



HAL
open science

Reduced models for fluid–structure interaction problems

Roger Ohayon

► **To cite this version:**

Roger Ohayon. Reduced models for fluid–structure interaction problems. *International Journal for Numerical Methods in Engineering*, 2004, 60 (1), pp.139-152. 10.1002/nme.957 . hal-03177895

HAL Id: hal-03177895

<https://hal.science/hal-03177895v1>

Submitted on 30 Oct 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Reduced models for fluid–structure interaction problems

Roger Ohayon^{*,†}

*Conservatoire National des Arts et Métiers (CNAM), Structural Mechanics and Coupled Systems Laboratory,
2 rue Conté 75003 Paris, France*

Several so-called reduced order formulations are reviewed for linear vibration analysis of bounded fluid–structure systems for low modal density situations. Compressibility effects in the fluid for interior structural–acoustic problems, and free surface gravity effects, for hydroelastic–sloshing interaction problems in the case of incompressible liquids, are examined. Reduced order models leading to symmetric matrix systems are then described using the static well-posed behaviour of the irrotational fluid. In this respect, the fluid–structure boundary value local equations, expressed in terms of fluid scalar field variables for the fluid (and displacement variables for the structure) are regularized for zero-frequency limit.

KEY WORDS: structural-acoustics; compressible hydroelasticity; reduced symmetric models

1. INTRODUCTION

We review in this paper reduced order models for modal analysis of elastic structures containing an inviscid fluid (gas or liquid, [1, 2]). These methods, based on the Ritz–Galerkin projection using appropriate Ritz vectors, allow us to construct *reduced models* expressed in terms of physical displacement vector field u in the structure, and generalized co-ordinates vector \mathbf{r} describing the behaviour of the fluid. Those reduced models lead to symmetric generalized eigenvalue matrix system involving a reduced number of degrees of freedom for the fluid. For this purpose, we construct symmetric matrix models of the fluid considered as a subsystem, by considering the response of the fluid to a prescribed normal displacement of the fluid–structure interface. Two distinct situations are analysed. On the one hand, we consider linear vibrations of an elastic structure completely filled with a compressible gas or liquid and on the other hand, we consider the case of an elastic structure containing an incompressible liquid with free surface effects due to gravity. The first case is a *structural–acoustic* situation with modal interaction between *structural modes in vacuo* for structure containing a gas or *hydroelastic modes including ‘static’ inertial and potential compressibility effects* for structure containing

*Correspondence to: Roger Ohayon, Conservatoire National des Arts et Métiers (CNAM), Structural Mechanics and Coupled Systems Laboratory, 2 rue Conté 75003 Paris, France.

†E-mail: ohayon@cnam.fr

liquids, with *acoustic modes in rigid motionless cavity*. Interface local fluid–structure dissipation through a local wall impedance can also be introduced easily in the formulations.

The second case is a *hydroelastic–sloshing* problem with modal interaction between *incompressible hydroelastic structural modes* with *incompressible liquid sloshing modes* in rigid motionless cavity, involving an *elastogravity operator* related to the wall normal displacement of the fluid–structure interface [3,4]. For the construction of reduced models, the static behaviour at zero frequency plays an important role. Therefore, we review ‘regularized’ variational formulations of the problem, in the sense that the limit zero-frequency static behaviour must be taken into account in the boundary value problem. Those ‘quasi-static’ potential and inertial contributions play a fundamental role in the Ritz–Galerkin procedure (error truncation). The general methodology corresponds to dynamic substructuring procedures adapted to fluid–structure modal analysis. For general presentations of computational methods using appropriate finite element and dynamic substructuring procedures applied to modal analysis of elastic structures containing inviscid fluids (sloshing, hydroelasticity and structural–acoustics), we refer the reader to References [3,5–10], taking into account fluid dissipation models for structural–acoustics [11,12].

2. STRUCTURAL–ACOUSTIC PROBLEM

Let us consider the linear vibrations of an elastic structure completely filled with an homogeneous, inviscid and compressible fluid. We also consider the particular case of a compressible liquid with a free surface, neglecting gravity effects.

After the derivation of the linearized equations of the fluid–structure coupled system, we introduce a linear constraint in order to obtain a regularized problem at zero frequency, and we then construct a reduced model of the fluid subsystem. Acoustic modes in rigid motionless cavity are introduced as Ritz projection vector basis, including the static solution of the coupled system. As this fluid–structure system has a resonant behaviour a direct finite element computation may lead to prohibitive time costs. That is why, starting from one of the possible variational formulations of the problem, convenient *reduced symmetric matrix models* are reviewed.

3. STRUCTURAL–ACOUSTIC EQUATIONS

3.1. Structure subjected to a fluid pressure loading

We consider an elastic structure occupying the domain Ω_S at equilibrium. The interior fluid domain is denoted as Ω_F and the fluid–structure interface is denoted Σ (see Figure 1). The angular frequency is denoted as ω . The chosen unknown field in the structure domain Ω_S is the displacement field u . The linearized deformation tensor is denoted as $\varepsilon_{ij}(u)$ and the corresponding stress tensor is denoted as $\sigma_{ij}(u)$. We denote by ρ_S the constant mass density at equilibrium and by n the unit normal, external to the structure domain Ω_S . Let δu be the test function, associated to u , belonging to the admissible space \mathcal{C}_u .

The weak variational formulation describing the response of the structure Ω_S to given harmonic forces of amplitude F^d on the external structure boundary $\partial\Omega_S \setminus \Sigma$, and to fluid pressure field p acting on the internal fluid–structure interface Σ is written as follows.

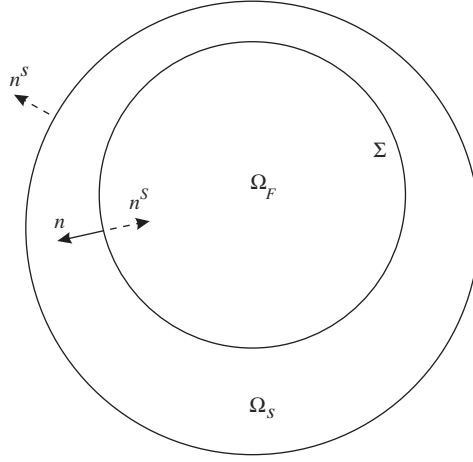


Figure 1. Elastic structure containing a gas.

For all real ω and $\forall \delta u \in \mathcal{C}_u$, find $u \in \mathcal{C}_u$ such that

$$\tilde{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \quad (1)$$

in which

$$\tilde{k}(u, \delta u) = k(u, \delta u) + k_G(u, \delta u) + k_{P_0}(u, \delta u) \quad (2)$$

and where $k(u, \delta u)$ is the mechanical elastic stiffness such that

$$k(u, \delta u) = \int_{\Omega_S} \sigma_{ij}(u) \varepsilon_{ij}(\delta u) \, dx \quad (3)$$

and where $k_G(u, \delta u)$ and $k_{P_0}(u, \delta u)$ are such that

$$k_G(u, \delta u) = \int_{\Omega_S} \sigma_{ij}^0 u_{l,i} \delta u_{l,j} \, dx, \quad k_{P_0} = \int_{\Sigma} P_0 n_1(u) \cdot \delta u \, d\sigma \quad (4)$$

In Equations (4) and (5), $k_G(u, \delta u)$ represents the classical structural prestress geometric symmetric bilinear form in which σ_{ij}^0 denotes the prestress tensor, and $k_{P_0}(u, \delta u)$ represents an additional prestress symmetric bilinear form due to rotation of normal n , in which P_0 denotes the initial pressure existing in the reference equilibrium configuration. Finally, $n_1(u)$ represents the variation of normal n between the reference configuration and the actual configuration.

3.2. Fluid subjected to a wall normal displacement

Since the fluid is inviscid, instead of describing the small motion of the fluid by a fluid displacement vector field u_F (see for instance Reference [13]), which requires an appropriate discretization of the fluid irrotationality constraint $\text{curl } u_F = 0$, we will use the pressure scalar field p . It should be noted that, the small movements corresponding to $\omega \neq 0$ are obviously

irrotational, but, in the static limit case, i.e. at zero frequency, we consider only fluids which exhibit a physical irrotational behaviour.

Let us denote by c the (constant) sound speed in the fluid, and by ρ_F , the (constant) mass density of the fluid at rest. We denote as Ω_F the domain occupied by the fluid at rest (which is taken as the equilibrium state). The local equations describing the harmonic response of the fluid to a prescribed arbitrary normal displacement $u \cdot n$ of the fluid–structure interface Σ are such that

$$\nabla p - \rho_F \omega^2 u_F = 0|_{\Omega_F} \quad (5)$$

$$p = -\rho_F c^2 \nabla \cdot u_f|_{\Omega_F} \quad (6)$$

$$u_F \cdot n = u \cdot n|_{\Sigma} \quad (7)$$

$$\text{curl } u_F = 0|_{\Omega_F} \quad (8)$$

Equation (5) corresponds to the linearized Euler equation in the fluid. Equation (6) corresponds to the constitutive equation of the barotropic fluid. Equation (7) corresponds to the wall slipping condition. Equation (8) corresponds to the irrotationality condition, only necessary in order to ensure that when $\omega \rightarrow 0$, u_F tends to static irrotational motion, which corresponds to the hypothesis that for $\omega = 0$, we only consider irrotational motions (for simply connected fluid domain).

Relation between static pressure p^s and $u \cdot n$. For $\omega = 0$, Equations (6) and (7) lead to the constant pressure static field which is related to the normal wall displacement by the relation (see References [3, 7])

$$p^s = -\frac{\rho_F c^2}{|\Omega_F|} \int_{\Sigma} u \cdot n \, d\sigma \quad (9)$$

in which $|\Omega_F|$ denotes the measure of the volume occupied by domain Ω_F .

3.3. Equations in terms of p and $u \cdot n$

The elimination of u_F from Equations (5)–(8) leads to

$$\Delta p + \frac{\omega^2}{c^2} p = 0|_{\Omega_F} \quad (10)$$

$$\frac{\partial p}{\partial n} = \rho_F \omega^2 u \cdot n|_{\Sigma} \quad (11)$$

with the constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, dx + \int_{\Sigma} u \cdot n \, d\sigma = 0 \quad (12)$$

Equation (10) is the classical Helmholtz equation expressed in terms of p . Equation (11) corresponds to the kinematic condition defined by Equation (7). The linear constraint defined by Equation (12) corresponds to the global mass conservation which ensures that the boundary problem defined by Equations (10)–(11) is equivalent to the problem defined by Equations (5)

to (8). In the absence of the condition defined by Equation (12), we would obtain a boundary value problem in terms of p which is not valid for $\omega = 0$ and which does not allow us to retrieve the value of p^s given by Equation (9).

3.4. Variational formulation in terms of (u, p)

Let δp be the test function, associated to p , belonging to the admissible space \mathcal{C}_p . The weak variational formulation corresponding to Equations (10)–(12) is obtained by the usual test-function method using Green's formula.

The weak variational formulation corresponding to the modal analysis of the structural–acoustic problem is then obtained by setting $F^d = 0$ in Equation (1). The eigenvalue structural–acoustic variational formulation is then stated as follows. Find ω^2 and $u \in \mathcal{C}_u$ and $p \in \mathcal{C}_p$, such that for all $\delta u \in \mathcal{C}_u$ and $\delta p \in \mathcal{C}_p$, we have

$$\tilde{k}(u, \delta u) - \int_{\Sigma} pn \cdot \delta u \, d\sigma = \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx \quad (13)$$

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, dx = \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} p \delta p \, dx + \omega^2 \int_{\Sigma} u \cdot n \delta p \, d\sigma \quad (14)$$

with the constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, dx + \int_{\Sigma} u \cdot n \, d\sigma = 0 \quad (15)$$

The variational formulation defined by Equations (13)–(15) is original due to the presence of the constraint defined by Equation (15) which regularizes the (u, p) formulation for $\omega = 0$ (see also References [8, 9]). In effect, usually, only Equations (13) and (14) are written, and as pointed out above, are not valid for $\omega = 0$. In the case of a direct finite element discretization of Equations (13)–(15), we obtain a matrix eigenvalue system of the type $\mathbf{A}\mathbf{Y} = \omega^2 \mathbf{B}\mathbf{Y}$, in which \mathbf{A} and \mathbf{B} are not symmetric. That is why various symmetric formulations using for the fluid pressure field p and displacement potential φ , defined up to an additive constant and such that $u_F = \nabla \varphi$, have been derived. The resulting *symmetric* formulations are then obtained by elimination of p or φ . In the present case, we are not considering a direct finite element approach of the variational formulation defined by Equations (13)–(15).

3.5. Symmetric reduced model

We will consider hereafter a dynamic substructuring approach through an appropriate decomposition of the admissible class into direct sum of admissible vector spaces (see Figure 2 and References [3, 6]).

Let us consider the following two basic problems. The first one corresponds to the acoustic modes in rigid motionless cavity and is obtained by setting $u = 0$ into Equations (14) and (15). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace \mathcal{C}_p^* of \mathcal{C}_p

$$\mathcal{C}_p^* = \left\{ p \in \mathcal{C}_p; \int_{\Omega_F} p \, dx = 0 \right\} \quad (16)$$

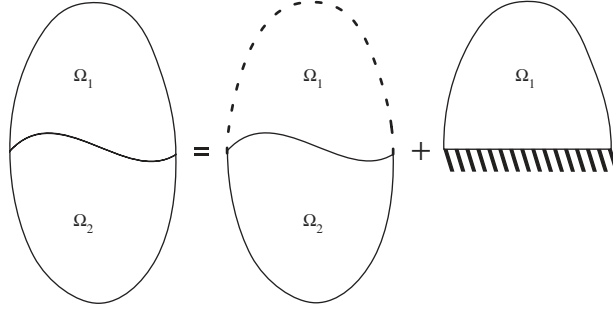


Figure 2. Dynamic substructuring scheme.

the variational formulation of acoustic modes is stated as follows: find $\omega^2 > 0$ and $p \in \mathcal{C}_p^*$ such that, for all $\delta p \in \mathcal{C}_p^*$, we have

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, dx = \omega^2 \frac{1}{\rho_F c^2} \int_{\Omega_F} p \delta p \, dx \quad (17)$$

with the constraint

$$\int_{\Omega_F} p \, dx = 0 \quad (18)$$

It should be noted that, in practice, if the constraint condition (18) is ‘omitted’, we only add a first *non-physical* zero frequency constant pressure mode, the other modes corresponding to $\omega \neq 0$ remaining the same as those defined by Equations (17) and (18). This zero frequency mode must not be retained in any Ritz–Galerkin projection analysis. In addition, we have the following orthogonality conditions:

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p_\alpha p_\beta \, dx = \mu_\alpha \delta_{\alpha\beta}, \quad \frac{1}{\rho_F} \int_{\Omega_F} \nabla p_\alpha \cdot \nabla p_\beta \, dx = \mu_\alpha \omega_\alpha^2 \delta_{\alpha\beta} \quad (19)$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement $u \cdot n$. The solution, denoted as $p^s(u \cdot n)$, is given by Equation (9). For any deformation $u \cdot n$ of the fluid–structure interface, $p^s(u \cdot n)$ belongs to a subset of \mathcal{C}_p , denoted as $\mathcal{C}^{u \cdot n}$

$$\mathcal{C}^{u \cdot n} = \left\{ p^s \in \mathcal{C}_p; p^s = -\frac{\rho_F c^2}{|\Omega_F|} \int_{\Sigma} u \cdot n \, d\sigma \right\} \quad (20)$$

In the variational formulation defined by Equations (13)–(15), p is searched under the form

$$p = p^s(u \cdot n) + \sum_{\alpha=1}^{N_p} r_\alpha p_\alpha \quad (21)$$

in which N_p denotes the number of retained acoustic modes. Decomposition (21) is unique. In addition, it should be noted that, since each eigenvector p_α corresponding to $\omega_\alpha \neq 0$, verifies the constraint defined by Equation (18), then, using Equation (9), we deduce that p and $u \cdot n$

satisfy the constraint defined by Equation (15). The decomposition defined by Equation (21) corresponds to a decomposition of the admissible class \mathcal{C}_p into the direct sum of the admissible classes defined, respectively, by Equations (20) and (16)

$$\mathcal{C}_p = \mathcal{C}^{u \cdot n} \oplus \mathcal{C}_p^* \quad (22)$$

Following Equation (21), the test function δp is then searched under the following form:

$$\delta p = p^s(\delta u \cdot n) + \sum_{\alpha=1}^{N_p} \delta r_\alpha p_\alpha \quad (23)$$

Variational formulation in δu defined by Equation (13) becomes

$$\tilde{k}(u, \delta u) + k^s(u, \delta u) - \sum_{\alpha=1}^{N_p} r_\alpha \int_{\Sigma} p_\alpha n \cdot \delta u \, d\sigma = \omega^2 \int_{\Omega_s} \rho_s u \cdot \delta u \, dx \quad (24)$$

in which $\tilde{k}(u, \delta u)$ is defined by Equation (2) and $k^s(u, \delta u)$ is such that

$$k^s(u, \delta u) = \frac{\rho_F c^2}{|\Omega_F|} \left(\int_{\Sigma} u \cdot n \, d\sigma \right) \left(\int_{\Sigma} \delta u \cdot n \, d\sigma \right) \quad (25)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of Equation (24) can be written as

$$[\tilde{\mathbf{K}} + \mathbf{K}^s] \mathbf{U} - \sum_{\alpha=1}^n \mathbf{C}_\alpha r_\alpha = \omega^2 \mathbf{M} \mathbf{U} \quad (26)$$

in which symmetric matrices $\tilde{\mathbf{K}}$ and \mathbf{K}^s correspond to finite element discretization of stiffness symmetric bilinear forms defined by Equations (2)–(4) and (25) respectively. In Equation (26), \mathbf{M} denotes the structural symmetric mass matrix and rectangular coupling matrix \mathbf{C}_α corresponds to the discretization of the coupling fluid–structure contribution $\int_{\Sigma} p \delta u \cdot n \, d\sigma$. The discretized form of Equation (14) in δp can then be written as

$$\omega_\alpha^2 \mu_\alpha r_\alpha = \omega^2 \mu_\alpha r_\alpha + \omega^2 \mathbf{C}_\alpha^T \mathbf{U} \quad (27)$$

From Equations (26) and (27), we obtain the following *symmetric matrix reduced model*:

$$\begin{pmatrix} \mathbf{K}^{\text{tot}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Diag} \mu_\alpha \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{r} \end{pmatrix} = \omega^2 \begin{pmatrix} \mathbf{M}^{\text{tot}} & \mathbf{D} \\ \mathbf{D}^T & \mathbf{Diag}(\mu_\alpha / \omega_\alpha^2) \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{r} \end{pmatrix} \quad (28)$$

in which \mathbf{r} denotes the vector of N generalized co-ordinates r_α , with $1 \leq \alpha \leq N_p$, and

$$\mathbf{K}^{\text{tot}} = \tilde{\mathbf{K}} + \mathbf{K}^s \quad (29)$$

$$\mathbf{M}^{\text{tot}} = \mathbf{M} + \sum_{\alpha=1}^{N_p} \frac{1}{\omega_\alpha^2 \mu_\alpha} \mathbf{C}_\alpha \mathbf{C}_\alpha^T \quad (30)$$

$$\mathbf{D}_\alpha = \sum_{\alpha=1}^{N_p} \frac{1}{\omega_\alpha^2} \mathbf{C}_\alpha \quad (31)$$

Further diagonalization of Equation (28) implies a projection of \mathbf{U} on the solutions of the following eigenvalue problem

$$\mathbf{K}^{\text{tot}}\mathbf{U}_\beta = \lambda_\beta\mathbf{M}^{\text{tot}}\mathbf{U}_\beta \quad (32)$$

Setting

$$\mathbf{U} = \sum_{\beta=1}^{N_u} q_\beta\mathbf{U}_\beta \quad (33)$$

in which q_β are the generalized co-ordinates describing the structure. Using the orthogonality conditions associated with the solutions of Equation (32), i.e. $\mathbf{U}_{\beta'}^T\mathbf{M}^{\text{tot}}\mathbf{U}_\beta = \mu_\beta^s\delta_{\beta\beta'}$ and $\mathbf{U}_{\beta'}^T\mathbf{K}^{\text{tot}}\mathbf{U}_\beta = \mu_\beta^s\lambda_\beta\delta_{\beta\beta'}$, Equation (28) becomes

$$\begin{pmatrix} \text{Diag } \lambda_\beta & 0 \\ 0 & \text{Diag } \omega_\alpha^2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \omega^2 \begin{pmatrix} I_{N_u} & [C_{\beta\alpha}] \\ [C_{\beta\alpha}]^T & I_{N_p} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} \quad (34)$$

It should be noted that two different situations must be considered hereafter.

Case of a liquid filling the enclosure. For the case of a heavy fluid, one must mandatory use the eigenmodes defined by Equation (32), i.e. *hydroelastic modes including ‘static’ inertial and potential compressibility effects*. In effect, if only the ‘dry’ structural modes were used, very slow (poor) convergence would be expected. Of course, some other alternatives are possible for slightly compressible liquids (this depends upon the frequency range and is particularly true for low modal density cases). One may use for instance incompressible hydroelastic modes combined with a variation of volume mode.

Case of a gas filling the enclosure. For the case of a light fluid, such as a gas filling the enclosure, instead of using the eigenmodes defined by Equation (32), one may use instead, the *in vacuo structural modes*, but, of course, the resulting matrix system would not be diagonal with respect to \mathbf{U} (in this case, fast convergence is expected).

Wall impedance condition. Referring to Reference [12] or for more detailed modelling, to Reference [11], wall impedance condition corresponds to a particular fluid–structure interface modelling. This interface is considered as a third medium with infinitesimal thickness, without mass, and with the following constitutive equation:

$$p = j\omega Z(\omega)(u \cdot n - u_F \cdot n) \quad (35)$$

in which $Z(\omega)$ denotes a complex impedance. Equations (7) and (11) must be replaced by Equation (35), using $\partial p / \partial n = \rho_F \omega^2 u_F \cdot n$.

Case of a liquid with a free surface. Let us consider a liquid with a free surface at rest denoted as Γ , If we neglect gravity effects, the boundary condition on Γ is such that

$$p = 0|_\Gamma \quad (36)$$

In this case, constraint condition (12) (or (15)) is replaced by Equation (36). Equation (9) is replaced by $p^s = 0$. Admissible space defined by Equation (16) becomes $\mathcal{C}_p^* = \{p \in \mathcal{C}_p; p = 0\}$.

In this case, the static problem defined Section 3, leads to a zero pressure field.

Let us remark that in this case, the ‘structural’ modal basis may be constituted by the hydroelastic incompressible modes using the classical added mass operator.

The reduced modal matrix models has been extended to the dissipative case (dissipative internal fluid with non-homogeneous local impedance wall condition) in Reference [11].

4. INCOMPRESSIBLE HYDROELASTIC–SLOSHING PROBLEM

We consider the linear vibrations of an elastic structure partially filled with an homogeneous, inviscid and incompressible liquid, taking into account gravity effects on the free surface Γ . We neglect in the present analysis compressibility effects of the liquid and we refer to Reference [14] for those aspects. After a derivation of the linearized equations of the fluid–structure coupled problem, introducing an appropriate linear constraint in order to obtain a ‘regularized’ problem at zero frequency, we construct a reduced model of the ‘liquid subsystem’. For this analysis, sloshing modes in rigid motionless cavity are introduced as Ritz projection vector basis, including the static solution of the coupled system. This interior fluid–structure system has a resonant behaviour. A direct finite element computation may lead therefore to prohibitive time costs. To avoid those drawbacks, reduced symmetric matrix models of this system are derived.

5. HYDROELASTIC–SLOSHING EQUATIONS

5.1. Structure subjected to a fluid pressure loading

The notations are the same that those defined in Section 3 adapted to liquid with a free surface at rest denoted Γ (see Figure 3 and Reference [3]).

The weak variational formulation describing the response of the structure Ω_S to given variation F^d of the applied forces with respect to the equilibrium state on the external structure boundary $\partial\Omega_S \setminus \Sigma$, and to fluid pressure field p acting on the internal fluid–structure interface Σ is written as follows.

For all real ω and $\forall \delta u \in \mathcal{C}_u$, find $u \in \mathcal{C}_u$ such that

$$\hat{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \quad (37)$$

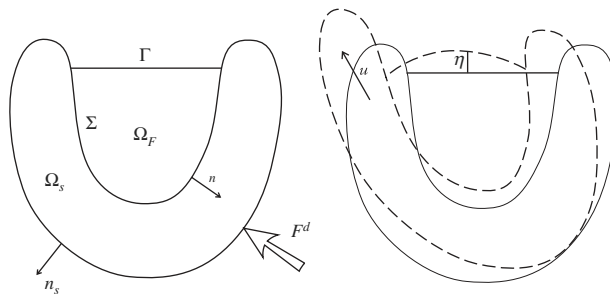


Figure 3. Structure containing a liquid.

in which

$$\hat{k} = \tilde{k} + k_{\Sigma} \quad (38)$$

In Equation (38), $\tilde{k}(u, \delta u)$ is defined by Equation (2), and k_{Σ} is the *elastogravity* symmetric bilinear form such that [3, 4]

$$k_{\Sigma}(u, \delta u) = -\frac{1}{2} \rho_{\text{F}} g \left\{ \int_{\Sigma} [zn_1(u) \cdot \delta u + u_z \delta u \cdot n] d\sigma + \int_{\Sigma} [zn_1(\delta u) \cdot u + \delta u_z u \cdot n] d\sigma \right\} \quad (39)$$

5.2. Fluid subjected to a wall normal displacement

We assume that the liquid is homogeneous, inviscid and incompressible. Free surface Γ is horizontal at equilibrium. We denote by z the external unit normal to Γ , and by g the gravity. The notations are similar to those of Section 3. The local equations describing the response of the fluid to a prescribed arbitrary normal displacement $u \cdot n$ of the fluid–structure interface Σ are such that

$$\nabla p - \rho_{\text{F}} \omega^2 u_{\text{F}} = 0|_{\Omega_{\text{F}}} \quad (40)$$

$$\nabla \cdot u_{\text{f}} = 0|_{\Omega_{\text{F}}} \quad (41)$$

$$u_{\text{F}} \cdot n = u \cdot n|_{\Sigma} \quad (42)$$

$$p = \rho_{\text{F}} g u_{\text{F}} \cdot n|_{\Gamma} \quad (43)$$

$$\text{curl } u_{\text{F}} = 0|_{\Omega_{\text{F}}} \quad (44)$$

Equation (41) corresponds to the incompressibility condition. Equation (43) is the constitutive equation on the free surface Γ due to gravity effects.

Relation between static pressure p^{s} and $u \cdot n$. For $\omega = 0$, Equations (41)–(43) lead to the constant pressure static field which is related to the normal wall displacement by the relation

$$p^{\text{s}} = -\frac{\rho_{\text{F}} g}{|\Gamma|} \int_{\Sigma} u \cdot n d\sigma \quad (45)$$

in which $|\Gamma|$ denotes the measure of the area of free surface Γ .

Equations in terms of p and $u \cdot n$. The elimination of u_{F} into Equations (40)–(44) leads to

$$\Delta p = 0|_{\Omega_{\text{F}}} \quad (46)$$

$$\frac{\partial p}{\partial n} = \rho_{\text{F}} \omega^2 u \cdot n|_{\Sigma} \quad (47)$$

$$\frac{\partial p}{\partial z} = \frac{\omega^2}{g} p|_{\Gamma} \quad (48)$$

with the constraint

$$\frac{1}{\rho_{\text{F}} g} \int_{\Gamma} p d\sigma + \int_{\Sigma} u \cdot n d\sigma = 0 \quad (49)$$

The linear constraint defined by Equation (49) ensures that the boundary problem defined by Equations (46)–(49) is equivalent to the problem defined by Equations (40)–(44). This condition is usually omitted in literature (see also References [8, 9]).

5.3. Variational formulation in terms of (u, p)

Let δp be the test function, associated to p , belonging to the admissible space \mathcal{C}_p . The weak variational formulation corresponding to Equations (46)–(49) is obtained by the usual test-function method using Green’s formula. Using Equation (37), the variational formulation of the *hydroelastic–sloshing* problem is then stated as follows. Find $u \in \mathcal{C}_u$ and $p \in \mathcal{C}_p$, such that for all $\delta u \in \mathcal{C}_u$ and $\delta p \in \mathcal{C}_p$, we have

$$\hat{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \quad (50)$$

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, dx = \frac{\omega^2}{\rho_F g} \int_{\Gamma} p \delta p \, dx + \omega^2 \int_{\Sigma} u \cdot n \delta p \, d\sigma \quad (51)$$

with the constraint

$$\frac{1}{\rho_F g} \int_{\Gamma} p \, d\sigma + \int_{\Sigma} u \cdot n \, d\sigma = 0 \quad (52)$$

The variational formulation defined by Equations (50)–(52) is original due to the presence of the constraint defined by Equation (49) which regularizes the (u, p) formulation for $\omega = 0$ (see also References [8, 9]).

5.4. Symmetric reduced matrix model

Let us consider the following two basic problems. The first one corresponds to the sloshing modes in rigid motionless cavity and is obtained by setting $u = 0$ into Equations (47) and (49). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace \mathcal{C}_p^* of \mathcal{C}_p

$$\mathcal{C}_p^* = \left\{ p \in \mathcal{C}_p; \int_{\Gamma} p \, d\sigma = 0 \right\} \quad (53)$$

the variational formulation of acoustic modes is stated as follows: find $\omega^2 > 0$ and $p \in \mathcal{C}_p^*$ such that, for all $\delta p \in \mathcal{C}_p^*$, we have

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, dx = \omega^2 \frac{1}{\rho_F g} \int_{\Gamma} p \delta p \, d\sigma \quad (54)$$

with the constraint

$$\int_{\Gamma} p \, d\sigma = 0 \quad (55)$$

It should be noted that, in practice, if the constraint condition (55) is ‘omitted’, we only add a first *non-physical* zero frequency constant pressure mode, the other modes corresponding to

$\omega \neq 0$ remaining the same as those defined by Equations (54) and (55). This zero frequency mode must not be retained in any Ritz–Galerkin projection analysis. In addition, we have the following orthogonality conditions:

$$\frac{1}{\rho_{\text{FG}}} \int_{\Gamma} p_{\alpha} p_{\beta} \, d\sigma = \mu_{\alpha} \delta_{\alpha\beta}, \quad \frac{1}{\rho_{\text{F}}} \int_{\Omega_{\text{F}}} \nabla p_{\alpha} \cdot \nabla p_{\beta} \, dx = \mu_{\alpha} \omega_{\alpha}^2 \delta_{\alpha\beta} \quad (56)$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement $u \cdot n$. The solution, denoted as $p^s(u \cdot n)$, is given by Equation (45). For any deformation $u \cdot n$ of the fluid–structure interface, $p^s(u \cdot n)$ belongs to a subset of \mathcal{C}_p , denoted as $\mathcal{C}^{u \cdot n}$

$$\mathcal{C}^{u \cdot n} = \left\{ p^s \in \mathcal{C}_p; p^s = -\frac{\rho_{\text{FG}}}{|\Gamma|} \int_{\Sigma} u c \dot{n} \, d\sigma \right\} \quad (57)$$

In the variational formulation defined by Equations (50)–(52), p is searched under the form

$$p = p^s(u \cdot n) + \sum_{\alpha=1}^{N_p} r_{\alpha} p_{\alpha} \quad (58)$$

in which N_p denotes the number of retained sloshing modes. Decomposition (58) is unique. In addition, it should be noted that, since each eigenvector p_{α} corresponding to $\omega_{\alpha} \neq 0$, verifies the constraint defined by Equation (55), then, using Equation (45), we deduce that p and $u \cdot n$ satisfy the constraint defined by Equation (52). The decomposition defined by Equation (58) corresponds to a decomposition of the admissible class \mathcal{C}_p into the direct sum of the admissible classes defined respectively by Equations (56) and (57), $\mathcal{C}_p = \mathcal{C}^{u \cdot n} \oplus \mathcal{C}_p^*$.

The variational formulation in u defined by Equation (50) becomes

$$\hat{k}(u, \delta u) + k^s(u, \delta u) - \sum_{\alpha=1}^{N_p} r_{\alpha} \int_{\Sigma} p_{\alpha} n \cdot \delta u \, d\sigma = \omega^2 \int_{\Omega_{\text{S}}} \rho_{\text{S}} u \cdot \delta u \, dx \quad (59)$$

in which $\hat{k}(u, \delta u)$ is defined by Equation (38) and $k^s(u, \delta u)$ is such that

$$k^s(u, \delta u) = \frac{\rho_{\text{FG}}}{|\Gamma|} \left(\int_{\Sigma} u \cdot n \, d\sigma \right) \left(\int_{\Sigma} \delta u \cdot n \, d\sigma \right) \quad (60)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of Equation (60) can be written as

$$[\hat{\mathbf{K}} + \mathbf{K}^s] \mathbf{U} - \sum_{\alpha=1}^n \mathbf{C}_{\alpha} r_{\alpha} - \omega^2 \mathbf{M} \mathbf{U} = \mathbf{F}^d \quad (61)$$

in which symmetric matrices $\hat{\mathbf{K}}$ and \mathbf{K}^s correspond to finite element discretization of stiffness symmetric bilinear forms defined by Equations (38) and (60), respectively. The discretized form of Equation (51) in δp can then be written as

$$\omega_{\alpha}^2 \mu_{\alpha} r_{\alpha} = \omega^2 \mu_{\alpha} r_{\alpha} + \omega^2 \mathbf{C}_{\alpha}^{\text{T}} \mathbf{U} \quad (62)$$

From Equations (61) and (62), we obtain a *symmetric matrix reduced model* whose expression is similar to the one given by expression (28).

Similarly to Section 3.5, further diagonalization can be obtained by setting

$$\mathbf{U} = \sum_{\beta=1}^{N_u} q_{\beta} \mathbf{U}_{\beta} \quad (63)$$

in which q_{β} are the generalized co-ordinates describing the structure and \mathbf{U}_{β} are the eigenmodes of an eigenvalue problem similar to the one described by Equation (32). We then obtain a similar matrix system than the one described by Equation (34)

$$\begin{pmatrix} \text{Diag } \lambda_{\beta} & 0 \\ 0 & \text{Diag } \omega_{\alpha}^2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} - \omega^2 \begin{pmatrix} I_{N_u} & [C_{\beta\alpha}] \\ [C_{\beta\alpha}]^T & I_{N_p} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^d \\ \mathbf{0} \end{pmatrix} \quad (64)$$

It should be noted that we can also use the *incompressible hydroelastic modes*, i.e. the modes of the coupled system constituted by the elastic structure containing an incompressible liquid, with $p = 0$ on Γ (through an *added mass operator*). In this case, the resulting matrix system is not completely diagonal with respect to \mathbf{U} variables [3, 6, 9].

6. CONCLUSION

In the present paper, we have reviewed appropriate formulations for low modal density frequency computations of the eigenmodes of elastic structures containing linear inviscid homogeneous fluids for structural–acoustics problems, using *structural modes in vacuo* for structure containing a gas or *hydroelastic modes including ‘static’ inertial and potential compressibility effects* for structure containing liquids, with *acoustic modes in rigid motionless cavity*. For incompressible hydroelastic–sloshing free surface problems, we have derived symmetric reduced order formulations using *incompressible hydroelastic modes including ‘static’ potential gravity effects with sloshing modes in rigid motionless tanks*. Those formulations, using various modal interaction schemes, with dynamic substructuring techniques lead to symmetric reduced matrix systems expressed in terms of generalized co-ordinates for the fluid–structure system.

REFERENCES

1. Abramson HN. The dynamic behaviour of liquids in moving containers. *SP-106*, NASA, 1966.
2. Liu WK, Uras RA. Variational approach to fluid–structure interaction with sloshing. *Nuclear Engineering and Design* 1992; **106**:69–85.
3. Morand H, Ohayon R. *Fluid–Structure Interaction*. Wiley: Chichester, 1995.
4. Schotté JS, Ohayon R. Effect of gravity on a free-free fluid–structure system. In *Fluid–Structure Interactions, Aeroelasticity, Flow-Induced Vibration and Noise*, Paidoussis MP (ed.), *Proceedings of the 5th International Symposium*, IMECE-02, New Orleans, LA, 2002.
5. Felippa CA, Ohayon R. Mixed variational formulation of finite element analysis of acoustoelastic/slosh fluid–structure interaction. *International Journal of Fluids and Structures* 1990; **4**:35–57.
6. Morand H, Ohayon R. Substructure variational analysis of the vibrations of coupled fluid–structure systems. Finite element results. *International Journal for Numerical Methods in Engineering* 1979; **14**:741–755.
7. Ohayon R, Felippa CA. The effect of wall motion on the governing equations of structures containing fluids. *Journal of Applied Mechanics* 1990; **57**: 783–785.
8. Ohayon R. Symmetric formulations for modal analysis of internal fluid structure systems. In *Fluid–Structure Interactions, Aeroelasticity, Flow-Induced Vibration and Noise*, Paidoussis MP (ed.), *Proceedings of the 4th International Symposium*, IMECE-1997, ASME, Fairfield, NJ, 1997; vol. AD **53-1**:67–74.

9. Ohayon R. Reduced symmetric models for modal analysis of internal structural–acoustic and hydroelastic–sloshing systems. *Computer Methods in Applied Mechanics and Engineering* 2001; **190**:3009–3019.
10. Ryan P. Eigenvalue and eigenfunction error estimates for finite element formulations of slosh–structure interaction. *Ph.D. Thesis*, Stanford University, USA, 1998.
11. Ohayon R, Soize C. *Structural Acoustics and Vibration*. Academic Press: New York, 1998.
12. Kehr-Candille V., Ohayon R. Elasto-acoustic damped vibrations. Finite element and modal reduction methods. In *New Advances in Computers and Structure Mechanics*, Zienkiewicz OC, Ladevèze P (eds). Elsevier: Amsterdam, 1992.
13. Bermudez A, Rodriguez R. Finite element computation of the vibration modes of a fluid-solid system. *Computer Methods in Applied Mechanics and Engineering* 1994; **119**:355–370.
14. Andrianarison O, Ohayon R. Linear vibrations of a fluid contained in a tank, with compressibility and gravity effects. In *Proceedings of the European Conference on Computational Mechanics ECCM-99*, Munich, 1999.