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Linearized formulation for fluid–structure interaction: Application to the linear dynamic response of a pressurized elastic structure containing a fluid with a free surface

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To control the linear vibrations of structures partially filled with liquids is of prime importance in various industries such as aerospace, naval, civil and nuclear engineering. It is proposed here to investigate a linearized formulation adapted to a rational computation of the vibrations of such coupled systems. Its particularity is to be fully Lagrangian since it considers the fluid displacement field with respect to a static equilibrium configuration as the natural variable describing the fluid motion, as classically done in structural dynamics. As the coupled system considered here is weakly damped in the low frequency domain (low modal density), the analysis of the vibrations of the associated undamped conservative system constitutes the main objective of this paper. One originality of the present formulation is to take into account the effect of the pressurization of the tank on the dynamics of the system, particularly in the case of a compressible liquid. We propose here a new way of deriving the linearized equations of the coupled problem involving a deformable structure and an inner inviscid liquid with a free surface. A review of the classical case considering a heavy incompressible liquid is followed by an application to the new case involving a light compressible liquid. A solution procedure in the frequency domain is proposed and a numerical discretization using the finite element method is discussed. In order to reduce the computational costs, an appropriate reduced order matrix model using modal synthesis approach is also presented.

1. Introduction

The case of structures partially filled with liquids (propellants, cooling liquids, liquid natural gas, etc.) is very common and many industrial domains are interested in this issue especially in the transport and aerospace industries (LNG vessels, aircrafts, liquid propelled launchers, satellites, etc.). This scientific domain is very active, as shown by the continuous interest in numerical and experimental investigations of this problem and the efforts invested in the coupling between computational fluid dynamics (CFD) and computation structural mechanics (CSM) [1–10]. Many different approaches exist to deal with this nonlinear fluid–structure interaction problem whose particularity is the interaction between the free surface flow and the flexibility of the structure (a short but complete review of those different approaches is given in the

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Nomenclature

E	Green–Lagrange strain tensor	$\boldsymbol{\varepsilon}$	linearized strain tensor
g	gravity field, $\mathbf{g} = -g\mathbf{i}_z$	ρ^F	mass density of the fluid
H	tensor of Hooke elastic coefficients	ρ^S	mass density of the structure
\mathcal{H}^1	Sobolev space	Σ_f	surface of application of external loads
I	identity tensor	Σ_g	gas–structure interface
$\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$	Cartesian coordinate system defined with unit vector \mathbf{i}_z colinear to \mathbf{g}	Σ_i	fluid–structure interface
\mathcal{L}^1	integrable function space	Σ_u	boundary with a prescribed displacement
\mathcal{L}^2	square-integrable function space	$\boldsymbol{\sigma}$	Cauchy stress tensor
\mathbf{n}_S	normal vector external to the structure	φ	fluid displacement potential field
\mathbf{n}_F	normal vector external to the fluid	Ω_F	fluid domain ($\partial\Omega_F = \Gamma \cup \Sigma_i$)
P	fluid pressure	Ω_G	tank ullage ($\partial\Omega_G = \Gamma \cup \Sigma_g$)
P^G	tank ullage gas pressure	Ω_S	structural domain ($\partial\Omega_S = \Sigma_i \cup \Sigma_g \cup \Sigma_f \cup \Sigma_u$)
P_0	hydrostatic pressure in the fluid	$ S $	area of surface S
p^{ext}	external pressure	$ \Omega $	volume of domain Ω
S	second symmetric Piola–Kirchhoff stress tensor	∇	nabla vector differential operator
\mathbf{u}^F	fluid displacement field	$\mathbf{D}_x(\mathbf{u})$	tensor of partial derivatives of vector \mathbf{u} with respect to vector coordinates \mathbf{x}
\mathbf{u}^S	structure displacement field	$\text{Div}_x \mathbf{A}$	divergence of a tensor A with respect to vector coordinates \mathbf{x}
\mathbf{v}^F	fluid velocity field	${}^t\mathbf{x}$	matrix or vector transpose
Γ	liquid free surface	\mathbf{x}^*	matrix or vector conjugate transpose
γ	heat capacity ratio of the gas	$\mathbf{x} \cdot \mathbf{y}$	scalar product
Δ	scalar Laplacian operator	$\text{tr} \mathbf{A}$	trace operator
		$\mathbf{A} : \mathbf{B}$	$= \text{tr}[{}^t\mathbf{A} \mathbf{B}] = \text{tr}[\mathbf{A} {}^t\mathbf{B}]$

introduction of [11]). Since most of these methods lead to an iterative resolution in time-domain, even the most efficient and most commonly used ones, like the arbitrary Lagrange Euler (ALE) method [12], remain very CPU-time consuming especially when applied to three-dimensional cases. Because, in many applications, conception engineers are concerned by the frequency response of the system, they are still interested in linearized models which can give, in very short times, a good estimation of the eigenfrequencies and eigenmodes characterizing the vibrational behavior of a fluid–structure coupled system [13–15].

For modal analysis of fluid–structure problems, the choice of the unknown variable field for the homogeneous fluid is of prime importance from a computational point of view. Generally, an Eulerian description of the fluid is adopted with the fluid velocity as main unknown. However, since we are interested in the low frequency domain where the system behavior can be characterized by its normal modes (which excludes from the analysis the medium and high frequency ranges), the fluid will be considered in a first step as inviscid. With this assumption, the velocity (or displacement) field in the fluid, supposed homogeneous at rest, is therefore irrotational and this irrotational constraint has to be imposed through a particular numerical analysis [16] or with Lagrange multipliers [17]. In order to use the minimal number of unknowns for the fluid, scalar fields, like pressure, have also been used as fluid variable, but this leads to nonsymmetric variational formulations for compressible acoustic fluids [18] as for incompressible sloshing liquids. Various symmetrization procedures have then been carried out [14,19–23].

The linearized fluid–structure formulation proposed here follows the methodology developed by Morand and Ohayon [14]. Its particularity is to be fully Lagrangian and to consider the fluid displacement with respect to a static equilibrium configuration as the natural variable describing the fluid motion, as classically done for the structure. As our main interest is for weakly damped fluid–structure problems in the low frequency domain (low modal density), the study of the conservative (undamped) associated system will be at the center of the discussion, and the fluid will be considered in a first step as inviscid. Taking into account this assumption (and several other simplifying hypotheses specified in the first section), the nonlinear local equations of the liquid–structure–gas system describing a closed tank containing a liquid with a free surface will be reminded (Section 2). After a brief description of the static reference configuration (Section 3), the linearization of the local equations will be carried out in the fourth section. We propose here a new derivation of these linearized equations that provides a promising framework for future developments. Its application to the classical case considering a heavy incompressible liquid (in Section 5) summarizes previous works of the same authors [24,25]. It will be followed by an application (in Section 6) to a new case involving a light compressible liquid, that can represent a cryogenic liquid for instance. One originality of the present formulation is to take into account the variations of the gas pressure p^G in the tank ullage in order to determine its effect on the dynamics of the system, particularly in the case of a compressible liquid. Once the variational formulations are established, a resolution method in the Fourier frequency domain will be proposed. A special care will be taken for the source terms associated to the initial conditions, in such a way that an inverse

Fourier transform will allow to express the (linearized) transient response of the fluid–structure system in the time domain. The integral equations will be finally solved numerically by a classical Rayleigh–Ritz finite element approach (Section 7). A modal synthesis method applied to the fluid will be presented for the particular case of liquid sloshing in a deformable tank (Section 8). The resulting matrix system will be in the same time symmetrized and drastically reduced in size, which will ensure low computation times. Finally some extensions to capillarity effects for microgravity environments and some works in progress to introduce the fluid viscosity will be touched on (Section 9).

2. Local equations

We suppose that the referential is a Galilean frame. Thermo-mechanical coupling is not considered here, therefore all mechanical parameters of the system are supposed given at a nominal temperature and are considered constant during the observation time. The dynamic state of this coupled problem is characterized by the structure displacement field $\mathbf{U}^S(\mathbf{x}, t)$, the fluid velocity field $\mathbf{v}^F(\mathbf{x}, t)$, the fluid pressure field $P(\mathbf{x}, t)$ and the tank ullage gas pressure $P^G(\mathbf{x}, t)$, which are the unknown fields of the problem. The description of the system is given by Fig. 1. Each domain will be designated with a ' when it refers to its time-dependent position.

2.1. Local equations of the fluid

We consider an inviscid heavy fluid with a free surface Γ inside a deformable structure. The tank ullage is filled by an inert pressurized gas (no phase or chemical transformations are taken into account). Effects of capillarity are supposed negligible.

$$\rho^F \frac{d\mathbf{v}^F}{dt} = -\nabla_{\mathbf{x}} P + \rho^F \mathbf{g} \quad \text{in } \Omega'_F \quad (1a)$$

$$\frac{d\rho^F}{dt} + \rho^F \nabla_{\mathbf{x}} \cdot \mathbf{v}^F = 0 \quad \text{in } \Omega'_F \quad (1b)$$

$$P = P^G \quad \text{on } \Gamma' \quad (1c)$$

$$\mathbf{v}^F \cdot \mathbf{n}'_F + \mathbf{v}^S \cdot \mathbf{n}_S = 0 \quad \text{on } \Sigma'_i \quad (1d)$$

Eq. (1a) is the balance momentum equation of the inviscid heavy fluid and Eq. (1b) expresses the mass conservation of the fluid (continuity equation). Eq. (1c) is the equilibrium equation of the fluid free surface when capillarity forces are neglected. Eq. (1d) yields the sliding condition on the structure wall for an inviscid fluid. In addition to this set of differential equations, Cauchy's initial conditions have to be added and they will be specified later on.

2.2. Local equations of the structure

The structure is supposed elastic (\mathbf{H} denotes the tensor of elastic coefficients) and submitted to external surface forces, including the external and internal pressures [26,27].

$$\text{Div}_{\mathbf{x}} \boldsymbol{\sigma} + \rho^S \mathbf{g} = \rho^S \mathbf{a} \quad \text{in } \Omega_S \quad (2a)$$

$$\boldsymbol{\sigma} = \mathbf{H} \boldsymbol{\varepsilon} \quad \text{in } \Omega_S \quad (2b)$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{D}_{\mathbf{x}}(\mathbf{U}^S) + {}^t\mathbf{D}_{\mathbf{x}}(\mathbf{U}^S)) \quad \text{in } \Omega_S \quad (2c)$$

$$\mathbf{U}^S = \mathbf{0} \quad \text{on } \Sigma'_u \quad (2d)$$

$$\boldsymbol{\sigma} \mathbf{n}_S = \mathbf{t}^{\text{ext}} - p^{\text{ext}} \mathbf{n}_S \quad \text{on } \Sigma'_f \quad (2e)$$

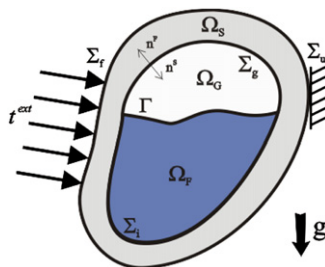


Fig. 1. Structure partially filled with liquid.

$$\boldsymbol{\sigma}\mathbf{n}_S = -P^G\mathbf{n}_S \quad \text{on } \Sigma'_g \quad (2f)$$

$$\boldsymbol{\sigma}\mathbf{n}_S = -P\mathbf{n}_S \quad \text{on } \Sigma'_i \quad (2g)$$

Eq. (2a) is the local balance of momentum (in a fixed frame of reference) where \mathbf{a} is the structure Eulerian acceleration. Eqs. (2b) and (2c) yield the classical definition of Cauchy's stress tensor $\boldsymbol{\sigma}$ and strain tensor $\boldsymbol{\varepsilon}$, and their relation through Hooke's law. In Eq. (2e), P^{ext} is the external pressure (the atmospheric pressure for instance) and \mathbf{t}^{ext} are additional local surface loads (thrust or aerodynamic forces for instance).

2.3. Local equations in the gas

The tank ullage being supposed to be pressurized by a gas, a simple gas behavior model will be used to represent its effect on the dynamics of the fluid–structure system. As the acoustic phenomena in the gas enclosed in the tank ullage are generally decoupled from the low frequency fluid–structure vibrations which are the frequency range of interest of the present paper, the acoustic field in the gas will be indeed represented by an uniform value $P^G(t)$ in Ω'_G and the isentropic law of transformation of a perfect gas will be used:

$$P^G|\Omega'_G|^\gamma = P_0^G|\Omega'_G|^\gamma \quad \text{in } \Omega'_G \quad (3)$$

where γ is the heat capacity ratio of the gas. This approach has been used in previous works [28,24] to model the influence of the tank ullage gas pressurization in the case of an elastic structure containing an incompressible liquid. In this paper, it will be extended to the compressible case.

3. Reference configuration

The geometry of the reference state will be described by the variable \mathbf{X} . In the reference configuration, the structure and fluid are supposed at rest. The local equations defining this reference state are then:

$$\mathbf{v}_0^F = \mathbf{0} \quad \text{in } \Omega_F \quad (4a)$$

$$\nabla_{\mathbf{X}}P_0 = \rho_0^F\mathbf{g} \quad \text{in } \Omega_F \quad (4b)$$

$$P_0 = P_0^G \quad \text{on } \Gamma \quad (4c)$$

$$\mathbf{v}_0^S = \mathbf{0} \quad \text{in } \Omega_S \quad (4d)$$

$$\text{Div}_{\mathbf{X}}\boldsymbol{\sigma}_0 + \rho_0^S\mathbf{g} = \mathbf{0} \quad \text{in } \Omega_S \quad (4e)$$

$$\boldsymbol{\sigma}_0 = \mathbf{H}\boldsymbol{\varepsilon}_{\mathbf{X}}(\mathbf{u}_0^S) \quad \text{in } \Omega_S \quad (4f)$$

$$\boldsymbol{\varepsilon}_{\mathbf{X}}(\mathbf{u}_0^S) = \frac{1}{2}(\mathbf{D}_{\mathbf{X}}(\mathbf{u}_0^S) + {}^t\mathbf{D}_{\mathbf{X}}(\mathbf{u}_0^S)) \quad \text{in } \Omega_S \quad (4g)$$

$$\mathbf{u}_0^S = \mathbf{0} \quad \text{on } \Sigma_u \quad (4h)$$

$$\boldsymbol{\sigma}_0\mathbf{n}_S = \mathbf{t}_0^{\text{ext}} - P_0^{\text{ext}}\mathbf{n}_S \quad \text{on } \Sigma_f \quad (4i)$$

$$\boldsymbol{\sigma}_0\mathbf{n}_S = -P_0^G\mathbf{n}_S \quad \text{on } \Sigma_g \quad (4j)$$

$$\boldsymbol{\sigma}_0\mathbf{n}_S = -P_0\mathbf{n}_S \quad \text{on } \Sigma_i \quad (4k)$$

where $\mathbf{t}_0^{\text{ext}}$ is the stationary part of the external loads. The external pressure P_0^{ext} is supposed to be a constant field. To solve this problem, a nonlinear iterative procedure is generally needed since these equations are written on domains whose position depends on the unknowns $(\mathbf{u}_0^S, \mathbf{u}_0^F)$ [29].

4. Linearized local equations

The solution of the static problem (4) is supposed known and, from now on, the position \mathbf{x} of a structural point or fluid particle (located in \mathbf{X} at the equilibrium) will be described with a Lagrangian point of view by $\mathbf{x}(\mathbf{X}, t) = \mathbf{X} + \mathbf{U}(\mathbf{X}, t)$, where \mathbf{U} is either the fluid or the structure displacement with respect to this equilibrium position.

We are interested here in the small amplitude response of the coupled system around this static reference state. For this purpose, we use a first order Taylor expansion of the nonlinear dynamic equations (1)–(3) around the equilibrium position [30]. The linear terms are obtained by differentiation (Fréchet derivative) in the direction of a small perturbation $(\mathbf{u}^S, \mathbf{u}^F)$. In the following, $\delta[h](\mathbf{u})$ will denote the Fréchet derivative of a function h with respect to the displacement \mathbf{U} , at the static position $\mathbf{U} = \mathbf{0}$ and in direction \mathbf{u} .

4.1. Linearization of the fluid equations

To linearize Eq. (1a), we first need to differentiate the gradient term $\nabla_{\mathbf{x}}P$ which is a spatial derivative with respect to the coordinates \mathbf{x} of the deformed fluid domain, depending, by definition, on the displacement \mathbf{U} since $\mathbf{x} = \mathbf{X} + \mathbf{U}$. Using the following relation:

$$\delta[\nabla_{\mathbf{x}}P](\mathbf{u}^F) = \nabla_{\mathbf{x}}\delta[P](\mathbf{u}^F) - \nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}P_0 \cdot \mathbf{u}^F) \quad (5)$$

and the value of $\nabla_{\mathbf{x}}P_0$ given by (4b), we finally obtain

$$\delta[\nabla_{\mathbf{x}}P](\mathbf{u}^F) = \nabla_{\mathbf{x}}p_{\varrho}(\mathbf{u}^F) - \nabla_{\mathbf{x}}(\rho_0^F \mathbf{g} \cdot \mathbf{u}^F) \quad (6)$$

where $p_{\varrho}(\mathbf{u}^F) = \delta[P](\mathbf{u}^F)$ is the Lagrangian pressure fluctuation in the liquid, i.e. the variation of the pressure observed when following a fluid particle.

Since the linearization of the acceleration term classically gives

$$\delta\left[\frac{d\mathbf{v}^F}{dt}\right](\mathbf{u}^F) = \frac{\partial^2 \mathbf{u}^F}{\partial t^2} \quad (7)$$

the linearized version of Euler's equation (1a) can be written as

$$\rho_0^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = \nabla_{\mathbf{x}}(\rho_0^F \mathbf{g} \cdot \mathbf{u}^F - p_{\varrho}(\mathbf{u}^F)) + \rho_{\varrho}^F(\mathbf{u}^F) \mathbf{g} \quad (8)$$

where $\rho_{\varrho}^F(\mathbf{u}^F) = \delta[\rho^F](\mathbf{u}^F)$ is the Lagrangian density fluctuation in the liquid.

Using the classical expression of the Lagrangian derivative, the mass conservation equation (1b) can be written as

$$\frac{\partial \rho^F}{\partial t} + \mathbf{v}^F \cdot \nabla_{\mathbf{x}} \rho^F + \rho^F \nabla_{\mathbf{x}} \cdot \mathbf{v}^F = 0 \quad (9)$$

The linearization of this equation can be achieved with the same approach as previously. Considering that $\mathbf{v}_0^F = \mathbf{0}$ (since the liquid is at rest in its equilibrium position), the differentiation of this relation simply yields

$$\delta[\nabla_{\mathbf{x}} \cdot \mathbf{v}^F](\mathbf{u}^F) = \nabla_{\mathbf{x}} \cdot \left(\frac{\partial \mathbf{u}^F}{\partial t} \right) \quad (10)$$

Then, the linearized form of Eq. (9) can be written as

$$\frac{\partial \rho_{\varrho}^F(\mathbf{u}^F)}{\partial t} + \frac{\partial \mathbf{u}^F}{\partial t} \cdot \nabla_{\mathbf{x}} \rho_0^F + \rho_0^F \cdot \delta[\nabla_{\mathbf{x}} \rho^F](\mathbf{u}^F) + \rho_{\varrho}^F(\mathbf{u}^F) \nabla_{\mathbf{x}} \cdot \mathbf{v}_0^F + \rho_0^F \nabla_{\mathbf{x}} \cdot \frac{\partial \mathbf{u}^F}{\partial t} = 0 \quad (11)$$

If we suppose that the fluid density in the reference state is homogeneous¹ ($\nabla_{\mathbf{x}} \rho_0^F \simeq 0$), the linearized mass conservation equation can be finally written as

$$\frac{\partial}{\partial t}(\rho_{\varrho}^F(\mathbf{u}^F) + \rho_0^F \nabla_{\mathbf{x}} \cdot \mathbf{u}^F) = 0 \quad (12)$$

The same approach applied to Eq. (1d) yields the following linearized equation:

$$\frac{\partial}{\partial t}(\mathbf{u}^F \cdot \mathbf{n}_F + \mathbf{u}^S \cdot \mathbf{n}_S) = 0 \quad (13)$$

We deduce therefore that the continuity of the normal displacement is satisfied at any time t as soon as it is satisfied by the initial conditions $\mathbf{u}^F(t=0)$ and $\mathbf{u}^S(t=0)$. The linearized equations of the fluid around the reference state are finally ($\forall t \in \mathbb{R}^+$):

$$\rho_0^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = \nabla_{\mathbf{x}}(\rho_0^F \mathbf{g} \cdot \mathbf{u}^F - p_{\varrho}(\mathbf{u}^F)) + \rho_{\varrho}^F(\mathbf{u}^F) \mathbf{g} \quad \text{in } \Omega_F \quad (14a)$$

$$\rho_{\varrho}^F(\mathbf{u}^F) + \rho_0^F \nabla_{\mathbf{x}} \cdot \mathbf{u}^F = 0 \quad \text{in } \Omega_F \quad (14b)$$

$$p_{\varrho} = p^G \quad \text{on } \Gamma \quad (14c)$$

$$\mathbf{u}^F \cdot \mathbf{n}_F + \mathbf{u}^S \cdot \mathbf{n}_S = 0 \quad \text{on } \Sigma_i \quad (14d)$$

where $p^G = \delta[P^G](\mathbf{u}^F, \mathbf{u}^S)$ is the pressure fluctuation in the gas when the fluid and structure are deformed. Eq. (12) has been simplified by supposing that initial conditions satisfy this mass conservation equation ($\rho_{\varrho}^F(t=0) = -\rho_0^F \nabla_{\mathbf{x}} \cdot \mathbf{u}_{\text{ini}}^F$).

To these equations we add some initial Cauchy conditions such that

$$\mathbf{u}^F(t=0) = \mathbf{u}_{\text{ini}}^F \quad (14e)$$

$$\mathbf{v}^F(t=0) = \mathbf{v}_{\text{ini}}^F \quad (14f)$$

¹ For an extension to a heterogeneous fluid (without tank pressurization), see [31].

4.2. Linearization of the structural equation

The local equations of the structure (2) are now linearized using the same approach. However, to simplify the derivation of the equations, before the differentiation, we use a Lagrangian transformation to write those equations in the reference configuration, as follows:

$$\text{Div}_{\mathbf{x}}(\mathbf{FS}) = \rho_0^S \frac{d^2 \mathbf{U}^S}{dt^2} - \rho_0^S \mathbf{g} \quad \text{in } \Omega_S \quad (15a)$$

$$\mathbf{S}(\mathbf{U}^S) = \mathbf{HE}(\mathbf{u}_0^S + \mathbf{U}^S) \quad (15b)$$

$$\mathbf{U}^S = \mathbf{0} \quad \text{on } \Sigma_u \quad (15c)$$

$$\mathbf{FSn}_S = \mathbf{t}^{\text{ext}} \det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{n}_S - p^{\text{ext}} \det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{n}_S \quad \text{on } \Sigma_f \quad (15d)$$

$$\mathbf{FSn}_S = -P^G \det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{n}_S \quad \text{on } \Sigma_g \quad (15e)$$

$$\mathbf{FSn}_S = -P \det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{n}_S \quad \text{on } \Sigma_i \quad (15f)$$

In these equations, \mathbf{E} is the (nonlinear) Green–Lagrange strain tensor and \mathbf{F} is the transformation gradient tensor, defined by $\mathbf{F} = \mathbf{D}_{\mathbf{x}}(\mathbf{x})$. To differentiate these equations, we now need to express the following term:

$$\delta[\mathbf{FS}](\mathbf{u}^S) = \mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \mathbf{HE}(\mathbf{u}_0^S) + \mathbf{H} \delta[\mathbf{E}](\mathbf{u}^S) \quad (16)$$

With the following notations:

$$\boldsymbol{\sigma}_0 = \mathbf{HE}(\mathbf{u}_0^S) \quad (17)$$

$$\boldsymbol{\varepsilon}_0(\mathbf{u}^S) = \delta[\mathbf{E}](\mathbf{u}^S) \quad (18)$$

the linearized version of Eq. (15a) can then be written as

$$\text{Div}_{\mathbf{x}}(\mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \boldsymbol{\sigma}_0 + \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S)) = \rho_0^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} \quad (19)$$

For the differentiation of Eq. (15f), the following term has to be computed:

$$\delta[p \det(\mathbf{F}) \mathbf{F}^{-1}](\mathbf{u}^S) = \delta[p](\mathbf{u}^S) \mathbf{I} + P_0 \delta[\det(\mathbf{F}) \mathbf{F}^{-1}](\mathbf{u}^S) \quad (20)$$

In the first term of this expression, we recognize a Lagrangian fluctuation of the pressure on the fluid–structure interface which is denoted by $p_{\underline{v}}^S(\mathbf{u}^S)$. Let us remark that this term is defined as a pressure fluctuation related to a structural point and then is different from the Lagrangian pressure fluctuation $p_x(\mathbf{u}^F)$ defined previously in Eq. (6) for a fluid particle. This difficulty is a consequence of the sliding of the inviscid liquid on the structure wall.²

To explicit the second term of (20), we use the following expressions:

$$\delta[\det(\mathbf{F})](\mathbf{u}) = \text{Tr}[\mathbf{D}_{\mathbf{x}} \mathbf{u}] \quad (21)$$

$$\delta[\mathbf{F}^{-1}](\mathbf{u}) = -\mathbf{F}^{-1} \mathbf{D}_{\mathbf{x}} \mathbf{u} \quad (22)$$

Then, we obtain

$$\left. \frac{d}{d\mathbf{U}^S} (\det(\mathbf{F}) \mathbf{F}^{-1}) \right|_0 (\mathbf{u}^S) = (\nabla_{\mathbf{x}} \cdot \mathbf{u}^S) \mathbf{I} - \mathbf{F}^{-1} \mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \quad (23)$$

Using this relation in (20) yields the linearized expression of (15f):

$$(\mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \boldsymbol{\sigma}_0 + \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S)) \mathbf{n}_S = -p_{\underline{v}}^S(\mathbf{u}^S) \mathbf{n}_S - P_0 \boldsymbol{\tau}(\mathbf{u}^S) \quad (24)$$

where the vector $\boldsymbol{\tau}$ is defined as

$$\boldsymbol{\tau}(\mathbf{u}^S) = (\nabla_{\mathbf{x}} \cdot \mathbf{u}^S) \mathbf{n}_S - \mathbf{F}^{-1} \mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \mathbf{n}_S \quad (25)$$

The comparison of this expression with the following one, that we would have obtained by a direct differentiation of the right-hand term of Eq. (2g):

$$\delta[-P \mathbf{n}_S d\Sigma'](\mathbf{u}^S) = -p_{\underline{v}}^S(\mathbf{u}^S) \mathbf{n}_S d\Sigma - P_0 \delta[\mathbf{n}_S d\Sigma'](\mathbf{u}^S) \quad (26)$$

shows that $\boldsymbol{\tau}$ can be interpreted as the linearization of the normal vector rotation:

$$\boldsymbol{\tau}(\mathbf{u}^S) d\Sigma = \delta[\mathbf{n}_S d\Sigma'](\mathbf{u}^S) \quad (27)$$

² This difference would disappear in case of a viscous liquid.

The same treatment can be applied to Eq. (15e). However, for Eq. (15d), we have to deal with the external force term \mathbf{t}^{ext} . The differentiation of this term yields

$$\delta[\mathbf{t}^{\text{ext}} \det(\mathbf{F}) \|\mathbf{F}^{-1} \mathbf{n}_S\|](\mathbf{u}^S) = \delta[\mathbf{t}^{\text{ext}}](\mathbf{u}^S) + \mathbf{t}_0^{\text{ext}} \delta[\det(\mathbf{F}) \|\mathbf{F}^{-1} \mathbf{n}_S\|](\mathbf{u}^S) \quad (28)$$

A direct computation of the second term can be avoided if we remember that

$$\delta[\det(\mathbf{F}) \|\mathbf{F}^{-1} \mathbf{n}_S\|](\mathbf{u}^S) dS = \delta[dS'](\mathbf{u}^S) \quad (29)$$

where dS and dS' are respectively the elementary surfaces on the reference and the actual geometries. Since, from (27), we have $\boldsymbol{\tau}(\mathbf{u}^S) dS = \delta[\mathbf{n}' dS'](\mathbf{u}^S) = \delta[\mathbf{n}'](\mathbf{u}^S) dS + \mathbf{n} \delta[dS'](\mathbf{u}^S)$ and $\mathbf{n} \cdot \delta[\mathbf{n}'](\mathbf{u}^S) = 0$, we can deduce from (25) that

$$\delta[\det(\mathbf{F}) \|\mathbf{F}^{-1} \mathbf{n}_S\|](\mathbf{u}^S) = \mathbf{n}_S \cdot \boldsymbol{\tau}(\mathbf{u}^S) = \nabla_{\mathbf{x}} \cdot \mathbf{u}^S - \mathbf{n}_S \cdot \mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \mathbf{n}_S \quad (30)$$

Finally, the linearized expression of (15d) can be written as

$$(\mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \boldsymbol{\sigma}_0 + \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S)) \mathbf{n}_S = \mathbf{t}^d + \delta[\mathbf{t}^{\text{ext}}](\mathbf{u}^S) + \mathbf{t}_0^{\text{ext}} \mathbf{n}_S \cdot \boldsymbol{\tau}(\mathbf{u}^S) - P^{\text{ext}} \boldsymbol{\tau}(\mathbf{u}^S) \quad (31)$$

where $\mathbf{t}^d = \mathbf{t}^{\text{ext}}(\mathbf{U} = \mathbf{0}) - \mathbf{t}_0^{\text{ext}}$ is the dynamic part of the external surface loads. Let us verify that if the external surface load \mathbf{t}^{ext} is a pressure, i.e. $\mathbf{t}^{\text{ext}} = -P_t \mathbf{n}_S$, then Eqs. (28) and (29) lead to coherent expressions of (31) and (24):

$$\delta[\mathbf{t}^{\text{ext}}] d\Sigma + \mathbf{t}_0^{\text{ext}} \mathbf{n}_S \cdot \boldsymbol{\tau} d\Sigma = \delta[-P_t \mathbf{n}_S] d\Sigma - P_{t0} \mathbf{n}_S \delta[d\Sigma'] = -\delta[P_t] \mathbf{n}_S d\Sigma - P_{t0} \delta[\mathbf{n}_S d\Sigma'] \quad (32)$$

By using (27), we denote then that expressions (24) and (31) are equivalent.

The linearized equations of the structure established in this section are summarized hereafter:

$$\text{Div}_{\mathbf{x}} \mathbf{s}(\mathbf{u}^S) = \rho_0^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} \quad \text{in } \Omega_S \quad (33a)$$

$$\mathbf{s}(\mathbf{u}^S) = \mathbf{D}_{\mathbf{x}}(\mathbf{u}^S) \boldsymbol{\sigma}_0 + \mathbf{H} \boldsymbol{\varepsilon}_0(\mathbf{u}^S) \quad \text{in } \Omega_S \quad (33b)$$

$$\mathbf{u}^S = \mathbf{0} \quad \text{on } \Sigma_u \quad (33c)$$

$$\mathbf{s} \mathbf{n}_S = \mathbf{t}^d + \delta[\mathbf{t}^{\text{ext}}](\mathbf{u}^S) + \mathbf{t}_0^{\text{ext}} \mathbf{n}_S \cdot \boldsymbol{\tau}(\mathbf{u}^S) - P^{\text{ext}} \boldsymbol{\tau}(\mathbf{u}^S) \quad \text{on } \Sigma_f \quad (33d)$$

$$\mathbf{s} \mathbf{n}_S = -p^G(\mathbf{u}^S, \mathbf{u}^F) \mathbf{n}_S - P_0^G \boldsymbol{\tau}(\mathbf{u}^S) \quad \text{on } \Sigma_g \quad (33e)$$

$$\mathbf{s} \mathbf{n}_S = -p_0^S(\mathbf{u}^S) \mathbf{n}_S - P_0 \boldsymbol{\tau}(\mathbf{u}^S) \quad \text{on } \Sigma_i \quad (33f)$$

where (33c) has been obtained by a trivial differentiation of (2d).

To these equations, we add the following Cauchy conditions:

$$\mathbf{u}^S(t=0) = \mathbf{u}_{\text{ini}}^S \quad (33g)$$

$$\mathbf{v}^S(t=0) = \mathbf{v}_{\text{ini}}^S \quad (33h)$$

with

$$\mathbf{u}_{\text{ini}}^S \cdot \mathbf{n}_S + \mathbf{u}_{\text{ini}}^F \cdot \mathbf{n}_F = 0 \quad (33i)$$

and

$$\mathbf{v}_{\text{ini}}^S \cdot \mathbf{n}_S + \mathbf{v}_{\text{ini}}^F \cdot \mathbf{n}_F = 0 \quad (33j)$$

4.3. Linearization of the gas equations

The differentiation of the gas equation (3) yields

$$\delta[P^G](\mathbf{u}^S, \mathbf{u}^F) |\Omega'_G|^\gamma + P_0^G \gamma |\Omega'_G|^{\gamma-1} \delta[|\Omega'_G|](\mathbf{u}^S, \mathbf{u}^F) = 0 \quad (34)$$

Since the gas volume $|\Omega'_G|$ is defined by

$$|\Omega'_G| = \int_{\Omega'_G} d\Omega' = \int_{\Omega_G} \det(\mathbf{F}) d\Omega \quad (35)$$

Using the relation (21) and Stokes theorem, we obtain

$$\delta[|\Omega'_G|](\mathbf{u}^S, \mathbf{u}^F) = \int_{\Omega_G} \nabla_{\mathbf{x}} \cdot \mathbf{u} d\Omega = - \int_{\Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_S d\Sigma - \int_{\Gamma} \mathbf{u}^F \cdot \mathbf{n}_F d\Gamma \quad (36)$$

Finally the linearized equation of the gas can be written as

$$p^G(\mathbf{u}^S, \mathbf{u}^F) = \frac{p_0^G \gamma}{|\Omega_G|} \left(\int_{\Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_S \, d\Sigma + \int_{\Gamma} \mathbf{u}^F \cdot \mathbf{n}_F \, d\Gamma \right) \quad \text{in } \Omega_G \quad (37)$$

Eqs. (14), (33) and (37) form the coupled problem we want to solve. In these equations, all the spatial derivatives are carried with respect to the coordinate vector \mathbf{X} and then, for the sake of convenience, this subscript \mathbf{X} will be omitted in the following.

In the fluid equations (14a)–(14d), after the elimination of $\rho_{\mathbf{v}}^F$ using Eq. (14b), it remains only one Eq. (14a) which contains two independent unknowns \mathbf{u}^F and $p_{\mathbf{v}}$. We must therefore specify now the constitutive equation of the fluid. Two cases will be considered: the incompressible fluids and the barotropic compressible fluids.

5. Incompressible fluid under pressure

In the case of an incompressible fluid, the Lagrangian fluctuation of density in the fluid is zero:

$$\text{incompressible fluid} \Rightarrow \rho_{\mathbf{v}}^F = 0 \quad (38)$$

From (14b), we have that

$$\nabla \cdot \mathbf{u}^F = 0 \quad (39)$$

which is the incompressibility condition expressed in terms of the fluid displacement.

Then Eq. (14a) is written as follows:

$$\rho_0^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = \nabla(\rho_0^F \mathbf{g} \cdot \mathbf{u}^F - p_{\mathbf{v}}(\mathbf{u}^F)) \quad (40)$$

Taking the curl of this vector equation, the following relation is obtained:

$$\rho_0^F \frac{\partial^2 (\nabla \wedge \mathbf{u}^F)}{\partial t^2} = \mathbf{0} \quad (41)$$

which shows that, if the initial conditions are such that $\nabla \wedge \mathbf{u}_{\text{ini}}^F = \mathbf{0}$ and $\nabla \wedge \mathbf{v}_{\text{ini}}^F = \mathbf{0}$, then the small displacements of an incompressible (homogeneous and inviscid) fluid are irrotational. In this case, the fluid displacement field can be represented³ by a scalar potential φ defined by

$$\nabla \wedge \mathbf{u}^F = \mathbf{0} \Rightarrow \mathbf{u}^F = \nabla \varphi \quad (42)$$

However, this relation does not ensure the uniqueness of φ . Therefore, it is necessary to specify an unicity condition on φ , generically written $\ell(\varphi) = 0$, where ℓ is a linear form such that $\ell(1) \neq 0$. This unicity condition will be chosen explicitly in the following.

This new variable φ can be substituted to \mathbf{u}^F in the fluid equations, in particular in (14a) which yields, after integration with respect to \mathbf{X} :

$$p_{\mathbf{v}} = \rho_0^F \mathbf{g} \cdot \nabla \varphi - \rho_0^F \frac{\partial^2 \varphi}{\partial t^2} + \pi \quad (43)$$

where π is an integration constant which represents a uniform pressure in the fluid whose value depends on the choice of the unicity condition $\ell(\varphi)$. The expression (43) of $p_{\mathbf{v}}$ as a function of φ allows to eliminate $p_{\mathbf{v}}$ from the equations. However, before establishing the variational formulation, we choose to introduce a scalar variable η to describe the normal component of the fluid free surface displacement⁴:

$$\eta = \mathbf{u}^F \cdot \mathbf{n}_F = \nabla \varphi \cdot \mathbf{n}_F \quad \text{on } \Gamma \quad (44)$$

Since $\mathbf{n}_F = \mathbf{i}_z$ on Γ , the local equations of the boundary problem (14) are now written for an incompressible fluid in terms of $(\varphi, \eta, \pi, p^G)$ as follows:

$$\Delta \varphi = 0 \quad \text{in } \Omega_F \quad (45a)$$

$$\mathbf{n}_F \cdot \nabla \varphi = \mathbf{u}^S \cdot \mathbf{n}_F \quad \text{on } \Sigma_i \quad (45b)$$

$$\mathbf{n}_F \cdot \nabla \varphi = \eta \quad \text{on } \Gamma \quad (45c)$$

$$\rho_0^F g \eta = -\rho_0^F \frac{\partial^2 \varphi}{\partial t^2} + \pi - p^G \quad \text{on } \Gamma \quad (45d)$$

³ This result is given by Poincaré lemma for a simply connected domain Ω_F .

⁴ To introduce η in the formulation is not mandatory since φ can be sufficient to describe the liquid behavior. However, the presence of η allows to generalize this formulation to liquid with surface tension more easily [32].

$$p^G = \frac{P_0^G \gamma}{|\Omega_G|} \int_{\Sigma_i \cup \Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_S \, d\Sigma \quad \text{in } \Omega_G' \quad (45e)$$

$$\ell(\varphi) = 0 \quad (45f)$$

$$\varphi(t=0) = \varphi_{\text{ini}} \quad (45g)$$

$$\frac{\partial \varphi}{\partial t}(t=0) = \Psi_{\text{ini}} \quad (45h)$$

where φ_{ini} and Ψ_{ini} are such that $\mathbf{u}_{\text{ini}}^F = \nabla \varphi_{\text{ini}}$, $\mathbf{v}_{\text{ini}}^F = \nabla \Psi_{\text{ini}}$ and $\ell(\varphi_{\text{ini}}) = \ell(\Psi_{\text{ini}}) = 0$, and Eq. (45e) has been obtained from (14c) and (37) by considering the following relation given by the integration of property (39) on the fluid domain:

$$0 = \int_{\Sigma_i} \mathbf{u}^F \cdot \mathbf{n}_F \, d\Sigma + \int_{\Gamma} \mathbf{u}^F \cdot \mathbf{n}_F \, d\Gamma \quad (46)$$

Eq. (45e) shows that, due to the incompressibility of the fluid, p^G is now a function depending only of the tank wall deformation \mathbf{u}^S .

From the local equations (45a)–(45c), we can write the following variational formulation for the liquid, where C_φ^ℓ is the admissible space of solutions φ defined by

$$C_\varphi^\ell = \{\varphi \in \mathcal{H}^1(\Omega_F) / \ell(\varphi) = 0\} \quad (47)$$

for $t \in \mathbb{R}^+$, $\exists \varphi \in C_\varphi^\ell$ such that $\forall \delta\varphi \in \mathcal{H}^1(\Omega_F)$,

$$F(\varphi, \delta\varphi) - B(\eta, \delta\varphi) = C(\delta\varphi, \mathbf{u}^S) \quad (48)$$

with

$$F(\varphi, \delta\varphi) = \rho_0^F \int_{\Omega_F} \nabla \varphi \cdot \nabla \delta\varphi \, d\Omega \quad (49)$$

$$B(\eta, \delta\varphi) = \rho_0^F \int_{\Gamma} \eta \delta\varphi \, d\Gamma \quad (50)$$

$$C(\delta\varphi, \mathbf{u}^S) = \rho_0^F \int_{\Sigma_i} \mathbf{u}^S \cdot \mathbf{n}_F \delta\varphi \, d\Sigma \quad (51)$$

In this expression, F is the symmetric positive semi-definite bilinear form associated with the fluid kinetic energy and C is the classical fluid–structure coupling operator.

Since it has been shown in [14] that the admissible space of (48) is the direct sum of two subspaces $\mathcal{H}^1(\Omega_F) = C_\varphi^\ell \oplus \mathbb{R}$, (48) can be split into two variational equations: the first one has exactly the same expression as (48), except that $\delta\varphi$ belongs to C_φ^ℓ and the second one is obtained by taking $\delta\varphi$ equal to a real constant. This latest yields a relation equivalent to (46), which expresses the incompressibility of the fluid:

$$-B(\eta, 1) = C(1, \mathbf{u}^S) \quad (52)$$

The variational formulation corresponding to the local equation (45d) is written as

for $t \in \mathbb{R}^+$, $\exists \eta \in C_\eta$ such that $\forall \delta\eta \in C_\eta$,

$$S_g(\eta, \delta\eta) + B \left(\delta\eta, \frac{\partial^2 \varphi}{\partial t^2} \right) - \frac{\pi}{\rho_0^F} B(\delta\eta, 1) + \frac{p^G}{\rho_0^F} B(\delta\eta, 1) = 0 \quad (53)$$

with

$$S_g(\eta, \delta\eta) = \rho_0^F g \int_{\Gamma} \eta \delta\eta \, d\Gamma \quad (54)$$

where C_η is the admissible space for η , defined as $C_\eta = \{\eta \in \mathcal{L}^2(\Gamma)\}$ and S_g is the symmetric positive definite bilinear form associated with the fluid gravity potential energy.

A final equation is obtained from the expression (45e) of p^G :

$$\chi^G \frac{p^G}{\rho_0^F} - B(\eta, 1) = C_g(1, \mathbf{u}^S) \quad (55)$$

with

$$C_g(\delta\varphi, \mathbf{u}^S) = \rho_0^F \int_{\Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_S \delta\varphi \, d\Sigma \quad (56)$$

and

$$\chi^G = \frac{(\rho_0^F)^2 |\Omega_G|}{P_0^G \gamma} \quad (57)$$

6. Barotropic compressible fluid under pressure

Since the temperature is supposed constant during the observation time and consequently its dynamic influence on the fluid–structure problem is supposed negligible, a barotropic model is suitable in case of a compressible fluid. This model suggests that the pressure fluctuation p_L in the fluid is directly proportional to the density fluctuation ρ_L^F , and it can be shown that, in this case, the proportionality coefficient is the square of the acoustic wave velocity in the fluid c :

$$p_L = c^2 \rho_L^F = -\rho_0^F c^2 \nabla \cdot \mathbf{u}^F \quad (58)$$

Using this relation and Eq. (14b), p_L and ρ_L^F can be eliminated from Eq. (14a) which can be written as

$$\rho_0^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = \nabla(\rho_0^F \mathbf{g} \cdot \mathbf{u}^F - \rho_0^F c^2 \nabla \cdot \mathbf{u}^F) - \rho_0^F \mathbf{g} \nabla \cdot \mathbf{u}^F \quad (59)$$

By taking the curl of this vector equation, we obtain the following relation:

$$\rho_0^F \frac{\partial^2 (\nabla \wedge \mathbf{u}^F)}{\partial t^2} = -\rho_0^F \nabla (\nabla \cdot \mathbf{u}^F) \wedge \mathbf{g} \quad (60)$$

which shows that, even if the displacement and velocity fields at the initial time are irrotational, this property is not generally maintained in a compressible fluid, except in the particular case where the gravity effects in the fluid could be neglected ($g \simeq 0$). In the following, we will do this assumption which can be justified by the fact that, in the case of compressible fluids, the compressibility potential energy is generally much greater than the gravity potential energy and then gravity effects can be neglected in a first approach.⁵ In this case, we speak about *light* fluids, by opposition to *heavy* fluid, for which the gravity potential energy cannot be neglected.

$$\text{barotropic compressible light fluid } (g = 0) \Rightarrow \mathbf{u}^F = \nabla \varphi \quad (61)$$

By neglecting the terms in \mathbf{g} , an integration of Eq. (14a) with respect to \mathbf{X} yields the expression of p_L :

$$p_L = -\rho_0^F \frac{\partial^2 \varphi}{\partial t^2} + \pi \quad (62)$$

This expression and (61) can be used in (58), (14c) and (14d) to eliminate p_L from the fluid equations. Moreover, as we did for the incompressible fluid, we introduce in the formulation the variable η to describe the normal component of the fluid free surface displacement. Finally, the local equations of the boundary problem (14) are now written for a compressible light fluid in terms of $(\varphi, \eta, \pi, p^G)$:

$$\Delta \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\pi}{\rho_0^F c^2} \quad \text{in } \Omega_F \quad (63a)$$

$$\mathbf{n}_F \cdot \nabla \varphi = \mathbf{u}^S \cdot \mathbf{n}_F \quad \text{on } \Sigma_i \quad (63b)$$

$$\mathbf{n}_F \cdot \nabla \varphi = \eta \quad \text{on } \Gamma \quad (63c)$$

$$-\rho_0^F \frac{\partial^2 \varphi}{\partial t^2} + \pi = p^G \quad \text{on } \Gamma \quad (63d)$$

$$p^G = \frac{P_0^G \gamma}{|\Omega_G|} \left(\int_{\Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_S \, d\Sigma + \int_{\Gamma} \eta \, d\Gamma \right) \quad \text{in } \Omega_G \quad (63e)$$

$$\ell(\varphi) = 0 \quad (63f)$$

with the initial conditions

$$\varphi(t = 0) = \varphi_{\text{ini}} \quad (63g)$$

$$\frac{\partial \varphi}{\partial t}(t = 0) = \Psi_{\text{ini}} \quad (63h)$$

Let us remark that, in the case of the so-called *light* fluid, we consider here that the free surface and the tank ullage exist. The case of a light acoustic fluid occupying the whole tank volume will not be treated here but can be found in [14].

⁵ A generalization to the case where both compressibility and gravity terms would be of same order can be found in [31,33].

Using the boundary conditions (63b) and (63c), the variational formulation corresponding to Eq. (63a) is written as

$$\begin{aligned} & \text{for } t \in \mathbb{R}^+, \exists \varphi \in C_\varphi^t \text{ such that } \forall \delta\varphi \in \mathcal{H}^1(\Omega_F), \\ & F(\varphi, \delta\varphi) + S_c \left(\frac{\partial^2 \varphi}{\partial t^2}, \delta\varphi \right) - B(\eta, \delta\varphi) - \frac{\pi}{\rho_0^F} S_c(1, \delta\varphi) = C(\delta\varphi, \mathbf{u}^S) \end{aligned} \quad (64)$$

with

$$S_c(\varphi, \delta\varphi) = \frac{\rho_0^F}{c^2} \int_{\Omega_F} \varphi \delta\varphi \, d\Omega \quad (65)$$

As previously done in Section 5, this variational formulation being valid for all $\delta\varphi$ in $\mathcal{H}^1(\Omega_F)$, it can be replaced by two variational equations: the first one has exactly the same expression as (64), except that $\delta\varphi$ belongs to C_φ^t and the second one is obtained by taking $\delta\varphi$ equal to a real constant. This latest yields the following relation:

$$S_c \left(\frac{\partial^2 \varphi}{\partial t^2}, 1 \right) - B(\eta, 1) - \chi^F \frac{\pi}{\rho_0^F} = C(1, \mathbf{u}^S) \quad (66)$$

with

$$\chi^F = \frac{\rho_0^F |\Omega_F|}{c^2} \quad (67)$$

The variational formulation corresponding to the local equation (63d) is written as

$$\begin{aligned} & \text{for } t \in \mathbb{R}^+, \exists \eta \in C_\eta \text{ such that } \forall \delta\eta \in C_\eta, \\ & B \left(\delta\eta, \frac{\partial^2 \varphi}{\partial t^2} \right) - \frac{\pi}{\rho_0^F} B(\delta\eta, 1) + \frac{p^G}{\rho_0^F} B(\delta\eta, 1) = 0 \end{aligned} \quad (68)$$

Finally, Eq. (63e) can be written as follows:

$$\chi^G \frac{p^G}{\rho_0^F} - B(\eta, 1) = C_g(1, \mathbf{u}^S) \quad (69)$$

7. Solution of the coupled fluid–structure problem

7.1. Variational formulation for the structure

From the local equations of the structure given in (33), a variational formulation for the structure can be written, where $C_u^0 = \{\mathbf{u}^S \in \mathcal{H}^1(\Omega_S)^3 / \mathbf{u}^S = \mathbf{0} \text{ on } \Sigma_u\}$ denotes the space of the kinematically admissible solutions. In the following, the fluid pressure P_0 in the reference configuration will be replaced by its expression obtained from (4b) and (4c):

$$P_0(\mathbf{X}) = P_0^G + \rho_0^F \mathbf{g} \cdot \mathbf{X} \quad (70)$$

where O is a point of the liquid free surface. Let us remark that this expression is also valid for a light fluid, since with $\mathbf{g} = \mathbf{0}$, $P_0(\mathbf{X}) = P_0^G$.

Then the variational principle for the structure is written as

$$\begin{aligned} & \exists \mathbf{u}^S \in C_u^0 \text{ such that } \forall \delta\mathbf{u} \in C_u^0, \\ & (K_E + K_G + K_P + K_t)(\mathbf{u}^S, \delta\mathbf{u}) + M \left(\frac{\partial^2 \mathbf{u}^S}{\partial t^2}, \delta\mathbf{u} \right) + \frac{p^G}{\rho_0^F} C_g(1, \delta\mathbf{u}) + \int_{\Sigma_i} p_L^S(\mathbf{u}^S) \mathbf{n}_S \cdot \delta\mathbf{u} + (\rho_0^F \mathbf{g} \cdot \mathbf{X}) \tau(\mathbf{u}^S) \cdot \delta\mathbf{u} \, d\Sigma = f^{\text{ext}}(\delta\mathbf{u}) \end{aligned} \quad (71)$$

with

$$K_E(\mathbf{u}^S, \delta\mathbf{u}) = \int_{\Omega_S} \mathbf{H}\boldsymbol{\varepsilon}_0(\mathbf{u}^S) : \mathbf{D}_x(\delta\mathbf{u}) \, d\Omega \quad (72)$$

$$K_G(\mathbf{u}^S, \delta\mathbf{u}) = \int_{\Omega_S} \mathbf{D}_x(\mathbf{u}^S) \boldsymbol{\sigma}_0 : \mathbf{D}_x(\delta\mathbf{u}) \, d\Omega \quad (73)$$

$$K_P(\mathbf{u}^S, \delta\mathbf{u}) = \int_{\Sigma_f} p^{\text{ext}} \tau(\mathbf{u}^S) \cdot \delta\mathbf{u} \, d\Sigma + \int_{\Sigma_g \cup \Sigma_i} P_0^G \tau(\mathbf{u}^S) \cdot \delta\mathbf{u} \, d\Sigma \quad (74)$$

$$K_t(\mathbf{u}^S, \delta\mathbf{u}) = - \int_{\Sigma_f} \delta[\mathbf{t}^{\text{ext}}](\mathbf{u}^S) \cdot \delta\mathbf{u} + \mathbf{n}_S \cdot \tau(\mathbf{u}^S) \mathbf{t}_0^{\text{ext}} \cdot \delta\mathbf{u} \, d\Sigma \quad (75)$$

$$\mathbf{M}(\mathbf{u}^S, \delta \mathbf{u}) = \int_{\Omega_S} \rho_0^S \mathbf{u}^S \cdot \delta \mathbf{u} \, d\Omega \quad (76)$$

$$f^{\text{ext}}(\delta \mathbf{u}) = \int_{\Sigma_f} \mathbf{t}^d \cdot \delta \mathbf{u} \, d\Sigma \quad (77)$$

where K_E is the elastic stiffness operator of the structure deformed in its static configuration. If these initial deformations \mathbf{u}_0^S are small, $\boldsymbol{\varepsilon}_0$ tends to $\boldsymbol{\varepsilon}$ and K_E is then the standard symmetric positive stiffness operator. \mathbf{M} is the classical symmetric positive definite operator associated to the mass of the structure and K_G is the symmetric operator, due to the initial prestress $\boldsymbol{\sigma}_0$ in the structure, classically known as the geometric stiffness. K_p is the stiffness operator arising from the follower force effect of the initial pressures applied on the structure external and internal walls (the external pressure P^{ext} and the pressurization P_0^G). K_t is a similar term due to the presence of a static external surface load \mathbf{t}^{ext} which can be also a follower force. f^{ext} is a linear form representing the dynamic part of this load.

We need now an expression of $p_L^S(\mathbf{u}^S)$ as a function of the unknowns \mathbf{u}^S and \mathbf{u}^F . This expression is obtained by considering a fluid particle in contact with a structural point of the fluid interface in the reference configuration. At time t , the pressure at this structural point is $P(\mathbf{U}^S)$ and the pressure of the same fluid particle is $P(\mathbf{U}^F)$. The difference can be written as follows:

$$P(\mathbf{U}^S) - P(\mathbf{U}^F) = \nabla_{\mathbf{x}} P \cdot (\mathbf{U}^S - \mathbf{U}^F) \quad (78)$$

By differentiating this expression at the reference configuration and in the direction $(\mathbf{u}^S, \mathbf{u}^F)$, we obtain

$$p_L^S(\mathbf{u}^S) - p_L(\mathbf{u}^F) = \nabla_{\mathbf{x}} P_0 \cdot (\mathbf{u}^S - \mathbf{u}^F) = \rho_0^F \mathbf{g} \cdot (\mathbf{u}^S - \mathbf{u}^F) \quad (79)$$

Since the expression of p_L is given either by (43) for an incompressible fluid or by (62) for a barotropic light fluid, we can write the following general expression (where \mathbf{g} must be taken equal to $\mathbf{0}$ in case of a light compressible fluid):

$$p_L^S(\mathbf{u}^S) = \rho_0^F \mathbf{g} \cdot \mathbf{u}^S - \rho_0^F \frac{\partial^2 \varphi}{\partial t^2} + \pi \quad (80)$$

By introducing this relation into the previously given variational formulation of the structure, we obtain

$$\begin{aligned} & \exists \mathbf{u}^S \in \mathcal{C}_u^0 \text{ such that } \forall \delta \mathbf{u} \in \mathcal{C}_u^0, \\ & K_T(\mathbf{u}^S, \delta \mathbf{u}) + \frac{P^G}{\rho_0^F} C_g(1, \delta \mathbf{u}) - \frac{\pi}{\rho_0^F} C(1, \delta \mathbf{u}) + \mathbf{M} \left(\frac{\partial^2 \mathbf{u}^S}{\partial t^2}, \delta \mathbf{u} \right) + C \left(\frac{\partial^2 \varphi}{\partial t^2}, \delta \mathbf{u} \right) = f^{\text{ext}}(\delta \mathbf{u}) \end{aligned} \quad (81)$$

with

$$K_T(\mathbf{u}^S, \delta \mathbf{u}) = (K_E + K_G + K_p + K_t + K_g)(\mathbf{u}^S, \delta \mathbf{u}) \quad (82)$$

$$K_g(\mathbf{u}^S, \delta \mathbf{u}) = \int_{\Sigma_i} (\rho_0^F \mathbf{g} \cdot \mathbf{u}^S) \mathbf{n}_S \cdot \delta \mathbf{u} + (\rho_0^F \mathbf{g} \cdot \mathbf{X}) \boldsymbol{\tau}(\mathbf{u}^S) \cdot \delta \mathbf{u} \, d\Sigma \quad (83)$$

where K_g is the stiffness operator related to the follower force effect of the fluid hydrostatic pressure on the structure wall (it can be shown that this operator is symmetric). Obviously, this operator being proportional to \mathbf{g} , it vanishes obviously in the case of a light fluid.

7.2. Solution procedure in the frequency domain

The solution in the frequency domain will be carried out using the following Fourier transform:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \quad (84)$$

in which f , defined at any time t in \mathbb{R} , belongs to L^1 , and where $\hat{f}(\omega)$ is a complex valued function defined in \mathbb{R} for any circular frequency ω .

After solving the problem in the frequency domain, the inverse Fourier transform will allow to recover the original function, defined in the time domain, by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega \quad (85)$$

The application of the Fourier transform to the equations of the problem requires to extend the functions $\mathbf{u}^S(\mathbf{x}, t)$, $\mathbf{u}^F(\mathbf{x}, t)$, $p_L(\mathbf{x}, t)$ (as well as the auxiliary unknowns $\varphi(\mathbf{x}, t)$, $\eta(\mathbf{x}, t)$, $\pi(t)$, $p^G(t)$, etc.), defined for $t \geq 0$ only, to extended functions $\tilde{\mathbf{u}}^S(\mathbf{x}, t)$ (respectively $\tilde{\mathbf{u}}^F(\mathbf{x}, t)$, $\tilde{p}_L(\mathbf{x}, t)$, etc.) defined $\forall t$ in \mathbb{R} such that $\tilde{\mathbf{u}}^S(\mathbf{x}, t < 0) = \mathbf{0}$ and $\tilde{\mathbf{u}}^S(\mathbf{x}, t \geq 0) = \mathbf{u}^S(\mathbf{x}, t)$ (idem for $\tilde{\mathbf{u}}^F(\mathbf{x}, t)$, $\tilde{p}_L(\mathbf{x}, t)$, etc.). Since, the extended functions are discontinuous at time $t=0$, their derivative with respect to time t are defined, in a distribution sense only, by

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = \left\{ \frac{\partial \tilde{\mathbf{u}}}{\partial t} \right\} + \mathbf{u}_{\text{ini}} \delta \quad (86a)$$

$$\frac{\partial^2 \hat{\mathbf{u}}}{\partial t^2} = \left\{ \frac{\partial^2 \hat{\mathbf{u}}}{\partial t^2} \right\} + \mathbf{u}_{\text{ini}} \delta' + \mathbf{v}_{\text{ini}} \delta \quad (86b)$$

where δ is the Dirac distribution, δ' its derivative (in the distribution sense) and $\{\partial/\partial t\}$ denotes the derivative in the function sense, defined for $t \neq 0$ [34].

By using this approach, the time derivatives in Eqs. (14a), (33a), (45d) or (63a), which are defined for $t > 0$, can be extended in the distribution sense on \mathbb{R} by using relations (86). Since all the equations of the linearized coupled problem are, by definition, linear, the Fourier transform of Eqs. (14), (33), (45) and (63) is easily obtained. Additional terms, given by the initial Cauchy conditions, are expressed by considering that $\hat{\delta} = 1$ and $\hat{\delta}' = i\omega$:

$$(14a), (14e), (14f) \Rightarrow -\omega^2 \rho_0^F \hat{\mathbf{u}}^F = \nabla(\rho_0^F \hat{\mathbf{g}} \cdot \hat{\mathbf{u}}^F - \hat{p}_L) + \hat{\rho}_L^F \hat{\mathbf{g}} + i\omega \rho_0^F \mathbf{u}_{\text{ini}}^F + \rho_0^F \mathbf{v}_{\text{ini}}^F \quad (87)$$

$$(33a), (33g), (33h) \Rightarrow \omega^2 \rho_0^S \hat{\mathbf{u}}^S + \text{Div} \hat{\mathbf{s}} = -i\omega \rho_0^S \mathbf{u}_{\text{ini}}^S - \rho_0^S \mathbf{v}_{\text{ini}}^S \quad (88)$$

$$(45d), (45g), (45h) \Rightarrow \rho_0^F g \hat{\eta} = \omega^2 \rho_0^F \hat{\phi} + \hat{p}^0 - \hat{p}^G + i\omega \rho_0^F \varphi_{\text{ini}} + \rho_0^F \Psi_{\text{ini}} \quad (89)$$

$$63a), (63f), (63g) \Rightarrow \Delta \hat{\phi} + \frac{\omega^2}{c^2} \hat{\phi} = -\frac{\hat{p}^0}{\rho_0^F c^2} - i\omega \frac{\varphi_{\text{ini}}}{c^2} - \frac{\Psi_{\text{ini}}}{c^2} \quad (90)$$

The new expressions of the variational formulations established from these local equations can be easily deduced (cf. section 7.3). To simplify the notations, the symbol $\hat{}$ will be dropped in the following, since, from now on, all the unknowns will be the Fourier transform of their temporal counterpart. However, let us remind that they are now complex functions and the operators constructed in the variational formulations must now be defined as sesquilinear Hermitian forms (respectively antilinear forms) instead of bilinear symmetric forms (respectively linear forms) in order to be physically associated to mechanical energies.

7.3. Finite element discretization and matrix form

The classical Rayleigh–Ritz finite element method can be applied to discretize the space part of the unknown fields. We consider here compatible meshes for the structure and the fluid. Since \mathbf{u}^S belongs to C_u^0 and φ to C_φ^1 , the kinematic conditions $\mathbf{u}^S = 0$ on Σ_u and $\ell(\varphi) = 0$ have to be taken into account in strong form. For $\mathbf{u}^S = 0$ the procedure is classic but for $\ell(\varphi) = 0$ it is more unusual. One way to do is to choose $\ell(\varphi)$ such that one component i of φ is equal to 0 ($\ell(\varphi) = \varphi_i = 0$). Then, all the row and columns associated to these zero values are removed from the matrix system. To simplify the notations, vectors of non-discarded nodal values will be denoted by the same letter as the associated continuous variable $\mathbf{u}^S(\mathbf{x}, \omega)$, $\varphi(\mathbf{x}, \omega)$ and $\eta(\mathbf{x}, \omega)$. In the same way, the (non-discarded part of) matrix operators⁶ associated to the sesquilinear forms will be denoted by the same capital letter, and the linear and antilinear forms by a lower case letter, as for example

$$B(\delta\eta, \varphi) \Rightarrow \delta\eta^* \mathbf{B}\varphi \quad \text{and} \quad B(\delta\eta, 1) \Rightarrow \delta\eta^* \mathbf{b} \quad (91)$$

7.3.1. Matrix equation in case of an incompressible heavy fluid

A Fourier transform and a finite element discretization of Eqs. (81), (48), (53), (52) and (55) lead to the following matrix system:

$$\left(\begin{bmatrix} \mathbf{K}_T & \mathbf{0} & \mathbf{0} & -\mathbf{c} & \mathbf{c}_g \\ -{}^t\mathbf{C} & \mathbf{F} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_g & -\mathbf{b} & \mathbf{b} \\ -{}^t\mathbf{c} & \mathbf{0} & -{}^t\mathbf{b} & \mathbf{0} & \mathbf{0} \\ {}^t\mathbf{c}_g & \mathbf{0} & {}^t\mathbf{b} & \mathbf{0} & -\chi^G \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} & \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^t\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \mathbf{u}^S \\ \varphi \\ \eta \\ \pi/\rho_0^F \\ p^G/\rho_0^F \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{\text{ext}} \\ \mathbf{0} \\ \mathbf{0} \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} \mathbf{M} & \mathbf{C} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^t\mathbf{B} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{v}_{\text{ini}}^S + i\omega \mathbf{u}_{\text{ini}}^S \\ \Psi_{\text{ini}} + i\omega \varphi_{\text{ini}} \end{pmatrix} \quad (92)$$

⁶ It can be shown that matrix operators obtained by the finite element discretization of sesquilinear Hermitian forms are real and symmetric [23].

7.3.2. Matrix equation in case of a compressible light fluid

The same transformations applied to Eqs. (81), (64), (68), (66) and (69) yield the following matrix system

$$\left(\begin{bmatrix} \mathbf{K}_T & \mathbf{0} & \mathbf{0} & -\mathbf{c} & \mathbf{c}_g \\ -{}^t\mathbf{C} & \mathbf{F} & -\mathbf{B} & -\mathbf{s}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{b} & \mathbf{b} \\ -{}^t\mathbf{c} & \mathbf{0} & -{}^t\mathbf{b} & -\chi^F & \mathbf{0} \\ {}^t\mathbf{c}_g & \mathbf{0} & {}^t\mathbf{b} & \mathbf{0} & -\chi^G \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} & \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^t\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^t\mathbf{s}_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \mathbf{u}^S \\ \varphi \\ \eta \\ \pi/\rho_0^F \\ p^G/\rho_0^F \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{\text{ext}} \\ \mathbf{0} \\ \mathbf{0} \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} \mathbf{M} & \mathbf{C} \\ \mathbf{0} & \mathbf{S}_c \\ \mathbf{0} & {}^t\mathbf{B} \\ \mathbf{0} & {}^t\mathbf{s}_c \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{v}_{\text{ini}}^S + i\omega\mathbf{u}_{\text{ini}}^S \\ \Psi_{\text{ini}} + i\omega\varphi_{\text{ini}} \end{pmatrix} \quad (93)$$

The solutions $(\mathbf{u}^S, \varphi, \eta, \pi, p^G)$ of Eqs. (92) and (93) can be obtained⁷ for all ω in \mathbb{R} by direct resolutions of these linear systems, however, the inversion of the dynamic stiffness matrices for each circular frequency⁸ can be very costly and a modal synthesis method will be proposed in the next section in order to reduce the computational costs.

8. Application to liquid sloshing in an elastic tank

In order to compute the vibrations of a pressurized elastic structure containing an inviscid incompressible liquid with a free surface, one has to solve Eq. (92) in the frequency domain (and then obtain the time response by an inverse Fourier transform). To reduce the computing times, a modal synthesis method is proposed here. It is based on the projection of the liquid equation onto its sloshing eigenmodes. This approach has already been applied to the standard (\mathbf{u}^S, φ) formulation (without pressurization) in previous works [14,25]. It is extended here to the new $(\mathbf{u}^S, \varphi, \eta, \pi, p^G)$ formulation presented in this paper.

8.1. Liquid sloshing modes

The harmonic vibrations of an incompressible heavy fluid in a rigid motionless cavity are called *sloshing modes*. They are the solutions $(\varphi_\alpha, \eta_\alpha, \pi_\alpha, p_\alpha^G) \cos(\omega_\alpha t)$ of (45) for $\mathbf{u}^S = \mathbf{0}$:

$$\Delta\varphi_\alpha = 0 \quad \text{in } \Omega_F \quad (94a)$$

$$\mathbf{n}_F \cdot \nabla\varphi_\alpha = \mathbf{0} \quad \text{on } \Sigma_i \quad (94b)$$

$$\mathbf{n}_F \cdot \nabla\varphi_\alpha = \eta_\alpha \quad \text{on } \Gamma \quad (94c)$$

$$\rho_0^F g \eta_\alpha = \rho_0^F \omega_\alpha^2 \varphi_\alpha + \pi_\alpha \quad \text{on } \Gamma \quad (94d)$$

$$p_\alpha^G = 0 \quad \text{in } \Omega_G \quad (94e)$$

$$\ell(\varphi_\alpha) = 0 \quad (94f)$$

The associated variational formulation is obtained directly from (48), (53) and (52):

$$\begin{aligned} \exists(\varphi_\alpha, \eta_\alpha, \pi_\alpha) \in \mathcal{C}_\varphi^\ell \times \mathcal{C}_\eta \times \mathbb{R} \text{ such that } \forall(\delta\varphi, \delta\eta) \in \mathcal{C}_\varphi^\ell \times \mathcal{C}_\eta, \\ \mathbf{F}(\varphi_\alpha, \delta\varphi) = \mathbf{B}(\delta\varphi, \eta_\alpha) \end{aligned} \quad (95a)$$

$$\mathbf{S}_g(\eta_\alpha, \delta\eta) - \omega_\alpha^2 \mathbf{B}(\delta\eta, \varphi_\alpha) - \frac{\pi_\alpha}{\rho_0^F} \mathbf{B}(\delta\eta, 1) = 0 \quad (95b)$$

$$\mathbf{B}(\eta_\alpha, 1) = 0 \quad (95c)$$

And after discretization by the finite element method, the associated eigenvalue matrix system is written as

$$\left(\begin{bmatrix} \mathbf{F} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_g & -\mathbf{b} \\ \mathbf{0} & -{}^t\mathbf{b} & \mathbf{0} \end{bmatrix} - \omega_\alpha^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^t\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \varphi_\alpha \\ \eta_\alpha \\ \pi_\alpha/\rho_0^F \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{pmatrix} \quad (96)$$

Since one row and one column of the matrix \mathbf{F} have been dropped when the condition $\ell(\varphi) = \varphi_1 = 0$ has been applied (cf. Section 7.3), \mathbf{F} becomes invertible⁹ and the first row of Eq. (96) can yield an expression of φ_α in function of η_α :

$$\varphi_\alpha = \mathbf{F}^{-1} \mathbf{B} \eta_\alpha \quad (97)$$

⁷ Let us remind that the solutions \mathbf{u}^S and φ of Eqs. (92) and (93) have to be completed by zero values to be reextended to their original size as explained previously in Section 7.3.

⁸ Due to the absence of dissipation in the model, the dynamic matrix of the system is not invertible for circular frequencies ω corresponding to the eigenfrequencies of the coupled problem. The introduction of a damping model for the structure (and eventually for the fluid) can solve this issue [35].

⁹ Despite operator \mathbf{F} is semi-definite, it has been shown in [14] that matrix \mathbf{F} becomes invertible as soon as at least one of its rows and columns has been eliminated.

The substitution of this relation in Eq. (96) yields

$$\left(\begin{bmatrix} \mathbf{S}_g & -\mathbf{b} \\ -\mathbf{b} & \mathbf{0} \end{bmatrix} - \omega_\alpha^2 \begin{bmatrix} {}^t\mathbf{B}\mathbf{F}^{-1}\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \boldsymbol{\eta}_\alpha \\ \pi_\alpha/\rho_0^F \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \quad (98)$$

If the matrix ${}^t\mathbf{B}\mathbf{F}^{-1}\mathbf{B}$ is denoted by \mathbf{M}_F , we deduce that the sloshing eigenmodes of the liquid are only described by their free surface normal displacement $\boldsymbol{\eta}_\alpha$ which are the solutions of the eigenvalue problem

$$\mathbf{S}_g \boldsymbol{\eta}_\alpha = \omega_\alpha^2 \mathbf{M}_F \boldsymbol{\eta}_\alpha \quad \text{with } {}^t\mathbf{b} \boldsymbol{\eta}_\alpha = 0 \quad (99)$$

where \mathbf{S}_g is the gravity potential energy operator, \mathbf{M}_F is the mass operator of the liquid in a rigid motionless cavity, condensed on its free surface, and π has the role of a Lagrange multiplier associated to the incompressibility condition ${}^t\mathbf{b} \boldsymbol{\eta}_\alpha = 0$ (which represents the fluid volume invariance). Since \mathbf{S}_g and \mathbf{M}_F are symmetric, the eigenmodes $\boldsymbol{\eta}_\alpha$ are real, orthogonal and form a basis for the liquid free surface deformations. Let us also remark that \mathbf{S}_g being nonsingular, there is no zero-frequency sloshing modes.

The orthogonality conditions between two sloshing modes can be obtained by writing relations (95a) and (95b) for two distinct modes $(\varphi_\alpha, \boldsymbol{\eta}_\alpha)$ and $(\varphi_\beta, \boldsymbol{\eta}_\beta)$ with $(\delta\eta, \delta\varphi)$ equal respectively to $(\varphi_\beta, \boldsymbol{\eta}_\beta)$ and $(\varphi_\alpha, \boldsymbol{\eta}_\alpha)$. We then obtain the following relations, $\forall (\alpha, \beta) \in \mathbb{N}^{*2}$:

$$F(\varphi_\alpha, \varphi_\beta) = B(\boldsymbol{\eta}_\alpha, \varphi_\beta) = B(\boldsymbol{\eta}_\beta, \varphi_\alpha) \quad (100)$$

$$S_g(\boldsymbol{\eta}_\alpha, \boldsymbol{\eta}_\beta) = \omega_\alpha^2 B(\boldsymbol{\eta}_\beta, \varphi_\alpha) = \omega_\beta^2 B(\boldsymbol{\eta}_\alpha, \varphi_\beta) \quad (101)$$

If $F(\varphi_\alpha, \varphi_\alpha)$ is denoted by μ_α , the following orthogonality relations can be deduced:

$$F(\varphi_\alpha, \varphi_\beta) = \mu_\alpha \delta_{\alpha\beta} = B(\boldsymbol{\eta}_\alpha, \varphi_\beta) = B(\boldsymbol{\eta}_\beta, \varphi_\alpha) \quad (102a)$$

$$S_g(\boldsymbol{\eta}_\alpha, \boldsymbol{\eta}_\beta) = \omega_\alpha^2 \mu_\alpha \delta_{\alpha\beta} \quad (102b)$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

8.2. Liquid static response

Since the sloshing modes span the space of the liquid movements in a rigid motionless cavity, it is necessary to complete this basis with a quasi-static response of the liquid corresponding to a tank wall static deformation. One possibility is to consider the static response $(\varphi_0, \eta_0, \pi_0, p_0^G)$ of the liquid to the displacement \mathbf{u}^S , i.e. the static solution of Eqs. (45) which satisfies

$$\Delta \varphi_0 = 0 \quad \text{in } \Omega_F \quad (103a)$$

$$\mathbf{n}_F \cdot \nabla \varphi_0 = \mathbf{u}^S \cdot \mathbf{n}_F \quad \text{on } \Sigma_i \quad (103b)$$

$$\mathbf{n}_F \cdot \nabla \varphi_0 = \eta_0 \quad \text{on } \Gamma \quad (103c)$$

$$\rho_0^F g \eta_0 = \pi_0 - p_0^G \quad \text{on } \Gamma \quad (103d)$$

$$p_0^G = \frac{P_0^G \gamma}{|\Omega_G|} \int_{\Sigma_i \cup \Sigma_g} \mathbf{u}^S \cdot \mathbf{n}_S \, d\Sigma \quad \text{in } \Omega_G \quad (103e)$$

$$\ell(\varphi_0) = 0 \quad (103f)$$

The variational equations (48), (53) and (52) can then be written for (φ_0, η_0) :

$$\exists (\varphi_0, \eta_0) \in C_\varphi^e \times C_\eta \quad \text{such that } \forall (\delta\varphi, \delta\eta) \in C_\varphi^e \times C_\eta,$$

$$F(\varphi_0, \delta\varphi) - B(\eta_0, \delta\varphi) = C(\delta\varphi, \mathbf{u}^S) \quad (104)$$

$$S_g(\eta_0, \delta\eta) - \frac{\pi_0}{\rho_0^F} B(\delta\eta, 1) + \frac{p_0^G}{\rho_0^F} B(\delta\eta, 1) = 0 \quad (105)$$

$$-B(\eta_0, 1) = C(1, \mathbf{u}^S) \quad (106)$$

Since π and p^G are scalar values, Eq. (103d) shows that η_0 is constant on Γ . Then, Eq. (106) yields

$$\eta_0 = -\frac{1}{\rho_0^F |\Gamma|} C(1, \mathbf{u}^S) \quad (107)$$

Using the variational relations (104) and (105), we can derive the conjugate relations between (φ_0, η_0) and $(\varphi_\beta, \eta_\beta)$. For this purpose, $\delta\varphi$ and $\delta\eta$ are chosen equal to φ_β and η_β . By using (95c), we obtain after some algebraic manipulations:

$$\forall \beta \in \mathbb{N}^*, \quad F(\varphi_0, \varphi_\beta) = C(\varphi_\beta, \mathbf{u}^S) - \frac{1}{\rho_0^F |\Gamma|} C(1, \mathbf{u}^S) B(1, \varphi_\beta) \quad (108)$$

$$S_g(\eta_0, \eta_\beta) = 0 \quad (109)$$

8.3. Construction of a fluid reduced matrix model

In a first step, we write φ and η as follows¹⁰:

$$\varphi(\omega) = \varphi_0(\mathbf{u}(\omega)) + \sum_{\alpha=1}^N q_\alpha(\omega) \varphi_\alpha \quad (110a)$$

$$\eta(\omega) = \eta_0(\mathbf{u}(\omega)) + \sum_{\alpha=1}^N \kappa_\alpha(\omega) \eta_\alpha \quad (110b)$$

where the coefficients q_α and κ_α represent the participation of each of the N selected sloshing modes to the liquid dynamic response.

In a second step, we introduce these decompositions in the first variational equation (48), and choosing $\delta\varphi$ equal to each mode φ_β , we obtain after simplifications due to the orthogonality relations (102):

$$\forall \beta \in \mathbb{N}^*, \quad F(\varphi_0, \varphi_\beta) + q_\beta \mu_\beta - B(\eta_0, \varphi_\beta) - \kappa_\beta \mu_\beta = C(\varphi_\beta, \mathbf{u}^S) \quad (111)$$

Using relation (104) applied to $\delta\varphi = \varphi_\beta$, we then obtain

$$\forall \beta \in \mathbb{N}^*, \quad q_\beta = \kappa_\beta \quad (112)$$

which shows that the modal participation coefficients are the same for φ and η .

The same derivation is now applied to the second variational equation (53). By choosing $\delta\eta$ equal to a each mode η_β , the following expression is obtained after some simplifications due to relations (102) and (109):

$$(\omega_\beta^2 - \omega^2) \mu_\beta q_\beta = \omega^2 B(\eta_\beta, \varphi_0) + B(\eta_\beta, \Psi_{ini} + i\omega\varphi_{ini}) \quad (113)$$

Eq. (95a) with $\delta\varphi = \varphi_0$ and conjugate relation (108) allow to express $B(\eta_\beta, \varphi_0)$ as follows:

$$B(\eta_\beta, \varphi_0) = F(\varphi_0, \varphi_\beta) = C(\varphi_\beta, \mathbf{u}^S) - \frac{1}{\rho_0^F |\Gamma|} C(1, \mathbf{u}^S) B(1, \varphi_\beta) \quad (114)$$

By introducing the function $\bar{\varphi}$ defined by

$$\forall \varphi \in \mathcal{C}_\varphi, \quad \bar{\varphi} = \varphi - \frac{1}{|\Gamma|} \int_\Gamma \varphi \, d\Gamma \quad (115)$$

the following relation can be easily deduced from (95c):

$$B(\eta_\beta, \bar{\Psi}_{ini} + i\omega\bar{\varphi}_{ini}) = B(\eta_\beta, \Psi_{ini} + i\omega\varphi_{ini}) \quad (116)$$

and then, we deduce from (113) the following reduced equation of the fluid:

$$\forall \beta \in \mathbb{N}^*, \quad (\omega_\beta^2 - \omega^2) \mu_\beta q_\beta = \omega^2 C(\bar{\varphi}_\beta, \mathbf{u}^S) + B(\eta_\beta, \bar{\Psi}_{ini} + i\omega\bar{\varphi}_{ini}) \quad (117)$$

Taking $\delta\eta = 1$ in Eq. (53), we obtain an expression of p_0 which can be simplified by considering that

$$S_g(\eta_0, 1) = \int_\Gamma \rho_0^F g \eta_0 \, d\Gamma = g B(\eta_0, 1) = -g C(1, \mathbf{u}^S) \quad (118)$$

$$\forall \alpha \in \mathbb{N}^*, \quad S_g(\eta_\alpha, 1) = \int_\Gamma \rho_0^F g \eta_\alpha \, d\Gamma = g B(\eta_\alpha, 1) = 0 \quad (119)$$

It finally yields

$$\rho_0^F |\Gamma| \frac{\pi}{\rho_0^F} = \rho_0^F |\Gamma| \frac{p_0^G}{\rho_0^F} - \omega^2 B(1, \varphi_0) - \omega^2 \sum_{\alpha=1}^N q_\alpha B(1, \varphi_\alpha) - g C(1, \mathbf{u}^S) - B(1, \Psi_{ini} + i\omega\varphi_{ini}) \quad (120)$$

¹⁰ In theory, the sum on the sloshing modes should be infinite but practically, only a small number N of sloshing modes is necessary to obtain a good representation of the solution.

The decomposition (110) of η is then used in Eq. (55), together with (95c) and (106), and yields the value of p^G :

$$\chi^G \frac{p^G}{\rho_0^F} = C_g(1, \mathbf{u}^S) - C(1, \mathbf{u}^S) \quad (121)$$

Concerning the structure, the influence of the fluid model reduction on the structure variational equation is obtained by introducing (110) in (81):

$$\begin{aligned} & \exists \mathbf{u}^S \in C_u^0 \text{ such that } \forall \delta \mathbf{u} \in C_u^0, \\ & K_T(\mathbf{u}^S, \delta \mathbf{u}) + \frac{p^G}{\rho_0^F} C_g(1, \delta \mathbf{u}) - \frac{\pi}{\rho_0^F} C(1, \delta \mathbf{u}) - \omega^2 M(\mathbf{u}^S, \delta \mathbf{u}) - \omega^2 C(\varphi_0, \delta \mathbf{u}) \\ & \dots - \omega^2 \sum_{\alpha=1}^N q_\alpha C(\varphi_\alpha, \delta \mathbf{u}) = f^{\text{ext}}(\delta \mathbf{u}) + M(\mathbf{v}_{\text{ini}}^S + i\omega \mathbf{u}_{\text{ini}}^S, \delta \mathbf{u}) + C(\Psi_{\text{ini}} + i\omega \varphi_{\text{ini}}, \delta \mathbf{u}) \end{aligned} \quad (122)$$

Relation (104) applied to $\mathbf{u}^S = \delta \mathbf{u}$ and $\delta \varphi = \varphi_0(\mathbf{u}^S)$ yields, using (107), the following expression of $C(\varphi_0, \delta \mathbf{u})$:

$$C(\varphi_0(\mathbf{u}^S), \delta \mathbf{u}) = F(\varphi_0(\mathbf{u}^S), \varphi_0(\delta \mathbf{u})) + \frac{1}{\rho_0^F |\Gamma|} C(1, \delta \mathbf{u}) B(1, \varphi_0(\mathbf{u}^S)) \quad (123)$$

Using this relation and replacing p_0 by its expression (120), the variational equation of the structure is written as

$$\begin{aligned} & \exists \mathbf{u}^S \in C_u^0 \text{ such that } \forall \delta \mathbf{u} \in C_u^0, \\ & K_T(\mathbf{u}^S, \delta \mathbf{u}) + \frac{g}{\rho_0^F |\Gamma|} C(1, \mathbf{u}^S) C(1, \delta \mathbf{u}) + \frac{p^G}{\rho_0^F} (C_g(1, \delta \mathbf{u}) - C(1, \delta \mathbf{u})) \\ & \dots - \omega^2 (M + M_0^F)(\mathbf{u}^S, \delta \mathbf{u}) - \omega^2 \sum_{\alpha=1}^N q_\alpha C(\bar{\varphi}_\alpha, \delta \mathbf{u}) = f^{\text{ext}}(\delta \mathbf{u}) \\ & \dots + M(\mathbf{v}_{\text{ini}}^S + i\omega \mathbf{u}_{\text{ini}}^S, \delta \mathbf{u}) + C(\bar{\Psi}_{\text{ini}} + i\omega \bar{\varphi}_{\text{ini}}, \delta \mathbf{u}) \end{aligned} \quad (124)$$

with

$$M_0^F(\mathbf{u}^S, \delta \mathbf{u}) = F(\varphi_0(\mathbf{u}^S), \varphi_0(\delta \mathbf{u})) \quad (125)$$

where M_0^F is called the fluid *hydrostatic* added mass matrix.

After discretization by the finite element method, Eqs. (124), (117) and (121) of the fluid–structure coupled problem are written as

$$\left(\begin{bmatrix} \mathbf{K}_T + \frac{g}{\rho_0^F |\Gamma|} \mathbf{c}^t \mathbf{c} & \mathbf{0} & \mathbf{c}_g - \mathbf{c} \\ \mathbf{0} & \mathbf{Y} \mathbf{\Omega}^2 & \mathbf{0} \\ \mathbf{c}_g - \mathbf{c} & \mathbf{0} & -\chi^G \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} + \mathbf{M}_0^F & \mathbf{C} \bar{\Phi} & \mathbf{0} \\ \mathbf{c}^t \bar{\Phi} & \mathbf{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \mathbf{u}^S \\ \mathbf{q} \\ p^G / \rho_0^F \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{\text{ext}} \\ \mathbf{0} \\ 0 \end{pmatrix} + \begin{bmatrix} \mathbf{M} & \mathbf{C} \\ \mathbf{0} & \mathbf{c}^t \mathbf{N}^t \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{v}_{\text{ini}}^S + i\omega \mathbf{u}_{\text{ini}}^S \\ \bar{\Psi}_{\text{ini}} + i\omega \bar{\varphi}_{\text{ini}} \end{pmatrix} \quad (126)$$

where each column of matrices $\bar{\Phi}$ and \mathbf{N} is one of the N selected sloshing modes (with zero-mean value on Γ) represented respectively by $\bar{\varphi}_\alpha$ and η_α . On the other hand, diagonal matrices \mathbf{Y} and $\mathbf{\Omega}$ contain respectively the N sloshing modal masses μ_α and circular frequencies ω_α .

With this modal synthesis approach, the fluid model size has been drastically reduced with respect to the full system (92) since $\dim(\mathbf{q}) = N \ll \dim(\varphi) + \dim(\eta)$, and the matrices of Eq. (126) are now symmetric. However, as we can see, this reduced model has one drawback since a new full stiffness matrix $\mathbf{c}^t \mathbf{c}$ must be assembled. One way to remove this drawback and to avoid its construction is to introduce an additional variable λ defined by

$$\lambda = \frac{1}{\chi^\Gamma} \int_{\Sigma_i} \mathbf{u}^S \cdot \mathbf{n}_F \, d\Sigma \quad (127)$$

with

$$\chi^\Gamma = \frac{\rho_0^F |\Gamma|}{g} \quad (128)$$

We then obtain the following matrix system:

$$\left(\begin{bmatrix} \mathbf{K}_T & \mathbf{0} & \mathbf{c}_g - \mathbf{c} & \mathbf{c} \\ \mathbf{0} & \mathbf{Y} \mathbf{\Omega}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{c}_g - \mathbf{c} & \mathbf{0} & -\chi^G & 0 \\ \mathbf{c} & \mathbf{0} & 0 & -\chi^\Gamma \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} + \mathbf{M}_0^F & \mathbf{C} \bar{\Phi} & \mathbf{0} & \mathbf{0} \\ \mathbf{c}^t \bar{\Phi} & \mathbf{Y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \mathbf{u}^S \\ \mathbf{q} \\ p^G / \rho_0^F \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{\text{ext}} \\ \mathbf{0} \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} \mathbf{M} & \mathbf{C} \\ \mathbf{0} & \mathbf{c}^t \mathbf{N}^t \mathbf{B} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{v}_{\text{ini}}^S + i\omega \mathbf{u}_{\text{ini}}^S \\ \bar{\Psi}_{\text{ini}} + i\omega \bar{\varphi}_{\text{ini}} \end{pmatrix} \quad (129)$$

Concerning \mathbf{M}_0^F , it can be computed by considering Eqs. (104) and (107):

$$\mathbf{F} \varphi_0(\mathbf{u}^S) = \left(\mathbf{c}^t - \frac{1}{\rho_0^F |\Gamma|} \mathbf{b}^t \mathbf{c} \right) \mathbf{u}^S \quad (130)$$

Applying the condition $\ell(\varphi) = \varphi_1 = 0$, the matrix \mathbf{F} is invertible and $\varphi_0(\mathbf{u}^S)$ can be obtained by solving a linear algebraic symmetric system. The same relation can be applied to calculate $\varphi_0(\delta\mathbf{u})$ and finally, from the definition of \mathbf{M}_0^F (125), the expression of \mathbf{M}_0^F is given by

$$\mathbf{M}_0^F = \mathbf{c} - \frac{1}{\rho_0^F |\Gamma|} \mathbf{c}^t \mathbf{b} \Big) \mathbf{F}^{-1} \Big(\mathbf{c} - \frac{1}{\rho_0^F |\Gamma|} \mathbf{b}^t \mathbf{c} \Big) \quad (131)$$

The solution of Eq. (129) can now be obtained either by a direct inversion of the matrix system or, if the size of the reduced model is still large, by a modal synthesis approach using the real fluid–structure coupled eigenmodes (which are the solution of the associated homogeneous equation) [14].

Finally, let us remark that the problem solving strategy illustrated here for the incompressible fluid case can be extended to the compressible equations (93) by using the liquid acoustic modes instead of its sloshing modes.

8.4. Application examples

Several examples of application of the linearized fluid–structure formulation presented here have been published during the past few years. The possibility to predict the coupling between the vibrations of the structure and the liquid sloshing using this formulation has been illustrated on test cases and on some more realistic examples [25,36] and then validated by comparison with experimental results [24]. For a study of the convergence of the reduced model (129) as a function of the number of selected sloshing modes, we refer the reader to [37]. An application to the specific case of free structures containing liquids and a validation by comparison with some benchmark results have been published in [38]. The nonlinear effect of the hydrostatic pressure on the hydroelastic vibrations of a plate has also been analyzed in [29]. Application examples concerning the effect of tank pressurization are in progress and will be published in a future paper.

9. Conclusions and perspectives

This paper presents a new way of deriving the linearized variational formulation for fluid–structure problems involving a deformable structure and an internal inviscid liquid with a free surface. It proposes a systematic approach by differentiation of the coupled nonlinear local equations around the static equilibrium solution. The complex prestress effects due to the fluid hydrostatic pressure on the structure are then automatically obtained. This approach gives a solid framework for future extensions to more complicated problems (stationary flow in the liquid, spinning tanks, meniscus in microgravity, etc.). One contribution of this paper was, for instance, to show how the formulation initially developed for incompressible heavy liquids could be generalized to the case of light compressible liquids. Moreover, the effect of the tank ullage pressurization on the dynamics of the coupled system was also easily added in both cases of incompressible or compressible liquids. Another specificity of the formulation proposed here is to keep the fluid free surface normal displacement in the state variables of the problem, with the aim of developing a model compatible with an extension to liquids with capillarity effects for which the free surface normal displacement is the main physical variable [39,32].

Finally, a numerical solution procedure in the frequency domain has been proposed with an appropriate finite element discretization. This makes the versatility of this approach which can be used for any arbitrary tank geometries, whereas many semi-analytical approaches are restricted to specific tank shapes (cylindrical, spherical, etc.) [40]. To reduce the computational costs, a modal synthesis approach has been proposed and has been illustrated on a particular application but could be generalized to other configurations in future works.

The introduction of the damping aspects in the fluid model is also a work in progress. As it has been seen, contrary to the case of an external fluid, there is no dissipation introduced by an internal liquid in the model when it is supposed inviscid (since the liquid domain is bounded). However, we could introduce in this formulation an *a posteriori* damping model for the liquid by affecting to each sloshing mode a damping coefficient that can be evaluated experimentally or numerically. It is known that the energy dissipation in a liquid is due mainly to four physical phenomena [41]: (i) the viscous dissipation in the boundary layers at the interface with the structure, (ii) the viscous dissipation at the liquid free surface due to a possible contamination by impurities (surfactants, etc.), (iii) the viscous dissipation in the bulk of the fluid and (iv) the capillary hysteresis due to the wetting phenomenon at the contact line between the liquid free surface and the structure. A simple model of each of these dissipation sources might be used to quantify numerically the energy ΔE_x dissipated during a vibration period of the system and then, a damping coefficient ζ_x associated to each sloshing mode could be deduced by [42,43]

$$\zeta_x \simeq - \frac{\Delta E_x}{2\pi\mu_x\omega_x^2} \quad (132)$$

By introducing this damping model in the fluid–structure equations, the coupled modes become complex but we can still use a modal synthesis approach to compute the frequency response (or the linearized time response) of the system to external excitations. These aspects will be developed in future publications [35].

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