Reduced models for modal analysis of fluid–structure systems taking into account compressibility and gravity effects

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It is proposed to investigate and discuss, from *theoretical point of view*, various reduced order variational formulations for modal analysis of linear vibrations of bounded fluid–structure systems with free surface, taking into account possible gravity/compressibility interactions, in a *non-homogeneous* fluid situation.

Those formulations are based on substructural synthesis techniques and various modal reduction schemes are then obtained.

In a previous paper, various original formulations have been exhibited associated to numerical validation. The present paper starts from those formulations in order to construct theoretically various reduced order models.

Keywords: Fluid–structure interactions; Gravity; Compressibility; Acoustic; Sloshing; Modal reduction; Substructuring

1. Introduction

Formulation, numerical analysis and computer implementation of fluid–structure interaction problems have attracted many researchers attention. In the context of linear vibration analysis of coupled internal fluid–structure interactions, it is customary to consider two distinct situations: linear vibrations of an elastic structure completely filled with a *homogeneous compressible* gas or liquid (without gravity effects) and linear vibrations of an elastic structure containing an *homogeneous incompressible* liquid with free surface effects due to gravity. Most of studies dealing with coupled fluid–structure interaction thus fall into these two classes according to the frequency domain of interest.

We consider here the case of a *non-homogeneous*, compressible liquid, with internal and free-surface gravity effects. Those effects must be quantified for instance for vibrations analysis of cryogenic fluids in liquid-propelled launchers. To the authors knowledge, this situation has only been analysed within the assumption that the internal gravity waves are negligible [7,12,14,16,17,19]. In Refs. [1,2], we have established the basic equations of the problem together with original appropriate variational formulations. The objective of this theoretical paper is to derive reduced order models in the non-homogeneous case from theoretical point of view. The numerical validation of those models will be the purpose of a forthcoming paper.

For that purpose, substructure synthesis techniques are used (see for instance, in structural dynamics, [6,9,15,3]).

Component mode analysis have also been established for fluid–structure vibration problems (see for instance [11,13]). In the approach presented in this paper, generalized coordinates are introduced to represent the liquid behaviour. The

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kinematically admissible space relative to the liquid domain (considered as *non-homogeneous*, compressible and subjected to gravity effects) is then decomposed into the direct sum of three subspaces: acoustic, sloshing and internal gravity waves subspaces. This decomposition of the solution is then substituted in different variational formulations and reduced symmetric matrix models are then obtained.

The outline of the paper is as follows: in Section 2, we introduce the basic equations and the associated variational principle relative to the general case of an inviscid *non-homogeneous compressible* liquid. In the following two sections we distinguish the cases of incompressible and compressible liquid: in Section 3, we state a modal reduction scheme for the case of a non-homogeneous *incompressible hydroelastic–sloshing* model and in Section 4, the case of a *compressible hydroelastic–sloshing* model is derived. Section 5 concerns modal reduction of the *homogeneous compressible hydroelastic–sloshing* case.

2. General case of an inviscid non-homogeneous liquid. Basic equations and variational formulation

The purpose of this section is to derive the variational formulation associated with a fluid–structure coupled problem subjected to gravity field [1]. The situation analysed in this section is that of a general *non-homogeneous* liquid. Local equations relative to this problem have already been derived in a earlier paper [2] through linearized Euler and mass conservation equations.

We adopt here a more direct approach: the variational formulation is stated through the classical principle of virtual work and the local equations are then obtained through linearization of the Piola–Lagrange stress tensor around a prestressed reference configuration.

2.1. Liquid subjected to a prescribed normal displacement

Let us consider a liquid domain of volume Ω_F. The boundary of Ω_F is supposed to satisfy

$$\partial\Omega_F = \Gamma \cup \Sigma \tag{1}$$

with $\Gamma \cap \Sigma = \emptyset$.

Γ denotes the liquid free surface and Σ denotes the fluid–structure interface. The outer unit normal vector to $\partial\Omega_F$ is denoted by $\mathbf{n}$ on the fluid–structure interface Σ and by $\mathbf{i}_z$ on the liquid free surface Γ (Fig. 1).

The weak variational formulation describing the response of the fluid to a prescribed normal displacement on the fluid–structure interface Σ is obtained through the principle of virtual work as follows:

$$\int_{\Omega_F} \mathrm{Div}\,\Theta \cdot \delta\mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta\mathbf{u}_F = -\omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta\mathbf{u}_F, \tag{2}$$

where Θ refers to the spherical Piola–Lagrange stress tensor defined by

$$\Theta_{(M,t)} = -JP_{(M',t)}\,\mathrm{Id}\,F^{-\mathrm{T}}, \tag{3}$$

$P_{(M',t)}$ denotes the instantaneous pressure in the actual configuration, F denotes the gradient of the transformation $M \to M'$, J is its jacobian and Id is the identity matrix.

In Eq. (2), $\delta\mathbf{u}_F$ denotes the virtual variation of the fluid displacement vector field $\mathbf{u}_F$, ρ_F^0 and $\mathbf{g}$ denote respectively the mass density and the gravity field (supposed constant) whereas ω refers to the angular frequency of vibrations.

In order to linearize Θ, the following quantities are considered:

- the eulerian pressure fluctuation

$$p(M,t) = P_{(M,t)} - P_{(M)}^0, \tag{4}$$

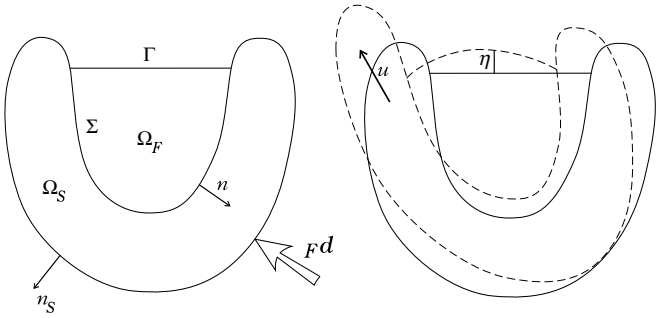

Fig. 1. Fluid–structure coupled system.

- the lagrangian pressure fluctuation

$$p_{\mathrm{L}}(M,t) = P_{(M',t)} - P^0_{(M)}, \tag{5}$$

- the eulerian density fluctuation

$$\rho(M,t) = \rho_F(M,t) - \rho^0_F(M), \tag{6}$$

- the lagrangian density fluctuation

$$\rho_{\mathrm{L}}(M,t) = \rho_F(M',t) - \rho^0_F(M), \tag{7}$$

where $P^0_{(M)}$ is the liquid hydrostatic pressure and $\rho^0_F(M)$ is the liquid density in the reference configuration.

It can be shown that these linearized quantities satisfy the following relations [2,11]:

$$\begin{cases} p_{\mathrm{L}} - p = \rho^0_F \mathbf{g}\cdot\mathbf{u}_F & \forall M \in \Omega_F, \\ \rho_{\mathrm{L}} - \rho = (\mathbf{u}_F\cdot\mathbf{i}_z)\dfrac{\partial\rho^0_F}{\partial z} & \forall M \in \Omega_f, \\ p_{\mathrm{L}} = c^2\rho_{\mathrm{L}} & \forall M \in \Omega_F, \\ \rho_{\mathrm{L}} = -\rho^0_F \operatorname{Div}\mathbf{u}_F & \forall M \in \Omega_F. \end{cases} \tag{8}$$

Notice that in the particular case of a *homogeneous weightless* (i.e. gravity neglected) *compressible* fluid or *homogeneous heavy incompressible* liquid, no distinction is to be made between p and p_{L} in Eqs. (4) and (5). Otherwise these two state variables are to be distinguished and combination of Eqs. (8) yields the following constitutive relation:

$$\frac{p}{c^2} + \rho^0_F \operatorname{Div}\mathbf{u}_F - \frac{\rho^0_F g}{c^2}(\mathbf{u}_F\cdot\mathbf{i}_z) = 0. \tag{9}$$

Considering now the first order Taylor expansion of J and $F^{-\mathrm{T}}$, it can be shown [8] that the Piola–Lagrange stress tensor defined in Eq. (2) writes:

$$\Theta = -p_{\mathrm{L}}\,\mathrm{Id} - P^0\,\mathrm{Id} + P^0\nabla^{\mathrm{T}}\mathbf{u}_F - P^0\,\mathrm{Id}\operatorname{Div}\mathbf{u}_F, \tag{10}$$

hence

$$\operatorname{Div}\Theta = -\nabla p_{\mathrm{L}} - \nabla P^0 - \nabla P^0 \operatorname{Div}\mathbf{u}_F + \nabla^{\mathrm{T}}\mathbf{u}_F\nabla P^0. \tag{11}$$

Eq. (2) then writes:

$$\int_{\Omega_F}\nabla p_{\mathrm{L}}\cdot\delta\mathbf{u}_F + \int_{\Omega_F}\nabla P^0\cdot\delta\mathbf{u}_F + \int_{\Omega_F}\nabla P^0\cdot\delta\mathbf{u}_F\operatorname{Div}\mathbf{u}_F - \int_{\Omega_F}\delta\mathbf{u}_F\cdot\nabla^{\mathrm{T}}\mathbf{u}_F\nabla P^0 - \int_{\Omega_F}\rho^0_F\mathbf{g}\cdot\delta\mathbf{u}_F = \omega^2\int_{\Omega_F}\rho^0_F\mathbf{u}_F\cdot\delta\mathbf{u}_F. \tag{12}$$

Moreover, the hydrostatic equation writes

$$\nabla P^0 = \rho^0_F\mathbf{g} \quad \text{in } \Omega_F \tag{13}$$

with its variational form

$$\int_{\Omega_F}\nabla P^0\cdot\delta\mathbf{u}_F = \int_{\Omega_F}\rho^0_F\mathbf{g}\cdot\delta\mathbf{u}_F. \tag{14}$$

Integrating by parts the first integral of Eq. (12), taking into account the constitutive Eqs. $(8)_1$ and $(8)_2$ and combining with Eq. (13), we have finally:

$$\begin{aligned} &\int_{\Omega_F}\rho^0_F c^2\operatorname{Div}\mathbf{u}_F\operatorname{Div}\delta\mathbf{u}_F + \int_{\Omega_F}\rho^0_F\mathbf{g}\cdot\delta\mathbf{u}_F\operatorname{Div}\mathbf{u}_F + \int_{\Omega_F}\rho^0_F\mathbf{g}\cdot\mathbf{u}_F\operatorname{Div}\delta\mathbf{u}_F - \int_{\Omega_F} g\frac{\partial\rho^0_F}{\partial z}(\mathbf{u}_F\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) \\ &\quad + \int_{\Gamma}\rho^0_F g(\mathbf{u}_F\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) + \int_{\Sigma}p\mathbf{n}\cdot\delta\mathbf{u}_F = \omega^2\int_{\Omega_F}\rho^0_F\mathbf{u}_F\cdot\delta\mathbf{u}_F. \end{aligned} \tag{15}$$

In Eq. (15), the free surface condition on Γ has been used as follows:

$$p = -\rho^0_F\mathbf{g}\cdot\mathbf{u}_F \quad \text{on } \Gamma. \tag{16}$$

Let us now introduce the following admissible spaces:

$$\mathscr{C}_F = \{\mathbf{u}_F \text{ regular in } \Omega_F \text{ such that: } \mathbf{u}_F\cdot\mathbf{n} = u_N \text{ on } \Sigma\} \tag{17}$$

and

$$\mathscr{C}_F^* = \{\mathbf{u}_F \text{ regular in } \Omega_F \text{ such that: } \mathbf{u}_F \cdot \mathbf{n} = 0 \text{ on } \Sigma\}, \tag{18}$$

where u_N denotes a prescribed normal displacement field of the fluid–structure interface Σ.

We can then state the following symmetric variational formulation in terms of $\mathbf{u}_F$:

$$\begin{cases} \text{for given } \omega \in \mathbb{R}^+, \text{ find } \mathbf{u}_F \in \mathscr{C}_F, \text{ such that } \forall \delta\mathbf{u}_F \in \mathscr{C}_F^*: \\ \quad \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div}\mathbf{u}_F \operatorname{Div}\delta\mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\delta\mathbf{u}_F \operatorname{Div}\mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\mathbf{u}_F \operatorname{Div}\delta\mathbf{u}_F \\ \quad - \displaystyle\int_{\Omega_F} g\frac{\partial\rho_F^0}{\partial z}(\mathbf{u}_F\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) + \int_{\Gamma} \rho_F^0 g(\mathbf{u}_F\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) \\ \quad\quad = \omega^2 \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{u}_F\cdot\delta\mathbf{u}_F. \end{cases} \tag{19}$$

The local boundary value problem is then derived from the previous variational formulation by using Green formula:

$$\begin{cases} \nabla p(\mathbf{u}_F) + \rho_F^0 \mathbf{g} \operatorname{Div}\mathbf{u}_F + \dfrac{\partial\rho_F^0}{\partial z}\mathbf{g}(\mathbf{u}_F\cdot\mathbf{i}_z) = \rho_F^0\omega^2\mathbf{u}_F & (\Omega_F), \\ p(\mathbf{u}_F) = -\rho_F^0 \mathbf{g}\cdot\mathbf{u}_F & (\Gamma), \\ \mathbf{u}_F\cdot\mathbf{n} = u_N & (\Sigma) \end{cases} \tag{20}$$

in which $p(\mathbf{u}_F)$ is defined by Eq. (9).

It is worth noting that formulations defined by Eqs. (19) and (20) contain *internal* gravity terms (contributions of gravity inside the domain Ω_F) which are often neglected in literature [10].

Namely, the situation where compressibility effects act in the fluid domain and gravity effects are confined on the free surface only can be considered as an approximated modeling. Its mechanical description can be stated by considering the limit case of constant mass density in Eqs. (19) and (20). We thus have the following variational formulation:

$$\begin{cases} \text{for given } \omega \in \mathbb{R}^+, \text{ find } \mathbf{u}_F \in \mathscr{C}_F, \text{ such that } \forall \delta\mathbf{u}_F \in \mathscr{C}_F^*: \\ \quad \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div}\mathbf{u}_F \operatorname{Div}\delta\mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\delta\mathbf{u}_F \operatorname{Div}\mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\mathbf{u}_F \operatorname{Div}\delta\mathbf{u}_F \\ \quad + \displaystyle\int_{\Gamma} \rho_F^0 g(\mathbf{u}_F\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) - \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F\cdot\delta\mathbf{u}_F = 0, \end{cases} \tag{21}$$

whereas the associated local boundary value problem writes:

$$\begin{cases} \nabla p(\mathbf{u}_F) + \rho_F^0 \mathbf{g} \operatorname{Div}\mathbf{u}_F = \rho_F^0\omega^2\mathbf{u}_F & (\Omega_F), \\ p(\mathbf{u}_F) = -\rho_F^0 \mathbf{g}\cdot\mathbf{u}_F & (\Gamma), \\ \mathbf{u}_F\cdot\mathbf{n} = u_N & (\Sigma). \end{cases} \tag{22}$$

2.2. Structure subjected to a liquid pressure loading

The unknown displacement field in the structure domain is denoted as $\mathbf{u}_S$, the associated linearized deformation tensor as $\varepsilon_{ij}(\mathbf{u}_S)$ and the corresponding stress tensor $\sigma_{ij}(\mathbf{u}_S)$. We also denote as ρ_S^0 the constant mass density of the structure at equilibrium and $\delta\mathbf{u}_S$ the test function associated to $\mathbf{u}_S$, belonging to the kinematically admissible space $\mathscr{C}_u^S$ of regular functions in Ω_S.

The weak variational formulation describing the response of the structure Ω_S to given harmonic forces of amplitude $\mathbf{F}$ on Σ_d, and to fluid pressure field p acting on Σ is written as follows:

$$\begin{cases} \text{for given } \mathbf{F} \text{ and } \omega \in \mathbb{R}^+, \text{ find } \mathbf{u}_S \in \mathscr{C}_u^S, \text{ such that } \forall \delta\mathbf{u}_S \in \mathscr{C}_u^S: \\ \quad \widehat{K}(\mathbf{u}_S, \delta\mathbf{u}_S) - \omega^2 \displaystyle\int_{\Omega_S} \rho_S^0 \mathbf{u}_S\cdot\delta\mathbf{u}_S - \int_{\Sigma} p\,\delta\mathbf{u}_S\cdot\mathbf{n} = \int_{\Sigma_d} \mathbf{F}\cdot\delta\mathbf{u}_S, \end{cases} \tag{23}$$

$\widehat{K}(\mathbf{u}_S, \delta\mathbf{u}_S)$ denotes the *elastogravity operator*. This operator takes into account the effect of gravity forces on the fluid–structure interface and is defined as

$$\widehat{K}(\mathbf{u}_S, \delta\mathbf{u}_S) = K(\mathbf{u}_S, \delta\mathbf{u}_S) + K_G(\mathbf{u}_S, \delta\mathbf{u}_S) + K_\Sigma(\mathbf{u}_S, \delta\mathbf{u}_S), \tag{24}$$

where

- $K(\mathbf{u}_S, \delta\mathbf{u}_S) = \int_{\Omega_S} \sigma_{ij}(\mathbf{u}_S)\varepsilon_{ij}(\delta\mathbf{u}_S)$ is the usual mechanical elastic stiffness,
- $K_G(\mathbf{u}_S, \delta\mathbf{u}_S) = \int_{\Omega_S} \sigma^0_{ij} u_{m,i}\,\delta u_{m,j}$ refers to the classical structural geometric prestress symmetric bilinear form in which σ^0_{ij} denotes the prestress tensor,
- $K_\Sigma(\mathbf{u}_S, \delta\mathbf{u}_S) = -\rho^0_F g \int_\Sigma [z\mathbf{n}_1(\mathbf{u}_S)\cdot\delta\mathbf{u}_S + (\mathbf{i}_z\cdot\mathbf{u}_S)(\delta\mathbf{u}_S\cdot\mathbf{n})]$ represents prestress symmetric bilinear form due to rotation of the external normal $\mathbf{n}$. z is the vertical coordinate. The outward normal vector $\mathbf{n}$ is dependent of the displacement $\mathbf{u}_S$ and $\mathbf{n}_1(\mathbf{u}_S)$ is its first order Taylor-expansion with respect to $\mathbf{u}_S$.

For detailed derivation of these operators, we refer the reader to [11,18].

2.3. The coupled problem

To write the variational formulation associated with the coupled problem subjected to given surface forces $\mathbf{F}$, consider the following admissible spaces:

- $\mathscr{C}^S_u = \{\mathbf{u}_S \text{ regular in } \Omega_S\}$, (25)
- $\mathscr{C}^F_u = \{\mathbf{u}_F \text{ regular in } \Omega_F\}$, (26)
- $\mathscr{C}_p = \{p \text{ regular in } \Omega_F\}$, (27)
- $\mathscr{C}^{S,F}_u = \{\mathbf{u}_S \in \mathscr{C}^S_u, \mathbf{u}_F \in \mathscr{C}^F_u : \mathbf{u}_S\cdot\mathbf{n} = \mathbf{u}_F\cdot\mathbf{n} \text{ on } \Sigma\}$. (28)

The variational formulation of the coupled problem is obtained by considering both Eqs. (21) and (23):

$$\left\{\begin{array}{l} \text{given } \mathbf{F} \text{ and } \omega\in\mathbb{R}^+,\ \text{find}(\mathbf{u}_S,\mathbf{u}_F)\in\mathscr{C}^{S,F}_u \text{ such that } \forall(\delta\mathbf{u}_S,\delta\mathbf{u}_F)\in\mathscr{C}^{S,F}_u: \\ \widehat{K}(\mathbf{u}_S,\delta\mathbf{u}_S) + \displaystyle\int_{\Omega_F}\rho^0_F c^2\,\mathrm{Div}\,\mathbf{u}_F\,\mathrm{Div}\,\delta\mathbf{u}_F - \int_{\Omega_F} g\frac{\partial\rho^0_F}{\partial z}(\mathbf{u}\cdot\mathbf{i}_z)(\delta\mathbf{u}\cdot\mathbf{i}_z) \\ + \displaystyle\int_\Gamma \rho^0_F g(\mathbf{u}_F\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) + \int_{\Omega_F}\rho^0_F\mathbf{g}\cdot\mathbf{u}_F\,\mathrm{Div}\,\delta\mathbf{u}_F \\ + \displaystyle\int_{\Omega_F}\rho^0_F\mathbf{g}\cdot\delta\mathbf{u}_F\,\mathrm{Div}\,\mathbf{u}_F - \omega^2\left\{\int_{\Omega_S}\rho^0_S\mathbf{u}_S\cdot\delta\mathbf{u}_S + \int_{\Omega_F}\rho^0_F\mathbf{u}_F\cdot\delta\mathbf{u}_F\right\} = \int_{\Sigma_d}\mathbf{F}\cdot\delta\mathbf{u}_S. \end{array}\right. \tag{29}$$

The kinematic compatibility condition between $\mathbf{u}_S$ and $\mathbf{u}_F$ on Σ may be relaxed by applying the Lagrange multipliers method. It can be shown that the Lagrange multiplier associated with this condition represents the pressure so that the following three-fields variational equation is obtained:

$$\left\{\begin{array}{l} \text{given } \mathbf{F} \text{ and } \omega\in\mathbb{R}^+,\ \text{find}(\mathbf{u}_S\in\mathscr{C}^S_u,\mathbf{u}_F\in\mathscr{C}^F_u,p\in\mathscr{C}_p) \text{ such that} \\ \forall(\delta\mathbf{u}_S,\delta\mathbf{u}_F;\delta p)\in\mathscr{C}^S_u\times\mathscr{C}^F_u\times\mathscr{C}_p: \\ \widehat{K}(\mathbf{u}_S,\delta\mathbf{u}_S) - \displaystyle\int_{\Omega_F} g\frac{\partial\rho^0_F}{\partial z}(\mathbf{u}\cdot\mathbf{i}_z)(\delta\mathbf{u}\cdot\mathbf{i}_z) + \int_{\Omega_F}\rho^0_F c^2\,\mathrm{Div}\,\mathbf{u}_F\,\mathrm{Div}\,\delta\mathbf{u}_F \\ + \displaystyle\int_\Gamma \rho^0_F g(\mathbf{u}_F\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) + \int_{\Omega_F}\rho^0_F\mathbf{g}\cdot\mathbf{u}_F\,\mathrm{Div}\,\delta\mathbf{u}_F + \int_{\Omega_F}\rho^0_F\mathbf{g}\cdot\delta\mathbf{u}_F\,\mathrm{Div}\,\mathbf{u}_F \\ + \displaystyle\int_\Sigma p\mathbf{n}\cdot(\delta\mathbf{u}_F-\delta\mathbf{u}_S) - \omega^2\left\{\int_{\Omega_S}\rho^0_S\mathbf{u}_S\cdot\delta\mathbf{u}_S + \int_{\Omega_F}\rho^0_F\mathbf{u}_F\cdot\delta\mathbf{u}_F\right\} = \int_{\Sigma_d}\mathbf{F}\cdot\delta\mathbf{u}_S \\ \displaystyle\int_\Sigma \delta p\mathbf{n}\cdot(\mathbf{u}_F-\mathbf{u}_S) = 0. \end{array}\right. \tag{30}$$

3. Modal reduction schemes for the non-homogeneous incompressible hydroelastic–sloshing problem

The inviscid liquid is considered as *non-homogeneous*, *incompressible*. We consider here only internal gravity waves (gravity effects are supposed to act only inside the liquid domain and not on the free surface).

The linearized local equations governing the response of the liquid subjected to a prescribed normal displacement u_N write

$$\begin{cases} \nabla p(\mathbf{u}_F) + \dfrac{\partial \rho_F^0}{\partial z}\mathbf{g}(\mathbf{u}_F \cdot \mathbf{i}_z) = \rho_F^0 \omega^2 \mathbf{u}_F & (\Omega_F), \\ \operatorname{Div} \mathbf{u}_F = 0 & (\Omega_F), \\ p(\mathbf{u}_F) = 0 & (\Gamma), \\ \mathbf{u}_F \cdot \mathbf{n} = u_N & (\Sigma). \end{cases} \tag{31}$$

The associated variational formulation writes

$$\begin{cases} \text{given } \omega \in \mathbb{R}^+, \text{ find } \mathbf{u}_F \in \mathscr{C}_G^d \text{ such that } \forall \delta\mathbf{u}_F \in \mathscr{C}_G^0: \\ \quad -\displaystyle\int_{\Omega_F} g \frac{\partial \rho_F^0}{\partial z}(\mathbf{n}_F \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta\mathbf{u}_F, \end{cases} \tag{32}$$

where

- $\mathscr{C}_G = \{\mathbf{u}_F \in \mathscr{C}_u^F, \operatorname{Div} \mathbf{n}_F = 0 \text{ in } \Omega_F\}$,
- $\mathscr{C}_G^d = \{\mathbf{u}_F \in \mathscr{C}_G \text{ such that } \mathbf{u}_F \cdot \mathbf{n} = u_N \text{ on } \Sigma\}$,
- $\mathscr{C}_G^0 = \{\mathbf{u}_F \in \mathscr{C}_G \text{ such that } \mathbf{u}_F \cdot \mathbf{n} = 0 \text{ on } \Sigma\}$.

3.1. Conjugate relations

Eigenmodes are obtained from Eq. (32) by letting $u_N = 0$. The associated variational formulation writes:

$$\begin{cases} \text{find } \omega_\alpha \in \mathbb{R}^+ \text{ and } \mathbf{u}_F^\alpha \in \mathscr{C}_G^0 \text{ such that } \forall \delta\mathbf{u}_F \in \mathscr{C}_G^0: \\ \quad -\displaystyle\int_{\Omega_F} g \frac{\partial \rho_F^0}{\partial z}(\mathbf{n}_F^\alpha \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) = \omega_\alpha^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \delta\mathbf{u}_F. \end{cases} \tag{33}$$

It can be shown that the eigenmodes $(\omega_\alpha, \mathbf{u}_F^\alpha)$ satisfy the following orthogonality relations (also called conjugate relations)

$$\begin{cases} -\displaystyle\int_{\Omega_F} g \frac{\partial \rho_F^0}{\partial z}(\mathbf{n}_F^\alpha \cdot \mathbf{i}_z)(\mathbf{u}_F^\beta \cdot \mathbf{i}_z) = \delta_{\alpha\beta}\omega_\alpha^2 \mu_\alpha, \\ \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \mathbf{n}_F^\beta = \delta_{\alpha\beta}\mu_\alpha, \end{cases} \tag{34}$$

where μ_α and $\omega_\alpha^2 \mu_\alpha$ are the generalized modal mass and rigidity of the α mode.

The static boundary functions $\mathbf{u}_F^0(u_N)$ are required to satisfy the following local equations

$$\begin{cases} \nabla p + \dfrac{\partial \rho_F^0}{\partial z}\mathbf{g}(\mathbf{n}_F^0 \cdot \mathbf{i}_z) = 0 & (\Omega_F), \\ \operatorname{Div} \mathbf{u}_F^0(u_N) = 0 & (\Omega_F), \\ p(\mathbf{u}_F^0) = 0 & (\Gamma), \\ \mathbf{u}_F^0 \cdot \mathbf{n} = u_N & (\Sigma) \end{cases} \tag{35}$$

and the corresponding variational formulation writes:

$$\begin{cases} \text{find } \mathbf{u}_F^0(u_N) \in \mathscr{C}_G^d \ \forall \delta\mathbf{u}_F \in \mathscr{C}_G^0: \\ \quad \displaystyle\int_{\Omega_F} \frac{\partial \rho_F^0}{\partial z} g(\mathbf{n}_F^0(u_N) \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) = 0. \end{cases} \tag{36}$$

The first conjugate relation is obtained by letting $\delta\mathbf{u}_F$ be $\delta\mathbf{u}_F = \mathbf{u}_F^\alpha$ in Eq. (36) so that we have

$$\int_\Omega \frac{\partial \rho_F^0}{\partial z} g(\mathbf{n}_F^0(u_N) \cdot \mathbf{i}_z)(\mathbf{u}_F^\alpha \cdot \mathbf{i}_z) = 0. \tag{37}$$

The second relation is stated by considering the modal reaction force $\mathbf{R}_\alpha$ corresponding to the eigenmode $(\omega_\alpha, \mathbf{u}_F^\alpha)$. This reaction force is such that

$$\int_\Sigma \mathbf{R}_\alpha \cdot \mathbf{u}_F^0(u_N) = -\omega_\alpha^2 \int_{\Omega_F} \rho_F^0 \mathbf{n}_F^\alpha \cdot \mathbf{u}_F^0(u_N) \tag{38}$$

hence the following conjugate relation holds:

$$\int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \mathbf{n}_F^0(u_N) = -\frac{1}{\omega_\alpha^2} \int_\Sigma \mathbf{R}_\alpha \cdot \mathbf{u}_F^0(u_N). \tag{39}$$

3.2. Internal gravity sloshing reduced model

In the static formulation defined by Eq. (36), we introduce $\mathscr{C}_G(\Sigma) = u_N(\mathbf{M}), \mathbf{M} \in \Sigma$. As the solution $\mathbf{u}_F^0$ of Eq. (36) depends linearly on u_N, we introduce the mapping $u_N \in \mathscr{C}_G(\Sigma) \to u_F^0(u_N) \in \mathscr{C}_G^d$ which is a static lifting operator of u_N and which is called in engineering the static boundary function. The image of $\mathscr{C}_G(\Sigma)$ in this mapping is denoted $\mathscr{C}_{G,\Sigma}^s$ and is called the space of static boundary functions.

To build a reduced model, let us therefore consider the following direct sum:

$$\mathscr{C} = \mathscr{C}_{G,\Sigma}^s \oplus \mathscr{C}_G^0 \tag{40}$$

so that the general solution can be searched in the form:

$$\mathbf{u}_F = \mathbf{u}_F^0(u_N) + \sum_{\alpha=1}^N q_\alpha \mathbf{u}_F^\alpha. \tag{41}$$

The proof of this assertion rests upon the following two points:

- $\mathbf{u}_F - \mathbf{u}_F^0(u_N) = 0$ on Σ thus we have $\mathbf{u}_F - \mathbf{u}_F^0(u_N) \in \mathscr{C}_G^0$; $\mathbf{u}_F - \mathbf{u}_F^0(u_N)$ can then be decomposed on the subspace $\mathscr{C}_G^0$.
- Conversely, $\forall \mathbf{v}_F \in \mathscr{C}$ such that $\mathbf{v}_N = \mathbf{v}_F|\Sigma$, then we have $\mathbf{v}_F - \mathbf{u}_F^0(v_N) \in \mathscr{C}_G^0$ if $\mathbf{u}_F^0(v_N)$ denotes the solution of Eq. (36) for $u_N = v_N$.

Combination of Eqs. (32), (34), (37) and (39) leads to the following reduced model:

$$\left\{ \begin{array}{l} \text{find } \mathbf{u}_F^0(u_N) \in \mathscr{C}_{G,\Sigma}^s, \text{ and } q_\alpha \text{ such that } \forall \mathbf{v}_F^0 \in \mathscr{C}_{G,\Sigma}^s: \\ \quad -\displaystyle\int_\Omega g \frac{\partial \rho_F^0}{\partial z} (\mathbf{u}_F^0 \cdot \mathbf{i}_z)(\mathbf{v}_F^0 \cdot \mathbf{i}_z) - \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F^0 \cdot \mathbf{v}_F^0 = -\omega^2 \displaystyle\sum_{\alpha=1}^N q_\alpha \int_\Sigma \frac{\mathbf{R}_\alpha}{\omega_\alpha^2} \cdot \mathbf{v}_F^0 \\ \quad (-\omega^2 + \omega_\alpha^2)\mu_\alpha q_\alpha = -\omega^2 \displaystyle\int_\Sigma \frac{\mathbf{R}_\alpha}{\omega_\alpha^2} \cdot \mathbf{u}_F^0, \end{array} \right. \tag{42}$$

where $\mathbf{v}_F^0 = \mathbf{u}_F^0(v_N)$ refers to the static boundary function associated to an arbitrary displacement v_N on Σ.

4. Modal reduction schemes for the non-homogeneous compressible hydroelastic–sloshing general case

It was shown earlier that the variational formulation associated to the coupling problem of an elastic structure in contact with a non-homogeneous heavy compressible liquid states:

$$\left\{ \begin{array}{l} \text{given } \omega \in \mathbb{R}^+ \text{ and } \mathbf{F}, \text{ find} (\mathbf{u}_S, \mathbf{u}_F) \in \mathscr{C}_u^{S,F} \text{ such that } \forall (\delta\mathbf{u}_S, \delta\mathbf{u}_F) \in \mathscr{C}_u^{S,F}: \\ \quad \widehat{K}(\mathbf{u}_S, \delta\mathbf{u}_S) + \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F \operatorname{Div} \delta\mathbf{u}_F - \int_{\Omega_F} g \frac{\partial \rho_F^0}{\partial z} (\mathbf{u}_F \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) \\ \quad + \displaystyle\int_\Gamma \rho_F^0 g (\mathbf{u}_F \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F \operatorname{Div} \delta\mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta\mathbf{u}_F \operatorname{Div} \mathbf{u}_F \\ \quad - \omega^2 \left\{ \displaystyle\int_{\Omega_S} \rho_S^0 \mathbf{u}_S \cdot \delta\mathbf{u}_S + \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta\mathbf{u}_F \right\} = \displaystyle\int_{\Sigma_d} \mathbf{F} \cdot \delta\mathbf{u}_S, \end{array} \right. \tag{43}$$

where the kinematically admissible space $\mathscr{C}_u^{S,F}$ refers to

$$\mathscr{C}_u^{S,F} = \{(\mathbf{u}_F, \mathbf{u}_S) \text{ regular: } \mathbf{u}_F \cdot \mathbf{n} = \mathbf{u}_S \cdot \mathbf{n} \text{ on } \Sigma\}.$$

The reduced model can be obtained by considering the following decomposition for the fluid displacement field:

$$\mathbf{u}_F = \mathbf{u}_{\mathrm{AC}} + \mathbf{u}_{\mathrm{SL}} + \mathbf{u}_{\mathrm{GW}}, \tag{44}$$

where $\mathbf{u}_{\mathrm{SL}}$ is the displacement relative to sloshing model, $\mathbf{u}_{\mathrm{AC}}$ is the displacement relative to compressible-hydroelastic model whereas $\mathbf{u}_{\mathrm{GW}}$ refers to the internal gravity waves displacements. It should be noted that this representation of the displacement field corresponds to the decomposition of $\mathscr{C}_u^F$ as

$$\mathscr{C}_u^F = \mathscr{C}_{\mathrm{AC}} \oplus \mathscr{C}_{\mathrm{SL}} \oplus \mathscr{C}_G. \tag{45}$$

Replacing $\mathbf{u}_F$ into the variational formulation (43) then leads to

$$\left\{\begin{array}{l} \text{given } \mathbf{F} \text{ and } \omega,\ \text{find}(\mathbf{u}_S, \mathbf{u}_{\mathrm{AC}}, \mathbf{u}_{\mathrm{SL}}, \mathbf{u}_{\mathrm{GW}}) \in \mathscr{C}_u^{S,F}, \forall (\delta\mathbf{u}_S, \delta\mathbf{u}_F) \in \mathscr{C}_u^{S,F}: \\ \widetilde{K}(\mathbf{u}_S, \delta\mathbf{u}_S) + \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \,\mathrm{Div}\,\mathbf{u}_{\mathrm{AC}}\,\mathrm{Div}\,\delta\mathbf{u}_F + \int_{\Gamma} \rho_F^0 g(\mathbf{u}_{\mathrm{SL}}\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) \\ + \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\delta\mathbf{u}_F\,\mathrm{Div}\,\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot(\mathbf{u}_{\mathrm{SL}}+\mathbf{u}_{\mathrm{AC}}+\mathbf{u}_{\mathrm{GW}})\,\mathrm{Div}\,\delta\mathbf{u}_F \\ - \displaystyle\int_{\Omega_F} g\frac{\partial\rho_F^0}{\partial z}(\mathbf{u}_{\mathrm{GW}}\cdot\mathbf{i}_z)(\delta\mathbf{u}_F\cdot\mathbf{i}_z) \\ - \omega^2\left\{\displaystyle\int_{\Omega_S}\rho_S^0\mathbf{u}_S\cdot\delta\mathbf{u}_S + \int_{\Omega_F}\rho_F^0\mathbf{u}_{\mathrm{SL}}\cdot\delta\mathbf{u}_F + \int_{\Omega_F}\rho_F^0\mathbf{u}_{\mathrm{AC}}\cdot\delta\mathbf{u}_F + \int_{\Omega_F}\rho_F^0\mathbf{u}_{\mathrm{GW}}\cdot\delta\mathbf{u}_F\right\} \\ \quad = \displaystyle\int_{\Sigma_d}\mathbf{F}\cdot\delta\mathbf{u}_S. \end{array}\right. \tag{46}$$

Incompressibility constraints $\mathrm{Div}\,\mathbf{u}_{\mathrm{SL}} = 0$ and $\mathrm{Div}\,\mathbf{u}_{\mathrm{GW}} = 0$ together with orthogonality conditions have been used to derive this variational formulation [4,5].

Similarly let us decompose the virtual displacement vector into

$$\delta\mathbf{u}_F = \delta\mathbf{u}_{\mathrm{AC}} + \delta\mathbf{u}_{\mathrm{SL}} + \delta\mathbf{u}_{\mathrm{GW}}. \tag{47}$$

By enforcing the continuity of the liquid and structure normal displacement on their interface, the variational formulation of the reduced model then rewrites, taking into account Eq. (47) and orthogonality properties between subspaces:

$$\left\{\begin{array}{l} \text{given } \omega \text{ and } \mathbf{F},\ \text{find}(\mathbf{u}_S, \mathbf{u}_{\mathrm{SL}}, \mathbf{u}_{\mathrm{AC}}, \mathbf{u}_{\mathrm{GW}}, p), \forall (\delta\mathbf{u}_S, \delta\mathbf{u}_{\mathrm{SL}}, \delta\mathbf{u}_{\mathrm{AC}}, \delta\mathbf{u}_{\mathrm{GW}}, \delta p): \\ \widetilde{K}(\mathbf{u}_S, \delta\mathbf{u}_S) + \int_{\Gamma}\rho_F^0 g(\mathbf{u}_{\mathrm{SL}}\cdot\mathbf{i}_z)(\delta\mathbf{u}_{\mathrm{SL}}\cdot\mathbf{i}_z) + \displaystyle\int_{\Omega_F}\rho_F^0 c^2\,\mathrm{Div}\,\mathbf{u}_{\mathrm{AC}}\,\mathrm{Div}\,\delta\mathbf{u}_{\mathrm{AC}} \\ - \displaystyle\int_{\Omega_F} g\frac{\partial\rho_F^0}{\partial z}(\mathbf{u}_{\mathrm{GW}}\cdot\mathbf{i}_z)(\delta\mathbf{u}_{\mathrm{GW}}\cdot\mathbf{i}_z) \\ + \displaystyle\int_{\Omega_F}\rho_F^0\mathbf{g}\cdot\delta\mathbf{u}_{\mathrm{AC}}\,\mathrm{Div}\,\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F}\rho_F^0\mathbf{g}\cdot\delta\mathbf{u}_{\mathrm{SL}}\,\mathrm{Div}\,\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F}\rho_F^0\mathbf{g}\cdot\delta\mathbf{u}_{\mathrm{GW}}\,\mathrm{Div}\,\mathbf{u}_{\mathrm{AC}} \\ + \displaystyle\int_{\Omega_F}\rho_F^0\mathbf{g}\cdot\mathbf{u}_{\mathrm{AC}}\,\mathrm{Div}\,\delta\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F}\rho_F^0\mathbf{g}\cdot\mathbf{u}_{\mathrm{SL}}\,\mathrm{Div}\,\delta\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F}\rho_F^0\mathbf{g}\cdot\mathbf{u}_{\mathrm{GW}}\,\mathrm{Div}\,\delta\mathbf{u}_{\mathrm{AC}} \\ - \omega^2\left\{\displaystyle\int_{\Omega_S}\rho_S^0\mathbf{u}_S\cdot\delta\mathbf{u}_S + \int_{\Omega_F}\rho_F^0\mathbf{u}_{\mathrm{SL}}\cdot\delta\mathbf{u}_{\mathrm{SL}} + \int_{\Omega_F}\rho_F^0\mathbf{u}_{\mathrm{AC}}\cdot\delta\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F}\rho_F^0\mathbf{u}_{\mathrm{GW}}\cdot\delta\mathbf{u}_{\mathrm{GW}}\right\} \\ + \displaystyle\int_{\Sigma} p\mathbf{n}\cdot(\delta\mathbf{u}_{\mathrm{SL}} + \delta\mathbf{u}_{\mathrm{AC}} + \delta\mathbf{u}_{\mathrm{GW}} - \delta\mathbf{u}_S) = \int_{\Sigma_d}\mathbf{F}\cdot\delta\mathbf{u}_S. \end{array}\right. \tag{48}$$

Let us now consider the modal decompositions of each subspace as discussed in the previous section. Using Eqs. (A.14), (A.25) and (40) (see Appendix A), we can write:

$$\left\{\begin{array}{l} \mathbf{u}_{\mathrm{SL}} = \mathbf{u}_{\mathrm{SL}}^0 + \displaystyle\sum_{i=1}^{N}\alpha_i\mathbf{u}_{\mathrm{SL}}^i, \\ \mathbf{u}_{\mathrm{AC}} = \mathbf{u}_{\mathrm{AC}}^0 + \displaystyle\sum_{i=1}^{N}\beta_i\mathbf{u}_{\mathrm{AC}}^i, \\ \mathbf{u}_{\mathrm{GW}} = \mathbf{u}_{\mathrm{GW}}^0 + \displaystyle\sum_{i=1}^{N}\gamma_i\mathbf{u}_{\mathrm{GW}}^i, \end{array}\right. \tag{49}$$

where $\mathbf{u}_{\mathrm{SL}}^i$ (resp. $\mathbf{u}_{\mathrm{AC}}^i$, $\mathbf{u}_{\mathrm{GW}}^i$) is the sloshing (resp. acoustic, internal gravity waves) modes and $\mathbf{u}_{\mathrm{SL}}^0$ (resp. $\mathbf{u}_{\mathrm{GW}}^0$, $\mathbf{u}_{\mathrm{AC}}^0$) is the static mode relative to sloshing (resp. internal gravity and acoustic waves) problems. By restricting $\delta\mathbf{u}_{\mathrm{SL}}$ to $\{\delta\mathbf{u}_{\mathrm{SL}}^0, \mathbf{u}_{\mathrm{SL}}^j\}$, $\delta\mathbf{u}_{\mathrm{AC}}$ to $\{\delta\mathbf{u}_{\mathrm{AC}}^0, \mathbf{u}_{\mathrm{AC}}^j\}$ and $\delta\mathbf{u}_{\mathrm{GW}}$ to $\{\delta\mathbf{u}_{\mathrm{GW}}^0, \mathbf{u}_{\mathrm{GW}}^j\}$ and taking into account the orthogonality and conjugate relations established in Section 3 and Appendix A, the different terms of the variational formulation are obtained and detailed in Appendix A.

It should be noted that the obtained variational formulation is symmetric and its discretization in terms of so-called reduced stiffness and mass matrices can be expressed as

$$\begin{vmatrix} \widetilde{K} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K^{00}_{\mathrm{SL,SL}} & K^{00}_{\mathrm{SL,AC}} & 0 & 0 & K^{0i}_{\mathrm{SL,AC}} & 0 \\ 0 & K^{00^{\mathrm{T}}}_{\mathrm{SL,AC}} & K^{00}_{\mathrm{AC,AC}} & K^{00}_{\mathrm{GW,AC}} & K^{i0}_{\mathrm{SL,AC}} & K^{i0}_{\mathrm{AC,AC}} & K^{i0}_{\mathrm{GW,AC}} \\ 0 & 0 & K^{00}_{\mathrm{GW,AC}} & K^{00}_{\mathrm{GW,GW}} & 0 & K^{0i}_{\mathrm{GW,AC}} & 0 \\ 0 & 0 & K^{j0}_{\mathrm{SL,AC}} & 0 & K^{jj}_{\mathrm{SL,SL}} & K^{ji}_{\mathrm{SL,AC}} & 0 \\ 0 & K^{0j}_{\mathrm{SL,AC}} & K^{j0}_{\mathrm{AC,AC}} & K^{0j}_{\mathrm{GW,AC}} & K^{ij}_{\mathrm{SL,AC}} & K^{jj}_{\mathrm{AC,AC}} & K^{ij}_{\mathrm{GW,AC}} \\ 0 & 0 & K^{j0}_{\mathrm{GW,AC}} & 0 & 0 & K^{ji}_{\mathrm{GW,AC}} & K^{jj}_{\mathrm{GW,GW}} \end{vmatrix},$$

$$\begin{vmatrix} M_S & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M^{00}_{\mathrm{SL,SL}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M^{00}_{\mathrm{AC,AC}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M^{00}_{\mathrm{GW,GW}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M^{jj}_{\mathrm{SL,SL}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M^{jj}_{\mathrm{AC,AC}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M^{jj}_{\mathrm{GW,GW}} \end{vmatrix}.$$

5. Modal reduction schemes for the particular case of the homogeneous compressible hydroelastic–sloshing problem

As mentioned in Section 1, one may be interested in an approximate model which consists in considering the liquid as *homogeneous*.

The first part of this section deals with a modal reduction scheme based on the displacement vector variable. The reduced model is obtained by a direct decomposition of the liquid displacement field into its parts relative to compressible hydroelastic and sloshing problems. This situation can be considered as a particular case of Section 4 by letting ρ_F^0 be constant.

The second part of this section consists in deriving a modal reduction scheme based on the pressure formulation.

5.1. Displacement based model

The considerations stated in the previous sections are used here to exhibit some modal reduction schemes of the compressible hydroelastic—sloshing problem in the case of *homogeneous* liquid.

The variational formulation associated to the coupling problem of an elastic structure in contact with a *homogeneous* heavy compressible liquid is obtained by letting ρ_F^0 be constant in Eq. (30). This formulation writes

$$\left\{ \begin{array}{l} \text{given } \mathbf{F} \text{ and } \omega \in \mathbb{R}^+, \ \text{find}(\mathbf{u}_S \in \mathscr{C}_u^S, \mathbf{u}_F \in \mathscr{C}_u^F, p \in \mathscr{C}_p) \text{ such that} \\ \quad \forall (\delta\mathbf{u}_S, \delta\mathbf{u}_F, \delta p) \in \mathscr{C}_u^S \times \mathscr{C}_u^F \times \mathscr{C}_p: \\ \quad \widehat{K}(\mathbf{u}_S, \delta\mathbf{u}_S) + \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div}\mathbf{u}_F \operatorname{Div}\delta\mathbf{u}_F + \int_\Gamma \rho_F^0 g(\mathbf{u}_F \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) \\ \quad + \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F \operatorname{Div}\delta\mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta\mathbf{u}_F \operatorname{Div}\mathbf{u}_F \\ \quad + \displaystyle\int_\Sigma p\mathbf{n} \cdot (\delta\mathbf{u}_F - \delta\mathbf{u}_S) - \omega^2 \left\{ \int_{\Omega_S} \rho_S^0 \mathbf{u}_S \cdot \delta\mathbf{u}_S + \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta\mathbf{u}_F \right\} = \int_{\Sigma_d} \mathbf{F} \cdot \delta\mathbf{u}_S \\ \quad \displaystyle\int_\Sigma \delta p\mathbf{n} \cdot (\mathbf{u}_F - \mathbf{u}_S) = 0. \end{array} \right. \tag{50}$$

The reduced model can be obtained by considering the following decomposition for the fluid displacement field:

$$\mathbf{u}_F = \mathbf{u}_{\mathrm{SL}} + \mathbf{u}_{\mathrm{AC}}, \tag{51}$$

where $\mathbf{u}_{\mathrm{SL}}$ is the displacement relative to sloshing model whereas $\mathbf{u}_{\mathrm{AC}}$ is the displacement relative to compressible-hydroelastic model. This representation of the liquid's displacement field also corresponds formally to decompose the admissible space as

$$\mathscr{C}_u^F = \mathscr{C}_{\mathrm{AC}} \oplus \mathscr{C}_{\mathrm{SL}}, \tag{52}$$

where $\mathscr{C}_{\mathrm{AC}}$ and $\mathscr{C}_{\mathrm{SL}}$ are defined in Appendix A.

Replacing $\mathbf{u}_F$ into the variational formulation (50) and decomposing the virtual displacement as $\delta\mathbf{u}_F = \delta\mathbf{u}_{\mathrm{SL}} + \delta\mathbf{u}_{\mathrm{AC}}$ lead to

$$\left\{\begin{array}{l} \text{given } \mathbf{F} \text{ and } \omega \text{ find } (\mathbf{u}_S, \mathbf{u}_{\mathrm{SL}}, \mathbf{u}_{\mathrm{AC}}, \mathbf{u}_{\mathrm{GW}}, p)\ \forall(\delta\mathbf{u}_S, \delta\mathbf{u}_{\mathrm{SL}}, \delta\mathbf{u}_{\mathrm{AC}}, \delta\mathbf{u}_{\mathrm{GW}}, \delta p): \\ \widetilde{K}(\mathbf{u}_S, \delta\mathbf{u}_S) + \displaystyle\int_\Gamma \rho_F^0 g(\mathbf{u}_{\mathrm{SL}}\cdot\mathbf{i}_z)(\delta\mathbf{u}_{\mathrm{SL}}\cdot\mathbf{i}_z) + \int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div}\mathbf{u}_{\mathrm{AC}} \operatorname{Div}\delta\mathbf{u}_{\mathrm{AC}} \\ + \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\delta\mathbf{u}_{\mathrm{AC}} \operatorname{Div}\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\delta\mathbf{u}_{\mathrm{SL}} \operatorname{Div}\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\delta\mathbf{u}_{\mathrm{GW}} \operatorname{Div}\mathbf{u}_{\mathrm{AC}} \\ + \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\mathbf{u}_{\mathrm{AC}} \operatorname{Div}\delta\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\mathbf{u}_{\mathrm{SL}} \operatorname{Div}\delta\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F} \rho_F^0 \mathbf{g}\cdot\mathbf{u}_{\mathrm{GW}} \operatorname{Div}\delta\mathbf{u}_{\mathrm{AC}} \\ -\omega^2\left\{\displaystyle\int_{\Omega_S} \rho_S^0 \mathbf{u}_S\cdot\delta\mathbf{u}_S + \int_{\Omega_F} \rho_F^0 \mathbf{u}_{\mathrm{SL}}\cdot\delta\mathbf{u}_{\mathrm{SL}} + \int_{\Omega_F} \rho_F^0 \mathbf{u}_{\mathrm{AC}}\cdot\delta\mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F} \rho_F^0 \mathbf{u}_{\mathrm{GW}}\cdot\delta\mathbf{u}_{\mathrm{GW}}\right\} \\ + \displaystyle\int_\Sigma p\mathbf{n}\cdot(\delta\mathbf{u}_{\mathrm{SL}} + \delta\mathbf{u}_{\mathrm{AC}} + \delta\mathbf{u}_{\mathrm{GW}} - \delta\mathbf{u}_S) = \int_{\Sigma_d} \mathbf{F}\cdot\delta\mathbf{u}_S. \end{array}\right. \tag{53}$$

After decomposing each displacement vector field as

$$\left\{\begin{array}{l} \mathbf{u}_{\mathrm{SL}} = \mathbf{u}_{\mathrm{SL}}^0 + \displaystyle\sum_{i=1}^N \alpha_i \mathbf{u}_{\mathrm{SL}}^i, \\ \mathbf{u}_{\mathrm{AC}} = \mathbf{u}_{\mathrm{AC}}^0 + \displaystyle\sum_{i=1}^N \beta_i \mathbf{u}_{\mathrm{AC}}^i \end{array}\right. \tag{54}$$

and taking into account the orthogonality and conjugate relations, we obtain the stiffness and mass reduced matrix models:

$$\left\|\begin{array}{ccccc} \widetilde{K} & 0 & 0 & 0 & 0 \\ 0 & K_{\mathrm{SL,SL}}^{00} & K_{\mathrm{SL,AC}}^{00} & 0 & K_{\mathrm{SL,AC}}^{0i} \\ 0 & K_{\mathrm{SL,AC}}^{00^{\mathrm{T}}} & K_{\mathrm{AC,AC}}^{00} & K_{\mathrm{SL,AC}}^{i0} & K_{\mathrm{AC,AC}}^{i0} \\ 0 & 0 & K_{\mathrm{SL,AC}}^{j0} & K_{\mathrm{SL,SL}}^{jj} & K_{\mathrm{SL,AC}}^{ji} \\ 0 & K_{\mathrm{SL,AC}}^{0j} & K_{\mathrm{AC,AC}}^{j0} & K_{\mathrm{SL,AC}}^{ij} & K_{\mathrm{AC,AC}}^{jj} \end{array}\right\|,$$

$$\left\|\begin{array}{ccccc} M_S & 0 & 0 & 0 & 0 \\ 0 & M_{\mathrm{SL,SL}}^{00} & 0 & 0 & 0 \\ 0 & 0 & M_{\mathrm{SL,SL}}^{jj} & 0 & 0 \\ 0 & 0 & 0 & M_{\mathrm{AC,AC}}^{00} & 0 \\ 0 & 0 & 0 & 0 & M_{\mathrm{AC,AC}}^{jj} \end{array}\right\|.$$

5.2. A modal reduction scheme for the pressure formulation

The purpose of this subsection is to derive an alternative reduced model of the coupled problem of a structure interacting with a *homogenous* fluid. Compared with the previous sections, the fluid is described here by the pressure variable.

The variational formulation describing the response of the structure to a fluid loading pressure writes

$$\left\{\begin{array}{l} \text{given } \omega \text{ and } \mathbf{F},\ \text{find } \mathbf{u}_S \in \mathscr{C}_u \text{ such that:} \\ \widehat{K}(\mathbf{u}_S, \delta\mathbf{u}_S) - \omega^2 \displaystyle\int_{\Omega_S} \rho_S^0 \mathbf{u}_S\cdot\delta\mathbf{u}_S - \int_\Sigma p\mathbf{n}\cdot\delta\mathbf{u}_S = \int_{\Sigma_d} \mathbf{F}\cdot\delta\mathbf{u}_S. \end{array}\right. \tag{55}$$

The local equations describing the response of the fluid to a prescribed normal wall displacement u^N write

$$\left\{\begin{array}{ll} \Delta p + \dfrac{\omega^2}{c^2} p = 0 & (\Omega_F), \\ \dfrac{\partial p}{\partial n} = \rho_F^0 \omega^2 u_N & (\Sigma), \\ \dfrac{\partial p}{\partial z} = \dfrac{\omega^2}{g} p & (\Gamma), \\ \displaystyle\int_\Sigma u^N + \int_\Gamma \frac{p}{\rho_F^0 g} + \int_{\Omega_F} \frac{p}{\rho_F^0 c^2} = 0. & \end{array}\right. \tag{56}$$

It must be noticed that the constraint equation $(56)_4$ is an additive relation which allows to state the well posedness of the boundary value problem (56) for $\omega = 0$. For more detailed considerations on this point, the reader is referred to [13].

The variational formulation associated to local Eqs. (56) states

$$\left\{\begin{array}{l} \text{given } \omega \text{ and } u_N, \text{ find } p \in \mathscr{C}_p \text{ such that:} \\ \displaystyle\int_{\Omega_F} \nabla p \cdot \nabla \delta p = \rho_F^0 \omega^2 \int_\Sigma u_N \, \delta p + \frac{\omega^2}{g} \int_\Gamma p \delta p + \frac{\omega^2}{c^2} \int_{\Omega_F} p \, \delta p \\ \text{with } \displaystyle\int_\Sigma u_N + \int_\Gamma \frac{p}{\rho_F^0 g} + \int_{\Omega_F} \frac{p}{\rho_F^0 c^2} = 0 \end{array}\right. \tag{57}$$

so that the coupled problem can be described with the following variational formulation:

$$\left\{\begin{array}{l} \text{given } \omega \text{ and } \mathbf{F}, \text{ find} (\mathbf{u}_S, p) \in \mathscr{C}_u \times \mathscr{C}_p \text{ such that:} \\ \widehat{K}(\mathbf{u}_S, \delta \mathbf{u}_S) - \omega^2 \displaystyle\int_{\Omega_S} \rho_S^0 \mathbf{u}_S \cdot \delta \mathbf{u}_S - \int_\Sigma p \mathbf{n} \cdot \delta \mathbf{u}_S = \int_{\Sigma_d} \mathbf{F} \cdot \delta \mathbf{u}_S \\ \displaystyle\int_{\Omega_F} \nabla p \cdot \nabla \, \delta p = \rho_F^0 \omega^2 \int_\Sigma \mathbf{u}_S \cdot \mathbf{n} \, \delta p + \frac{\omega^2}{g} \int_\Gamma p \, \delta p + \frac{\omega^2}{c^2} \int_{\Omega_F} p \, \delta p \\ \displaystyle\int_\Sigma \mathbf{u}_S \cdot \mathbf{n} + \int_\Gamma \frac{p}{\rho_F^0 g} + \int_{\Omega_F} \frac{p}{\rho_F^0 c^2} = 0. \end{array}\right. \tag{58}$$

Let us define

$$\mathscr{C}_p^{\mathrm{AC}^*} = \{p \in \mathscr{C}^p : p = 0 \text{ on } (\Gamma)\},$$

$\mathscr{C}_p^{\mathrm{AC}^*}$ is the vector space spanned by the acoustic modes which satisfy the following orthogonality relations:

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega_F} \nabla p_i \cdot \nabla p_j = \frac{\omega_{i,AC}^2}{c^2} \mu_{ij,\mathrm{AC}} \delta_{ij}, \\ \displaystyle\int_{\Omega_F} p_i \, \delta p_j = \mu_{ij,\mathrm{AC}} \delta_{ij}. \end{array}\right. \tag{59}$$

In the same way, it can be shown that for a *homogeneous heavy incompressible* liquid, the following decomposition holds:

$$\mathscr{C}_p^{\mathrm{SL}} = \mathscr{C}_{p_s}^{\mathrm{SL}} \oplus \mathscr{C}_p^{\mathrm{SL}^*}, \tag{60}$$

where

$$\mathscr{C}_{p_s}^{\mathrm{SL}} = \left\{p_s \in \mathscr{C}^p : p_s = -\frac{\rho_F^0 g}{|\Gamma|} \int_\Sigma u_N\right\},$$

$$\mathscr{C}_p^{\mathrm{SL}^*} = \left\{p \in \mathscr{C}^p : \int_\Gamma p = 0\right\},$$

$\mathscr{C}_p^{\mathrm{SL}^*}$ refers to the vectorial space associated to the sloshing modes which satisfy the following orthogonality relations:

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega_F} \nabla p_i \cdot \nabla p_j = \frac{\omega_{i,\mathrm{SL}}^2}{g} \mu_{ij,\mathrm{SL}} \delta_{ij}, \\ \displaystyle\int_\Gamma p_i \, \delta p_j = \mu_{ij,\mathrm{SL}} \delta_{ij}. \end{array}\right. \tag{61}$$

In order to establish the reduced model for the complete problem, we consider the following decomposition:

$$\mathscr{C}_p = \mathscr{C}_{p_s}^{\mathrm{SL}} \oplus \mathscr{C}_p^{\mathrm{SL}^*} \oplus \mathscr{C}_p^{\mathrm{AC}^*} \tag{62}$$

that is the pressure variable p is searched in the following form:

$$p = p_s^{\mathrm{SL}} + \sum_i r_i p_i^{\mathrm{SL}} + \sum_i r_i p_i^{\mathrm{AC}}. \tag{63}$$

Let this expression of p be replaced in the variational formulation Eq. (58) and let us restrict the virtual pressure to $\delta p = \{\delta p_s^{\mathrm{SL}}, p_j^{\mathrm{AC}}, p_j^{\mathrm{SL}}\}$ by taking into account the orthogonality relations. We have:

$$
\begin{cases}
\widehat{K}(\mathbf{u}_S, \delta\mathbf{u}_S) + \frac{\rho_F g}{|I|}\left(\int_\Sigma \mathbf{u}_S\cdot\mathbf{n}\right)\left(\int_\Sigma \delta\mathbf{u}_S\cdot\mathbf{n}\right) - \sum_\alpha r_\alpha^{\mathrm{SL}}\int_\Sigma p_\alpha^{\mathrm{SL}}\mathbf{n}\cdot\delta\mathbf{u}_S \\
\quad - \sum_\beta r_\alpha^{\mathrm{AC}}\int_\Sigma p_\beta^{\mathrm{AC}}\mathbf{n}\cdot\delta\mathbf{u}_S = \int_{\Sigma_d}\mathbf{F}\cdot\delta\mathbf{u}_S \\
\quad \omega_\alpha^{\mathrm{SL}^2}\mu_\alpha^{\mathrm{SL}} r_\alpha^{\mathrm{SL}} = \omega^2\int_\Sigma \mathbf{u}_S\cdot\mathbf{n}p_\alpha^{\mathrm{SL}} + \omega^2\mu_\alpha^{\mathrm{SL}} r_\alpha^{\mathrm{SL}} \\
\quad \omega_\beta^{\mathrm{AC}^2}\mu_\beta^{\mathrm{AC}} r_\beta^{\mathrm{AC}} = \omega^2\int_\Sigma \mathbf{u}_S\cdot\mathbf{n}p_\beta^{\mathrm{AC}} + \omega^2\mu_\beta^{\mathrm{AC}} r_\beta^{\mathrm{AC}}.
\end{cases}
\tag{64}
$$

The associated finite element discretizations then writes:

$$
\begin{cases}
\left[\widehat{K} + K^s\right]U - \sum_\alpha C_\alpha^{\mathrm{SL}} r_\alpha^{\mathrm{SL}} - \sum_\beta C_\beta^{\mathrm{AC}} r_\beta^{\mathrm{AC}} - \omega^2 M U = \mathbf{F}, \\
\omega_\alpha^{\mathrm{SL}^2}\mu_\alpha^{\mathrm{SL}} r_\alpha^{\mathrm{SL}} = \omega^2 C_\alpha^{\mathrm{SL}^{\mathrm{T}}} r_\alpha^{\mathrm{SL}} + \omega^2\mu_\alpha^{\mathrm{SL}} r_\alpha^{\mathrm{SL}}, \\
\omega_\beta^{\mathrm{AC}^2}\mu_\beta^{\mathrm{AC}} r_\beta^{\mathrm{AC}} = \omega^2 C_\beta^{\mathrm{AC}^{\mathrm{T}}} r_\beta^{\mathrm{AC}} + \omega^2\mu_\beta^{\mathrm{AC}} r_\beta^{\mathrm{AC}}
\end{cases}
\tag{65}
$$

and

$$
\begin{pmatrix}
K^{\mathrm{tot}} - \omega^2 M^{\mathrm{tot}} & \omega^2 D_{\mathrm{SL}} & \omega^2 D_{\mathrm{AC}} \\
\omega^2 D_{\mathrm{SL}}^{\mathrm{T}} & [\mu^{\mathrm{SL}}] - \omega^2\left[\frac{\mu^{\mathrm{SL}}}{\omega^2_{\mathrm{SL}}}\right] & 0 \\
\omega^2 D_{\mathrm{AC}}^{\mathrm{T}} & 0 & [\mu^{\mathrm{AC}}] - \omega^2\left[\frac{\mu^{\mathrm{AC}}}{\omega^2_{\mathrm{AC}}}\right]
\end{pmatrix}
\begin{pmatrix} U \\ [r^{\mathrm{SL}}] \\ [r^{\mathrm{AC}}] \end{pmatrix} =
\begin{pmatrix} \mathbf{F} \\ 0 \\ 0 \end{pmatrix},
\tag{66}
$$

where

$$
\begin{aligned}
K^{\mathrm{tot}} &= \widehat{K} + K^s, \\
M^{\mathrm{tot}} &= M + \sum_\alpha \frac{1}{\omega_\alpha^{\mathrm{SL}^2}} C_\alpha^{\mathrm{SL}} C_\alpha^{\mathrm{SL}^{\mathrm{T}}} + \sum_\beta \frac{1}{\omega_\beta^{\mathrm{AC}^2}} C_\beta^{\mathrm{AC}} C_\beta^{\mathrm{AC}^{\mathrm{T}}}, \\
D_\alpha^{\mathrm{SL}} &= \sum_\alpha \frac{1}{\omega_\alpha^{\mathrm{SL}^2}} C_\alpha^{\mathrm{SL}}, \\
D_\beta^{\mathrm{AC}} &= \sum_\beta \frac{1}{\omega_\beta^{\mathrm{AC}^2}} C_\beta^{\mathrm{AC}}.
\end{aligned}
$$

6. Conclusion

This paper presents theoretical derivation for dynamic substructuring analysis of fluid–structure interaction problems taking into account both gravity and compressibility effect in order to construct reduced order models.

The key point of our approach is that the generalized coordinates of the compressible liquid can be decomposed into the direct sum of those relative to structural acoustics and incompressible hydroelastic–sloshing problems. Using this decomposition into the variational formulations, symmetric reduced matrix systems expressed in terms of generalized coordinates of the fluid are then obtained. Investigation are presently undertaken in order to validate numerically the theoretical results and will be the subject of future publications.

Appendix A. Basic reduced fluid–structure problems

We will recall in this appendix basic results which will be useful to construct reduced order models in the general *non-homogeneous* fluid–structure problem. For detailed aspects of the adopted methodology, we refer the reader to [11,13].

The two elementary situations we consider are the following:

- structural acoustic problem with modal interaction between structural modes in vacuo or incompressible hydroelastic modes and acoustic modes in rigid cavity,
- hydroelastic–sloshing problem with modal interaction between incompressible hydroelastic structural modes and incompressible liquid sloshing modes in rigid cavity.

A.1. Structural acoustic

The first situation which is considered here is the structural acoustic one. A fluid domain Ω_F at equilibrium is considered. The fluid is supposed to be homogeneous, inviscid and gravity effects are not considered. Its boundary consists of two parts: Σ is the part that is in contact with the structure domain and is submitted to a prescribed normal displacement u_N whereas Γ is the free surface.

The local equations describing the harmonic response of the fluid to a prescribed arbitrary normal displacement u_N write:

$$\begin{cases} \nabla p = \rho_F^0 \omega^2 \mathbf{u}_F & (\Omega_F), \\ p = -\rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F & (\Omega_F), \\ \mathbf{u}_F \cdot \mathbf{n} = u_N & (\Sigma), \\ p = 0 & (\Gamma). \end{cases} \tag{A.1}$$

Eq. (A.1) may be rewritten in terms of the displacement vector $\mathbf{u}_F$. We obtain:

$$\begin{cases} -\rho_F^0 c^2 \nabla \operatorname{Div} \mathbf{u}_F = \rho_F^0 \omega^2 \mathbf{u}_F & (\Omega_F), \\ \mathbf{u}_F \cdot \mathbf{n} = u_N & (\Sigma), \\ p(\mathbf{u}_F) = 0 & (\Gamma). \end{cases} \tag{A.2}$$

It should be noted that the solution of this boundary value problem satisfies automatically the irrotationality constraint $\operatorname{Curl} \mathbf{u}_F = 0$ for the strictly positive values of ω.

The test function method allows us to derive the variational formulation associated to Eq. (A.2) as

$$\begin{cases} \text{given } \omega \in \mathbb{R}^+, \text{ find } \mathbf{u}_F \in \mathscr{C}_{\mathrm{AC}}^d \text{ such that } \forall \delta \mathbf{u}_F \in \mathscr{C}_{\mathrm{AC}}^0: \\ \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F \operatorname{Div} \delta \mathbf{u}_F = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F, \end{cases} \tag{A.3}$$

where

- $\mathscr{C}_{\mathrm{AC}} = \{\mathbf{u}_F \in \mathscr{C}_u^F, \operatorname{Curl} \mathbf{u}_F = 0 \text{ in } \Omega_F\}$,
- $\mathscr{C}_{\mathrm{AC}}^d = \{\mathbf{u}_F \in \mathscr{C}_{\mathrm{AC}} \text{ such that } \mathbf{u}_F \cdot \mathbf{n} = u_N \text{ on } \Sigma\}$,
- $\mathscr{C}_{\mathrm{AC}}^0 = \{\mathbf{u}_F \in \mathscr{C}_{\mathrm{AC}} \text{ such that } \mathbf{u}_F \cdot \mathbf{n} = 0 \text{ on } \Sigma\}$.

A.1.1. Conjugate relations

Conjugate relations relate the acoustic modes in rigid motionless cavity $\mathbf{u}_F^\alpha$ with the static boundary functions $\mathbf{u}_F^0(u_N)$. Acoustic modes are obtained from Eq. (A.3) with $u_N = 0$. The associated weak form then states:

$$\begin{cases} \text{find } \omega_\alpha \in \mathbb{R}^+, \text{ and } \mathbf{u}_F^\alpha \in \mathscr{C}_{\mathrm{AC}}^0 \text{ such that } \forall \delta \mathbf{u}_F \in \mathscr{C}_{\mathrm{AC}}^0: \\ \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F^\alpha \operatorname{Div} \delta \mathbf{u}_F = \omega_\alpha^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \delta \mathbf{u}_F. \end{cases} \tag{A.4}$$

It can be shown that the acoustic modes $(\omega_\alpha, \mathbf{u}_F^\alpha)$ satisfy the following orthogonality relations:

$$\begin{cases} \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F^\alpha \operatorname{Div} \mathbf{u}_F^\beta = \delta_{\alpha\beta} \omega_\alpha^2 \mu_\alpha, \\ \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \mathbf{u}_F^\beta = \delta_{\alpha\beta} \mu_\alpha, \end{cases} \tag{A.5}$$

where μ_α and $\omega_\alpha^2 \mu_\alpha$ are the generalized modal mass and rigidity of the α mode.

The local equations relative to the static boundary functions are obtained from Eq. (A.1) by letting $\omega = 0$. The static boundary functions $\mathbf{u}_F^0(u_N)$ thus satisfy:

$$\begin{cases} \nabla p = 0 & (\Omega_F), \\ p = -\rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F^0 & (\Omega_F), \\ \mathbf{u}_F^0 \cdot \mathbf{n} = u_N & (\Sigma), \\ p = 0 & (\Gamma). \end{cases} \tag{A.6}$$

Eq. (A.6)$_1$, in conjunction with Eq. (A.6)$_4$ leads to

$$p = p^0 = 0. \tag{A.7}$$

The fluid displacement $\mathbf{u}_F^0(u_N)$ thus satisfies:

$$\begin{cases} \operatorname{Div} \mathbf{u}_F^0 = 0 & (\Omega_F), \\ \mathbf{u}_F^0 \cdot \mathbf{n} = u_N & (\Sigma), \\ \operatorname{Div} \mathbf{u}_F^0 = 0 & (\Gamma). \end{cases} \tag{A.8}$$

It can be shown that the solution $\mathbf{u}_F^0(u_N)$ of Eq. (A.8) is unique as far as we are concerned with static irrotational motion. The corresponding variational formulation Eq. (A.8) writes:

$$\begin{cases} \text{find } \mathbf{u}_F^0(u_N) \in \mathscr{C}_{\mathrm{AC}}^d \ \forall \delta\mathbf{u}_F \in \mathscr{C}_{\mathrm{AC}}^0: \\ \quad \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F^0(u_N) \operatorname{Div} \delta\mathbf{u}_F = 0. \end{cases} \tag{A.9}$$

The first conjugate relation is obtained by letting $\delta\mathbf{u}_F$ to be $\delta\mathbf{u}_F = \mathbf{u}_F^\alpha$ in Eq. (A.9). This leads to

$$\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F^0(u_N) \operatorname{Div} \mathbf{u}_F^\alpha = 0. \tag{A.10}$$

In order to state the second conjugate relation, let us consider the modal reaction force $\mathbf{R}_\alpha$ corresponding to the eigenmode $(\omega_\alpha, \mathbf{u}_F^\alpha)$. This reaction force obeys the following property:

$$\int_\Sigma \mathbf{R}_\alpha \cdot \delta\mathbf{u}_F = \int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F^\alpha \operatorname{Div} \delta\mathbf{u}_F - \omega_\alpha^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \delta\mathbf{u}_F. \tag{A.11}$$

The second conjugate relation is obtained by stating $\delta\mathbf{u}_F = \mathbf{u}_F^0(u_N)$ in Eq. (A.11) so that:

$$\int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \mathbf{u}_F^0(u_N) = -\frac{1}{\omega_\alpha^2} \int_\Sigma \mathbf{R}_\alpha \cdot \mathbf{u}_F^0(u_N). \tag{A.12}$$

A.1.2. Structural acoustic reduced model

To derive the reduced model associated to structural acoustic, the solution $\mathbf{u}_F$ of Eq. (A.3) is decomposed into

$$\mathbf{u}_F = \mathbf{u}_F^0(u_N) + \sum_{\alpha=1}^N q_\alpha \mathbf{u}_F^\alpha, \tag{A.13}$$

where N denotes the number of retained acoustic modes. It can be shown that this decomposition is unique and it actually corresponds to the following decomposition:

$$\mathscr{C}_{\mathrm{AC}} = \mathscr{C}_{\mathrm{AC}}^s \oplus \mathscr{C}_{\mathrm{AC}}^0, \tag{A.14}$$

where $\mathscr{C}_{\mathrm{AC}}^s$ refers to the admissible class associated to the static boundary functions $\mathbf{u}_F^0(u_N)$.

Inserting Eq. (A.13) into Eq. (A.3) and taking into account of the conjugate relations (A.10) and (A.12), the following system is obtained:

$$\begin{cases} \text{find } \mathbf{u}_F^0(u_N) \in \mathscr{C}_{\mathrm{AC}}^s \ \forall \mathbf{v}_F^0 \in \mathscr{C}_{\mathrm{AC}}^s: \\ \quad \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \operatorname{Div} \mathbf{u}_F^0 \operatorname{Div} \mathbf{v}_F^0 - \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F^0 \cdot \mathbf{v}_F^0 = -\omega^2 \sum_{\alpha=1}^N q_\alpha \int_\Sigma \frac{\mathbf{R}_\alpha}{\omega_\alpha^2} \cdot \mathbf{v}_F^0 \\ \quad (-\omega^2 + \omega_\alpha^2)\mu_\alpha q_\alpha = -\omega^2 \displaystyle\int_\Sigma \frac{\mathbf{R}_\alpha}{\omega_\alpha^2} \cdot \mathbf{u}_F^0, \end{cases} \tag{A.15}$$

where $\mathbf{v}_F^0 = \mathbf{u}_F^0(v_N)$ refers to a static boundary function associated to an arbitrary displacement v_N on Σ.

A.2. Incompressible hydroelastic sloshing

The second elementary situation which is to be considered is the incompressible hydroelastic problem. The liquid is supposed to be inviscid homogeneous and incompressible. The liquid possesses a free surface Γ on which gravity effects act and a fluid–structure interface Σ where a prescribed displacement u_N is applied.

The local equations governing the small movements of such a liquid are

$$\begin{cases} \nabla p = \rho_F^0 \omega^2 \mathbf{u}_F & (\Omega_F), \\ \mathrm{Div}\,\mathbf{u}_F = 0 & (\Omega_F), \\ p = \rho_F^0 g(\mathbf{u}_F \cdot \mathbf{i}_z) & (\Gamma), \\ \mathbf{u}_F \cdot \mathbf{n} = u_N & (\Sigma). \end{cases} \tag{A.16}$$

The test function method allows us to write the corresponding variational formulation as

$$\begin{cases} \text{given } \omega \in \mathbb{R}^+, \text{ find } \mathbf{u}_F \in \mathscr{C}_{\mathrm{SL}}^d \text{ such that } \forall \delta\mathbf{u}_F \in \mathscr{C}_{\mathrm{SL}}^0: \\ \quad \displaystyle\int_\Gamma \rho_F^0 g(\mathbf{u}_F \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta\mathbf{u}_F, \end{cases} \tag{A.17}$$

where

- $\mathscr{C}_{\mathrm{SL}} = \{\mathbf{u}_F \in \mathscr{C}_u^F, \mathrm{Div}\,\mathbf{u}_F = 0 \text{ in } \Omega_F, \mathrm{Curl}\,\mathbf{u}_F = 0 \text{ in } \Omega_F\}$,
- $\mathscr{C}_{\mathrm{SL}}^d = \{\mathbf{u}_F \in \mathscr{C}_{\mathrm{SL}} \text{ such that } \mathbf{u}_F \cdot \mathbf{n} = u_N \text{ on } \Sigma\}$,
- $\mathscr{C}_{\mathrm{SL}}^0 = \{\mathbf{u}_F \in \mathscr{C}_{\mathrm{SL}} \text{ such that } \mathbf{u}_F \cdot \mathbf{n} = 0 \text{ on } \Sigma\}$.

A.2.1. Conjugate relations

Conjugate relations relate the sloshing modes with the static boundary functions $\mathbf{u}_F^0(u_N)$. Sloshing modes are obtained from Eq. (A.17) by letting $u_N = 0$. The associated variational formulation writes:

$$\begin{cases} \text{find } \omega_\alpha \in \mathbb{R}^+, \text{ and } \mathbf{u}_F^\alpha \in \mathscr{C}_{\mathrm{SL}}^0 \text{ such that } \forall \delta\mathbf{u}_F \in \mathscr{C}_{\mathrm{SL}}^0: \\ \quad \displaystyle\int_\Gamma \rho_F^0 g(\mathbf{u}_F^\alpha \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) = \omega_\alpha^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \delta\mathbf{u}_F, \end{cases} \tag{A.18}$$

It can easily be shown that the sloshing modes $(\omega_\alpha, \mathbf{u}_F^\alpha)$ satisfy the following orthogonality relations:

$$\begin{cases} \displaystyle\int_\Gamma \rho_F^0 g(\mathbf{u}_F^\alpha \cdot \mathbf{i}_z)(\mathbf{u}_F^\beta \cdot \mathbf{i}_z) = \delta_{\alpha\beta}\omega_\alpha^2 \mu_\alpha, \\ \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \mathbf{u}_F^\beta = \delta_{\alpha\beta}\mu_\alpha, \end{cases} \tag{A.19}$$

where μ_α and $\omega_\alpha^2 \mu_\alpha$ are the generalized modal mass and rigidity of the α mode.

Following the same process as in the structural acoustic problem, the local equations relative to the static boundary functions are obtained from Eq. (A.16) by letting $\omega = 0$. The static boundary functions $\mathbf{u}_F^0(u_N)$ thus satisfy:

$$\begin{cases} \nabla p = 0 & (\Omega_F), \\ \mathrm{Div}\,\mathbf{u}_F^0(u_N) = 0 & (\Omega_F), \\ p = \rho_F^0 g(\mathbf{u}_F^0(u_N) \cdot \mathbf{i}_z) & (\Gamma), \\ \mathbf{u}_F^0 \cdot \mathbf{n} = u_N & (\Sigma) \end{cases} \tag{A.20}$$

and the corresponding variational formulation states

$$\begin{cases} \text{find } \mathbf{u}_F^0(u_N) \in \mathscr{C}_{\mathrm{SL}}^d \ \forall \delta\mathbf{u}_F \in \mathscr{C}_{\mathrm{SL}}^0: \\ \quad \displaystyle\int_\Gamma \rho_F^0 g(\mathbf{u}_F^0(u_N) \cdot \mathbf{i}_z)(\delta\mathbf{u}_F \cdot \mathbf{i}_z) = 0. \end{cases} \tag{A.21}$$

The first conjugate relation is obtained by letting $\delta\mathbf{u}_F$ be $\delta\mathbf{u}_F = \mathbf{u}_F^\alpha$ in Eq. (A.21) so that we have

$$\int_\Gamma \rho_F^0 g(\mathbf{u}_F^0(u_N) \cdot \mathbf{i}_z)(\mathbf{u}_F^\alpha \cdot \mathbf{i}_z) = 0. \tag{A.22}$$

In order to state the second conjugate relation, the reaction force $\mathbf{R}_\alpha$ corresponding to the eigenmode $(\omega_\alpha, \mathbf{u}_F^\alpha)$ needs to be expressed. This reaction force satisfies the following property:

$$\int_\Sigma \mathbf{R}_\alpha \cdot \mathbf{u}_F^0(u_N) = -\omega_\alpha^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \mathbf{u}_F^0(u_N), \tag{A.23}$$

hence the following conjugate relation holds:

$$\int_{\Omega_F} \rho_F^0 \mathbf{u}_F^\alpha \cdot \mathbf{u}_F^0(u_N) = -\frac{1}{\omega_\alpha^2} \int_\Sigma \mathbf{R}_\alpha \cdot \mathbf{u}_F^0(u_N). \tag{A.24}$$

A.2.2. Hydroelastic–sloshing reduced model

The admissible class $\mathscr{C}_{\mathrm{SL}}$ can be considered as the direct sum of two subsets $\mathscr{C}_{\mathrm{SL}}^s$ and $\mathscr{C}_{\mathrm{SL}}^0$ so that:

$$\mathscr{C}_{\mathrm{SL}} = \mathscr{C}_{\mathrm{SL}}^d \oplus \mathscr{C}_{\mathrm{SL}}^0. \tag{A.25}$$

The solution $\mathbf{u}_F$ can then be searched under the form:

$$\mathbf{u}_F = \mathbf{u}_F^0(u_N) + \sum_{\alpha=1}^{N} q_\alpha \mathbf{u}_F^\alpha, \tag{A.26}$$

where N is the number of retained sloshing modes.

This decomposition of $\mathbf{u}_F$, in conjunction with the conjugate relations (A.22) and (A.24) leads to the following reduced model:

$$\begin{cases} \text{find } \mathbf{u}_F^0(u_N) \in \mathscr{C}_{\mathrm{SL}}^d, \text{ and } q_\alpha \in \mathscr{C}_{\mathrm{SL}}^0 \ \forall \mathbf{v}_F^0 \in \mathscr{C}_{\mathrm{SL}}^s: \\ \quad \displaystyle\int_\Gamma \rho_F^0 (\mathbf{u}_F^0 \cdot \mathbf{i}_z)(\mathbf{v}_F^0 \cdot \mathbf{i}_z) - \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F^0 \cdot \mathbf{v}_F^0 = -\omega^2 \sum_{\alpha=1}^{N} q_\alpha \int_\Sigma \frac{\mathbf{R}_\alpha}{\omega_\alpha^2} \cdot \mathbf{v}_F^0 \\ \quad (-\omega^2 + \omega_\alpha^2)\mu_\alpha q_\alpha = -\omega^2 \displaystyle\int_\Sigma \frac{R_\alpha}{\omega_\alpha^2} \cdot \mathbf{u}_F^0, \end{cases} \tag{A.27}$$

where $\mathbf{v}_F^0 = \mathbf{u}_F^0(v_N)$ refers to a static boundary function associated to an arbitrary displacement v^N on Σ.

Appendix B. Development of the bilinear forms for the reduced model

The different bilinear forms describing the reduced model are splitted, using the decomposition of the virtual displacement fields given corresponding to Eq. (49) (taking for $\delta\mathbf{u}_{\mathrm{SL}}$, successively $\delta\mathbf{u}_{\mathrm{SL}}^0$, $\mathbf{u}_{\mathrm{SL}}^j, j = 1, n$, and similarly for the other virtual displacement fields).

We give hereafter some of the decomposed bilinear forms.

$$\int_\Gamma \rho_F^0 g(\mathbf{u}_{\mathrm{SL}} \cdot \mathbf{i}_z)(\delta\mathbf{u}_{\mathrm{SL}} \cdot \mathbf{i}_z) \Rightarrow \begin{cases} \displaystyle\int_\Gamma \rho_F^0 g(\mathbf{u}_{\mathrm{SL}}^0 \cdot \mathbf{i}_z)(\delta\mathbf{u}_{\mathrm{SL}}^0 \cdot \mathbf{i}_z) \\ \text{and} \\ \alpha_i \delta_{ij} \mu_j \omega_{j,\mathrm{SL}}^2, \end{cases} \tag{B.1}$$

$$\int_{\Omega_F} \rho_F^0 c^2 \nabla \cdot \mathbf{u}_{\mathrm{AC}} \nabla \cdot \delta\mathbf{u}_{\mathrm{AC}} \Rightarrow \begin{cases} \displaystyle\int_{\Omega_F} \rho_F^0 c^2 \nabla \cdot \mathbf{u}_{\mathrm{AC}}^0 \nabla \cdot \delta\mathbf{u}_{\mathrm{AC}}^0 \\ \text{and} \\ \alpha_i \delta_{ij} \mu_j \omega_{j,\mathrm{AC}}^2, \end{cases} \tag{B.2}$$

$$\int_{\Omega_F} g \frac{\partial \rho_F^0}{\partial z} (\mathbf{u}_{\mathrm{GW}} \cdot \mathbf{i}_z)(\delta\mathbf{u}_{\mathrm{GW}} \cdot \mathbf{i}_z) \Rightarrow \begin{cases} \displaystyle\int_{\Omega_F} g \frac{\partial \rho_F^0}{\partial z} (\mathbf{u}_{\mathrm{GW}}^0 \cdot \mathbf{i}_z)(\delta\mathbf{u}_{\mathrm{GW}}^0 \cdot \mathbf{i}_z) \\ \text{and} \\ \alpha_i \delta_{ij} \mu_j \omega_{j,\mathrm{GW}}^2, \end{cases} \tag{B.3}$$

$$\begin{aligned} &\int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta\mathbf{u}_{\mathrm{AC}} \nabla \cdot \mathbf{u}_{\mathrm{AC}} + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_{\mathrm{AC}} \nabla \cdot \delta\mathbf{u}_{\mathrm{AC}} \\ &\Rightarrow \begin{cases} \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta\mathbf{u}_{\mathrm{AC}}^0 \nabla \cdot \mathbf{u}_{\mathrm{AC}}^0 + \beta_i \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta\mathbf{u}_{\mathrm{AC}}^0 \nabla \cdot \mathbf{u}_{\mathrm{AC}}^i + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_{\mathrm{AC}}^0 \nabla \cdot \delta\mathbf{u}_{\mathrm{AC}}^0 + \beta i \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_{\mathrm{AC}}^i \nabla \cdot \delta\mathbf{u}_{\mathrm{AC}}^0 \\ \text{and} \\ \displaystyle\int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_{\mathrm{AC}}^j \nabla \cdot \mathbf{u}_{\mathrm{AC}}^0 + \beta_i \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_{\mathrm{AC}}^j \nabla \cdot \mathbf{u}_{\mathrm{AC}}^i + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_{\mathrm{AC}}^0 \nabla \cdot \mathbf{u}_{\mathrm{AC}}^j + \beta_i \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_{\mathrm{AC}}^i \nabla \cdot \mathbf{u}_{\mathrm{AC}}^j. \end{cases} \end{aligned} \tag{B.4}$$

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