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A multiscale “Trefftz” computational method for medium-frequency vibrations of assemblies of heterogeneous plates with uncertainties

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A new approach, called the “Variational Theory of Complex Rays”, is being developed for the calculation of the vibrations of slightly damped elastic plates in the medium-frequency range. In this approach, the resolution of a small system of equations which does not result from a refined spatial discretization of the structure leads to the evaluation of effective quantities (deformation energy, vibration amplitude, . . .). This theory has already been validated for assemblies of homogeneous substructures and extended to the case of heterogeneous substructures. In this paper, we propose some improvements to deal with sensitivity analysis. The objective is to evaluate the influence of discontinuities with uncertain positions, sizes or directions on the structural response. A specific partial inversion of the constrained system of equations enables multiple resolutions for combinations of the parameters defining the possible shapes of the discontinuity to be carried out at low cost.

Keywords: Vibrations; Medium-frequency range; Complex rays; Heterogeneous structures; Uncertainties

1. Introduction

The modeling and analysis of the vibrational response of elastic structures are undoubtedly among the key issues which arise in the design of satellites or car chassis. Today, at least concerning modeling and analysis, there remain no major difficulties in the low frequency range, even for complex structures [6]. Concerning high frequencies, computational tools quite distinct from those used for low frequencies do exist, particularly the SEA method [26,27].

Conversely, the modeling and analysis of medium-frequency vibrations, which constitute the topic of this paper, continue to cause certain problems. Difficulties are experienced in extending the SEA method, which is appropriate mostly for high frequencies. In particular, the scale on which space is described is too coarse for medium-

frequency analysis since the spatial aspect disappears almost entirely. The difficulty in attempting to extend low frequency methods to this context is that the lengths of variation of the phenomena being studied are very small compared to the characteristic dimension of the structure. Therefore, the finite element calculations involved would require an unreasonable number of degrees of freedom. Apart from these serious numerical difficulties, the quantities involved in the calculations remain associated with small variation lengths which are not very significant. These are not “effective” quantities, i.e., they are not representative of the vibratory response of the structure on the time and space scales being considered. Therefore, they lead to results which are very sensitive to inaccuracies in the data. Nevertheless, various improved finite element approaches have been studied, in particular in [1,14,5,2,9,15,16,12,13,34–36,24,25,30].

There have been only few works dedicated specifically to the calculation of medium-frequency vibrations. For rods

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and beams, the problem was solved in particular in [7,11,29]. For more complex structures, such as plates or shells, boundary element methods were proposed in [8] and methods based on the use of specific reduced bases can be found in [28,32,33]. Most of these works are closely related to finite element methods and the quantities calculated are still not effective quantities. Therefore, in our opinion, these are not “true” medium-frequency methods. On the contrary, the theory initiated by Belov and Ryback [3] is built upon the concepts of effective energy density and effective power flow. However, despite the improvements proposed in [17], this theory still faces theoretical obstacles.

The approach followed here is the Variational Theory of Complex Rays (VTCR), a predictive tool designed specifically to deal with medium-frequency problems which was first introduced by Ladevèze [19]. This is a true medium-frequency method in the sense that the calculations are performed on effective quantities. Previous works have already validated this strategy for two-dimensional and three-dimensional assemblies of elastic homogeneous plates with low damping [20–22].

However, most industrial structures present some kind of structural heterogeneity, whether intentional (such as portholes or equipment connections) or unintentional (such as cracks). Their vibratory response in the medium-frequency range is very sensitive to these structural discontinuities. Although the VTCR was originally based on the assumption that the structure is an assembly of homogeneous substructures, an extension to the VTCR enabling the designer to take heterogeneity into account has already been proposed [23]. This paper goes one step further and deals with three-dimensional assemblies of homogeneous and heterogeneous substructures in an uncertain context.

2. The basic aspects of the VTCR

2.1. Principle

In the general case of a structure composed of several substructures, the first step of the VTCR consists in associating each substructure with a superelement described by degrees of freedom which correspond to local basic modes. These basic modes, which are defined on two scales and verify the dynamic equations (local equilibrium and constitutive relation) exactly, have a strong mechanical meaning and are called complex rays. They can be divided into families related to interior, edge or corner zones. In the vicinity of a point, the solution is assumed to be properly described locally as the superimposition of an infinite number of such local vibration modes. The unknowns are large-wavelength quantities, each representing the amplitude of the basic mode with which it is associated.

Next, an appropriate variational formulation enables the boundary and transmission conditions to be verified in an average sense. Our choice of reasoning on a superelement’s level requires the formulation to allow approximations within the substructures which are *a priori*

independent of one another. These approximations must not necessarily verify the transmission conditions *a priori*. In practice, the variational formulation associates each superelement with an elementary matrix which represents the interaction among the fields and a right-hand side which represents the interaction between the fields and the boundary conditions. The transmission conditions at the interface are taken into account automatically during the assembly stage.

The last characteristic of the VTCR is that from the calculated discretized amplitudes it retains only effective quantities related to the elastic energy, the kinetic energy, the dissipation work, etc.

2.2. The reference problem

Here, in order to simplify the presentation, the problem will be formulated for an assembly of only two substructures, but this can easily be generalized to an assembly of n substructures. For the purpose of illustration, let us consider two thin, isotropic and elastic Kirchhoff–Love plates. First, we assume that they are homogeneous.

Let us study the steady-state vibrations of this assembly. Classically, all quantities are defined in the complex domain: an amplitude $Q(\underline{X})$ is associated with $Q(\underline{X}) \cdot \exp(i\omega t)$.

Let S_1 and S_2 be two plates, ∂S_1 and ∂S_2 their respective boundaries solicited harmonically at a fixed angular frequency ω and Γ_{12} the perfect interface between S_1 and S_2 . The boundary conditions are as follows, with $l = (1, 2)$: deflection w_l^d on part $\partial_{wad}S_l$ of ∂S_l ; slope w_{nl}^d on $\partial_{wnd}S_l$; bending moment M_l^d on $\partial_{Md}S_l$; finally, Kirchhoff shear K_l^d on $\partial_{Kd}S_l$ (Fig. 1). The quantities of interest are the deflections w_l and the moments \mathbf{M}_l .

For S_l , let us introduce the space \mathcal{S}_{adl} of the displacement–stress pairs $s_l = (w_l, \mathbf{M}_l)$ defined on S_l such that:

$$s_l \in \mathcal{S}_{adl} \iff \begin{cases} s_l \in \mathcal{U} \times \mathcal{S} \text{ (set of finite-energy fields)} \\ \Delta \cdot \Delta w_l - k_l^4 w_l = 0 \text{ on } S_l \\ \mathbf{M}_l = (1 + i\eta_l) \mathbf{K}_{PSl} \mathcal{X}(w_l) \\ \text{for } l = 1, 2 \\ \text{with } k_l^4 = \frac{3\rho_l \omega^2 (1 - \nu_l^2)}{(1 + i\eta_l) E_l h_l^3} \end{cases} \quad (1)$$

where k_l is the wave number and ρ_l , ν_l , η_l , E_l and h_l designate respectively the density, Poisson’s ratio, structural damping coefficient, Young’s modulus and thickness of plate S_l . \mathbf{K}_{PSl} is Hooke’s tensor for plane stress and \mathcal{X} is

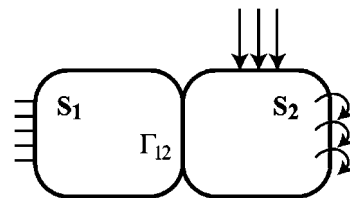


Fig. 1. The reference problem.

the curvature operator. Let us express \mathbf{M}_l and \mathcal{X} in cartesian coordinates:

$$[\mathbf{M}_l] = \begin{bmatrix} M_{lx} \\ M_{ly} \\ M_{lxy} \end{bmatrix} \quad (2)$$

$$[\mathcal{X}] = - \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (3)$$

then, \mathbf{K}_{PSl} is defined by:

$$[\mathbf{K}_{PSl}] = \frac{E_l h_l^3}{12(1 - \nu_l^2)} \begin{bmatrix} 1 & \nu_l & 0 \\ \nu_l & 1 & 0 \\ 0 & 0 & \frac{1-\nu_l}{2} \end{bmatrix} \quad (4)$$

The fields of \mathcal{S}_{adl} are admissible in the sense that they verify both the local equilibrium and the constitutive relation exactly. The reference problem is:

Problem 1. Find s_l such that:

$$\left\{ \begin{array}{l} s_l \in \mathcal{S}_{adl} \\ w_l = w_l^d \quad \text{on } \partial_{wd} S_l \\ w_{l,n_l} = w_{n_l}^d \quad \text{on } \partial_{wnd} S_l \\ M_{ln_l} = M_l^d \quad \text{on } \partial_{Md} S_l \\ K_{ln_l} = K_l^d \quad \text{on } \partial_{Kd} S_l \\ \text{with } l = 1, 2 \\ w_1 = w_2 \quad \text{on } \Gamma_{12} \\ w_{1,n_1} = w_{2,n_2} \quad \text{on } \Gamma_{12} \\ M_{1n_1} + M_{2n_2} = 0 \quad \text{on } \Gamma_{12} \\ K_{1n_1} + K_{2n_2} = 0 \quad \text{on } \Gamma_{12} \\ \text{with } M_{ln_l} = \underline{n}_l \mathbf{M}_l \underline{n}_l \\ \text{and } K_{ln_l} = \underline{n}_l \text{div}[\mathbf{M}_l] + (\underline{t}_l \mathbf{M}_l \underline{n}_l)_{,l}. \end{array} \right. \quad (5)$$

The reference problem has, at the most, one solution if:

- Hooke's tensors are positive definite.
- $\eta_l > 0$.

2.3. The variational formulation associated with the VTCT

The VTCT is primarily a global formulation of the boundary conditions and transmission conditions in terms of both displacements and forces across Γ_{12} . With this understanding, the problem becomes:

Problem 2. Find s_l such that:

$$\left\{ \begin{array}{l} s_l \in \mathcal{S}_{adl} \\ \sum_l (A_{\partial S_l}(\delta s_l, s) - L_{\partial S}(\delta s_l)) + \sum_{\substack{l,m \\ l < m}} C_{\Gamma_{lm}}(\delta s_l, s_l, \delta s_m, s_m) = 0 \\ \forall \delta s_l \in \mathcal{S}_{adl} \\ \text{for } l = 1, 2 \quad m = 1, 2 \end{array} \right. \quad (6)$$

with the boundary conditions:

$$\begin{aligned} & A_{\partial S_l}(\delta s_l, s_l) - L_{\partial S_l}(\delta s_l) \\ &= \text{Re} \left\{ -i\omega \left(- \int_{\partial_{wd} S_l} \delta K_{ln_l} (w_l - w_l^d)^* dL \right. \right. \\ &\quad + \int_{\partial_{wnd} S_l} \delta \underline{n}_l \mathbf{M}_l \underline{n}_l (w_{l,n_l} - w_{l,n_l}^d)^* dL \\ &\quad + \int_{\partial_{Md} S_l} (\underline{n}_l \mathbf{M}_l \underline{n}_l - M_l^d) \delta w_{l,n_l}^* dL \\ &\quad \left. \left. - \int_{\partial_{Kd} S_l} (K_{ln_l} - K_l^d) \delta w_l^* dL \right) \right\} \quad (7) \end{aligned}$$

and with the transmission conditions:

$$\begin{aligned} & C_{\Sigma_{lm}}(\delta s_l, s_l, \delta s_m, s_m) \\ &= \text{Re} \left\{ -i\omega \left(\int_{\Gamma_{lm}} \frac{1}{2} [-\delta(K_{ln_l} - K_{mn_m})(w_l - w_m)^* \right. \right. \\ &\quad + \delta(\underline{n}_l \mathbf{M}_l \underline{n}_l - \underline{n}_m \mathbf{M}_m \underline{n}_m)(w_{l,n_l} - w_{m,n_m})^* \\ &\quad + (\underline{n}_l \mathbf{M}_l \underline{n}_l + \underline{n}_m \mathbf{M}_m \underline{n}_m) \delta(w_{l,n_l} + w_{m,n_m})^* \\ &\quad \left. \left. - (K_{ln_l} + K_{mn_m}) \delta(w_l + w_m)^* \right] dL \right) \right\}. \quad (8) \end{aligned}$$

$\text{Re}[Q]$ designates the real part of Q and Q^* designates the conjugate of Q . Note that the terms which appear in this formulation are powers.

In practice, the VTCT procedure consists of two stages: first, admissible fields are built; then, the boundary conditions are taken into account through the discretization of the variational formulation associated with a set of admissible fields chosen among the elements of \mathcal{S}_{adl} .

2.3.1. Construction of admissible fields

In practice, the first stage of the VTCT consists in building admissible fields. Let us define a subset of \mathcal{S}_{adl} which can contain local interior vibration modes described using complex rays of the n th order. If the damping coefficient η_l ($\eta_l \ll 1$) is small, the basic mode associated with the wave vector \underline{P}_l can be written as:

$$\left\{ \begin{array}{l} w_l(\underline{X}, \underline{P}_l) = e^{i\sqrt{\omega} \underline{P}_l \underline{X}} e^{\frac{\eta_l}{4} \sqrt{\omega} \underline{P}_l \underline{X}} [\mathbf{V}(\underline{X}, \underline{P}_l)] [a_l(\underline{P}_l)] \\ \quad = [\mathbf{W}(\underline{X}, \underline{P}_l)] [a_l(\underline{P}_l)], \\ \mathbf{M}_l(\underline{X}, \underline{P}_l) = (1 + i\eta_l) \mathbf{K}_{PSl} \mathcal{X}(w(\underline{X}, \underline{P}_l)), \\ \underline{P}_l \in \mathcal{C}_{adl}. \end{array} \right.$$

The admissibility relation requires that the locus of the end of \underline{P}_l be a curve \mathcal{C}_{adl} characteristic of the material. $[\mathbf{V}(\underline{X}, \underline{P}_l)] [a_l(\underline{P}_l)]$ is a polynomial expression of degree n in \underline{X} . The small-length part appears explicitly. The unknown generalized amplitudes $a_l(\underline{P}_l)$ are independent of \underline{X} and, therefore, are large-length quantities. The edge and corner modes are built in the same way.

For the interior zone, we take into account all the admissible directions of the rays. The solution w_l^{sol} is written as:

$$w_l^{\text{sol}}(\underline{X}) = \int_{\mathcal{C}_{adl}} w_l(\underline{X}, \underline{P}_l) d\mathcal{C}. \quad (9)$$

The integrals are calculated along the curve \mathcal{C}_{adl} . Clearly, w_l^{sol} depends on the generalized amplitude \underline{a}_l relative to \underline{P}_l defined on \mathcal{C}_{adl} .

2.3.2. The discretized form of the VTCR

In order to derive approximations from the VTCR, one discretizes the curve \mathcal{C}_{adl} using finite elements; for example, $\underline{P}_l \rightarrow \underline{a}_l(\underline{P}_l)$ is assumed to be constant within each element. In other terms, \underline{a}_l^h is associated with $\mathcal{C}_{adl}^{r_l}$ (i.e., \mathcal{C}_{adl} discretized with r_l elements). The subspace \mathcal{S}_{adl}^h of the approximations deduced from \mathcal{S}_{adl} yields a discretized variational formulation leading to a system of equations of dimension $\sum_l r_l$ in the complex domain:

$$[\mathbf{A}^h][\underline{a}^h] = [\underline{L}_d^h], \quad (10)$$

$[\underline{a}^h]$ designates the column of all the amplitudes \underline{a}_l^h .

2.4. Effective quantities

For relatively high frequencies, only effective quantities associated with the generalized amplitudes are meaningful. Indeed, the precise spatial distribution of s_l^h is meaningless from a mechanical point of view. On a given substructure S with a characteristic dimension L , the quantities involved are expressed as $q(\underline{X})$, which can represent, for example, the elastic energy, the kinetic energy or the dissipation work; in the vicinity of any test point \underline{X}_0 , the associated effective quantities are defined on a square patch $S_{\underline{X}_0}$ with side L by:

$$q_{\text{eff}}(\underline{X}_0) = \frac{1}{L^2} \int_{S_{\underline{X}_0}} q(\underline{X}) dS. \quad (11)$$

These effective quantities are large-wavelength quantities. Compared to classical quantities such as the displacements of points, which can vary considerably due to slight variations of input data, they are much less sensitive. Effective quantities can also be defined on the edge of S .

2.5. Example

A three-dimensional assembly of plates is used to model a part of the hull of a ship (Fig. 2). The 13 plates are made of sheet steel. Their mechanical properties are: $E = 210$ GPa; $\eta = 2\%$; $\nu = 0.3$; $\rho = 7800$ kg/m³; $h = 2$ mm. The harmonic excitation is a distributed momentum M_d on the edge $P_{17}P_{18}$: $M_d = 1$ Nm/m; $f = 500$ Hz. All boundaries are simply supported. We used the program CORAY-MF (Complex RAYs for Medium Frequencies) developed by Rouch at LMT-Cachan to compute the VTCR solution. This solution and the solution obtained with finite elements are very similar (Fig. 3).

3. Extension of the VTCR to heterogeneous structures

3.1. The reference problem

Here, we focus on the construction of the superelement associated with an isolated heterogeneous substructure (Fig. 4). Let us consider the steady-state vibrations of a thin, isotropic and elastic Kirchhoff–Love plate with local heterogeneity. Each type of heterogeneity imposes specific conditions on its boundary. In order to test our strategy, let us consider the case of a plate with an arbitrarily-shaped hole. Its boundary Γ_0 is assumed to be free. Therefore, the boundary conditions on Γ_0 are $M^d = 0$ and $K^d = 0$. The notations are the same as before, except for dropping the suffix l in order to simplify the equations. The space \mathcal{S}_{ad} is defined exactly like \mathcal{S}_{adl} in the first part. The reference problem becomes :

Problem 3. Find s such that:

$$\left\{ \begin{array}{l} s \in \mathcal{S}_{ad} \\ w = w^d \quad \text{on } \partial_{wd}S \\ w_{,n} = w_{,n}^d \quad \text{on } \partial_{wnd}S \\ M_n = M^d \quad \text{on } \partial_{Md}S \\ K_n = K^d \quad \text{on } \partial_{Kd}S \\ M_n = M^d = 0 \quad \text{on } \Gamma_0 \\ K_n = K^d = 0 \quad \text{on } \Gamma_0 \\ \text{with } M_n = \underline{n} \mathbf{M} \underline{n} \\ \text{and } K_n = \underline{n} \text{div}[\mathbf{M}] + (\underline{t} \mathbf{M} \underline{n})_{,t} \end{array} \right. \quad (12)$$

3.2. Direct application of the VTCR

3.2.1. The variational formulation associated with the VTCR

The first option is to treat Γ_0 like ∂S . This leads to Problem 4:

Problem 4. Find s such that:

$$\left\{ \begin{array}{l} s \in \mathcal{S}_{ad} \\ (A_{\partial S}(\delta s, s) + A_{\Gamma_0}(\delta s, s)) - (L_{\partial S}(\delta s) + L_{\Gamma_0}(\delta s)) = 0 \\ \forall \delta s \in \mathcal{S}_{ad} \end{array} \right. \quad (13)$$

with:

$$\begin{aligned} & A_{\Gamma_0}(\delta s, s) - L_{\Gamma_0}(\delta s) \\ & = \text{Re} \left\{ -i\omega \left(\int_{\Gamma_0} \underline{n} \mathbf{M} \underline{n} \delta w_{,n}^* dL - \int_{\Gamma_0} K_n \delta w^* dL \right) \right\}. \end{aligned}$$

The term $(A_{\partial S}(\delta s, s) - L_{\partial S}(\delta s))$ was developed while defining Problem 2 in Eq. (7).

3.2.2. Limitations of the direct formalism

While the influence of heterogeneity is taken into account through the terms integrated over Γ_0 , we would like the treatment of local heterogeneity to be distinguished from that of the master structure. Indeed, from a mechanical

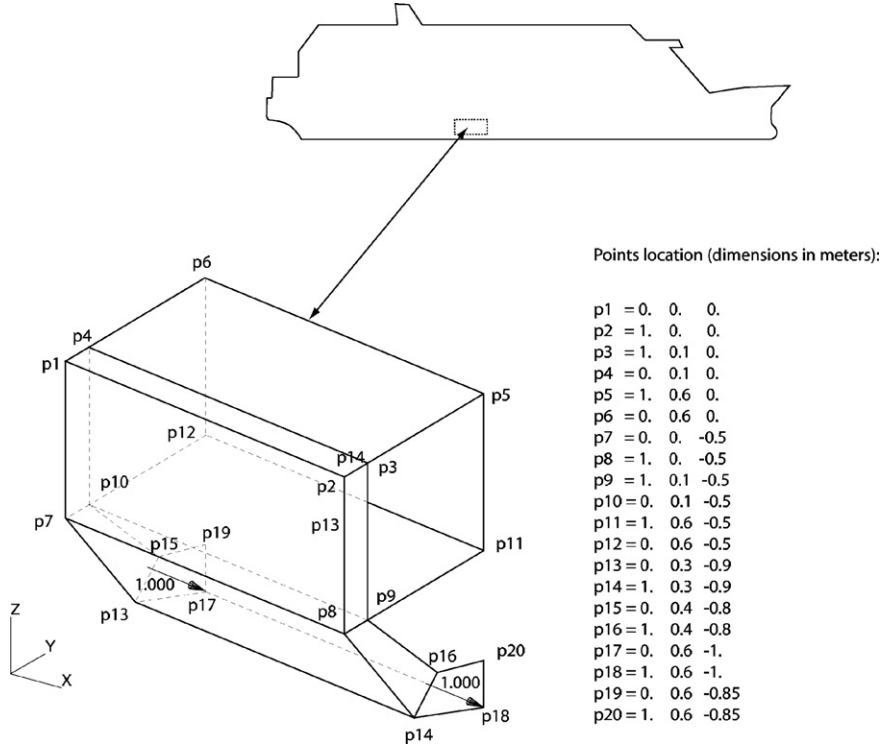


Fig. 2. Definition of the geometry.

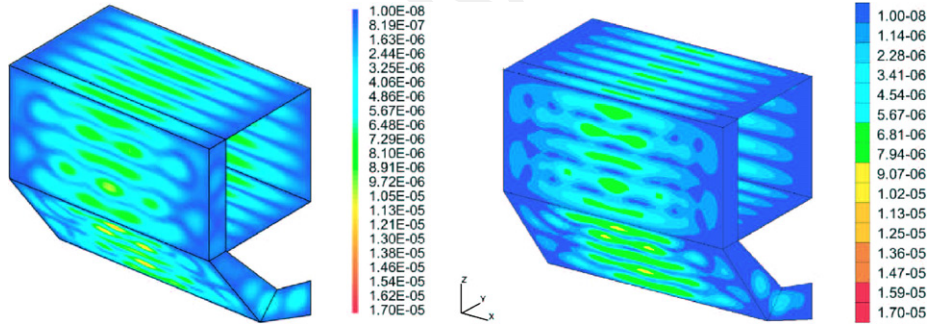


Fig. 3. VTCR solution (left) and FE solution (right): $|w|$ (m).

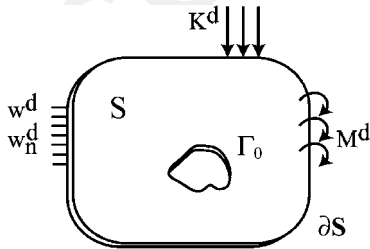


Fig. 4. The reference problem.

point of view, the hole creates only a perturbation of the vibratory response. One should note that this interpretation is not valid for all types of heterogeneity: for example, a long stiffener would behave differently. The case of stiffeners will be developed in a subsequent paper.

3.3. Modification of the VTCR in order to deal with local heterogeneity

3.3.1. The new modified formulation

The modification consists in taking the perturbation due to local heterogeneity into account as soon as the admissible fields of \mathcal{S}_{ad} have been built. The idea is to develop *a priori*, once and for all, families of admissible fields which verify the boundary conditions resulting from each type of heterogeneity. Therefore, let us define the set \mathcal{S}_{ad}^c (i.e., corrected \mathcal{S}_{ad}) of the pairs s such that:

$$s \in \mathcal{S}_{ad}^c \iff \begin{cases} s \in \mathcal{S}_{ad} \\ M_n = M^d = 0 & \text{on } \Gamma_0 \\ K_n = K^d = 0 & \text{on } \Gamma_0 \end{cases} \quad (14)$$

This leads to Problem 5:

Problem 5. Find s such that:

$$\begin{cases} s \in \mathcal{S}_{ad}^c \\ A_{\partial S}(\delta s, s) - L_{\partial S}(\delta s) = 0 \\ \forall \delta s \in \mathcal{S}_{ad}^c \end{cases} \quad (15)$$

3.3.2. Principle of construction of the corrected admissible fields

We want to build fields which correspond to the new definition of admissibility. Thus, we seek $s^c \in \mathcal{S}_{ad}^c$ among the admissible pairs \bar{s} in the sense of \mathcal{S}_{ad} which already exist. \bar{s} is a n th complex ray which results from the linear combination of elementary complex rays $\bar{s}_e = (\bar{w}_e, \bar{\mathbf{M}}_e)$, associated with the amplitudes a_e :

$$\bar{w}(\underline{X}, \underline{P}) = \sum_{e=0}^n \bar{w}_e(\underline{X}, \underline{P}) a_e(\underline{P}). \quad (16)$$

Each \bar{s}_e has a unitary amplitude. The corresponding correction field \tilde{s}_e is calculated so that the sum s_e^c of \bar{s}_e and \tilde{s}_e verifies the boundary conditions resulting from heterogeneity. Of course, this correction must belong to \mathcal{S}_{ad} , too. One can view each s^c as the superimposition of an incident field \bar{s} and a diffracted field \tilde{s} due to heterogeneity:

$$(w_e^c, \mathbf{M}_e^c) = (\bar{w}_e, \bar{\mathbf{M}}_e) + (\tilde{w}_e, \tilde{\mathbf{M}}_e). \quad (17)$$

3.3.3. Practical construction of the corrected admissible fields

For our reference problem, the corrections $\tilde{s}_e(\underline{X}, \underline{P})$ associated with an incident field of unit amplitude $\bar{s}_e(\underline{X}, \underline{P})$ are defined by Problems 6 and 7:

Problem 6. Find \tilde{s}_e such that:

$$\begin{cases} \tilde{s}_e \in \mathcal{S}_{ad}, \\ \tilde{M}_{en} = -\bar{M}_{en}|_{\Gamma_0}, \\ \tilde{K}_{en} = -\bar{K}_{en}|_{\Gamma_0}. \end{cases} \quad (18)$$

In order to take into account the radiating aspect of the wave diffracted due to heterogeneity, we wish to calculate \tilde{s}_e in polar coordinates with an origin located at the ‘‘center of gravity’’ of the hole. In order to do that, we define an ‘‘equivalent circular hole’’ with appropriate boundary conditions: this is an imaginary circular contour Γ_{0c} with radius R , centered at the origin of the polar coordinates system and containing the heterogeneous property (Fig. 5). Then, \tilde{w}_e is sought in the classical form:

$$\tilde{w}_e(r, \theta) = \sum_{\substack{m \in \mathbf{Z} \\ j=1,2}} C_{jm}(\tilde{w}) H_{jm}(r, \theta) \quad (19)$$

$$\text{with } H_{jm} = H_m^{(j)}(\alpha_j kr) e^{im\theta} \quad \text{and } \alpha_1 = i, \alpha_2 = 1, \quad (20)$$

where $H_m^{(1)}$ and $H_m^{(2)}$ denote the Hankel functions of the first and second kind respectively. The solution is a combination of an outbound propagating wave and a boundary wave which decreases rapidly. Both waves are emitted by

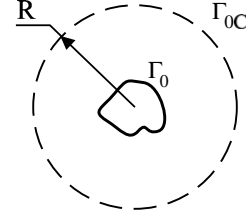


Fig. 5. Definition of the imaginary circular contour.

Γ_0 and neither reflects at infinity. The coefficients C_{jm} are determined through the boundary conditions on Γ_0 , which are relocated on Γ_{0c} . These relocated boundary conditions are the restrictions to the contour Γ_{0c} of the solution to Problem 6. In fact, outside Γ_{0c} , the solution to Problem 6 is the same as the solution to Problem 7.

Problem 7. Find \tilde{s}_e such that:

$$\begin{cases} \tilde{s}_e \in \mathcal{S}_{ad} \\ \tilde{w}_e = \tilde{w}_e^d|_{\Gamma_{0c}} \\ \tilde{w}_{e,n} = \tilde{w}_{en}^d|_{\Gamma_{0c}} \end{cases} \quad (21)$$

Problem 6 can be solved by a boundary element method (e.g., [18]) or an extension of the VTCT to unbounded media which is currently under development. Then, $\tilde{w}_e^d|_{\Gamma_{0c}}$ and $\tilde{w}_{en}^d|_{\Gamma_{0c}}$ are expanded into Fourier series:

$$\begin{cases} \tilde{w}_e^d(R, \theta) = \sum_{m \in \mathbf{Z}} \tilde{w}_e^{dm}(R) e^{im\theta} \\ \tilde{w}_{e,n}^d(R, \theta) = \sum_{m \in \mathbf{Z}} \tilde{w}_{e,n}^{dm}(R) e^{im\theta} \end{cases} \quad (22)$$

$$\text{with } \tilde{w}_e^{dm}(R) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} \tilde{w}_e^d(R, \theta) d\theta, \quad (23)$$

$$\text{and } \tilde{w}_{e,n}^{dm}(R) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} \tilde{w}_{e,n}^d(R, \theta) d\theta.$$

The coefficients C_{jm} are calculated by inverting linear systems of dimension 2:

$$\begin{cases} C_{1m} H_m^{(1)}(ikr) + C_{2m} H_m^{(2)}(kr) = \tilde{w}_e^{dm}(R), \\ C_{1m} H_m^{(1)'}(ikr) + C_{2m} H_m^{(2)'}(kr) = \tilde{w}_{e,n}^{dm}(R). \end{cases} \quad (24)$$

This can be expressed in matrix form by introducing the operator $\mathbf{C}_{\Gamma_0/\Gamma_{0c}}$: it can be considered as a projection operator of the correction on the basis of the functions $\mathbf{H}_{\Gamma_{0c}}$ which are associated with the contour Γ_{0c} :

$$\tilde{w}_e(r, \theta) = [\mathbf{H}_{\Gamma_{0c}}(r, \theta)] [\mathbf{C}_{\Gamma_0/\Gamma_{0c}}(\tilde{w}_e)]. \quad (25)$$

The expression of the field w_e^c , adapted from the elementary complex ray \bar{w}_e , is:

$$w_e^c(\underline{X}, \underline{P}) = \bar{w}_e(\underline{X}, \underline{P}) + [\mathbf{H}_{\Gamma_{0c}}(r, \theta)] [\mathbf{C}_{\Gamma_0/\Gamma_{0c}}(\tilde{w}_e)], \quad (26)$$

and the complete corrected complex ray can be expressed as:

$$w^c(\underline{X}, \underline{P}) = ([\bar{\mathbf{W}}(\underline{X}, \underline{P})] + [\mathbf{H}_{\Gamma_{0c}}(r, \theta)] [\mathbf{C}_{\Gamma_0/\Gamma_{0c}}(\bar{\mathbf{W}})]) [\underline{a}(\underline{P})]. \quad (27)$$

In short, the superelement corresponding to a heterogeneous substructure includes a family of additional basic modes, weighted in order to verify conditions on the

boundary of the discontinuity *a priori*. The weights corresponding to ranges of non-dimensional parameters associated with each type of local heterogeneity are processed and stored in advance in sorts of “libraries”. This approach can be generalized to two or more zones containing local heterogeneities.

3.3.4. Discretized expression of the modified VTTCR

The expression of the solution w^{sol} for the heterogeneous structure is:

$$w^{\text{sol}}(\underline{X}) = \int_{\mathcal{C}_{ad}} w^c(\underline{X}, \underline{P}) d\mathcal{C}. \quad (28)$$

The integral is evaluated over the admissibility domain \mathcal{C}_{ad} of the field \bar{w} . Before discretizing the variational formulation, all the contributions of each correction function to the correction of all the elementary complex rays defined over the admissibility domain are put together. Then, the generalized amplitudes \hat{a} associated with the correction functions are defined. These generalized amplitudes, considered as the new unknowns, are linked with the $\underline{a}(\underline{P})$ by the relation:

$$[\hat{a}] = \int_{\mathcal{C}_{ad}} [\mathbf{C}_{\Gamma_0/\Gamma_{0c}}(\bar{\mathbf{W}}(\underline{X}, \underline{P}))][\underline{a}(\underline{P})] d\mathcal{C}. \quad (29)$$

The discretization is performed as follows: an approximate expression of \bar{w} is obtained by discretizing \mathcal{C}_{ad} according to the process described in Section 2.3.2; an approximate expression of \tilde{w} is obtained by truncating the series of correction functions to $2 \times (2M + 1)$ terms, M being the maximum order of the harmonics kept; finally, the subspace \mathcal{S}_{ad}^{ch} of the approximations deduced from \mathcal{S}_{ad}^c is defined by:

$$s^{ch} \in \mathcal{S}_{ad}^{ch} \iff \begin{cases} w^{ch}(\underline{X}) = [\bar{\mathbf{W}}^h(\underline{X})][\underline{a}^h] + [\mathbf{H}_{\Gamma_{0c}}^h(r, \theta)][\hat{a}^h] \\ \mathbf{M}^{ch}(\underline{X}) = [\bar{\Lambda}^h(\underline{X})][\underline{a}^h] + [\bar{\Sigma}_{\Gamma_{0c}}^h(r, \theta)][\hat{a}^h] \\ [\hat{a}^h] = [\mathbf{D}_{\Gamma_0/\Gamma_{0c}}^h][\underline{a}^h] \end{cases} \quad (30)$$

$\mathbf{D}_{\Gamma_0/\Gamma_{0c}}^h$, linking the generalized amplitudes of uncorrected fields to the ones of correction functions, designates the correction operator. The modified variational formulation defined by Eq. (15) and the subspace \mathcal{S}_{ad}^{ch} defined by Eq. (30) lead to a constrained linear system of equations of dimension $r + r^c$ in the complex domain, where r^c denotes the number of correction fields:

$$\begin{bmatrix} \bar{\mathbf{A}}^{ch} & \tilde{\mathbf{A}}^{ch} \\ \tilde{\mathbf{A}}^{ch} & \tilde{\mathbf{A}}^{ch} \end{bmatrix} \begin{bmatrix} \underline{a}^h \\ \hat{a}^h \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{L}}_d^{ch} \\ \tilde{\mathbf{L}}_d^{ch} \end{bmatrix} \quad (31)$$

$$\text{with } [\hat{a}^h] = [\mathbf{D}_{\Gamma_0/\Gamma_{0c}}^h][\underline{a}^h]$$

This formalism enables us to associate each heterogeneous plate superelement with a stiffness matrix and a right-hand side relative to uncorrected fields and to correction functions which enrich the space of the basis functions. The correction functions are adjusted to the discontinuity through the correction operator $\mathbf{D}_{\Gamma_0/\Gamma_{0c}}$ which links the generalized

amplitudes of the uncorrected fields to those of the correction functions.

The constraint on the generalized amplitudes can be taken into account during the resolution, for example by introducing r^c Lagrange multipliers which form the vector μ . The new linear system to be solved has the dimension $r + 2 \times r^c$ in the complex domain, and its expression is:

$$\begin{bmatrix} \bar{\mathbf{A}}^{ch} & \tilde{\mathbf{A}}^{ch} & -\mathbf{D}_{\Gamma_0/\Gamma_{0c}}^{hT} \\ \tilde{\mathbf{A}}^{ch} & \tilde{\mathbf{A}}^{ch} & \mathbf{I} \\ -\mathbf{D}_{\Gamma_0/\Gamma_{0c}}^h & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{a}^h \\ \hat{a}^h \\ \mu \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{L}}_d^{ch} \\ \tilde{\mathbf{L}}_d^{ch} \\ \mathbf{0} \end{bmatrix}. \quad (32)$$

3.4. Example

This method can be applied, for example, to the problem of a rectangular steel plate with length $L = 1$ m, width $l = 0.8$ m and thickness $h = 0.8$ mm containing a circular hole 0.11 m in diameter at the center. The material's properties are $E = 210$ GPa, $\eta = 0.001$, $\nu = 0.3$ and $\rho = 7800$ kg/m³. The harmonic excitation is a distributed force F_d on one side: $f = 330$ Hz; $F_d = 1$ N/m; the plate is clamped on the opposite side. The modified VTTCR solution was calculated using 84 interior modes, 48 edge modes and eight harmonics of the correction functions (Fig. 6). This solution and the solution obtained using finite elements (Fig. 7) are very similar.

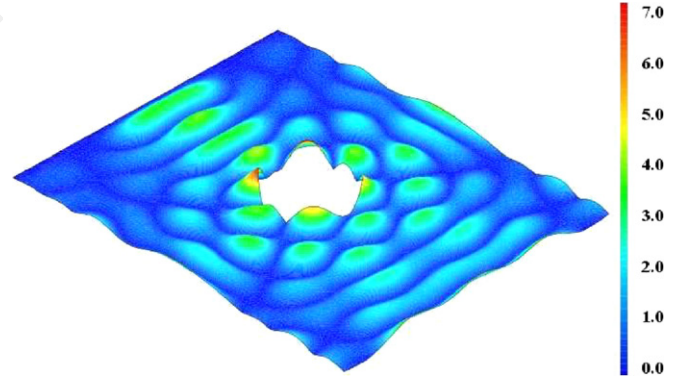


Fig. 6. VTTCR solution: $|w|$ (μm).

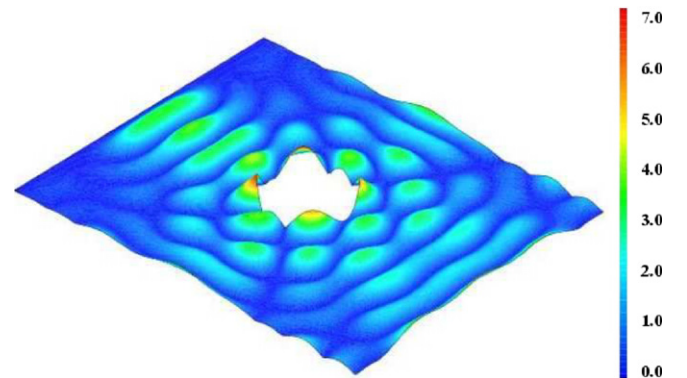


Fig. 7. Finite element solution: $|w|$ (μm).

4. Sensitivity analysis

4.1. Definition of the problem

The objective is to evaluate the influence of discontinuities with uncertain positions, sizes or directions on the structural response. The point of departure is the library of corrected fields defined in Section 3.3.3. This library includes the elements, stored in non-dimensional form for each family of discontinuities (e.g., circular holes), which are necessary to rebuild the specific corrections dedicated to particular members of the family (for example, the hole whose diameter is twice the wavelength of the flexural waves in the plate). These elements, defined in the local coordinate system associated with the equivalent discontinuity, constitute the basis of the correction functions and the projection operator of the correction.

In the case of a perfectly known structure (i.e., in the absence of uncertainties), using the library to calculate the superelement requires the positioning of the equivalent discontinuity's imaginary circular contour in the global coordinate system of the structure: then, the polar coordinate system associated with the correction functions is centered and oriented with respect to the boundaries of the structure.

When the parameters of the discontinuity vary about their nominal values, the imaginary circular contour also varies in the global coordinate system: this may affect its size (e.g., when the diameter of a hole changes), its position (when the center of the hole moves) or its orientation (when the axis of symmetry of an elliptical hole tilts). The expressions of the correction functions in the global coordinate system depend on the evolution of the imaginary contour; consequently, the matrix of the interactions among the basis functions and the right-hand side vector of the interactions between the fields and the boundary conditions also vary. Then, each set of parameters requires a full calculation. Because many very similar problems are considered, the sensitivity analysis is very expensive. It is of paramount importance to choose a well suited and efficient calculation method.

4.2. Principle of the method: use of a single imaginary circular contour

In the framework of structural mechanics, a multiresolution strategy has been developed: it is dedicated to the LATIN method which deals with non-linear evolution problems [4]. The fundamental idea is to exploit the same basis of functions to describe the solution of similar problems.

An analogous principle is introduced here. It consists in using a single imaginary circular contour for all the discontinuities corresponding to the range of uncertain parameters we wish to consider (Fig. 8). The basis functions which are associated with this contour do not change when the parameters vary. The interaction matrix and the right-hand side remain unchanged, which drastically reduces the

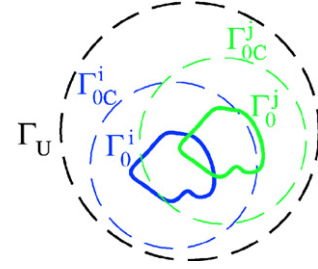


Fig. 8. Definition of the single imaginary contour.

time needed for calculating the matrices defining the super-element of the heterogeneous substructure.

The only change concerns the constraint relations between the generalized amplitudes of the corrected fields and those of the correction functions: these are deduced from the constraint relations which were established when the equivalent discontinuity was defined in order to build the library of corrected fields.

4.3. Practical aspects involved in building a model of uncertain local heterogeneity

The single imaginary contour Γ_U is chosen such that it includes all the imaginary contours associated with the equivalent discontinuities envisaged. The set of parameters relative to each discontinuity is evaluated using, for example, the Monte Carlo method. The origin of the common coordinate system can be chosen such that it coincides with the “center of gravity” of the discontinuity with nominal parameters.

The field s^c (an adjusted version of field \bar{s} for the particular discontinuity marked i), whose contour is Γ_0^i and whose equivalent discontinuity has the contour Γ_{0C}^i , is projected onto the basis functions associated with the common imaginary contour through an operator $\mathbf{C}_{\Gamma_{0C}^i/\Gamma_U}$ defined as follows:

$$w^c(\underline{X}, \underline{P}) = \left([\bar{\mathbf{W}}(\underline{X}, \underline{P})] + [\mathbf{H}_{\Gamma_U}(r, \theta)] \left[\mathbf{C}_{\Gamma_{0C}^i/\Gamma_U}(\mathbf{H}_{\Gamma_{0C}^i}) \right] \times \left[\mathbf{C}_{\Gamma_0^i/\Gamma_{0C}^i}(\bar{\mathbf{W}}) \right] \right) [\underline{a}(\underline{P})]. \quad (33)$$

$\mathbf{C}_{\Gamma_{0C}^i/\Gamma_U}$ can be viewed as an operator which projects the basis functions relative to contour Γ_{0C} onto the basis functions relative to the common contour Γ_U . The process of constructing $\mathbf{C}_{\Gamma_{0C}^i/\Gamma_U}$ is analogous in every respect to the process described in Section 3.3.3 to express the projection of \tilde{w} onto the basis of the functions $\mathbf{H}_{\Gamma_{0C}}$.

When one discretizes the variational formulation, the formalism introduced in Section 3.3.4 leads to the use of generalized amplitudes associated with common correction functions \mathbf{H}_{Γ_U} defined by:

$$\begin{aligned} [\hat{a}] &= \int_{\mathcal{C}_{ad}} \left[\mathbf{C}_{\Gamma_{0C}^i/\Gamma_U}(\mathbf{H}_{\Gamma_{0C}^i}) \right] \left[\mathbf{C}_{\Gamma_0^i/\Gamma_{0C}^i}(\bar{\mathbf{W}}(\underline{X}, \underline{P})) \right] [\underline{a}(\underline{P})] ds \\ &= \left[\mathbf{D}_{\Gamma_0^i/\Gamma_U} \right] [\underline{a}(\underline{P})] \end{aligned} \quad (34)$$

The specificity of the corrections relative to discontinuity i are taken into account through the operator $\mathbf{D}_{\Gamma_0^i/\Gamma_U}$. System (32) can be partially inverted so that $\underline{\mu}$ and $\mathbf{D}_{\Gamma_0^i/\Gamma_U}$ remain parameters (Schur method):

$$\begin{bmatrix} \underline{d}^h \\ \underline{\hat{q}}^h \end{bmatrix} = \begin{bmatrix} \bar{\bar{\mathbf{A}}}^{ch} & \bar{\bar{\mathbf{A}}}^{ch} \\ \bar{\bar{\mathbf{A}}}^{ch} & \bar{\bar{\mathbf{A}}}^{ch} \end{bmatrix}^{-1} \left(\begin{bmatrix} \bar{\bar{L}}_d^{ch} \\ \bar{\bar{L}}_d^{ch} \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_{\Gamma_0^i/\Gamma_{0c}}^{hT} \\ \mathbf{I} \end{bmatrix} [\underline{\mu}] \right) \quad (35)$$

with

$$[\underline{\mu}] = [\mathbf{M}]^{-1}[\mathbf{V}] \quad (36)$$

where

$$[\mathbf{M}] = \begin{bmatrix} -\mathbf{D}_{\Gamma_0^i/\Gamma_{0c}}^h & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\bar{\mathbf{A}}}^{ch} & \bar{\bar{\mathbf{A}}}^{ch} \\ \bar{\bar{\mathbf{A}}}^{ch} & \bar{\bar{\mathbf{A}}}^{ch} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{D}_{\Gamma_0^i/\Gamma_{0c}}^{hT} \\ \mathbf{I} \end{bmatrix} \quad (37)$$

and

$$[\mathbf{V}] = \begin{bmatrix} -\mathbf{D}_{\Gamma_0^i/\Gamma_{0c}}^h & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\bar{\mathbf{A}}}^{ch} & \bar{\bar{\mathbf{A}}}^{ch} \\ \bar{\bar{\mathbf{A}}}^{ch} & \bar{\bar{\mathbf{A}}}^{ch} \end{bmatrix}^{-1} \begin{bmatrix} \bar{\bar{L}}_d^{ch} \\ \bar{\bar{L}}_d^{ch} \end{bmatrix}. \quad (38)$$

From here on, for each discontinuity, the only calculation consists in building the operator $\mathbf{D}_{\Gamma_0^i/\Gamma_U}$ using the data previously stored in the library: thus, the generalized amplitudes are obtained almost instantaneously through the inversion of matrix \mathbf{M} which is cheap compared to the inversion of the whole system (32).

The formalism associated with the parametric model of local heterogeneity lays the foundations of a multiresolution strategy which is fitted for the VTCR. Coupled with a response surface technique [10,31], it makes it possible to deal with many kinds of problems linked to the study of the sensitivity of the response to a variation of heterogeneity parameters: parametric analysis, Monte Carlo simulation, reliability analysis.

4.4. Example

Let us consider a plate with a hole, using the same dimensions and material characteristics as in the example of Section 3.4. We wish to know the influence of a variation in the position of the hole (due, e.g., to manufacturing inaccuracies) on the vibrational response (Fig. 9).

For the sake of simplicity, let us consider only the possibility of a offset y parallel to the width of the plate. The frequency of the harmonic excitation is $f = 277$ Hz; the diameter of the hole is $D = 0.11$ m (the ratio between D and the wavelength λ is 0.58). Let us assume that y varies about its nominal value according to a normal centered Gaussian distribution with standard deviation $\sigma = 0.03$ m. A sampling of 50000 possible values of y is obtained through a Monte Carlo draw (\bar{y} designates the reduced variable associated with y). Distribution tails are truncated at $\pm 3\sigma$.

First, the response curve giving the global effective displacement $|w|$ as a function of y is calculated using the

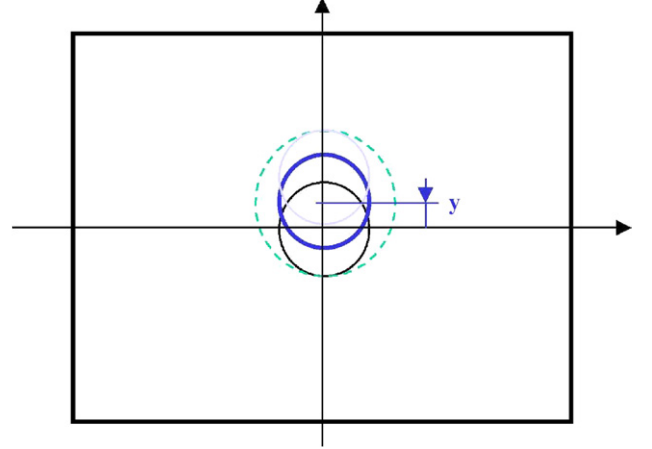


Fig. 9. Variation in the position of the hole.

multiple resolution process we just defined (Fig. 10). It is an interpolation function built on a small number of values of the diameter of the hole. This number, of the order of a few tens, is considered to be enough when the precision of the interpolated global effective displacement does not improve significantly by further refining.

Fig. 11 shows the probability density function $f_{|w|}$ associated with $|w|$. It reveals three peaks of values of $|w|$ which are more probable than others around $2.5 \mu\text{m}$, $3.75 \mu\text{m}$ and $8.75 \mu\text{m}$. These values correspond to local extrema of $|w|$, hence a possible accumulation of the samples of $f_{\bar{y}}$ associated with these values (the maximum of $|w(y)|$ at $17.5 \mu\text{m}$ is reached only for values situated in the tail of $f_{\bar{y}}$, that is why the accumulation effect does not happen there). In other respects, there is no value below $2.5 \mu\text{m}$ and there is very little chance of getting a global effective displacement above $8.75 \mu\text{m}$.

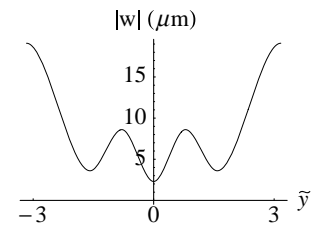


Fig. 10. Response curve of $|w|$ to an offset of the hole.

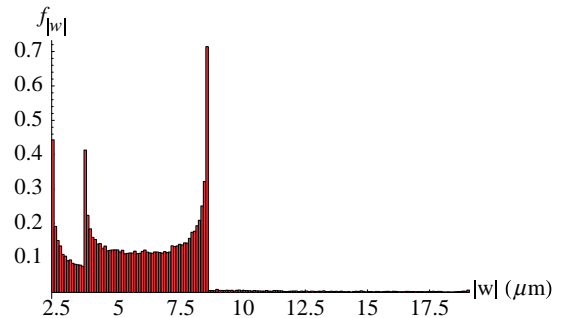


Fig. 11. Approximate probability density function associated with $|w|$.

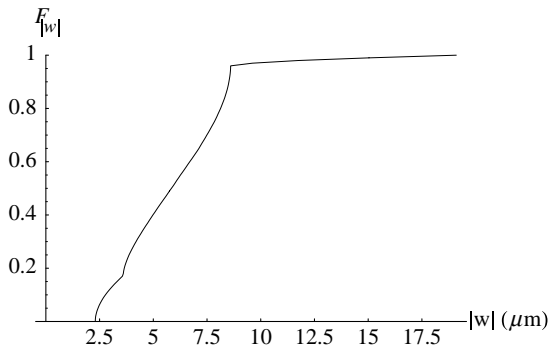


Fig. 12. Approximate cumulative density function associated with $|w|$.

The cumulative distribution function $F_{|w|}$ associated with $|w|$ (Fig. 12) makes it possible to evaluate the probability of exceeding a critical threshold: then only 2% of the values of $|w|$ are above $8.75 \mu\text{m}$.

5. Conclusion

The VTCR initially introduced to calculate the vibrations of slightly damped elastic plates in the medium-frequency range is a very general approach with a strong mechanical basis. This tool is not only efficient for assemblies of homogeneous plates, but also flexible enough to deal with locally heterogeneous substructures economically. Moreover, using an appropriate formalism, the designer can even take uncertainty into account in the context of a sensitivity analysis.

In the near future, the enriched superelement and the sensitivity analysis procedure will be implemented in the program CORAY-MF for more general applications. Afterwards, the capabilities of the VTCR will be extended from local heterogeneity to other complex cases, such as multi-stiffened substructures.

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