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A multiscale computational method for medium-frequency vibrations of assemblies of heterogeneous plates

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A new approach, called the “variational theory of complex rays” has been developed in order to calculate the vibrations of slightly damped elastic plates in the medium-frequency range. The resolution of a small system of equations which does not result from a fine spatial discretization of the structure leads to the evaluation of effective quantities (deformation energy, vibration amplitude,...). Here, we extend this approach, which was already validated for assemblies of homogeneous substructures, to the case of heterogeneous substructures.

Keywords: Vibrations; Medium-frequency range; Complex rays; Heterogeneous structures

1. Introduction

The modeling and analysis of the vibrational response of elastic structures are undoubtedly among the key issues which appear in the design of satellites or car chassis. Today, at least concerning modeling and analysis, there remain no major difficulties in the low-frequency range, even for complex structures [1]. Concerning high frequencies, computational tools quite distinct from those used for low frequencies do exist, particularly the SEA method [2–5].

Conversely, the modeling and analysis of medium-frequency vibrations, which constitute the topic of this paper, continue to raise certain problems. The difficulty in attempting to extend the low-frequency methods to this case is that the length of variation of the phenomena being studied is very small compared to the characteristic dimension of the structure. Therefore, the finite element calculation involved would require an unreasonable number of degrees of freedom. Moreover, as the frequency increases, a pollution error which affects the accuracy of the finite element solution [6–8] arises and

must be taken into account. Apart from these serious numerical difficulties, the quantities involved in the calculation remain associated with small variation lengths which are not very significant: these are not “effective” quantities, i.e. they are not representative of the vibratory response of the structure on the time and space scales considered. Therefore, they lead to results which are very sensitive to imprecisions in the data. Nevertheless, various improved finite element approaches have been studied, in particular in [9–27].

Difficulties are also experienced in trying to extend the SEA method, which is appropriate mostly for high frequencies. In particular, the spatial aspect disappears almost entirely: thus, the scale on which space is described is too coarse for medium-frequency analysis.

There have been few works dedicated specifically to the calculation of medium-frequency vibrations. For rods and beams, the problem was solved, in particular, in [28–30]. For more complex structures such as plates or shells, boundary element methods were proposed in [31] and methods based on the use of specific reduced bases can be found in [32–35]. Most of these works are closely related to finite element methods and the quantities calculated are still not “effective”. Therefore, in our opinion, they are not “true” medium-frequency methods. On the contrary, the theory initiated by Belov and Ryback [36,37] derives from the SEA method: it is

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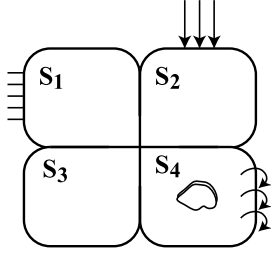


Fig. 1. An “industrial” problem.

built upon the concepts of “effective energy density” and “effective power flow”, which are extremely attractive. However, despite improvements made primarily by Ichchou et al. [38], this theory still encounters obstacles which are difficult to circumvent.

The approach followed here is the variational theory of complex rays (VTCR), a predictive tool designed specifically to deal with medium-frequency problems, which was introduced by Ladevèze in [39]. This approach is a “true” medium-frequency method in the sense that the calculations are performed on “effective” quantities. Previous works already validated this strategy for two-dimensional assemblies of elastic homogeneous plates with low damping [40] and for three-dimensional assemblies [41,42]. However, most industrial structures present some kind of structural heterogeneity, whether intentional (such as portholes or equipment connections) or unintentional (such as cracks). The vibratory response in the medium-frequency range is very sensitive to these structural discontinuities. While the VTCR was originally based on the assumption that the structure is an assembly of homogeneous substructures, this paper goes one step further and proposes an extension of the VTCR which enables the designer to take heterogeneity into account.

Our objective is to deal with three-dimensional assemblies of homogeneous and heterogeneous substructures (Fig. 1). In the first part of this paper, we will present the fundamental aspects of the VTCR and explain how it can be used with assemblies of homogeneous substructures. Then, we will address the difficulty in two stages: in the second part, we will focus on the modeling of a heterogeneous substructure alone; in the third part, we will present the principle of a method to reduce the size of the model associated with a heterogeneous substructure.

2. Basic aspects of the VTCR

2.1. Principle

Let us review the basic aspects of the VTCR [41,42]. In the general case of a structure made of several sub-

structures, the first step of the VTCR consists of associating each substructure with a superelement described by degrees of freedom which correspond to local basic modes, defined on two scales, which satisfy the dynamic equations (local equilibrium and constitutive relation) exactly. These modes are called complex rays. Besides having a strong mechanical meaning, they can be divided into families related to interior, edge or corner zones. In the vicinity of a point, the solution is assumed to be properly described locally as the superimposition of an infinite number of such local vibration modes. A remarkable fact is that, in this process, all wave directions are taken into account. The degrees of freedom are the unknowns of the problem, each representing the amplitude of the basic mode with which it is associated. These are large-wavelength quantities.

Next, an appropriate variational formulation enables us to verify, on the average, the boundary and transmission conditions. Our choice of reasoning on a superelement level requires the formulation to allow approximations which are a priori independent within substructures: these approximations do not necessarily need to verify the transmission conditions a priori, whether in terms of displacements or stresses. In practice, the variational formulation associates each superelement with, on the one hand, an elementary matrix which represents the interaction of the fields with one another and, on the other hand, a right-hand side which represents the interaction between the fields and the boundary conditions. The transmission conditions at the interface are taken into account automatically at the assembly stage.

The last characteristic of the VTCR is that, from the calculated discretized amplitudes, it retains only effective quantities related to the elastic energy, the kinetic energy, the dissipation work, etc.

2.2. The reference problem

Here, in order to simplify the presentation, the problem will be formulated for an assembly of only two substructures, but this can easily be generalized to an assembly of n substructures. For the purpose of illustration, let us consider two thin, isotropic and elastic Kirchhoff–Love plates. First, we assume that they are homogeneous.

Let us study the steady-state vibrations of this assembly. Classically, all quantities are defined in the complex domain: an amplitude $\underline{Q}(\underline{X})$ is associated with $\underline{Q}(\underline{X}) \cdot \exp(i\omega t)$.

Let S_1 and S_2 be two plates, ∂S_1 and ∂S_2 their respective boundaries solicited harmonically at a fixed angular frequency ω and Γ_{12} the perfect interface between S_1 and S_2 . The boundary conditions are as follows, with $l = (1, 2)$: deflection w_l^d on part $\partial_{wd} S_l$ of ∂S_l ;

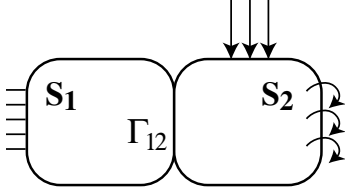


Fig. 2. The reference problem.

slope $w_{n_l}^d$ on $\partial_{wnd}S_l$; bending moment M_l^d on $\partial_{Md}S_l$; finally, Kirchhoff shear K_l^d on $\partial_{Kd}S_l$ (Fig. 2). The quantities of interest are the deflections w_l and the moments \mathbf{M}_l .

For S_l , let us introduce the space \mathcal{S}_{adl} of the displacement–stress pairs $s_l = (w_l, \mathbf{M}_l)$ defined on S_l such that:

$$s_l \in \mathcal{S}_{adl} \iff \begin{cases} s_l \in \mathcal{U} \times \mathcal{S} \quad (\text{set of finite-energy fields}) \\ \Delta \Delta w_l - k_l^4 w_l = 0 \quad \text{on } S_l \\ \mathbf{M}_l = \frac{2h_l^3}{3} (1 + i\eta_l) \mathbf{K}_{PSl} \mathcal{X}(w_l) \end{cases} \quad \text{for } l = 1, 2 \quad (1)$$

$$\text{with } k_l^4 = \frac{3\rho_l \omega^2 (1 - \nu_l^2)}{(1 + i\eta_l) E_l h_l^2}$$

where k_l is the wave number and ρ_l , ν_l , η_l , E_l and h_l designate respectively the density, Poisson's ratio, structural damping coefficient, Young's modulus and thickness of plate S_l . \mathbf{K}_{PSl} is Hooke's tensor for plane stress and \mathcal{X} is the curvature operator. The fields of \mathcal{S}_{adl} are admissible in the sense that they verify both the local equilibrium and the constitutive relation exactly. The reference problem is:

Problem 1. Find s_l such that:

$$\begin{cases} s_l \in \mathcal{S}_{adl} \\ w_l = w_l^d \quad \text{on } \partial_{wd}S_l \\ w_{l,n_l} = w_{l,n_l}^d \quad \text{on } \partial_{wnd}S_l \\ M_{ln_l} = M_{ln_l}^d \quad \text{on } \partial_{Md}S_l \\ K_{ln_l} = K_{ln_l}^d \quad \text{on } \partial_{Kd}S_l \end{cases} \quad (2)$$

$$\begin{cases} w_1 = w_2 \quad \text{on } \Gamma_{12} \\ w_{1,n_1} = w_{2,n_2} \quad \text{on } \Gamma_{12} \\ M_{1n_1} = M_{2n_2} \quad \text{on } \Gamma_{12} \\ K_{1n_1} = K_{2n_2} \quad \text{on } \Gamma_{12} \end{cases}$$

$$\text{with } l = 1, 2, M_{ln_l} = \underline{n}_l \mathbf{M}_l \underline{n}_l$$

$$\text{and } K_{ln_l} = \underline{n}_l \operatorname{div}[\mathbf{M}_l] + (\underline{t}_l \mathbf{M}_l \underline{n}_l)_{,t_l}$$

The reference problem has, at the most, one solution if:

- Hooke's tensors are positive definite,
- $\eta_l \geq 0$.

2.3. The variational formulation associated with the VTCR

The VTCR is primarily a global formulation of the boundary conditions and transmission conditions in terms of both displacements and forces across Γ_{12} . With this understanding, the problem becomes:

Problem 2. Find s_l such that:

$$\begin{cases} s_l \in \mathcal{S}_{adl} \\ \sum_l (A_{\partial S_l}(\delta s_l, s) - L_{\partial S}(\delta s_l)) \\ + \sum_{\substack{l,m \\ l < m}} C_{\Gamma_{lm}}(\delta s_l, s_l, \delta s_m, s_m) = 0 \\ \forall \delta s_l \in \mathcal{S}_{adl} \end{cases} \quad \text{for } l = 1, 2, m = 1, 2 \quad (3)$$

with the boundary conditions:

$$\begin{aligned} & A_{\partial S_l}(\delta s_l, s_l) - L_{\partial S_l}(\delta s_l) \\ & = \operatorname{Re} \left\{ -i\omega \left(- \int_{\partial_{wd}S_l} \delta K_{ln_l} (w_l - w_l^d)^* dL \right. \right. \\ & \quad + \int_{\partial_{wnd}S_l} \delta \underline{n}_l \mathbf{M}_l \underline{n}_l (w_{l,n_l} - w_{l,n_l}^d)^* dL \\ & \quad + \int_{\partial_{Md}S_l} (\underline{n}_l \mathbf{M}_l \underline{n}_l - M_l^d) \delta w_{l,n_l}^* dL \\ & \quad \left. \left. - \int_{\partial_{Kd}S_l} (K_{ln_l} - K_l^d) \delta w_l^* dL \right) \right\} \end{aligned} \quad (4)$$

and with the transmission conditions:

$$\begin{aligned} & C_{\Gamma_{lm}}(\delta s_l, s_l, \delta s_m, s_m) \\ & = \operatorname{Re} \left\{ -i\omega \left(\int_{\Gamma_{lm}} \frac{1}{2} [-\delta(K_{l,n_l} - K_{m,n_m})(w_l - w_m)^* \right. \right. \\ & \quad + \delta(\underline{n}_l \mathbf{M}_l \underline{n}_l + \underline{n}_m \mathbf{M}_m \underline{n}_m)(w_{l,n_l} + w_{m,n_m})^* \\ & \quad + (\underline{n}_l \mathbf{M}_l \underline{n}_l - \underline{n}_m \mathbf{M}_m \underline{n}_m) \delta(w_{l,n_l} - w_{m,n_m})^* \\ & \quad \left. \left. - (K_{ln_l} + K_{mn_m}) \delta(w_l + w_m)^* \right] dL \right) \right\} \end{aligned} \quad (5)$$

$\operatorname{Re}[Q]$ designates the real part of Q and Q^* designates the conjugate of Q . Note that the terms which appear in this formulation are powers.

In practice, the VTCR procedure consists of two stages: first, admissible fields are built; then, the boundary conditions are taken into account through the discretization of the variational formulation associated with a set of admissible fields chosen among the elements of \mathcal{S}_{ad} .

2.4. Construction of admissible fields

Let us define a subset of \mathcal{S}_{adl} . The VTCR uses two-scale approximations. For example, in the vicinity of a point \underline{X} of S_l , the solution is described by the superimposition of modes which can be written in the following manner:

$$w_l(\underline{X}, \underline{P}) = e^{i\sqrt{\omega}\underline{P}\cdot\underline{X}} U_l(\underline{X}, \underline{P}) \quad (6)$$

The moments are deduced from the constitutive relation:

$$\mathbf{M}_l(\underline{X}, \underline{P}) = (1 + i\eta_l) \mathbf{K}_{PS} \mathcal{X}(w_l(\underline{X}, \underline{P})) \quad (7)$$

$e^{i\sqrt{\omega}\underline{P}\cdot\underline{X}}$ represents the ‘‘fast’’ part of the field, which is defined explicitly. $U(\underline{X}, \underline{P})$ is the ‘‘slow’’ part. \underline{P} designates a vector characterizing the local vibration mode and chosen so that the mode verifies the dynamic equations. There are many possible choices depending on the zone whose vibrations are particularly sought; the VTCR distinguishes among the interior zone, the edge zone and the corner zone.

For example, let us concentrate on the properties of n th-order complex rays for the interior zone. Thus, U includes an n th-order polynomial in \underline{X} . If the damping factor is small, U can be expressed as

$$U_l(\underline{X}, \underline{P}) = e^{\frac{\eta_l}{4}\sqrt{\omega}\underline{P}\cdot\underline{X}} \mathbf{V}(\underline{X}, \underline{P}) [a_l(\underline{P})] \quad (8)$$

The coefficients $a_l(\underline{P})$ of the polynomial $\mathbf{V}(\underline{X}, \underline{P}) [a_l(\underline{P})]$ are unknowns; they form a column $[a_l(\underline{P})]$ of ‘‘generalized amplitudes’’. Consequently, all unknowns are large-wavelength quantities. The damping of the wave being modeled is represented by the expression $e^{\frac{\eta_l}{4}\sqrt{\omega}\underline{P}\cdot\underline{X}}$. These fields can be considered to be vibration modes in an infinite domain with the same mechanical properties as substructure S_l ; therefore, they belong to \mathcal{S}_{adl} . In the case of a 0th-order complex ray, the admissibility relation

$$\Delta \Delta w_l - k_l^4 w_l = 0 \quad \text{on } S_l \quad (9)$$

requires

$$(\underline{P}\cdot\underline{P})^2 = r_l^4 \quad \text{with } r_l^4 = \frac{3\rho_l(1 - \nu_l^2)}{E_l h_l^2} \quad (10)$$

Therefore, the locus of the end of the admissible wave vector \underline{P} for the interior zone is a circle \mathcal{C}_l of radius r_l . This curve is a characteristic of the material. The definitions of corner and edge modes are very similar.

For the interior zone, we take into account all the admissible directions of the rays. The solution w^{sol} is written as

$$w_l^{sol}(\underline{X}) = \int_{\mathcal{C}_l} w(\underline{X}, \underline{P}) \, d\underline{s} \quad (11)$$

The integrals are calculated along the curve \mathcal{C}_l . Clearly, w_l^{sol} depends on the generalized amplitude $[a]$ relative to \underline{P} defined on \mathcal{C}_l .

2.5. The discretized form of the VTCR

All that is needed in order to derive approximations from the VTCR is for one to define the subspace \mathcal{S}_{adl}^h from \mathcal{S}_{adl} . In practice, one discretizes the curve \mathcal{C}_l using finite elements; for example $\underline{P} \rightarrow \underline{a}_l(\underline{P})$ is assumed to be constant in each element. Then

$$\underline{a}_l(\underline{P}) = \mathbf{B} \underline{a}_l^h \quad \text{for } \underline{P} \in \mathcal{C}_l \quad (12)$$

In other terms, \underline{a}_l^h is associated with $\mathcal{C}_l^{r_l}$ (i.e. \mathcal{C}_l discretized with r_l elements). Then, the subspace \mathcal{S}_{adl}^h of the approximations deduced from \mathcal{S}_{adl} is defined by

$$s_l^h \in \mathcal{S}_{adl}^h \iff \begin{cases} w_l^h(\underline{X}) = \mathbf{W}_l^h(\underline{X}) [\underline{a}_l^h] \\ \mathbf{M}_l^h(\underline{X}) = \mathbf{\Lambda}_l^h(\underline{X}) [\underline{a}_l^h] \\ \underline{a}_l^h \in \mathcal{C}_l^{r_l} \end{cases} \quad \text{for } l = 1, 2 \quad (13)$$

The same procedure can be applied to the corner and edge zones.

In the case of an assembly of two substructures S_1 and S_2 , the variational formulation and the subspace \mathcal{S}_{adl}^h defined by Eq. (13) lead to a system of equations of dimension $\sum_l r_l$ in the complex domain:

$$[\mathbf{A}^h] [\underline{a}_l^h] = [\underline{L}_d^h] \quad \text{for } l = 1, 2 \quad (14)$$

2.6. Effective quantities

For relatively high frequencies, only effective quantities associated with the generalized amplitudes are meaningful. Indeed, the precise spatial distribution of s_l^h is meaningless from a mechanical point of view. On a given substructure S with a characteristic dimension L , the quantities involved are expressed as $q(\underline{X})$, which can represent, for example, the elastic energy, the kinetic energy or the dissipation work; in the vicinity of any test point \underline{X}_0 , the associated effective quantities are defined on a square patch S_{X_0} with side L by

$$q_{eff}(\underline{X}_0) = \frac{1}{L^2} \int_{S_{X_0}} q(\underline{X}) \, d\underline{S} \quad (15)$$

These effective quantities are large-wavelength quantities. Compared to classical quantities such as the dis-

placements of points, which can vary considerably due to slight variations of input data, they are much less sensitive. Effective quantities can also be defined on the edge of S .

2.7. Examples

A three-dimensional assembly of plates is used to model the front part of the body-in-white of a car, made of sheet steel. The mechanical properties of the 54 plates are: $E = 210$ GPa; $\eta = 0.001$; $\nu = 0.3$; $\rho = 7800$ kg/m³; $h = 0.8$ mm. The harmonic excitation is a distributed force F_d on the front side frame: $F_d = 1$ N/m; $f = 650$ Hz; $\omega = 2\pi f$ rad/s. Symmetry conditions are prescribed on the appropriate boundaries; the rear boundaries are fixed. The VTCR solution (Fig. 3) was obtained using 40 interior modes and 9 edge modes per plate. Since 0th-order complex rays were used, there were 152 degrees of freedom per plate, which is a very small number for this frequency. Thus, we had to solve a (6536×6536) linear problem. We used the program CORAY-MF (Complex RAYs for Medium Frequencies) developed by Ph. Rouch at LMT-Cachan. The effective displacements on patches were post-processed in order to compare this VTCR solution with experimental data (Fig. 4).

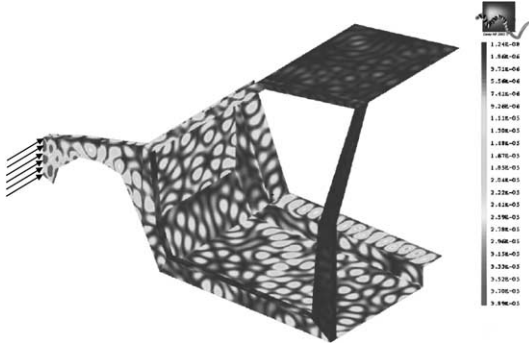


Fig. 3. VTCR solution: $|w|$ (m).

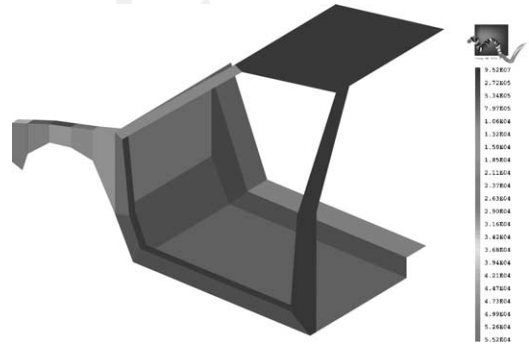


Fig. 4. Post-processing of the effective displacements $|w_{eff}|$ (m).

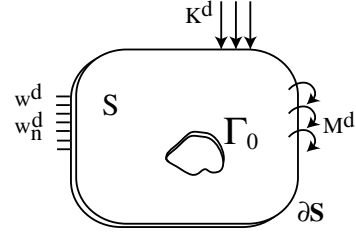


Fig. 5. The reference problem.

3. Extension of the VTCR to heterogeneous structures

3.1. The reference problem

Now, we will focus on the construction of the super-element associated with an isolated heterogeneous sub-structure (Fig. 5). Then, this superelement will be available to be assembled with other existing superelements.

Let us consider the vibrations of a locally heterogeneous isotropic plate. Each type of heterogeneity results in specific limitations on the deflection, slope, bending moment and Kirchhoff's shear at its boundary. In order to test the strategy, let us consider the case of a plate with an arbitrarily shaped hole. The boundary Γ_0 is assumed to be free. Therefore, the boundary conditions on Γ_0 are $M^d = 0$ and $K^d = 0$. The notations are the same as before, except for dropping the suffix l in order to simplify the equations.

The space \mathcal{S}_{ad} is defined exactly like \mathcal{S}_{adl} in the first part. The reference problem becomes:

Problem 3. Find s such that:

$$\begin{cases} s \in \mathcal{S}_{ad} \\ w = w^d & \text{on } \partial_{w^d} S \\ w_n = w_n^d & \text{on } \partial_{w_n^d} S \\ M_n = M^d & \text{on } \partial_{M^d} S \\ K_n = K^d & \text{on } \partial_{K^d} S \\ M_n = M^d = 0 & \text{on } \Gamma_0 \\ K_n = K^d = 0 & \text{on } \Gamma_0 \end{cases} \quad (16)$$

3.2. Direct application of the VTCR

The first option is to treat Γ_0 like ∂S . Thus, the problem becomes:

Problem 4. Find s such that:

$$\begin{cases} s \in \mathcal{S}_{ad} \\ (A_{\partial S}(\delta s, s) - L_{\partial S}(\delta s)) + (A_{\Gamma_0}(\delta s, s) - L_{\Gamma_0}(\delta s)) = 0 \\ \forall \delta s \in \mathcal{S}_{ad} \end{cases} \quad (17)$$

The term $(A_{\partial S}(\delta s, s) - L_{\partial S}(\delta s))$ was developed while defining Problem 2 in Eq. (4). The term $(A_{\Gamma_0}(\delta s, s) - L_{\Gamma_0}(\delta s))$ is written as

$$\begin{aligned} & A_{\Gamma_0}(\delta s, s) - L_{\Gamma_0}(\delta s) \\ &= \text{Re} \left\{ -i\omega \left(\int_{\Gamma_0} \underline{n} \underline{M} \underline{n} \delta w_n^* dL - \int_{\Gamma_0} K_n \delta w^* dL \right) \right\} \end{aligned} \quad (18)$$

Such a method can be applied, for example, to the problem of a square steel plate of side $L = 1$ m with a center square hole of side $L/3$. The mechanical properties are the same as for the example in Section 2.7. The harmonic excitation is a distributed force F_d on one side: $f = 400$ Hz; $\omega = 2\pi f$ rad/s; $F_d = 1$ N/m. The plate is clamped on the opposite side. The solution obtained with the VTCR (Fig. 7) and the solution obtained with finite elements (Fig. 6) are very similar.

However, the influence of heterogeneity is taken into account through the terms integrated on Γ_0 , whereas we would like the treatment of local heterogeneity to be distinguished from that of the master structure. Indeed, from a mechanical point of view, the hole creates only a

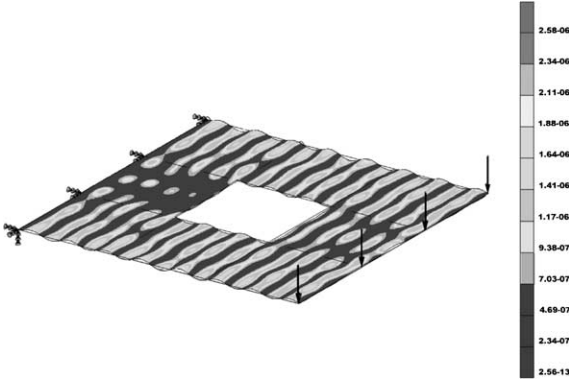


Fig. 6. Finite element solution: $|w|$ (m).

perturbation of the vibratory response. One should note that this interpretation is not valid for all types of heterogeneity: for example, a long stiffener would behave differently. The case of stiffeners requires a specific treatment which will be developed in a subsequent paper.

3.3. Adaptation of the VTCR to the treatment of local heterogeneity

Our proposal consists of taking the perturbation due to local heterogeneity into account as soon as the admissible fields of \mathcal{S}_{ad} have been built. The purpose is to develop ahead of time, and only once, families of admissible fields which satisfy the boundary conditions resulting from each type of heterogeneity a priori. These fields will then be used as the need arises while modeling heterogeneous structures.

Therefore, let us define the set \mathcal{S}_{ad}^c (i.e. corrected \mathcal{S}_{ad}) of the pairs s such that:

$$s \in \mathcal{S}_{ad}^c \iff \begin{cases} s \in \mathcal{S}_{ad} \\ M_n = M^d = 0 \quad \text{on } \Gamma_0 \\ K_n = K^d = 0 \quad \text{on } \Gamma_0 \end{cases} \quad (19)$$

Then, the problem becomes:

Problem 5. Find s such that:

$$\begin{cases} s \in \mathcal{S}_{ad}^c \\ A_{\partial S}(\delta s, s) - L_{\partial S}(\delta s) = 0 \\ \forall \delta s \in \mathcal{S}_{ad}^c \end{cases} \quad (20)$$

3.4. Construction of the corrected admissible fields

We want to build fields which correspond to the new definition of admissibility. Thus, we seek $s^c \in \mathcal{S}_{ad}^c$ among the pairs \bar{s} admissible in the sense of \mathcal{S}_{ad} which already exist, to which we apply a correction \tilde{s} so that s^c verifies the boundary conditions resulting from heterogeneity. Of course, this correction must belong to \mathcal{S}_{ad} also. One can view each s^c as the superimposition of an incident field \bar{s} and a diffracted field \tilde{s} due to heterogeneity:

$$(w^c, \mathbf{M}^c) = (\bar{w}, \bar{\mathbf{M}}) + (\tilde{w}, \tilde{\mathbf{M}}) \quad (21)$$

Let us give a generic expression of \bar{w} and \tilde{w} :

$$\bar{w}(\underline{X}, P) = \bar{\mathbf{W}}(\underline{X}, P) [\bar{a}(P)] \quad (22)$$

$$\tilde{w}(\underline{X}, P) = \tilde{\mathbf{W}}(\underline{X}, P) [\tilde{a}(P)] \quad (23)$$

The fields s^c of \mathcal{S}_{ad}^c are deduced from those of \mathcal{S}_{ad} :

$$\begin{aligned} & s^c \in \mathcal{S}_{ad}^c \iff \\ & \begin{cases} w^c(\underline{X}, P) = \bar{\mathbf{W}}(\underline{X}, P) + [\mathbf{H}(\underline{X})][\mathbf{C}_r(\bar{\mathbf{W}}(\underline{X}, P))] [\bar{a}(P)] \\ \mathbf{M}^c(\underline{X}, P) = (1 + i\eta) \mathbf{K}_{PS} \mathcal{X}(w^c) \end{cases} \end{aligned} \quad (24)$$

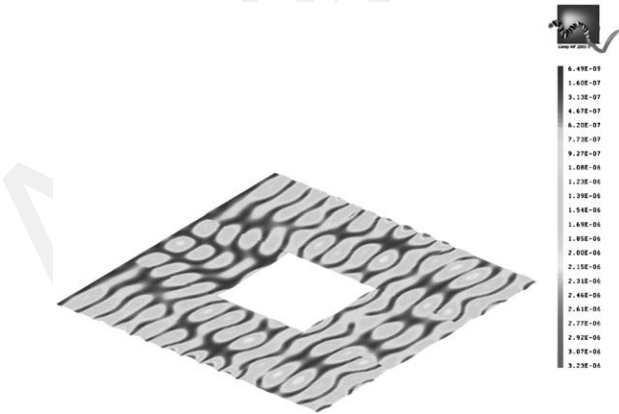


Fig. 7. VTCR solution: $|w|$ (m).

with

$$\widetilde{\mathbf{W}}(\underline{X}, \underline{P}) = [\mathbf{H}(\underline{X})] \quad (25)$$

$$[\underline{\tilde{a}}(\underline{P})] = [C_r(\widetilde{\mathbf{W}}(\underline{X}, \underline{P}))][\underline{\tilde{a}}(\underline{P})] \quad (26)$$

Thus, the correction associated with each \tilde{s} is built individually as a linear combination of functions H , which are additional basic modes; these modes are weighted by coefficients C_r adjusted to the incident field. In the final calculation, each correction function enriches the basic modes and the associated generalized amplitude becomes a new unknown connected to the amplitudes associated with \tilde{s} .

The solution w^{sol} for the heterogeneous structure can be written as

$$w^{sol}(\underline{X}) = \int_{\mathcal{G}} w^c(\underline{X}, \underline{P}) ds \quad (27)$$

3.5. The discretized form of the adjusted VTCT

To derive approximations from the adjusted VTCT, we follow the same reasoning as in Section 2.5. Given a single heterogeneous substructure, the variational formulation and the subspace \mathcal{S}_{ad}^{ch} lead to a constrained system of equations of dimension $r + r^c$ in the complex domain, where r^c designates the number of correction fields:

$$[\mathbf{A}^{ch}] \begin{bmatrix} \underline{\tilde{a}}^h \\ \underline{\tilde{a}}^h \end{bmatrix} = [\underline{L}_d^{ch}] \quad (28)$$

with

$$[\underline{\tilde{a}}^h] = [C_r^h][\underline{\tilde{a}}^h] \quad (29)$$

3.6. Example of the construction of corrected admissible fields

In the case of the reference problem, the correction $\tilde{s}(\underline{X}, \underline{P})$ associated with an incident field of unit amplitude $\tilde{s}(\underline{X}, \underline{P})$ is the solution to:

Problem 6. Find \tilde{s} such that:

$$\begin{cases} \tilde{s} \in \mathcal{S}_{ad} \\ \widetilde{M}_n = -\overline{M}_n|_{\Gamma_0} \\ \widetilde{K}_n = -\overline{K}_n|_{\Gamma_0} \end{cases} \quad (30)$$

In order to take into account the radiating aspect of the wave diffracted due to heterogeneity, we wish to calculate \tilde{s} in polar coordinates with an origin located at the ‘‘center of gravity’’ of the hole. Therefore, we define an ‘‘equivalent circular hole’’ with appropriate boundary conditions. This hole is an imaginary circular contour

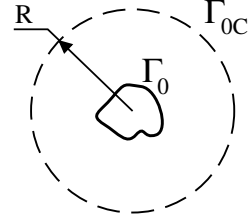


Fig. 8. Definition of the imaginary circular contour.

Γ_{0C} of radius R , centered at the origin of the polar coordinates system and containing the heterogeneous property (Fig. 8).

Then, \tilde{w} is sought in the classical form [43]:

$$\tilde{w}(r, \theta) = \sum_{m \in \mathbb{Z}} (C_{1m} H_m^{(1)}(ikr) + C_{2m} H_m^{(2)}(kr)) e^{im\theta} \quad (31)$$

where $H_m^{(1)}$ and $H_m^{(2)}$ denote the Hankel functions of the first and second kind respectively. Thus, the solution is a combination of an outbound propagating wave and a boundary wave which decreases rapidly. Both waves are emitted by Γ_0 and neither reflects at infinity. The coefficients C_{fm} are determined through the boundary conditions on Γ_0 , which are relocated on Γ_{0C} . These relocated boundary conditions are the restrictions to the contour Γ_{0C} of the solution to Problem 6. In fact, outside Γ_{0C} , the solution to Problem 6 is the same as the solution to Problem 7:

Problem 7. Find \tilde{s} such that:

$$\begin{cases} \tilde{s} \in \mathcal{S}_{ad} \\ \tilde{w} = \tilde{w}^d|_{\Gamma_{0C}} \\ \tilde{w}_{,n} = \tilde{w}_{,n}^d|_{\Gamma_{0C}} \end{cases} \quad (32)$$

Problem 6 is solved by a boundary element method, e.g. [44], or an extension of the VTCT to unbounded media which is currently under development. Then, $\tilde{w}^d|_{\Gamma_{0C}}$ and $\tilde{w}_{,n}^d|_{\Gamma_{0C}}$ are expanded into Fourier series:

$$\begin{cases} \tilde{w}^d(R, \theta) = \sum_{m \in \mathbb{Z}} \tilde{w}^{dm}(R) e^{im\theta} \\ \tilde{w}_{,n}^d(R, \theta) = \sum_{m \in \mathbb{Z}} \tilde{w}_{,n}^{dm}(R) e^{im\theta} \end{cases} \quad (33)$$

$$\text{with } \tilde{w}^{dm}(R) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} \tilde{w}^d(R, \theta) d\theta \quad (34)$$

$$\text{and } \tilde{w}_{,n}^{dm}(R) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} \tilde{w}_{,n}^d(R, \theta) d\theta$$

The coefficients C_{fm} are calculated by inverting linear systems of dimension 2:

$$\begin{cases} C_{1m} H_m^{(1)}(ikr) + C_{2m} H_m^{(2)}(kr) = \tilde{w}^{dm}(R) \\ C_{1m} H_m^{(1)'}(ikr) + C_{2m} H_m^{(2)'}(kr) = \tilde{w}_{,n}^{dm}(R) \end{cases} \quad (35)$$

$$\text{or } [\mathcal{H}_m][\underline{C}_m(\bar{w})] = [\underline{F}_{dm}(\bar{w})] \quad (36)$$

One obtains an approximate solution by truncating the series, i.e.:

$$\tilde{w}(r, \theta) \simeq \sum_{\substack{m \in \mathbb{Z}, |m| \leq M \\ f=1,2}} C_{fm}(\bar{w}) H_{fm}(r, \theta) \quad (37)$$

$$\text{with } H_{fm} = H_m(\alpha_f kr) e^{im\theta} \quad (38)$$

For any incident pair \bar{s} , the correction \tilde{s} is expressed on the same basis of $(2M + 1)$ local fields H_{fm} . Let \underline{C} be the vector of correction coefficients associated with the vector \underline{H} of functions H_{fm} . Then, (37) can be expressed as

$$\tilde{w}(r, \theta) \simeq [\underline{C}(\bar{w})]^T [\underline{H}(r, \theta)] \quad (39)$$

and \underline{C} is the solution to the following linear system, which results from the assembly of systems of the same type as (36):

$$[\mathcal{H}][\underline{C}(\bar{w})] = [\underline{F}_d(\bar{w})] \quad (40)$$

Then, all the \underline{C} s obtained while correcting each incident field are assembled to build the operator \mathbf{C}_r defined in Section 3.5.

In short, a superelement corresponding to a heterogeneous substructure includes a family of additional basic modes weighted in order to verify conditions on the boundary of the discontinuity a priori. The weights for ranges of nondimensional parameters corresponding to each type of local heterogeneity are processed and stored in advance. This approach can be generalized to two or more zones containing local heterogeneity.

4. Reduction method for the VTCR

In the case of minor heterogeneity where the enrichment functions are localized inside the substructure and negligible on its boundary, the size of the heterogeneous discretized model is identical to that of the homogeneous discretized model. In all other cases, for example in Section 3.3 when the superelement of a heterogeneous plate was enriched with cylindrical basis functions, the number of discretized degrees of freedom associated with the substructure becomes large. In such cases, it is time-consuming to calculate all the terms coupling the degrees of freedom of the superelement associated with the finely described substructure with those of the superelements associated with the (normally meshed) neighboring substructures. Therefore, it would be advantageous first to define an equivalent superelement to be assembled with the neighboring superelements. The dynamic behavior on the boundary of this equivalent superelement, seen from the outside, would be the same

as that of the refined superelement in terms of power exchanged.

For this purpose, one associates two models with the heterogeneous substructure S . These models differ in the description of the subspaces in which the solution is sought. The larger model includes the fields associated with the degrees of freedom of the corresponding homogeneous substructure \bar{s} as well as the enrichment fields \tilde{s} .

$$w(\underline{X}, P) = \bar{w} + \tilde{w} = \overline{\mathbf{W}}(\underline{X}, P)[\underline{\bar{a}}(P)] + \widetilde{\mathbf{W}}(\underline{X}, P)[\underline{\tilde{a}}(P)] \quad (41)$$

The equivalent reduced model also includes \bar{s} , but the enrichment fields are replaced by their projections on the fields \bar{s} .

$$w_{eq}(\underline{X}, P) = \bar{w} + \tilde{w}_p = \overline{\mathbf{W}}(\underline{X}, P)[\underline{\bar{a}}(P)] + \mathbf{P}\overline{\mathbf{W}}(\underline{X})[\underline{\bar{a}}(P)] \quad (42)$$

In order to identify the projection operator \mathbf{P} , both models are subjected to the same test load cases: in displacement and slope with $\underline{u} = (w^d, w_n^d)$; in force and moment with $\underline{f} = (K^d, M^d)$ on the boundary ∂S (Fig. 9).

The problem is formulated identically in both cases:

Problem 8. Find s such that:

$$\begin{cases} s \in \mathcal{S}_{ad} \\ A_{\partial S}^{(\underline{u}, \underline{f})}(\delta s, s) - L_{\partial S}^{(\underline{u}, \underline{f})}(\delta s) = 0 \\ \forall \delta s \in \mathcal{S}_{ad} \end{cases} \quad (43)$$

with

$$\begin{aligned} & A_{\partial S}^{(\underline{u}, \underline{f})}(\delta s, s) - L_{\partial S}^{(\underline{u}, \underline{f})}(\delta s) \\ &= \frac{1}{2} \text{Re} \left\{ -i\omega \int_{\partial S} \left(\begin{bmatrix} w \\ w_n \end{bmatrix} - \underline{u} \right) \delta \begin{bmatrix} K_n \\ \underline{n}\mathbf{M}\underline{n} \end{bmatrix} \right. \\ & \quad \left. + \left(\begin{bmatrix} K_n \\ \underline{n}\mathbf{M}\underline{n} \end{bmatrix} - \underline{f} \right) \delta \begin{bmatrix} w \\ w_n \end{bmatrix} \right\} dL \end{aligned} \quad (44)$$

The power exchanged through the boundary between the large problem and the reduced problem is the same regardless of the external loads: this property is used to identify \mathbf{P} . The test load cases are chosen accordingly. These can be, for example, the traces on the boundary of the classical fields of a homogeneous substructure. The

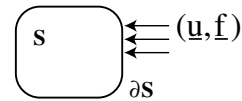


Fig. 9. Test load cases for the reduction process.

relation between \underline{u} and \underline{f} is $(\underline{u}, \underline{f}) \in \partial \mathbf{A}_d$. The power equivalence is expressed as

$$\frac{1}{2} \text{Re} \left\{ -i\omega \int_{\partial S} \left(\begin{bmatrix} \tilde{w} \\ \tilde{w}_n \end{bmatrix} - \begin{bmatrix} \tilde{w}_p \\ \tilde{w}_{np} \end{bmatrix} \right) \delta \underline{f}^{*} + \left(\begin{bmatrix} \tilde{K}_n \\ \underline{n} \tilde{\mathbf{M}} \underline{n} \end{bmatrix} - \begin{bmatrix} \tilde{K}_{np} \\ \underline{n} \tilde{\mathbf{M}}_p \underline{n} \end{bmatrix} \right) \delta \underline{u}^{*} \right\} dL = 0$$

$$\forall (\delta \underline{u}, \delta \underline{f}) \in \partial \mathbf{A}_d$$

$$\tilde{w}(\underline{X}, P) = \tilde{\mathbf{W}}(\underline{X}, P) [\underline{a}(P)]; \quad \tilde{w}_p(\underline{X}, P) = \mathbf{P} \tilde{\mathbf{W}}(\underline{X}) [\underline{a}(P)] \quad (45)$$

In order to reduce the size of the large discretized problem significantly, $[\underline{a}^h]$ is expressed as a function of $[\underline{\tilde{a}}^h]$. The test loads are discretized as follows:

$$\begin{aligned} \underline{u}^h(\underline{X}) &= \mathbf{U}^h(\underline{X}) [a_u^h] \\ \underline{f}^h(\underline{X}) &= \mathbf{F}^h(\underline{X}) [a_f^h] \end{aligned} \quad (46)$$

Then, the linear system of equations associated with the large discretized problem is

$$\begin{bmatrix} \overline{\mathbf{A}}^h & \overline{\mathbf{A}}^h \\ \underline{\mathbf{A}}^h & \underline{\mathbf{A}}^h \end{bmatrix} \begin{bmatrix} \underline{\tilde{a}}^h \\ \underline{\tilde{a}}^h \end{bmatrix} = \begin{bmatrix} \underline{\tilde{L}}^h \\ \underline{\tilde{L}}^h \end{bmatrix} \begin{bmatrix} a_u^h \\ a_f^h \end{bmatrix} \quad (47)$$

The enrichment degrees of freedom are eliminated and the linear system is condensed:

$$\begin{aligned} & \left(\overline{\mathbf{A}}^h - \overline{\mathbf{A}}^h [\underline{\tilde{\mathbf{A}}^h}]^{-1} \overline{\mathbf{A}}^h \right) [\underline{\tilde{a}}^h] \\ &= \left(\underline{\tilde{L}}^h - \underline{\tilde{L}}^h [\underline{\tilde{\mathbf{A}}^h}]^{-1} \underline{\tilde{L}}^h \right) \begin{bmatrix} a_u^h \\ a_f^h \end{bmatrix} \end{aligned} \quad (48)$$

$[\underline{\tilde{a}}^h]$ can be deduced from $[\underline{\tilde{a}}^h]$ through the relation:

$$[\underline{\tilde{a}}^h] = [\underline{\tilde{\mathbf{A}}^h}]^{-1} \left(\underline{\tilde{L}}^h \begin{bmatrix} a_u^h \\ a_f^h \end{bmatrix} - \overline{\mathbf{A}}^h [\underline{\tilde{a}}^h] \right) \quad (49)$$

A simplifying hypothesis consists of neglecting the average power exchanged on the boundary due to the interaction between the enrichment fields and the test loads compared to the average energy exchanged due to the interaction between the enrichment fields and the classical fields. This can be justified by the usually large length of variation of the boundary or transmission conditions compared to the small length of variation of the enrichment fields. Consequently, one can assume a proportional relation between $[\underline{\tilde{a}}^h]$ and $[\underline{\tilde{a}}^h]$:

$$[\underline{\tilde{a}}^h] = \mathbf{C}_{\partial S}^h [\underline{\tilde{a}}^h] \quad (50)$$

Then, the equivalent superelement is described by the fields s_{eq} :

$$\begin{cases} \mathbf{w}_{eq}^h(\underline{X}) = \left(\overline{\mathbf{W}}^h(\underline{X}) + \mathbf{P}^h \overline{\mathbf{W}}^h(\underline{X}) \mathbf{C}_{\partial S}^h \right) [\underline{\tilde{a}}^h] = \mathbf{W}_{eq}^h(\underline{X}) [a_{eq}^h] \\ \mathbf{M}_{eq}^h(\underline{X}) = \mathbf{\Lambda}_{eq}^h(\underline{X}) [a_{eq}^h] \end{cases} \quad (51)$$

Such a superelement is intrinsic and, therefore, can be assembled with other superelements at a cost comparable to that of a homogeneous substructure.

5. Conclusion

The ‘‘variational theory of complex rays’’, initially introduced to calculate the vibrations of slightly damped elastic plates in the medium-frequency range, is a very general approach with a strong mechanical basis. Its application to beam assemblies leads to the exact solution both in traction and in bending. For complex structures, the VTCR predicts the correct vibration level without difficulty, even if the number of substructures is relatively large. Moreover, this tool is flexible enough to deal with locally heterogeneous substructures in an efficient way.

In the near future, the capabilities of the VTCR will be extended in several directions: from plates to shells; from local heterogeneity to more complex cases, e.g. multistiffened substructures. In addition, an appropriate approach will be proposed to deal with wide-band frequency excitation.

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