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Compressibility and gravity effects in internal fluid–structure vibrations: Basic equations and appropriate variational formulations

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Various variational formulations for the linear vibrations of bounded fluid–structure systems, taking into account possible gravity/acoustic interactions are investigated and discussed. The system consists of a tank partially filled with an inviscid liquid. Two situations corresponding to the cases of a *homogeneous* and a *non-homogeneous* liquid are considered. In this respect, the basic equations describing the interactions between acoustic and gravity waves are derived and appropriate variational formulations are constructed.

Keywords: Fluid–structure interactions; Gravity; Compressibility; Acoustic; Sloshing

1. Introduction

Within the context of linear vibrations analysis of coupled internal fluid–structure interactions, it is customary to consider two distinct situations: linear vibrations of an elastic structure filled with an inviscid *homogeneous compressible* fluid neglecting gravity effects for structural-acoustic problems and vibrations of an elastic structure containing an inviscid *homogeneous incompressible* liquid with free surface effects due to gravity. Various symmetric variational formulations based on scalar (pressure and/or displacement potential and/or velocity potential) fields have been derived leading to finite element discretization for

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interior problems and to reduced order symmetric matrix models through adapted component mode analysis (see for instance [11,12,15,14]).

It should be noted that in those formulations, fluid–structure interaction taking into account simultaneously acoustic and gravity internal waves (specially for intermediate frequency ranges) have not been generally considered. Such interaction have been only considered through physical assumptions which amount to consider gravity effects only on the boundary of the fluid domain (liquid free surface and fluid–structure interface). Those studies are motivated by the consideration of cryogenics liquids contained in elastic tanks for aerospace industry—liquid propelled launchers and satellites [13].

The standard linearized Euler equations for linearized vibrations of inviscid fluid must be reconsidered in order to take into account coupling between acoustic and gravity waves. On the other hand, greater attention must be paid to the kinematics of the liquid. As it will be shown, in this case of full coupling between gravity and compressibility, the liquid must not any more be considered as homogeneous. The usual description of the liquid displacement or velocity in terms of a scalar potential must then be reconsidered because the classical irrotationality property does not hold any more. It is shown that this property is replaced by an *in-plane* irrotationality condition.

The objective and the originality of this paper is to present the variational framework for modeling fluid–structure interaction problems where possible couplings between internal gravity and compressibility effects may occur. After the derivation of the local equations, appropriate symmetric variational formulations are constructed. In a first step, the general case of an inviscid *non-homogeneous* liquid is analyzed. In a second step, the particular case of an inviscid *homogeneous* stratified liquid is derived. Preliminary numerical results are then presented.

2. General case of an inviscid non-homogeneous liquid

The purpose of this section is to derive the variational formulations associated with a fluid–structure coupled problem subjected to gravity field [1]. Basic linear equations are first derived, then it is shown that combination of gravity field and non-homogeneity leads to a particular kinematically admissible space, which can be exploited to derive original variational formulations.

Let us consider an elastic structure occupying at rest the volume Ω_S containing a liquid occupying the volume Ω_F with a free surface Γ . The fluid–structure interface is denoted by Σ . \mathbf{n} and \mathbf{i}_z denote the outward unit normal of the fluid–structure interface and the liquid free surface. The system is submitted to prescribed forces \mathbf{F}_d on $\Sigma_d(\partial\Omega_S \setminus \Sigma)$. The notations are those of Fig. 1.

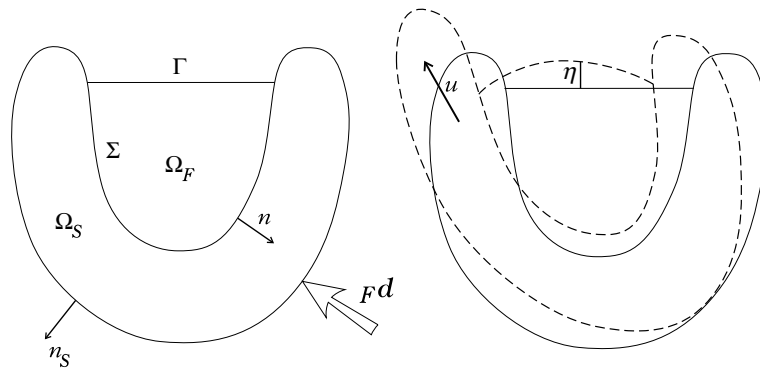


Fig. 1. Fluid–structure coupled system.

2.1. Basic linear local equations

In order to derive the linear local equations, the following quantities are considered:

- The Eulerian pressure fluctuation

$$p(M, t) = P_{(M, t)} - P_{(M)}^0 \quad (1)$$

in which $P_{(M)}^0$ denotes the hydrostatic pressure at rest.

- The Lagrangian pressure fluctuation in the liquid

$$p_{\mathcal{L}}(M, t) = P_{(M', t)} - P_{(M)}^0. \quad (2)$$

- The Eulerian density fluctuation

$$\rho(M, t) = \rho_F(M, t) - \rho_F^0(M). \quad (3)$$

- The Lagrangian density fluctuation in the liquid

$$\rho_{\mathcal{L}}(M, t) = \rho_F(M', t) - \rho_F^0(M). \quad (4)$$

It can be shown that these linearized quantities satisfy the following relations [11]:

$$\begin{cases} p_{\mathcal{L}} - p = \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F & \forall M \in \Omega_F, \\ p_{\mathcal{L}} = c^2 \rho_{\mathcal{L}} & \forall M \in \Omega_F, \\ \rho_{\mathcal{L}} = -\rho_F^0 \text{Div} \mathbf{u}_F & \forall M \in \Omega_F, \end{cases} \quad (5)$$

where $\rho_F^0(M)$ is the liquid density in the reference configuration.

Combination of Eq. (5)₂ with Eq. (5)₃ yields the following constitutive relation:

$$\frac{p}{c^2} + \rho_F^0 \text{Div} \mathbf{u}_F - \frac{\rho_F^0 \mathbf{g}}{c^2} (\mathbf{u}_F \cdot \mathbf{i}_z) = 0 \quad (6)$$

in which c is the velocity of sound.

Performing a first order Taylor expansion, the following relation is moreover obtained:

$$\rho_{\mathcal{L}} - \rho = \rho_F'(M', t) - \rho_F(M, t) \simeq \nabla \rho_F^0 \cdot \mathbf{u}_F = (\mathbf{u}_F \cdot \mathbf{i}_z) \frac{\partial \rho_F^0}{\partial z}. \quad (7)$$

In Eq. (7), M' (belonging to the actual configuration Ω'_F), and M (belonging to the reference configuration at rest Ω_F), are related by

$$M' = M + \mathbf{u}_F(M). \quad (8)$$

Moreover the last equality of Eq. (7) will be justified in the following subsection where it is shown that the fluid density ρ_F^0 depends only on the vertical coordinate z .

Taking into consideration Eqs. (5) and (7) the following constitutive relation is also obtained:

$$\rho_{(M, t)} = \frac{p}{c^2} - \left(\frac{\partial \rho_F^0}{\partial z} + \rho_F^0 \frac{\mathbf{g}}{c^2} \right) (\mathbf{u}_F \cdot \mathbf{i}_z). \quad (9)$$

To proceed further, the dynamical equilibrium equations governing the liquid domain behaviour can be obtained through the Euler equations:

$$\begin{cases} \nabla P_{(M')} = \rho'_{(M', t)} [\mathbf{g} - \gamma] & (M' \in \Omega'_F), \\ \nabla P^0 = \rho_F^0 \mathbf{g} & (M \in \Omega_F), \end{cases} \quad (10)$$

where γ is the acceleration of the fluid particle.

Linearization of Eq. (10) in the framework of vibrations, in conjunction with definitions (1)–(4) gives

$$\nabla p = \rho_{(M)} \mathbf{g} + \omega^2 \rho_F^0 \mathbf{u}_F, \quad (11)$$

where ω is the angular frequency.

On account of the constitutive relation (9), the previous equilibrium equation turns into

$$\nabla p = \frac{p}{c^2} \mathbf{g} - \left[\frac{\partial \rho_F^0}{\partial z} + \frac{\rho_F^0 g}{c^2} \right] g(\mathbf{u}_F \cdot \mathbf{i}_z) + \rho_F^0 \omega^2 \mathbf{u}_F. \quad (12)$$

Eq. (12) may then be written in a more convenient way by defining the following quantity:

$$N^2 = - \left[g \frac{\partial \rho_F^0}{\partial z} + \frac{\rho_F^0 g^2}{c^2} \right]. \quad (13)$$

N^2 is called the Brunt–Väisälä frequency and it can be shown that its positiveness is a necessary condition for the liquid to be stable under the action of gravity field [9, Chapter 4.1].

Note that the constitutive relation (6) is still to be considered here so that the two following equations state:

$$\begin{cases} \nabla p = \frac{p}{c^2} \mathbf{g} - \rho_F^0 N^2 (\mathbf{u}_F \cdot \mathbf{i}_z) + \rho_F^0 \omega^2 \mathbf{u}_F & (\Omega_F), \\ \frac{p}{c^2} = -\rho_F^0 \text{Div} \mathbf{u}_F + \frac{\rho_F^0 g}{c^2} (\mathbf{u}_F \cdot \mathbf{i}_z) & (\Omega_F). \end{cases} \quad (14)$$

Eqs. (14) may then be recasted as follows:

$$\begin{cases} \nabla p = -\rho_F^0 \mathbf{g} \text{Div} \mathbf{u}_F + g \frac{\partial \rho_F^0}{\partial z} (\mathbf{u}_F \cdot \mathbf{i}_z) \mathbf{i}_z + \rho_F^0 \omega^2 \mathbf{u}_F, \\ \frac{p}{c^2} = -\rho_F^0 \text{Div} \mathbf{u}_F + \frac{\rho_F^0 g}{c^2} (\mathbf{u}_F \cdot \mathbf{i}_z). \end{cases} \quad (15)$$

2.2. Considerations on the admissible space

Attention is focused here on the kinematically admissible space in which the fluid displacement field is sought. It is shown that the fluid has a so-called *in-plane irrotationality*.

Let us consider the hydrostatic equation

$$\nabla P^0 = \rho_F^0(M) \mathbf{g}. \quad (16)$$

Taking the rotational of Eq. (16) leads to

$$\text{curl} \rho_F^0(M) \mathbf{g} = 0 \quad (17)$$

or

$$\begin{cases} g \frac{\partial \rho_F^0}{\partial y} = 0, \\ g \frac{\partial \rho_F^0}{\partial x} = 0. \end{cases} \quad (18)$$

It follows from Eq. (18) that:

$$\rho_F^0(M) = \rho_F^0(z) \quad (19)$$

which means that under the gravity field the liquid is horizontally stratified.

Consider now the dynamical equilibrium equation (15). Projecting this equation onto the horizontal plane, and taking into account (19), results in

$$\nabla_\alpha p = \rho_F^0 \omega^2 u_\alpha^F, \quad (20)$$

where the Greek indices α refers to horizontal coordinates (1, 2).

As a consequence, the admissible space of the fluid displacement field is

$$\mathcal{V}_u^F = \{\mathbf{u}_F \in \Omega_F / \text{curl } u_\alpha^F = 0\}. \quad (21)$$

Therefore an in-plane *horizontal* displacement potential ψ , defined up to an additive constant, can be introduced

$$u_\alpha^F = \nabla_\alpha \psi. \quad (22)$$

2.3. Symmetric variational formulations in terms of $(\mathbf{u}_S, p, \psi, \zeta)$

A mixed variational formulation for the liquid-structure vibrations is now derived, taking into account gravity and compressibility effects for non-homogeneous fluid.

Let us first introduce the internal energy \mathcal{W} of the liquid.

It can be shown (see [9]) that this internal energy can be expressed as follows:

$$\mathcal{W} = \int_{\Omega_F} \frac{p^2}{2\beta} + \int_{\Omega_F} \frac{1}{2} \rho_F^0 N^2 (\mathbf{u}_F \cdot \mathbf{i}_z)^2 + \int_\Gamma \frac{1}{2} \rho_F^0 g (\mathbf{u}_F \cdot \mathbf{i}_z)^2, \quad (23)$$

where the Brunt–Väisälä frequency N^2 has been defined in Eq. (13) and the parameter β is defined by $\beta = \rho_F^0 c^2$.

The internal energy consists of

- a potential energy due to acoustic waves,
- a potential energy due to the so-called internal gravity waves,
- a free-surface potential energy.

It should be emphasized that gravity acts in two different ways: inside the liquid domain, which corresponds to the second volume integral of Eq. (23) and on the free-surface (third surface integral in Eq. (23)). It should be also noticed that the compressibility-gravity coupling aspects are contained in the Brunt–Väisälä frequency.

Two types of variational formulations can be derived, a so-called stiffness coupling formulation where the coupling terms between the field variables are not multiplied by ω^2 and a so-called mass coupling formulation (those considerations may be easily seen through a matrix discretization of the formulation in the framework of a finite element procedure for instance) (see [11,19]).

2.3.1. Stiffness coupling formulation

The stiffness coupling formulation can be derived by considering the following Lagrangian functional for the fluid part (introducing $\zeta = \mathbf{u}_F \cdot \mathbf{i}_z$ and omitting, in the sequel, for sake of brevity, the subscript F for the liquid displacement notation \mathbf{u}_F):

$$\begin{aligned} \mathcal{L}_f(u_\alpha, \zeta, p, \lambda) = & \int_{\Omega_F} \left(\frac{p^2}{2\beta} + \frac{1}{2} \rho_F^0 N^2 \zeta^2 \right) d\tau + \int_\Gamma \frac{1}{2} \rho_F^0 g \zeta^2 d\sigma - \int_{\Omega_F} \lambda \left[\frac{p}{\beta} + \nabla \cdot \mathbf{u}_\alpha + \partial_z \zeta - \frac{g}{c^2} \zeta \right] d\tau \\ & - \int_{\Omega_F} \frac{1}{2} \rho_F^0 \omega^2 (u_\alpha^2 + \zeta^2) d\tau, \end{aligned} \quad (24)$$

where, as can be seen, the displacement vector has been split into in-plane (referred to by Greek indices) and out of plane components. Notice also that the constitutive relation is enforced through a Lagrange multiplier λ .

Stationarity conditions with respect to u_α lead to

$$-\int_{\Omega_F} \lambda \nabla \cdot \delta u_\alpha d\tau - \int_{\Omega_F} \rho_F^0 \omega^2 u_\alpha \cdot \delta u_\alpha d\tau = 0. \quad (25)$$

Applying the Green formula, we obtain the local equation

$$\nabla_\alpha \lambda = \rho_F^0 \omega^2 u_\alpha \quad (26)$$

or (for $\omega \neq 0$)

$$\nabla_\alpha \left(\frac{\lambda}{\rho_F^0 \omega^2} \right) = u_\alpha \quad (27)$$

so that an in-plane *horizontal* displacement potential ψ can be defined as

$$u_\alpha = \nabla_\alpha \psi. \quad (28)$$

In the same way, writing the stationarity condition with respect to the pressure variable allows us to define the Lagrange multiplier λ as the pressure p . Substituting back the expressions of λ and u_α in Eq. (24), the Lagrangian reduces to

$$\begin{aligned} \mathcal{L}_f(\psi, \zeta, p) = & \int_{\Omega_F} \left(-\frac{p^2}{2\beta} + \frac{1}{2} \rho_F^0 N^2 \zeta^2 \right) d\tau + \int_\Gamma \frac{1}{2} \rho_F^0 g \zeta^2 d\sigma \\ & - \int_{\Omega_F} p \left[\nabla \cdot \nabla_\alpha \psi + \partial_z \zeta - \frac{g}{c^2} \zeta \right] d\tau - \int_{\Omega_F} \frac{1}{2} \rho_F^0 \omega^2 (\nabla_\alpha \psi^2 + \zeta^2) d\tau. \end{aligned} \quad (29)$$

Considering again the variational formulation Eq. (64) associated to the response of the structure subjected to a fluid pressure loading, and taking into account the kinematic condition at the fluid–structure interface, one obtains the following stiffness coupling formulation:

$$\left\{ \begin{array}{l} \text{given } \omega \in \mathbb{R}^+, \text{ and } \mathbf{F}_d, \text{ find } (\mathbf{u}_S, p, \psi, \zeta) \text{ such that } \forall (\delta \mathbf{u}_S, \delta p, \delta \psi, \delta \zeta) : \\ \hat{K}(\mathbf{u}_S, \delta \mathbf{u}_S) - \int_\Sigma \delta \mathbf{u}_S \cdot \mathbf{n} p d\sigma - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u}_S \cdot \delta \mathbf{u}_S d\tau = \int_{\Sigma_d} \delta \mathbf{u}_S \cdot \mathbf{F}_d d\sigma, \\ \int_{\Omega_F} \nabla_\alpha p \cdot \nabla_\alpha \delta \psi d\tau = \omega^2 \int_{\Omega_F} \rho_F^0 \nabla_\alpha \psi \cdot \nabla_\alpha \delta \psi d\tau, \\ \int_{\Omega_F} \rho_F^0 N^2 \zeta \delta \zeta d\tau + \int_\Gamma \rho_F^0 g \zeta \delta \zeta d\sigma + \int_{\Omega_F} \frac{g}{c^2} p \delta \zeta d\tau - \int_{\Omega_F} p \partial_z \delta \zeta d\tau \\ \quad \cdots + \int_\Sigma p \delta \zeta (\mathbf{i}_z \cdot \mathbf{n}) d\sigma = \omega^2 \int_{\Omega_F} \rho_F^0 \zeta \delta \zeta d\tau, \\ \int_{\Omega_F} -\frac{p \delta p}{\beta} d\tau + \int_{\Omega_F} \frac{g}{c^2} \delta p \zeta d\tau - \int_{\Omega_F} \delta p \partial_z \zeta d\tau + \int_{\Omega_F} \nabla_\alpha \psi \cdot \nabla_\alpha \delta p d\tau \\ \quad \cdots + \int_\Sigma \zeta \delta p (\mathbf{i}_z \cdot \mathbf{n}) d\sigma - \int_\Sigma \delta p \mathbf{n} \cdot \mathbf{u}_S d\sigma = 0. \end{array} \right. \quad (30)$$

2.3.2. Mass coupling formulation

It was established in the previous section that the splitting of the displacement field into in-plane and normal components allows us to define a displacement potential ψ that accounts for the in-plane irrotationality condition. Let us now consider the following Lagrangian functional:

$$\mathcal{L}_f(\psi, \zeta, p) = \int_{\Omega_F} \left(\frac{p^2}{2\beta} + \frac{1}{2} \rho_F^0 N^2 \zeta^2 \right) d\tau + \int_{\Gamma} \frac{1}{2} \rho_F^0 g \zeta^2 d\sigma - \int_{\Omega_F} \frac{1}{2} \rho_F^0 \omega^2 (\nabla_\alpha \psi^2 + \zeta^2) d\tau, \quad (31)$$

where the variables (ψ, ζ, p) are sought to satisfy the relations

$$\begin{cases} \frac{p}{\beta} + \nabla \cdot \nabla_\alpha \psi + \partial_z \zeta - \frac{g}{c^2} \zeta = 0 & (\Omega_F), \\ \partial_n \psi + \zeta (\mathbf{i}_z \cdot \mathbf{n}) = \mathbf{u}_S \cdot \mathbf{n} & (\Sigma). \end{cases} \quad (32)$$

Applying a partial Legendre–Fenchel transform (see [16] for a similar derivation of the mass-coupling formulation and [3,17] for details on Legendre transforms) to the third integral

$$\int_{\Omega_F} \frac{1}{2} \rho_F^0 (\nabla_\alpha \psi^2 + \zeta^2) d\tau$$

of Eq. (31) and using the expression of $\nabla \cdot \nabla_\alpha \psi$ extracted from Eq. (32), one obtains

$$- \int_{\Omega_F} \frac{1}{2} \rho_F^0 \nabla_\alpha \psi^2 d\tau - \int_{\Omega_F} \rho_F^0 \psi \left[-\frac{p}{\beta} - \partial_z \zeta + \frac{g}{c^2} \zeta \right] d\tau \cdots + \int_{\Sigma} \rho_F^0 \psi \mathbf{u}_S \cdot \mathbf{n} d\sigma + \int_{\Omega_F} \frac{1}{2} \rho_F^0 \zeta^2 d\tau. \quad (33)$$

Hence the Lagrangian for the fluid part writes

$$\begin{aligned} \mathcal{L}_f(\psi, \zeta, p) = & \int_{\Omega_F} \left(\frac{p^2}{2\beta} + \frac{1}{2} \rho_F^0 N^2 \zeta^2 \right) d\tau + \int_{\Gamma} \frac{1}{2} \rho_F^0 g \zeta^2 d\sigma \\ & \cdots + \omega^2 \left\{ \int_{\Omega_F} \frac{1}{2} \rho_F^0 \nabla_\alpha \psi^2 d\tau + \int_{\Omega_F} \rho_F^0 \psi \left[-\frac{p}{\beta} - \partial_z \zeta + \frac{g}{c^2} \zeta \right] d\tau \right\} \\ & \cdots - \omega^2 \left\{ \int_{\Sigma} \rho_F^0 \psi \mathbf{u}_S \cdot \mathbf{n} d\sigma + \int_{\Omega_F} \frac{1}{2} \rho_F^0 \zeta^2 d\tau \right\}. \end{aligned} \quad (34)$$

The mass coupling variational formulation can finally be obtained by writing the stationarity condition of \mathcal{L}_f combined with the response of the structure to a pressure loading. One then obtains

$$\left\{ \begin{array}{l} \text{Given } \omega \in \mathbb{R}^+, \text{ and } \mathbf{F}_d, \text{ find } (\mathbf{u}_S, p, \psi, \zeta) \text{ such that } \forall (\delta \mathbf{u}_S, \delta p, \delta \psi, \delta \zeta) : \\ \widehat{K}(\mathbf{u}_S, \delta \mathbf{u}_S) - \omega^2 \int_{\Sigma} \rho_F^0 \psi \delta \mathbf{u}_S \cdot \mathbf{n} d\sigma - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u}_S \cdot \delta \mathbf{u}_S d\tau = \int_{\Sigma_d} \delta \mathbf{u}_S \cdot \mathbf{F}_d d\sigma, \\ -\omega^2 \left\{ - \int_{\Omega_F} \rho_F^0 \nabla_\alpha \psi \cdot \nabla_\alpha \delta \psi + \int_{\Omega_F} \rho_F^0 \delta \psi \frac{p}{\beta} + \int_{\Omega_F} \rho_F^0 \delta \psi \partial_z \zeta - \int_{\Omega_F} \frac{\rho_F^0 g}{c^2} \zeta \delta \psi \right\} \\ \cdots - \omega^2 \int_{\Sigma} \rho_F^0 \delta \psi \mathbf{u}_S \cdot \mathbf{n} = 0, \\ \int_{\Omega_F} \rho_F^0 N^2 \zeta \delta \zeta + \int_{\Gamma} \rho_F^0 g \zeta \delta \zeta - \omega^2 \left\{ \int_{\Omega_F} \rho_F^0 \zeta \delta \zeta - \int_{\Omega_F} \rho_F^0 \psi \frac{g}{c^2} \delta \zeta + \int_{\Omega_F} \rho_F^0 \psi \partial_z \delta \zeta \right\} = 0, \\ \int_{\Omega_F} \frac{p \delta p}{\beta} - \omega^2 \int_{\Omega_F} \frac{\psi \delta p}{c^2} = 0. \end{array} \right. \quad (35)$$

3. Particular case of an inviscid homogeneous liquid

This section is devoted to the analysis of fluid–structure interaction problems, taking into account both gravity and compressibility effects [1]. The liquid is considered here as *homogeneous*.

The linear local equations and variational formulations can obviously be derived from the previous section by considering a constant density.

Nevertheless, as it is shown below, a more direct approach is possible. The local equations are obtained through linearization of the Piola–Lagrange stress tensor around a prestressed reference configuration whereas the variational formulation is derived through the classical principle of virtual work.

3.1. Liquid subjected to a prescribed normal displacement

The variational formulation describing the response of the fluid to an imposed normal displacement on the fluid–structure interface Σ is obtained through the principle of virtual work

$$\int_{\Omega_F} \text{Div } \Theta \cdot \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F \mathbf{g} \cdot \delta \mathbf{u}_F = -\omega^2 \int_{\Omega_F} \rho_F \mathbf{u}_F \cdot \delta \mathbf{u}_F, \quad (36)$$

where Θ refers to the Piola–Lagrange stress tensor defined by

$$\Theta(M, t) = -JP_{(M', t)} Id F^{-T}, \quad (37)$$

where $P_{(M', t)}$ denotes the instantaneous pressure in the actual configuration, F denotes the gradient of the transformation $M \rightarrow M'$, J is its jacobian and Id is the identity matrix.

In Eq. (36), $\delta \mathbf{u}_F$ denotes the virtual variation of the fluid displacement vector field \mathbf{u}_F , ρ_F and \mathbf{g} denote respectively the mass density and the gravity (supposed constant) and ω denotes the angular frequency of the vibrations.

In order to linearize Θ , we consider the quantities defined in Eqs. (1)–(4) under the assumption that $\rho_F^0(M)$ is constant.

Notice that in the particular case of a *homogeneous weightless (gravity neglected) compressible* fluid or *homogeneous heavy incompressible* liquid, p and $p_{\mathcal{F}}$ are identical in Eq. (5)₁. Otherwise these two state variables are to be distinguished and combination of Eq. (5)₂ with Eq. (5)₃ leading to constitutive relation (6) is still valid.

Considering now the first order Taylor expansion of J and F^{-T} , it can be shown [7] that the Piola–Lagrange stress tensor defined in Eq. (37) writes

$$\Theta = -p_{\mathcal{F}} Id - P^0 Id + P^0 \nabla^T \mathbf{u}_F - P^0 Id \text{ Div } \mathbf{u}_F. \quad (38)$$

Applying Green formula to Eq. (36) and letting Θ be replaced by its above stated expression, the latter Eq. (36) turns into

$$\int_{\Omega_F} \Theta \cdot \nabla \delta \mathbf{u}_F - \int_{\partial \Omega_F} \Theta \mathbf{n} \cdot \delta \mathbf{u}_F - \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F \quad (39)$$

and

$$\begin{aligned} & \int_{\Omega_F} \rho_F^0 c^2 \text{Div } \mathbf{u}_F \text{ Div } \delta \mathbf{u}_F + \int_{\Omega_F} P^0 \nabla^T \mathbf{u}_F \cdot \nabla \delta \mathbf{u}_F - \int_{\Omega_F} P^0 \text{Div } \mathbf{u}_F \text{ Div } \delta \mathbf{u}_F - \int_{\partial \Omega_F} \Theta \mathbf{n} \cdot \delta \mathbf{u}_F \\ & - \int_{\Omega_F} P^0 \text{Div } \delta \mathbf{u}_F - \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F. \end{aligned} \quad (40)$$

Moreover the hydrostatic equation writes

$$\nabla P^0 = \rho_F^0 \mathbf{g}. \quad (41)$$

Applying test function method to Eq. (41) leads to

$$\int_{\Omega_F} P^0 \text{Div } \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F - \int_{\partial \Omega_F} P^0 \mathbf{n} \cdot \delta \mathbf{u}_F = 0. \quad (42)$$

Hence combination of Eq. (40) and Eq. (42) finally gives

$$\begin{aligned} & \int_{\Omega_F} \rho_F^0 c^2 \text{Div } \mathbf{u}_F \text{Div } \delta \mathbf{u}_F - \int_{\Omega_F} P^0 \text{Div } \mathbf{u}_F \text{Div } \delta \mathbf{u}_F + \int_{\Omega_F} P^0 \nabla^T \mathbf{u}_F \cdot \nabla \delta \mathbf{u}_F \\ & - \int_{\partial \Omega_F} (\Theta + P^0) \mathbf{n} \cdot \delta \mathbf{u}_F = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F. \end{aligned} \quad (43)$$

Alternatively, another formulation of this problem can be derived by expressing the Piola–Lagrange tensor Θ in terms of the Eulerian fluctuation pressure p .

Namely, substitution of Eq. (5)₁ into Eq. (38) leads to

$$\Theta = -(p + \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F) Id - P^0 Id + P^0 \nabla^T \mathbf{u}_F - P^0 \text{Div } \mathbf{u}_F Id. \quad (44)$$

Hence

$$\text{Div } \Theta = -\nabla p - \nabla P^0 - \rho_F^0 \mathbf{g} \text{Div } \mathbf{u}_F. \quad (45)$$

Equilibrium Eq. (36) then writes

$$-\int_{\Omega_F} \nabla p \cdot \delta \mathbf{u}_F - \int_{\Omega_F} \nabla P^0 \cdot \delta \mathbf{u}_F - \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F \text{Div } \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F = -\omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F. \quad (46)$$

Applying Green formula to the first integral of Eq. (46) and taking into account Eq. (5) and Eq. (40), one obtains

$$\begin{aligned} & \int_{\Omega_F} \rho_F^0 c^2 \text{Div } \mathbf{u}_F \text{Div } \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F \text{Div } \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F \text{Div } \delta \mathbf{u}_F \\ & + \int_{\partial \Omega_F} p \mathbf{n} \cdot \delta \mathbf{u}_F = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F. \end{aligned} \quad (47)$$

In terms of the Lagrangian fluctuation, the free surface condition on Γ writes

$$p_{\mathcal{L}} = 0 \quad (48)$$

which can be written in terms of the Eulerian fluctuation as

$$p = \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F. \quad (49)$$

The boundary integral in Eq. (47) then writes

$$\int_{\partial \Omega_F} p \mathbf{n} \cdot \delta \mathbf{u}_F = \int_{\Sigma} p \mathbf{n} \cdot \delta \mathbf{u}_F + \int_{\Gamma} \rho_F^0 g (\mathbf{u}_F \cdot \hat{\mathbf{i}}_z) (\delta \mathbf{u}_F \cdot \hat{\mathbf{i}}_z). \quad (50)$$

Eq. (47) then leads to

$$\begin{aligned} & \int_{\Omega_F} \rho_F^0 c^2 \text{Div } \mathbf{u}_F \text{Div } \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F \text{Div } \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F \text{Div } \delta \mathbf{u}_F + \int_{\Gamma} \rho_F^0 g \mathbf{u}_F \cdot \delta \mathbf{u}_F \\ & + \int_{\Sigma} p \mathbf{n} \cdot \delta \mathbf{u}_F = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F \end{aligned} \quad (51)$$

from which, the following local equations in Ω_F are easily derived (by integrations by parts):

$$\nabla p + \rho_F^0 \mathbf{g} \text{Div } \mathbf{u}_F = \rho_F^0 \omega^2 \mathbf{u}_F \quad (52)$$

in which p is expressed as a function of \mathbf{u}_F as

$$-p = \rho_F^0 c^2 \text{Div } \mathbf{u}_F + \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F. \quad (53)$$

Remark. It can be seen that the *Eulerian* pressure fluctuation depends upon gravity while its expression in terms of *Lagrangian* pressure fluctuation is independent of gravity (see also Eq. (5)).

From Eq. (52) we notice that, by applying the curl operator, generally

$$\text{curl} \mathbf{u}_F \neq 0. \quad (54)$$

If we neglect internal gravity waves (i.e. gravity terms in Eq. (52) and Eq. (53)), keeping only gravity effects on the free surface Γ , then for $\omega \neq 0$, \mathbf{u}_F satisfies the usual *irrotationality* condition

$$\text{curl} \mathbf{u}_F = 0. \quad (55)$$

Let us now introduce the following admissible spaces:

$$\mathcal{C}_F = \{\mathbf{u}_F / \mathbf{u}_F \cdot \mathbf{n} = u_N \text{ on } \Sigma\}, \quad (56)$$

$$\mathcal{C}_F^* = \{\mathbf{u}_F / \mathbf{u}_F \cdot \mathbf{n} = 0 \text{ on } \Sigma\}, \quad (57)$$

where u_N denotes a prescribed normal displacement field of the fluid–structure interface Σ .

We can then state the following symmetric variational formulation in terms of \mathbf{u}_F :

$$\left\{ \begin{array}{l} \text{for given } \omega \in \mathbb{R}^+, \text{ find } \mathbf{u}_F \in \mathcal{C}_F, \text{ such that } \forall \delta \mathbf{u}_F \in \mathcal{C}_F^* : \\ \int_{\Omega_F} \rho_F^0 c^2 \text{Div} \mathbf{u}_F \text{Div} \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F \text{Div} \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F \text{Div} \delta \mathbf{u}_F \\ + \int_{\Gamma} \rho_F^0 \mathbf{g} \mathbf{u}_F \cdot \delta \mathbf{u}_F - \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F = 0. \end{array} \right. \quad (58)$$

The local boundary value problem is then derived from the previous variational formulation as follows (using Green formula):

$$\left\{ \begin{array}{ll} \nabla p(\mathbf{u}_F) + \rho_F^0 \mathbf{g} \text{Div} \mathbf{u}_F = \rho_F^0 \omega^2 \mathbf{u}_F & (\Omega_F), \\ p(\mathbf{u}_F) = \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F & (\Gamma), \\ \mathbf{u}_F \cdot \mathbf{n} = u_N & (\Sigma) \end{array} \right. \quad (59)$$

in which $p(\mathbf{u}_F)$ is defined by Eq. (53).

The formulations defined by Eqs. (58) and (59) contain internal gravity terms (contributions of gravity inside the domain Ω_F) which are often neglected.

In this last case, in which compressibility effects (in the domain Ω_F) and gravity effects (on the free surface Γ) are taken into account, the irrotationality condition holds and scalar formulations for the fluid together with the introduction of the elevation η of the fluid surface can be derived [16,10,8,5,18].

More specifically, the curl-free condition may be dealt through a displacement potential φ taken as an auxiliary variable in place of the displacement field \mathbf{u}_F . Hence, following a procedure described in [11], it can then be shown that the fluid domain may be completely described by three independent *scalar* fields: the pressure p , a displacement potential φ and the free-surface elevation η . The choice of these three fields ensures the variational formulation to satisfy *exactly* the curl-free condition on one hand and to be symmetric on the other hand. Consistent with the above description of the fluid behaviour, two types of $(\mathbf{u}_S, p, \varphi, \eta)$ variational formulations can be constructed: a mass coupling or a stiffness coupling formulation [16,6].

As a particular case, under the assumption that the fluid is *homogeneous*, *compressible* and *weightless* (gravity neglected), Eq. (58), restricted to irrotational motions for \mathbf{u}_F , reduces to

$$\int_{\Omega_F} \rho_F^0 c^2 \text{Div} \mathbf{u}_F \text{Div} \delta \mathbf{u}_F = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F. \quad (60)$$

The associated local equations are then

$$\begin{cases} \nabla p = \rho_F^0 \omega^2 \mathbf{u}_F & (\Omega_F), \\ -p = \rho_F^0 c^2 \text{Div} \mathbf{u}_F & (\Omega_F), \\ p = 0 & (\Gamma), \\ \mathbf{u}_F \cdot \mathbf{n} = u_N & (\Sigma). \end{cases} \quad (61)$$

Another particular case consists in considering the fluid as *homogeneous, incompressible* and with gravity acting on free surface. Eq. (58), restricted to irrotational motions for \mathbf{u}_F , and also to the incompressibility condition, then reduces to

$$\int_{\Gamma} \rho_F^0 \mathbf{g} \mathbf{u}_F \cdot \delta \mathbf{u}_F = \omega^2 \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F \quad (62)$$

with the corresponding local equations

$$\begin{cases} \nabla p = \rho_F^0 \omega^2 \mathbf{u}_F & (\Omega_F), \\ \text{Div} \mathbf{u}_F = 0 & (\Omega_F), \\ p = \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F & (\Gamma), \\ \mathbf{u}_F \cdot \mathbf{n} = u_N & (\Sigma). \end{cases} \quad (63)$$

For those cases, one can use, for the fluid, either displacement type of unknown as in [4] or scalar field as in [11].

3.2. Structure subjected to a liquid pressure loading

The unknown displacement field in the structure domain is denoted as \mathbf{u}_S , the associated linearized deformation tensor as $\varepsilon_{ij}(\mathbf{u}_S)$ and the corresponding stress tensor $\sigma_{ij}(\mathbf{u}_S)$. We also denote as ρ_S the constant mass density of the structure at equilibrium and $\delta \mathbf{u}_S$ the test function associated to \mathbf{u}_S , belonging to the kinematically admissible space \mathcal{C}_u^S , classical for linear elasticity (regular functions in Ω_S).

The weak variational formulation describing the response of the structure Ω_S to given harmonic forces of amplitude \mathbf{F}_d on Σ_d , and to fluid pressure field p acting on Σ is written as follows:

$$\begin{cases} \text{for given } \omega \in \mathbb{R}^+, \text{ find } \mathbf{u}_S \in \mathcal{C}_u^S, \text{ such that } \forall \delta \mathbf{u}_S \in \mathcal{C}_u^S : \\ \widehat{K}(\mathbf{u}_S, \delta \mathbf{u}_S) - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u}_S \cdot \delta \mathbf{u}_S - \int_{\Sigma} p \delta \mathbf{u}_S \cdot \mathbf{n} = \int_{\Sigma_d} \mathbf{F}_d \cdot \delta \mathbf{u}_S, \end{cases} \quad (64)$$

$\widehat{K}(\mathbf{u}_S, \delta \mathbf{u}_S)$ denotes the *elastogravity operator*. This operator takes into account the effect of gravity forces on the fluid–structure interface and is defined as

$$\widehat{K}(\mathbf{u}_S, \delta \mathbf{u}_S) = k(\mathbf{u}_S, \delta \mathbf{u}_S) + k_G(\mathbf{u}_S, \delta \mathbf{u}_S) + k_{\Sigma}(\mathbf{u}_S, \delta \mathbf{u}_S), \quad (65)$$

where

- $k(\mathbf{u}_S, \delta \mathbf{u}_S) = \int_{\Omega_S} \sigma_{ij}(\mathbf{u}_S) \varepsilon_{ij}(\delta \mathbf{u}_S)$ is the usual mechanical elastic stiffness,
- $k_G(\mathbf{u}_S, \delta \mathbf{u}_S) = \int_{\Omega_S} \sigma_{ij}^0 u_{m,i} \delta u_{m,j}$ refers to the classical structural geometric prestress symmetric bilinear form in which σ_{ij}^0 denotes the prestress tensor,
- $k_{\Sigma}(\mathbf{u}_S, \delta \mathbf{u}_S) = -\rho_F^0 \mathbf{g} \int_{\Sigma} [z \mathbf{n}_1(\mathbf{u}_S) \cdot \delta \mathbf{u}_S + (\mathbf{i}_z \cdot \mathbf{u}_S)(\delta \mathbf{u}_S \cdot \mathbf{n})]$ represents prestress symmetric bilinear form due to rotation of the external normal \mathbf{n} . z is the vertical coordinate. \mathbf{n} is dependent of the displacement \mathbf{u}_S and $\mathbf{n}_1(\mathbf{u}_S)$ is its first order Taylor-expansion with respect to \mathbf{u}_S .

For a detailed derivation of these operators, see [11,20,21].

3.3. The coupled problem

To write the weak variational formulation associated with the coupled problem, consider the following kinematically admissible space:

- $\mathcal{C}_u^S = \{\mathbf{u}_S \text{ regular in } \Omega_S\},$ (66)

- $\mathcal{C}_u^F = \{\mathbf{u}_F \text{ regular in } \Omega_F\},$ (67)

- $\mathcal{C}_p = \{p \text{ regular in } \Omega_F\},$ (68)

- $\mathcal{C}_u^{S,F} = \{\mathbf{u}_S \in \mathcal{C}_u^S, \mathbf{u}_F \in \mathcal{C}_u^F \text{ such that } \mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_F \cdot \mathbf{n} \text{ on } \Sigma\}.$ (69)

The variational formulation of the coupled problem is obtained by considering both Eqs. (58) and (64):

$$\left\{ \begin{array}{l} \text{given } \omega \in \mathbb{R}^+, \text{ find } (\mathbf{u}_S, \mathbf{u}_F) \in \mathcal{C}_u^{S,F} \text{ such that } \forall (\delta \mathbf{u}_S, \delta \mathbf{u}_F) \in \mathcal{C}_u^{S,F} : \\ \widehat{K}(\mathbf{u}_S, \delta \mathbf{u}_S) + \int_{\Omega_F} \rho_F^0 c^2 \text{Div } \mathbf{u}_F \text{Div } \delta \mathbf{u}_F \\ + \int_{\Gamma} \rho_F^0 g \mathbf{u}_F \cdot \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F \text{Div } \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F \text{Div } \mathbf{u}_F \\ - \omega^2 \left\{ \int_{\Omega_S} \rho_S \mathbf{u}_S \cdot \delta \mathbf{u}_S + \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F \right\} = \int_{\Sigma_d} \mathbf{F}_d \cdot \delta \mathbf{u}_S. \end{array} \right. \quad (70)$$

The kinematic compatibility condition between \mathbf{u}_F and \mathbf{u}_S on Σ may be relaxed by applying the Lagrange multipliers method. It can be shown that the Lagrange multiplier associated with the jump condition identifies itself to the pressure so that the following three-field variational equation is obtained:

$$\left\{ \begin{array}{l} \text{given } \omega \in \mathbb{R}^+, \text{ find } (\mathbf{u}_S \in \mathcal{C}_u^S, \mathbf{u}_F \in \mathcal{C}_u^F, p \in \mathcal{C}_p) \text{ such that} \\ \forall (\delta \mathbf{u}_S, \delta \mathbf{u}_F, \delta p) \in \mathcal{C}_u^S \times \mathcal{C}_u^F \times \mathcal{C}_p : \\ \widehat{K}(\mathbf{u}_S, \delta \mathbf{u}_S) + \int_{\Omega_F} \rho_F^0 c^2 \text{Div } \mathbf{u}_F \text{Div } \delta \mathbf{u}_F \\ + \int_{\Gamma} \rho_F^0 g \mathbf{u}_F \cdot \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \mathbf{u}_F \text{Div } \delta \mathbf{u}_F + \int_{\Omega_F} \rho_F^0 \mathbf{g} \cdot \delta \mathbf{u}_F \text{Div } \mathbf{u}_F \\ + \int_{\Sigma} p \mathbf{n} \cdot (\delta \mathbf{u}_F - \delta \mathbf{u}_S) - \omega^2 \left\{ \int_{\Omega_S} \rho_S \mathbf{u}_S \cdot \delta \mathbf{u}_S + \int_{\Omega_F} \rho_F^0 \mathbf{u}_F \cdot \delta \mathbf{u}_F \right\} = \int_{\Sigma_d} \mathbf{F}_d \cdot \delta \mathbf{u}_S, \\ \int_{\Sigma} \delta p \mathbf{n} \cdot (\mathbf{u}_F - \mathbf{u}_S) = 0. \end{array} \right. \quad (71)$$

4. Numerical results

Preliminary numerical results are presented here to validate the derived formulations. The considered model is a two-dimensional square box of dimensions $a = 1$ m with a liquid height $h = 1$ m. All the walls are taken to be rigid. A free surface is considered where gravity effects act.

Exact analytical solutions can be obtained in that case of a incompressible non-homogeneous liquid contained in a rigid cavity. The derivation of the analytical expression of the eigenvalues are not detailed here. It can be shown that they may be obtained by solving the local equations in terms of a stream function ψ .

Table 1

Analytical/numerical circular eigenfrequencies comparison for internal gravity waves

<i>Eigenvalues IGW</i>				
Mode	(1, 2)	(1, 3)	(2, 2)	(3, 1)
Analytical (rad/s)	0.4470	0.3161	0.7069	0.9485
Computed (rad/s)	0.4509	0.3255	0.7066	0.9453

Table 2

Analytical/numerical circular eigenfrequencies comparison for sloshing gravity waves

<i>Eigenvalues SW</i>				
Mode	(1, 1)	(2, 1)	(2, 2)	(3, 1)
Analytical (rad/s)	5.5383	7.8469	9.6105	11.0973
Computed (rad/s)	5.4958	7.6128	9.0091	9.9753

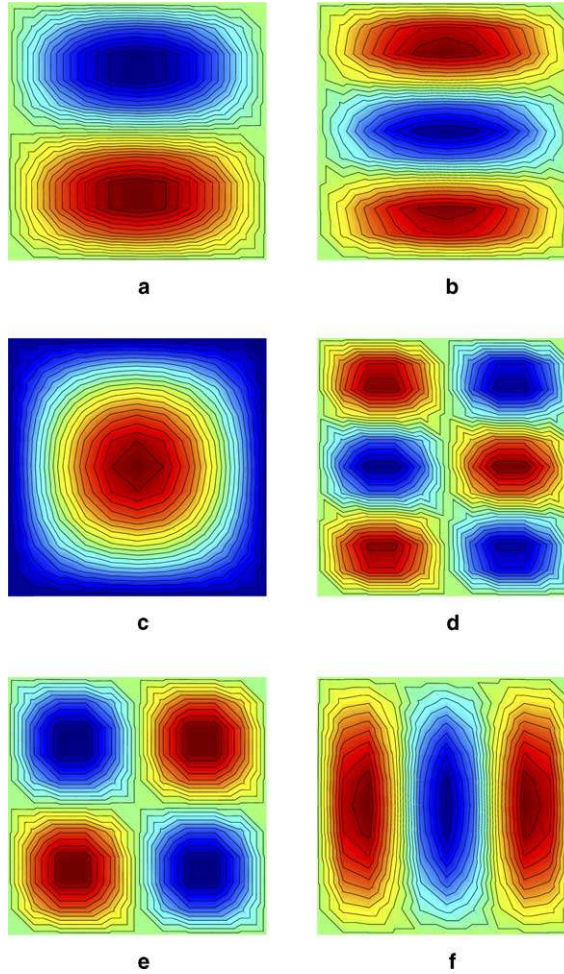


Fig. 2. Internal gravity modes (eigenfrequencies in rad/s) (a) 0.4509 (1, 2), (b) 0.3255 (1, 3), (c) 0.7063 (1, 1), (d) 0.5502 (2, 3), (e) 0.7066 (2, 2) and (f) 0.9453 (3, 1).

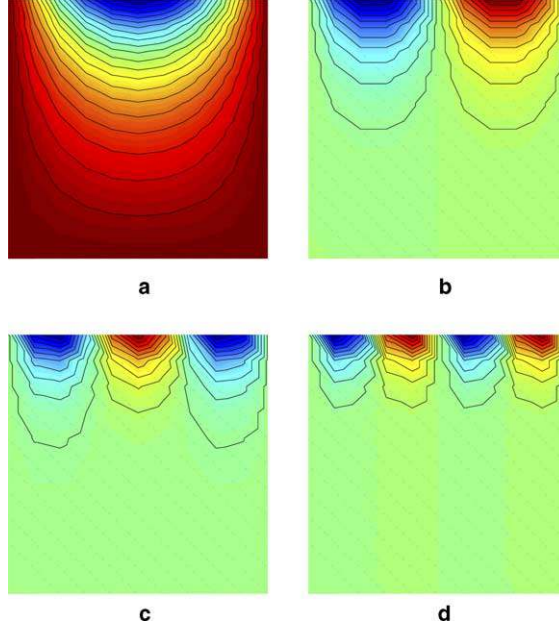


Fig. 3. Sloshing modes (eigenfrequencies in rad/s) (a) 5.4958, (b) 7.6128, (c) 9.0091 and (d) 9.9753.

Analytical eigenfrequencies for internal gravity waves and sloshing waves are given as follows:

$$\begin{cases} \omega_{ij}^{\text{IGW}} = N \frac{i\pi}{a} \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{h} \right)^2 + \frac{\beta^2}{4} \right]^{-1/2}, \\ \omega_i^{\text{SW}} = \sqrt{g \left(\frac{i\pi}{a} \right) \tanh \left(\frac{i\pi}{a} \right)}. \end{cases} \quad (72)$$

The geometric and physical characteristics of the problem are:

$$\begin{cases} \rho_F^0 = 1000 \text{ kg/m}^3, \\ N^2 = 1 \text{ s}^{-2}, \\ a = 1 \text{ m}, \\ h = 1 \text{ m}. \end{cases} \quad (73)$$

Tables 1 and 2 sum up the first analytical eigenfrequencies and their corresponding wave numbers for internal gravity waves.

The first eigenfrequencies of sloshing waves propagation are presented in Table 2.

In Figs. 2 and 3, we represent the computed eigenmodes corresponding to the previous analytical ones. The finite elements mesh consists of 20×20 triangle elements. For the incompressible situation considered here, the previous variational formulations reduce to a displacement-based formulation which is discretized in the framework of Raviat–Thomas finite elements method.

5. Conclusion

This paper presents the theoretical framework for analysis of fluid–structure interaction interior linear vibration problems taking into account both internal gravity and compressibility effects. Local equations

are derived and original symmetric variational formulations are constructed in the case of an inviscid *homogeneous* then a *non-homogeneous* liquid.

Those symmetric formulations lead to possible various finite element discretization. Current work is undertaken for the construction of various symmetric matrix reduced order models using adapted component mode analysis [2].

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