, which gives an exact representation of gravity effects. A non-standard stiffness operator, depending on gravity, has been exhibited. This so-called elastogravity operator has been proven to be symmetric which ensures the symmetry of this formulation. To validate the interest of this approach, we have confronted our results with experimental measurements and verified that this linear model was able to represent the coupling between the sloshing of the liquid free surface and the hydroelastic deformations of the structure (see also [20]). Furthermore, the extension of this formulation to free-free fluid-structure systems had shown that the elastogravity operator gives a precise representation of rigid body modes [11].

We are now carrying on the validation of the formulation and the design of several experiments is in progress. The perspective of these works is to develop reduced models from this formulation for applications in the industrial domain [21]. We consider for example to use this model to study the influence of fuel sloshing on the aeroelastic stability of aircrafts or to elaborate the control algorithm of space launchers. These reduced models will also be used to investigate the vibration reduction of internal fluid-structure systems either with baffles (see for instance [22,23]) or with damping structural treatment (passive, see for instance [24], or hybrid passive-active with smart materials). The introduction of nonlinear sloshing in this modelling, to evaluate its effect on the system behavior, will be the purpose of further investigations (see for instance [25]).

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