



**HAL**  
open science

# Unbounded growth of the energy density associated to the Schrödinger map and the binormal flow

Valeria Banica, Luis Vega

► **To cite this version:**

Valeria Banica, Luis Vega. Unbounded growth of the energy density associated to the Schrödinger map and the binormal flow. *Annales de l'Institut Henri Poincaré (C), Analyse non linéaire (Nonlinear Analysis)*, 2022, 39, pp.927-946. <hal-03177240>

**HAL Id: hal-03177240**

**<https://hal.science/hal-03177240v1>**

Submitted on 23 Mar 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# UNBOUNDED GROWTH OF THE ENERGY DENSITY ASSOCIATED TO THE SCHRÖDINGER MAP AND THE BINORMAL FLOW

VALERIA BANICA AND LUIS VEGA

ABSTRACT. We consider the binormal flow equation, which is a model for the dynamics of vortex filaments in Euler equations. Geometrically it is a flow of curves in three dimensions, explicitly connected to the 1-D Schrödinger map with values on the 2-D sphere, and to the 1-D cubic Schrödinger equation. Although these equations are completely integrable we show the existence of an unbounded growth of the energy density. The density is given by the amplitude of the high frequencies of the derivative of the tangent vectors of the curves, thus giving information of the oscillation at small scales. In the setting of vortex filaments the variation of the tangent vectors is related to the derivative of the direction of the vorticity, that according to the Constantin-Fefferman-Majda criterion plays a relevant role in the possible development of singularities for Euler equations.

## 1. INTRODUCTION

In this paper we show the existence of an unbounded flux of the energy density of the solutions of two related partial differential equations which are completely integrable and have a connection with fluid mechanics. The first equation is the Schrödinger map in one dimension with values on the 2-D sphere, known in the physics literature as the classical continuous Heisenberg chain model in ferromagnetism:

$$(1) \quad T_t = T \times T_{xx}.$$

Here  $t$  will represent time and  $x$  the spatial variable. Equation (1) can be also written in divergence form. This is due to the fact that (1) can be obtained by simple differentiation in the spatial variable from the following second equation on curves  $\chi(t, x)$  in  $\mathbb{R}^3$ :

$$(2) \quad \chi_t = \chi_x \times \chi_{xx} \quad \chi_x = T, |T| = 1.$$

This latter equation, known as the Localized Induction Approximation (LIA), and also as the Vortex Filament Equation (VFE) and as the binormal flow (BF), appears naturally as a formal approximation (see [15], [41], [2],[9]), after a renormalization of time, of the location evolution of vortex filaments that move according to Euler equations. This model is conjectured to give the right dynamics of vortex filaments in certain situations (see [30] and the references therein). Through this model, at any given time  $t$ , the curve  $\chi(t, \cdot)$  represents the location of the vortex filament, and the tangent vector to the curve  $T = \chi_x$  represents the direction of the vorticity.

A simple use of the Frenet equations immediately gives that (2) can be written as

$$(3) \quad \chi_t = c\mathbf{b},$$

which explains the binormal flow name, and that

$$(4) \quad T_x = cn,$$

where  $n$  is the normal vector,  $b$  the binormal vector, and  $c$  the curvature. It is also easy to see that  $c^2(t, x)dx$  is an energy density that from (3) describes the kinetic energy of the filament and from (4) the interaction energy of the chain. More precisely

$$(5) \quad \int |\chi_t(t, x)|^2 dx = \int |T_x(t, x)|^2 dx = \int c^2(t, x) dx,$$

and for smooth solutions these quantities are conserved in time if they are finite.

For analytical reasons it is much more convenient to use instead of the classical Frenet frame given by the tangent, normal and the binormal vectors, the one given by parallel frames  $(T, e_1, e_2)$  constructed as solutions of

$$(6) \quad \begin{cases} T_x = \alpha e_1 + \beta e_2, \\ e_{1x} = -\alpha T, \\ e_{2x} = -\beta T. \end{cases}$$

Above  $\alpha$  and  $\beta$  are real scalars. A further simplification can be made defining the complex vector  $N = e_1 + ie_2 \in \mathbb{S}^2 + i\mathbb{S}^2$  and the complex scalar

$$(7) \quad u = \alpha + i\beta$$

to obtain

$$(8) \quad \begin{cases} T_x = \Re(\bar{u} N), \\ N_x = -u T. \end{cases}$$

It was proved in [29] that in order the constraint  $T_{xt} = T_{tx}$  to hold,  $u$  has to solve the one dimensional focusing non-linear Schrödinger equation (NLS)

$$(9) \quad iu_t + u_{xx} + \frac{1}{2}(|u|^2 - a(t))u = 0,$$

with  $a(t)$  a real scalar, and the tangent and normal vector have to satisfy the linear system

$$(10) \quad \begin{cases} T_t = \Im(\bar{u}_x N), \\ N_t = -iu_x T + \frac{i}{2}(|u|^2 - a(t))N. \end{cases}$$

Finally, (2) writes

$$(11) \quad \chi_t = \Im(\bar{u} N).$$

It is a well known fact that equation (9) is completely integrable and belongs to the so called AKNS-ZS hierarchy ([49],[48],[1]; for (1) see [45], [50]). The geometric meaning of  $u$  is clarified when we write (7) in polar form

$$(12) \quad u(t, x) = \rho(t, x)e^{i\theta(t, x)}$$

as  $\rho(t)$  and  $\theta_x(t)$  are the curvature and the torsion respectively of the curve  $\chi(t)$  ([29]).

Conversely, given a solution  $u$  of (9),  $t_0, x_0 \in \mathbb{R}$  and  $\mathcal{B}$  an orthonormal basis of  $\mathbb{R}^3$ , one can construct a solution of (1) by imposing  $\{T, \Re N, \Im N\}(t_0, x_0) = \mathcal{B}$  and solving (10) for  $(t, x_0)$  and then (6) for  $(t, x)$ . Then, given a point  $P \in \mathbb{R}^3$  a solution of (2) is constructed by imposing  $\chi(t_0, x_0) = P$  and solving  $\chi_t = T \wedge T_x$  for  $(t, x_0)$  and then  $\chi_x = T$  for  $(t, x)$ . We shall call Hasimoto's method this way of constructing a solution of (1)-(2) from a solution of (9).

For  $x$  either in the real line  $\mathbb{R}$  or in the torus  $\mathbb{T}$ , the well-posedness theory of the initial value problem associated to (9) was established in the function spaces  $L^2(\mathbb{R})$  in [46] and  $L^2(\mathbb{T})$  in [6]. Observe that (9) is invariant under the scaling  $u_\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x)$ , and that according to this scaling  $L^2(\mathbb{R})$  is subcritical. Moreover, among the homogeneous Sobolev spaces,  $\dot{H}^{-\frac{1}{2}}$  is the one invariant with respect to the scaling, thus there is a gap of  $1/2$  derivative between  $L^2$  and the critical space  $\dot{H}^{-\frac{1}{2}}$ . Starting in [47] a lot of attention has been devoted to extend the well-posedness theory to function and distribution spaces, not necessarily given by the Sobolev class, to make this gap as small as possible. As observed in [22] a good choice is to consider the so called Fourier-Lebesgue spaces that are defined using the  $L^p$  norm of the Fourier transform of the solution. Therefore, they are invariant under translation in phase space or, equivalently, under the so-called Galilean symmetries. The Fourier-Lebesgue space of functions with Fourier transform in  $L^\infty$  is also invariant with respect to the scaling. Several results about ill-posedness, either in the sense that the map datum-solution is not uniformly continuous, or showing what is known as the norm inflation phenomena has been proved ([32],[12],[8],[34],[42]). On the other hand, (local) well-posedness holds for data with Fourier transform in  $L^p$  spaces, for all  $2 < p < +\infty$  ([22],[11], [23]). This result can be proved using perturbation techniques and a fixed point argument. Making a strong use of the complete integrability the gap to the critical space has been also reduced, even in the quite remarkable case of the non-homogeneous Sobolev class and for global well-posedness as recently proved in [28] to all the subcritical cases, see also [33],[36], and [43] for the global in time result in the Fourier-Lebesgue class. As a consequence, no possible unbounded flux in the size of the Fourier transform of the solution can happen in this subcritical regime. In this paper we focus our attention in the critical case.

Geometrically, critical regularity for (2) means the possibility of having either corners or logarithmic spirals. We will concentrate ourselves in the particular case of corners that implies the existence of jumps for the corresponding tangent vectors. The case of logarithmic spirals has been considered in [24],[25] [39], and [40] and it is poorly understood. Nevertheless, we think that it is a quite relevant question to what extent the results showed in this work can be extended to that of the logarithmic spirals.

The simplest way of obtaining a corner is to look for selfsimilar solutions of (2). That is to say, solutions that can be written as  $\chi(t, x) = \sqrt{t}G(x/\sqrt{t})$  for some well chosen  $G$ . A simple computation gives that such a curve  $G$  has to solve the non-linear ODE

$$(13) \quad \frac{1}{2}G - \frac{y}{2}G' = G' \times G''.$$

In [37],[38], and [7] it is proved that a solution of (13) is characterized by the property that the curvature has to be a constant  $c = \alpha$ , and the torsion  $\tau$  has to be  $\tau(y) = y/2$ . Thanks to (12) this amounts to say that

$$u_\alpha(t, x) = \frac{\alpha}{\sqrt{t}} e^{i\frac{x^2}{4t}}.$$

As a consequence, if in (9) we take  $a(t) = \frac{|\alpha|^2}{t}$  we observe that  $u_\alpha$  solves (9) with initial condition

$$u_\alpha(0, x) = \sqrt{i}\alpha\delta(x).$$

Here  $\delta$  is the Dirac- $\delta$  function located at the origin.

Observe that (2) is invariant under rotations. Therefore, it is enough to give the Frenet frame of the curve given by  $G$  at say  $x = 0$  to construct all the solutions of (13). Take this frame,  $\{T(0), n(0), b(0)\}$ , to be the canonical orthonormal basis of  $\mathbb{R}^3$ , and call  $G_\alpha$  the corresponding solution. It was proved in [20] that if

$$\chi_\alpha(t, x) = \sqrt{t}G_\alpha\left(\frac{x}{\sqrt{t}}\right),$$

then  $\chi_\alpha$  solves (2) for  $t > 0$  and there exists  $\chi_\alpha(0, x)$  such that

$$|\chi_\alpha(t, x) - \chi_\alpha(0, x)| \leq 2\alpha\sqrt{t}, \quad t > 0.$$

Moreover,  $\chi_\alpha(0, x)$  is given by two half lines joined at a corner at the origin. Calling  $\theta_\alpha$  the corresponding interior angle, it is proved in [20] that

$$(14) \quad \sin \frac{\theta_\alpha}{2} = e^{-\frac{\pi}{2}|\alpha|^2}.$$

In our previous works we have considered two different scenarios for finding a functional setting that includes this example and such that at least a local well-posedness result can be established for the corresponding initial value problem.

The first scenario is when the polygonal line given by  $\chi_\alpha(0, x)$  is perturbed in such a way that the angle remains to be  $\theta_\alpha$  but outside of the corner location the curve is smooth and tends to two, possibly different, lines at infinity. To find these solutions we study first (9) with  $a(t) = \frac{|\alpha|^2}{t}$ , then we use Hasimoto's method for positive times and eventually we deal with the limit curves at  $t = 0$ . Regarding (9) we use the pseudo-conformal transformation. More concretely, we look for  $v$  with  $u = \mathcal{T}(v)$  where

$$(15) \quad \mathcal{T}(v)(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} \bar{v}\left(\frac{1}{t}, \frac{x}{t}\right).$$

Observe that above we also make a change of variables so that the time interval  $(0, 1)$  becomes  $(1, \infty)$ . A standard calculation gives that  $v$  has to solve

$$(16) \quad iv_t + v_{xx} + \frac{1}{2t}(|v|^2 - \tilde{a}(t))v = 0, \quad \tilde{a}(t) = \frac{1}{t}a\left(\frac{1}{t}\right) = |\alpha|^2.$$

Solutions of the above equation (16) formally preserve the  $L^2$  norm

$$(17) \quad \int |v|^2 dx,$$

and

$$(18) \quad E(v)(t) := \frac{1}{2} \int |v_x(t)|^2 dx - \frac{1}{4t} \int (|v|^2 - |\alpha|^2)^2 dx$$

satisfies

$$\partial_t E(v)(t) = \frac{1}{4t^2} \int (|v|^2 - |\alpha|^2)^2 dx.$$

Finally

$$\Im \int x \bar{v}_x v dx$$

is also a conserved quantity. We could also consider any of the infinitely many conserved quantities of (9) but observe that from the definition of (15) it would be necessary to assume regularity and decay on  $v$  for these quantities to be finite.

Notice that  $v_\alpha(t, x) := \alpha$  is a particular solution of (16), and the corresponding binormal flow solution is  $\chi_\alpha$ . In a series of papers, see the introduction of [3] for a survey of the results, we prove well posedness and small data modified scattering results for  $v - v_\alpha$ ,  $t \geq 1$ , in some appropriate function spaces such that  $E(t)$  given in (18) is finite.

The second scenario was started in [4] and considers solutions of (2) that at time  $t = 0$  are given by a skew polygonal line  $\chi_0(x)$  that tends to two lines when  $x \rightarrow \pm\infty$ . The corners are all located at integers  $j \in \mathbb{Z}$ . We use the Hasimoto's method, and at the level of (9) this problem is related to consider data

$$\sum_j \alpha_j \delta(x - j)$$

with some appropriate conditions in the size of  $\alpha_j$ . Following [35]<sup>1</sup> we look for solutions of (9) with  $a(t) = M/t$ , where  $M := \sum_j |\alpha_j|^2$ , for  $t > 0$ , of the type

$$(19) \quad u(t, x) = \sum_j A_j(t) \frac{e^{i \frac{(x-j)^2}{4t}}}{\sqrt{t}},$$

with

$$(20) \quad A_j(t) = e^{-i(|\alpha_j|^2 - M) \log \sqrt{t}} (\alpha_j + R_j(t)),$$

and  $R_j(t)$  satisfying decay properties as  $t$  goes to zero. The construction is performed for  $\{\alpha_j\}$  such that

$$(21) \quad \sum_j |j|^{2s} |\alpha_j|^2 < \infty \quad \text{for } s > 1/2.$$

---

<sup>1</sup> The authors acknowledge Tohru Ozawa for having pointed to them this article.

Using Hasimoto's method we construct a solution of the binormal flow for  $t > 0$  such that we recover at time  $t = 0$  the curve  $\chi_0$ , provided that we choose  $\alpha_j$  in a precise way determined by the curvature and torsion angles of  $\chi_0$  at  $x = j$ . In particular, as (14) for self-similar solutions, we choose

$$(22) \quad \sin \frac{\theta_j}{2} = e^{-\frac{\pi}{2}|\alpha_j|^2}.$$

The need of weights in (21) comes from integrating (10) because the coefficients of that system involve  $u_x$ . Notice that making the expansion of the square phases in (19) one immediately computes which is the pseudo-conformal transformation (15) of  $u$ . Indeed, we can write  $u = \mathcal{T}(v)$  with  $v$  the  $2\pi$ -periodic function in the  $x$ -variable:

$$v(t, x) = \sum_j A_j\left(\frac{1}{t}\right) e^{-itj^2 + ijx},$$

solution of (16) with  $\tilde{a}(t) = M$ . Observe that (17), the  $L^2$  conservation law of (16), and (20) give that for  $t > 0$

$$(23) \quad \sum_j |A_j(t)|^2 = M.$$

Recall that from (6)

$$|T_x(t, x)|^2 = |u(t, x)|^2 = \frac{1}{t} \left| v\left(\frac{1}{t}, \frac{x}{t}\right) \right|^2.$$

Hence for any  $t > 0$  the function  $|T_x(t, x)|^2$  is a  $2\pi t$ -periodic function in the  $x$ -variable and by (23) the integral on each of the periods is  $M$ . Nevertheless, this ‘‘conservation law’’ does not give any information about  $\hat{T}_x$ , the Fourier transform of  $T_x$ . We proved in [5] that (23) can be also understood as a kind of scattering energy of  $\hat{T}_x$  for the solutions of (1) and (2) that we constructed in [4]. More precisely, if

$$(24) \quad \Xi(T(t)) := \lim_{k \rightarrow \infty} \int_k^{k+1} |\widehat{T}_x(t, \xi)|^2 d\xi,$$

then for  $t > 0$  we have the following conservation law:

$$(25) \quad \Xi(T(t)) = 4\pi \sum_j |\alpha_j|^2.$$

It was also proved in [5] that there is a jump discontinuity of  $\Xi(T(t))$  at  $t = 0$ . From (24) we can see  $|\widehat{T}_x(t, \xi)|^2 d\xi$  as an asymptotic energy density in phase space. The main result of this paper is to prove that this energy density can grow in time at specific Fourier modes.

The procedure used in [4] to construct the solution  $u$  with the shape given in (19) is to solve the infinite dimensional non-homogeneous dynamical system generated by  $A_j(t)$ . The choice (23) kills all the resonant frequencies except those generated by the interaction of any mode  $j$  with itself. This interaction is easily absorbed by a logarithmic modification of the phase of  $A_j(t)$  that has been already incorporated in (20). Hence, a fixed point

argument can be performed to solve the system and to obtain the decay properties of  $R_j(t)$  mentioned above. As a consequence, there is no possible growth for  $A_j(t)$ .

The appearance of the logarithmic correction in the phases mentioned above is analogous to the long-range modified scattering that smooth small solutions of (9) satisfy ([44]). This modified scattering is behind the growth results of high Sobolev norms proved in [27] for the scalar cubic NLS on  $\mathbb{R} \times \mathbb{T}^d$  with  $d \geq 2$ . In that setting, which is a mixture of periodic and continuous variables, the authors prove a *loglog* growth in time of the amplitudes of the Fourier modes. The key ingredient for this growth is that, differently to what happens in one dimension, for  $d \geq 2$  the corresponding infinite dimensional system has a non-trivial resonant subsystem that generates solutions whose high Sobolev norms grow in time - see [13], [18], [26]. All these equations are not integrable. At this purpose we recall that growth of Sobolev norms for an integrable equation was proved in the case of the cubic Szegő equation ([17], see also [19]).

In Theorem 1.1 below we obtain a precise logarithmic growth in time for  $\hat{T}_x$ , being  $T$  a solution of (1) and the tangent vector of a curve that evolves according to (2). This curve at time  $t = 0$  is a polygonal line with just two corners of the same angle that are located at  $x = 1$  and  $x = -1$ . Recall that  $T_x$  represents the variation of the direction of the vorticity that as proved in [14] plays a crucial role in the possible formation of singularities of Euler equations.

**Theorem 1.1.** *Consider a polygonal line  $\chi_0(x)$  with two corners of angle  $\theta$  located at  $x \in \{-1, 1\}$ . Let  $\chi(t, x)$  be its evolution by the binormal flow (2) as explained above and let  $T(t, x)$  be its tangent vector.*

*There exist  $t_\theta, \tilde{t}_\theta \in (0, 1)$  and  $n_\theta \in \mathbb{N}$  such that for  $n \in \mathbb{N}, n \geq n_\theta, t \in (\frac{\tilde{t}_\theta}{n^2 \log^2 n}, \frac{t_\theta}{n^2})$  and  $\xi$  satisfying either  $|\xi - \frac{1}{2\pi t}| \leq \frac{1}{n}$  or  $|\xi + \frac{1}{2\pi t}| \leq \frac{1}{n}$  the following growth holds:*

$$(26) \quad |\widehat{T}_x(t, \xi) - V \log n| \leq \frac{1}{2}|V| \log n,$$

*where  $V$  is the non-null vector  $i(-\frac{2}{\pi}) \log(\sin \frac{\theta}{2})(T^{-\infty} - 2T^0 + T^{+\infty})$ , and the vectors  $T^{-\infty}, T^0, T^{+\infty}$  are the directions of the polygonal line  $\chi_0(x)$  on  $x < -1, -1 < x < 1$  and  $1 < x$  respectively.*

*As a consequence, for  $t \in (0, \frac{t_\theta}{n_\theta^2})$  there exists  $C_\theta > 0$  such that*

$$(27) \quad \sup_{\xi} |\widehat{T}_x(t, \xi)| \geq C_\theta \log t.$$

*Finally for  $\xi$  satisfying  $|\xi - \frac{1}{2\pi t}| \geq \frac{3}{8\pi t}$  and  $|\xi + \frac{1}{2\pi t}| \geq \frac{3}{8\pi t}$  and  $t \in (0, \frac{1}{4\pi})$  we have an upper-bound of  $|\widehat{T}_x(t, \xi)|$  depending only on  $\theta$ .*

Let us first note that our result concerns the growth of  $\|\widehat{T}_x\|_\infty$ , the  $L^\infty$  norm of the Fourier transform of  $T_x$ , and therefore we are looking at a critical norm in the class of Fourier-Lebesgue spaces.

The proof Theorem 1.1, which is given in §2, is based on the computation of  $\hat{T}_x(\xi)$  using (8) with  $u = \alpha + i\beta$  satisfying (19). This generates a first sum in  $j$  with the corresponding

$\bar{A}_j$  and their quadratic phases. Then, we use again (8) to integrate by parts, and a second sum appears with some new amplitudes  $A_r$  and new quadratic phases. It was observed in [5] that a resonance can happen if  $j - r \in \mathbb{Z}$  is properly chosen. It is easy to obtain a logarithmic upper bound for this resonance. It involves a small set of frequencies  $\xi$  which does not prevent (24) to hold. Our purpose in this paper to obtain a lower bound.

In §3 we prove the extension of Theorem 1.1 to the case of polygonal lines with many corners. The exhibited logarithmic growth is a hint that the numerical computations given in [16] about the unboundedness of  $\|\widehat{T}_x\|_\infty$ , are correct - see equation (8) in §5 in that paper. In that case the initial condition of (2) is a planar regular polygon. The dynamics becomes then periodic also in time and exhibits a Talbot effect, in the sense that at rational  $p/q$  multiples of the time period, skew polygons emerge with typically as many sides as  $q$ , see [31]. So this (numerical) logarithmic growth also happens at these rational times. A rigorous proof of this fact is a very challenging question that we propose to address in the future.

We shall denote systematically by  $C(\|A_j(t)\|_{l^1})$  constants depending only on universal constants and on a finite number of positive powers of  $\|A_j(t)\|_{l^1}$ .

## 2. PROOF OF THEOREM 1.1

Let  $n \in \mathbb{N}^*$ . First we recall that for  $s > \frac{1}{2}, 0 < \gamma < 1$ , it was proved in [4] that equation

$$(28) \quad iu_t + u_{xx} + \frac{1}{2}(|u|^2 - \frac{\sum_j |\alpha_j|^2}{t})u = 0,$$

has an unique local solution for  $t \in (0, \mathcal{T})$  of type

$$(29) \quad u(t, x) = \sum_{j \in \mathbb{Z}} e^{-i(|\alpha_j|^2 - M) \log \sqrt{t}} (\alpha_j + R_j(t)) \frac{e^{i \frac{(x-j)^2}{4t}}}{\sqrt{t}},$$

with

$$(30) \quad \sup_{0 < t < \mathcal{T}} t^{-\gamma} \|R_j(t)\|_{l^{2,s}} + t \|\partial_t R_j(t)\|_{l^{2,s}} < C(\gamma) \|\alpha_j\|_{l^{2,s}}^3.$$

The time of existence  $\mathcal{T}$  is in terms of  $s, \gamma, \|\alpha_j\|_{l^{2,s}}$ .

As explained in the Introduction, the evolution  $\chi(t)$  of  $\chi_0$  on  $(0, \mathcal{T})$ , is constructed by Hasimoto's method from the solution (29) of (28) with, in view of (22),

$$\alpha_j = \sqrt{\left(-\frac{2}{\pi}\right) \log\left(\sin \frac{\theta}{2}\right)} =: \alpha \text{ for } j \in \{\pm 1\}, \quad \alpha_j = 0 \text{ otherwise.}$$

**2.1. A general analysis on locating possible growth scenarios.** We shall start with a lemma that highlights the part of  $\widehat{T}_x(t, \xi)$  that can grow for small times, in general cases of polygonal lines.

**Lemma 2.1.** *Let  $\{\alpha_j\} \in l^{2, \frac{1}{2}+}$  and let  $\chi(t)$  be the evolution through the binormal flow of the corresponding polygonal line. For  $t \in (0, \frac{1}{4\pi n^2})$  and  $\xi \in \mathbb{R}$  the tangent vector of  $\chi(t)$  satisfies*

$$\left| \widehat{T}_x(t, \xi) - i \sum_{|j-r+4\pi t\xi| < 2nt} \overline{A_j(t)} A_r(t) e^{-i\frac{j^2-r^2}{4t}} \right. \\ \left. \times \int_{|x-j-4\pi t\xi| > \frac{1}{n}, |x-r+4\pi t\xi| > \frac{1}{n}} e^{i\frac{x(j-r+4\pi t\xi)}{2t}} \left( \frac{1}{x-j-4\pi t\xi} - \frac{1}{x-r+4\pi t\xi} \right) T(t, x) dx \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.$$

*Proof.* From (8) we have  $T_x(t, x) = \Re(\bar{u}N)(t, x)$  so

$$\begin{aligned} \widehat{T}_x(t, \xi) &= \int_{-\infty}^{\infty} e^{i2\pi x\xi} \Re(\bar{u}N)(t, x) dx = \int_{-\infty}^{\infty} e^{i2\pi x\xi} \Re\left(\sum_j \overline{A_j(t)} \frac{e^{-i\frac{(x-j)^2}{4t}}}{\sqrt{t}} N(t, x)\right) dx \\ &= \frac{e^{i4\pi^2 t\xi^2}}{2\sqrt{t}} \sum_{\pm, j} e^{i2\pi j\xi} \overline{A_j(t)} \int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}} N(t, x) dx \\ &\quad + \frac{e^{-i4\pi^2 t\xi^2}}{2\sqrt{t}} \sum_{\pm, j} e^{i2\pi j\xi} A_j(t) \int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}} \overline{N(t, x)} dx. \end{aligned}$$

We start by removing bounded pieces of the integral centered in  $j \pm 4\pi t\xi$ , by using a cut-off function  $\psi_n$  vanishing on  $B(0, \frac{1}{4n})$  and valued 1 on  ${}^cB(0, \frac{1}{2n})$ . These pieces are easy to estimate by  $C\frac{\|A_j(t)\|_{l^1}}{n\sqrt{t}}$ , as the integrands are of constant modulus. On the remaining pieces we integrate by parts:

$$\left| \widehat{T}_x(t, \xi) + i\sqrt{t}e^{i4\pi^2 t\xi^2} \sum_{\pm, j} e^{i2\pi j\xi} \overline{A_j(t)} \int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}} \left( \frac{N(t, x)\psi_n(x-j-4\pi t\xi)}{x-j-4\pi t\xi} \right)_x dx \right. \\ \left. - i\sqrt{t}e^{-i4\pi^2 t\xi^2} \sum_{\pm, j} e^{i2\pi j\xi} A_j(t) \int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}} \left( \frac{\overline{N(t, x)\psi_n(x-j+4\pi t\xi)}}{x-j+4\pi t\xi} \right)_x dx \right| \leq C\frac{\|A_j(t)\|_{l^1}}{n\sqrt{t}}.$$

When the derivative falls on  $\psi_n$  and on the denominator we get terms bounded by  $Cn\sqrt{t}\|\{A_j(t)\}\|_{l^1}$ , and  $n\sqrt{t} \leq \frac{1}{n\sqrt{t}}$  as  $t \in (0, \frac{1}{4\pi n^2})$ . We are left with the part from  $N_x = -uT$ . We get then

$$\left| \widehat{T}_x(t, \xi) - i \sum_j \overline{A_j(t)} \int_{-\infty}^{\infty} \frac{\sum_r A_r(t) e^{i\frac{x(j-r+4\pi t\xi)}{2t}} e^{-i\frac{j^2-r^2}{4t}}}{x-j-4\pi t\xi} T(t, x) \psi_n(x-j-4\pi t\xi) dx \right. \\ \left. + i \sum_j A_j(t) \int_{-\infty}^{\infty} \frac{\sum_r \overline{A_r(t)} e^{i\frac{x(r-j+4\pi t\xi)}{2t}} e^{i\frac{j^2-r^2}{4t}}}{x-j+4\pi t\xi} T(t, x) \psi_n(x-j+4\pi t\xi) dx \right| \leq C\frac{\|A_j(t)\|_{l^1}}{n\sqrt{t}}.$$

For  $|\frac{\pm(j-r)+4\pi t\xi}{2t}| \geq n$  we perform an integration by parts using the linear phase and get terms bounded by  $Ct\|\{A_j(t)\}\|_{l^1}^2$  when the derivative falls on  $\psi_n$  and on the denominator.

When the derivative falls on  $T$  we get  $\Re(\bar{u}N)$  which yields a quadratic phase. We complete to a quadratic phase incorporating the linear one and we remove bounded pieces localized where the quadratic phases cancel, which are upper-bounded by  $C\sqrt{t}\|\{A_j(t)\}\|_{l^1}^3$ . Then we perform an integration by parts from the quadratic phase and get terms upper-bounded by  $Cnt\sqrt{t}\|\{A_j(t)\}\|_{l^1}^3 + Cnt\|\{A_j(t)\}\|_{l^1}^4$ . Summarizing we have obtained for all  $\xi \in \mathbb{R}$

$$\left| \widehat{T}_x(t, \xi) - i \sum_{|j-r+4\pi t\xi| < 2nt} \overline{A_j(t)} A_r(t) I^+(t, \xi, j, r) + i \sum_{|-(j-r)+4\pi t\xi| < 2nt} A_j(t) \overline{A_r(t)} I^-(t, \xi, j, r) \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}},$$

where

$$I^\pm(t, \xi, j, r) := e^{\mp i \frac{j^2 - r^2}{4t}} \int_{|x - j \mp 4\pi t\xi| > \frac{1}{n}} \frac{e^{i \frac{x(\pm(j-r)+4\pi t\xi)}{2t}}}{x - j \mp 4\pi t\xi} T(t, x) dx.$$

The first discrete summation holds for  $(j, r)$  if and only if the second discrete summation holds for  $(r, j)$ , so

$$\left| \widehat{T}_x(t, \xi) - i \sum_{|j-r+4\pi t\xi| < 2nt} \overline{A_j(t)} A_r(t) (I^+(t, \xi, j, r) - I^-(t, \xi, r, j)) \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.$$

We then have

$$\left| \widehat{T}_x(t, \xi) - i \sum_{|j-r+4\pi t\xi| < 2nt} \overline{A_j(t)} A_r(t) e^{-i \frac{j^2 - r^2}{4t}} \times \int_{|x - j - 4\pi t\xi| > \frac{1}{n}, |x - r + 4\pi t\xi| > \frac{1}{n}} e^{i \frac{x(j-r+4\pi t\xi)}{2t}} \left( \frac{1}{x - j - 4\pi t\xi} - \frac{1}{x - r + 4\pi t\xi} \right) T(t, x) dx \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.$$

□

**2.2. An analysis on locating particular solutions that can exhibit growth.** To get the logarithmic growth of the theorem we shall restrict to a particular class of polygonal lines and we shall look for values of  $\xi_n$  such that  $|4\pi \frac{t}{n^2} \xi_n|$  is close to the distance between the corners. We note that in the case of a single corner we have  $\alpha_1 \alpha_{-1} = 0$  so the following Lemma ensures us that  $\widehat{T}_x(\frac{t}{n^2}, \xi_n)$  is bounded. Therefore the logarithmic growth comes from the interaction of several corners.

**Lemma 2.2.** *Let  $\{\alpha_j\}$  such that*

$$(31) \quad \alpha_j = 0 \text{ for } |j| > 1.$$

We have for all  $t \in (0, \frac{1}{4\pi n^2})$  and  $|\delta| < \frac{1}{n}$ :

$$(32) \quad \left| \widehat{T}_x(t, \frac{1}{2\pi t} + \delta) - i\overline{\alpha_{-1}}\alpha_1 \int_{|x-1| > \frac{1}{n}, |x+1| > \frac{1}{n}, |x| < 2} \left( \frac{1}{x-1} - \frac{1}{x+1} \right) T(t, x) dx \right| \\ \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.$$

A similar estimate holds at  $-\frac{1}{2\pi t} + \delta$ .

*Proof.* Denoting  $\xi = \frac{1}{2\pi t} + \delta$  implies  $4\pi t\xi = 2 + 4\pi t\delta$  so we have

$$|j - r + 4\pi t\xi| < 2nt < \frac{1}{2\pi n} \iff r = j + 2,$$

and Lemma 2.1 gives us

$$(33) \quad \left| \widehat{T}_x(t, \xi) - i \sum_j \overline{A_j(t)} A_{j+2}(t) e^{i\frac{(j+2)^2 - j^2}{4t}} \right. \\ \left. \times \int_{|x-j-2-4\pi t\delta| > \frac{1}{n}, |x-j+4\pi t\delta| > \frac{1}{n}} e^{ix2\pi\delta} \left( \frac{1}{x-j-2-4\pi t\delta} - \frac{1}{x-j+4\pi t\delta} \right) T(t, x) dx \right| \\ \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.$$

First we note that, as  $|4\pi t\delta| < \frac{1}{n}$ , we obtain by Cauchy-Schwarz

$$\int_{|x-j-2-4\pi t\delta| > \frac{1}{n}, |x-j+4\pi t\delta| > \frac{1}{n}, |x-j-1| > 2} \left| \frac{1}{x-j-2-4\pi t\delta} - \frac{1}{x-j+4\pi t\delta} \right| dx \\ = \int_{|y-1-4\pi t\delta| > \frac{1}{n}, |y+1+4\pi t\delta| > \frac{1}{n}, |y| > 2} \left| \frac{2 + 8\pi t\delta}{(y-1-4\pi t\delta)(y+1+4\pi t\delta)} \right| dy \leq C$$

Thus we can reduce the integration in (33) to  $|x-j-1| < 2$ . The remaining integrals are upper-bounded by  $\log n$ , and as  $|x2\pi\delta| \in (0, \frac{4\pi}{n})$  and  $|4\pi t\delta| < \frac{1}{n}$  we obtain

$$\left| \widehat{T}_x(t, \xi) - i \sum_j \overline{A_j(t)} A_{j+2}(t) e^{i\frac{(j+2)^2 - j^2}{4t}} \right. \\ \left. \times \int_{|x-j-2| > \frac{1}{n}, |x-j| > \frac{1}{n}, |x-j-1| < 2} \left( \frac{1}{x-j-2} - \frac{1}{x-j} \right) T(t, x) dx \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.$$

Now using the fact that  $\alpha_j = 0$  for  $|j| > 1$  and again that the integrals are upper-bounded by  $\log n$  we have

$$\left| \widehat{T}_x(t, \xi) - i\overline{\alpha_{-1}}\alpha_1 \int_{|x-1| > \frac{1}{n}, |x+1| > \frac{1}{n}, |x| < 2} \left( \frac{1}{x-1} - \frac{1}{x+1} \right) T(t, x) dx \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}$$

$$+ \sum_{|j| \leq 1} |\alpha_j| |R_{j+2}(t)| \log n + \sum_{|j+2| \leq n} |R_j(t)| |\alpha_{j+2}| \log n + \sum_j |R_j(t)| |R_{j+2}(t)| \log n.$$

By using the decay (30) of  $R_j(t)$  with  $\gamma > \frac{1}{2}$ , we get

$$(34) \quad |R_j(t)| \leq \|R_k(t)\|_{l^{2,s}} \leq C(\gamma) t^\gamma \|\alpha_k\|_{l^{2,s}}^3 \leq \frac{C}{n} \|\alpha_k\|_{l^{2,s}}^3,$$

thus the conclusion (32) of the Lemma. For  $\xi = -\frac{1}{2\pi t} + d$  we proceed the same. □

**2.3. Proof of the logarithmic growth.** Lemma 2.2 ensures us that for  $\xi_n = \frac{n^2}{2\pi t} + \delta$  with  $t \in (0, \frac{1}{4\pi})$  and  $|\delta| < \frac{1}{n}$  we have:

$$\left| \widehat{T}_x\left(\frac{t}{n^2}, \xi_n\right) - i|\alpha|^2 \int_{|x-1| > \frac{1}{n}, |x+1| > \frac{1}{n}, |x| < 2} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) T\left(\frac{t}{n^2}, x\right) dx \right| \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}.$$

Now we note that we can further reduce to

$$\left| \widehat{T}_x\left(\frac{t}{n^2}, \xi_n\right) - i|\alpha|^2 \int_{\frac{1}{3} > |x-1| > \frac{1}{n}, \frac{1}{3} > |x+1| > \frac{1}{n}} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) T^n\left(\frac{t}{n^2}, x\right) dx \right| \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}.$$

Therefore

$$\left| \widehat{T}_x\left(\frac{t}{n^2}, \xi_n\right) - i|\alpha|^2 \int_{\frac{1}{3} > |x-1| > \frac{1}{n}, \frac{1}{3} > |x+1| > \frac{1}{n}} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) T\left(\frac{t}{n^2}, x\right) dx \right| \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}.$$

We recall also that from Lemma 4.1 in [4] we have the convergence

$$|T\left(\frac{t}{n^2}, x\right) - T(0, x)| \leq C(\|\alpha_j\|_{l^{1,1}})(1 + |x|) \sqrt{\frac{t}{n^2}} \left(\frac{1}{d(x, \frac{1}{2}\mathbb{Z})} + \frac{1}{d(x, \mathbb{Z})}\right),$$

so

$$\begin{aligned} & \left| \widehat{T}_x\left(\frac{t}{n^2}, \xi_n\right) - i|\alpha|^2 \int_{\frac{1}{3} > |x-1| > \frac{1}{n}, \frac{1}{3} > |x+1| > \frac{1}{n}} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) T(0, x) dx \right| \\ & \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}} + C|\alpha|^3 \sqrt{t} \log n. \end{aligned}$$

As  $\chi_0$  is a polygonal line with  $T^{-\infty}, T^0$  and  $T^{+\infty}$  the directions on  $x < -1, -1 < x < 1$  and  $1 < x$  respectively,

$$\begin{aligned} & \int_{\frac{1}{3} > |x-1| > \frac{1}{n}, \frac{1}{3} > |x+1| > \frac{1}{n}} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) T(0, x) dx \\ & = T^{-\infty} \left[ \log \frac{|x-1|}{|x+1|} \right]_{-\frac{4}{3}}^{-1-\frac{1}{n}} + T^0 \left[ \log \frac{|x-1|}{|x+1|} \right]_{-1-\frac{1}{n}}^{-\frac{2}{3}} + T^0 \left[ \log \frac{|x-1|}{|x+1|} \right]_{\frac{2}{3}}^{1-\frac{1}{n}} + T^{+\infty} \left[ \log \frac{|x-1|}{|x+1|} \right]_{1+\frac{1}{n}}^{\frac{4}{3}}, \end{aligned}$$

and a logarithmic growth in  $n$  comes from the boundary terms at  $-1 - \frac{1}{n}, -1 + \frac{1}{n}, 1 - \frac{1}{n}$  and  $1 + \frac{1}{n}$ . Therefore we obtain

$$\left| \widehat{T}_x\left(\frac{t}{n^2}, \xi_n\right) - i|\alpha|^2(T^{-\infty} - 2T^0 + T^{+\infty}) \log n \right| \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}} + C|\alpha|^3\sqrt{t} \log n.$$

Now we compute:

$$(35) \quad |T^{-\infty} - 2T^0 + T^{+\infty}| = 2(1 - \cos(\pi - \theta)).$$

The smallest values of this modulus appears for  $\theta$  close to  $\pi$ , equivalent to  $\alpha$  close to zero in view of (22):

$$1 - \frac{(\pi - \theta)^2}{8} \stackrel{\theta \rightarrow \pi}{\approx} \cos \frac{\pi - \theta}{2} = \sin \frac{\theta}{2} = e^{-\frac{\pi}{2}|\alpha|^2} \stackrel{\alpha \rightarrow 0}{\approx} 1 - \frac{\pi}{2}|\alpha|^2,$$

thus

$$|T^{-\infty} - 2T^0 + T^{+\infty}| = 2(1 - \cos(\pi - \theta)) \stackrel{\theta \rightarrow \pi}{\approx} (\pi - \theta)^2 \approx 4\pi|\alpha|^2.$$

We then choose  $t$  small enough such that

$$C|\alpha|^3\sqrt{t} < \frac{1}{8}|\alpha|^2|T^{-\infty} - 2T^0 + T^{+\infty}|,$$

that reduces to

$$\sqrt{t} < C|\alpha|,$$

and large enough such that

$$\frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}} < \frac{1}{8}|\alpha|^2|T^{-\infty} - 2T^0 + T^{+\infty}| \log n,$$

that is implied, in view of (34) by

$$\frac{C(|\alpha| + \frac{|\alpha|^3}{n})}{|\alpha|^4 \log n} < \sqrt{t}.$$

Therefore the conditions on  $t$  and  $n$  with respect to  $\alpha$  are

$$(36) \quad \frac{C(|\alpha| + \frac{|\alpha|^3}{n})}{|\alpha|^4 \log n} < \sqrt{t} < \min\{C|\alpha|, \frac{1}{4\pi}\}.$$

We note that the upper and lower condition on  $t$  imply that  $n$  has to be chosen large with respect to  $\alpha$ .

Summarizing we have obtained the existence of  $t_\theta, \tilde{t}_\theta \in (0, 1)$  and  $n_\theta \in \mathbb{N}$  such that for  $n \in \mathbb{N}, n \geq n_\theta$  and  $t \in (\frac{\tilde{t}_\theta}{\log^2 n}, t_\theta)$  the following growth holds

$$|\widehat{T}_x(\frac{t}{n^2}, \frac{n^2}{2\pi t} + \delta) - i|\alpha|^2(T^{-\infty} - 2T^0 + T^{+\infty}) \log n| \leq \frac{1}{2}|\alpha|^2|T^{-\infty} - 2T^0 + T^{+\infty}| \log(n), \forall \delta, |\delta| < \frac{1}{n}.$$

This yields (26) in Theorem 1.1. For the analysis at  $-\frac{n^2}{2\pi t} + \delta$  we proceed the same way.

As a consequence we get the existence of  $C_\theta > 0$  such that for  $n \geq n_\theta$  and  $t \in (\frac{\tilde{t}_\theta}{n^2}, \frac{t_\theta}{n^2})$  we have

$$\sup_{\xi} |\widehat{T}_x(t, \xi)| \geq C_\theta \log n.$$

We can choose  $n_\theta$  large enough such that for all  $\tau \in (\tilde{t}_\theta, t_\theta)$

$$|\log \tau| < \frac{1}{2} \left| \log \frac{t_\theta}{n_\theta^2} \right| < \frac{1}{2} \left| \log \frac{\tau}{n_\theta^2} \right|.$$

Then, for  $t = \frac{\tau}{n^2} \in (\frac{\tilde{t}_\theta}{n^2}, \frac{t_\theta}{n^2})$

$$\sup_{\xi} |\widehat{T}_x(t, \xi)| \geq \frac{C_\theta}{2} (|\log t| - |\log \tau|) \geq \frac{C_\theta}{4} \log t.$$

By choosing moreover  $n_\theta$  large enough such that

$$\frac{t_\theta}{(n+1)^2} > \frac{\tilde{t}_\theta}{n^2},$$

we have that  $(0, \frac{t_\theta}{n_\theta^2}) = \cup_{n \geq n_\theta} (\frac{\tilde{t}_\theta}{n^2}, \frac{t_\theta}{n^2})$  so (27) holds.

**2.4. Bounds away from the growth zone.** Finally we consider for  $n \in \mathbb{N}^*$  those  $\xi$  such that  $|\xi - \frac{n^2}{2\pi t}| \geq \frac{3n^2}{2\pi t}$  and  $|\xi + \frac{n^2}{2\pi t}| \geq \frac{3n^2}{2\pi t}$ , which means  $|4\pi \frac{t}{n^2} \xi - 2| \geq \frac{3}{4}$  and  $|4\pi \frac{t}{n^2} \xi + 2| \geq \frac{3}{4}$ . We denote  $m \in \frac{\mathbb{Z}}{2}$  and  $d \in [-\frac{1}{2}, \frac{1}{2}[$  the numbers such that  $4\pi \frac{t}{n^2} \xi = 2m + d$ , that in particular implies  $m \notin \{\pm 1\}$ . Then

$$|j - r + 4\pi \frac{t}{n^2} \xi| < 2n \frac{t}{n^2} \iff r = j + 2m,$$

and Lemma 2.1 gives us for  $t \in (0, \frac{1}{4\pi})$

$$\begin{aligned} |\widehat{T}_x(\frac{t}{n^2}, \xi)| &\leq \sum_j |A_j(\frac{t}{n^2})| |A_{j+2m}(\frac{t}{n^2})| \int_{|x-j-2m-d| > \frac{1}{n}, |x-j+d| > \frac{1}{n}} \left| \frac{1}{x-j-2m-d} - \frac{1}{x-j+d} \right| dx \\ &\quad + \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}. \end{aligned}$$

The piece of integration  $|x - j - m| > 4m$  is bounded and the remaining part is upper-bounded by  $C \log(\max\{m, n\})$ . Since  $m \notin \{\pm 1\}$  then

$$\begin{aligned} \sum_j |A_j(\frac{t}{n^2})| |A_{j+2m}(\frac{t}{n^2})| &\leq \sum_{|j| \leq 1} |\alpha_j| |R_{j+2m}(\frac{t}{n^2})| + \sum_{|j+2m| \leq 1} |R_j(\frac{t}{n^2})| |\alpha_{j+2m}| + \sum_j |R_j(\frac{t}{n^2})| |R_{j+2m}(\frac{t}{n^2})| \\ &= |\alpha| (|R_{-1+2m}(\frac{t}{n^2})| + |R_{1+2m}(\frac{t}{n^2})| + |R_{-1-2m}(\frac{t}{n^2})| + |R_{1-2m}(\frac{t}{n^2})|) + \sum_j |R_j(\frac{t}{n^2})| |R_{j+2m}(\frac{t}{n^2})|. \end{aligned}$$

From (30) we have:

$$\sum_j |j|^{2s} |R_j(\frac{t}{n^2})|^2 \leq C \frac{|\alpha|^3}{n},$$

so for all  $j \in \mathbb{Z}$

$$|R_j(\frac{t}{n^2})| \leq \frac{C|\alpha|^3}{\sqrt{n}\langle j \rangle^s}.$$

Thus

$$|\widehat{T}_x(\frac{t}{n^2}, \xi)| \leq C(\alpha) \log(\max\{\langle m \rangle, n\}) \frac{1}{\sqrt{n}\langle m \rangle^s} + \frac{C(\alpha)}{\sqrt{t}}.$$

Therefore for  $\xi$  such that  $|\xi - \frac{n^2}{2\pi t}| \geq \frac{3n^2}{2\pi t}$  and  $|\xi + \frac{n^2}{2\pi t}| \geq \frac{3n^2}{2\pi t}$  and  $t \in (0, \frac{1}{4\pi})$  we obtain

$$|\widehat{T}_x(\frac{t}{n^2}, \xi)| \leq \frac{C(\alpha)}{\sqrt{t}},$$

and the proof of Theorem 1.1 is completed.

### 3. SEVERAL CORNERS

Instead of 2 corners we consider a planar polygonal line with  $2N$  corners located at  $-2N+1, \dots, 2N-1$ , with same angle  $\theta$ . Let  $m \in \{1, \dots, N\}$ . Proceeding similarly as in the proof of Lemma 2.2 we obtain (with constants that can depend on  $N$ ) for  $|\delta| < \frac{1}{n}$ :

$$\begin{aligned} & \left| \widehat{T}_x(\frac{t}{n^2}, \frac{mn^2}{2\pi t} + \delta) - i|\alpha|^2 \sum_{j \in \{-2N+1, \dots, (2N-2m-1)\}} \right. \\ & \times e^{\frac{i(j+2m)^2 - j^2}{4\frac{t}{n^2}}} \int_{\frac{1}{n} < |x - (j+2m)| < \frac{1}{3}, \frac{1}{n} < |x-j| < \frac{1}{3}} \left( \frac{1}{x - (j+2m)} - \frac{1}{x-j} \right) T(t, x) dx \left. \right| \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}. \end{aligned}$$

Now we restrict to  $t$  and  $n$  such that  $\frac{n^2}{t} \in 8\pi\mathbb{Z}$  to get rid of the phases in front of the integrals. Arguing as for the end of the proof of Theorem 1.1 in §2.3, we get

$$\left| \widehat{T}_x(\frac{t}{n^2}, \frac{mn^2}{2\pi t} + \delta) - V_m \log n \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{\sqrt{t}} + C|\alpha|^3 \sqrt{t} \log n.$$

where

$$\begin{aligned} (37) \quad V_m &= i\left(-\frac{2}{\pi}\right) \log\left(\sin \frac{\theta}{2}\right) \sum_{j \in \{-2N+1, \dots, (2N-2m-1)\}} (T(j^-) - T(j^+) - T((j+2m)^-) + T((j+2m)^+)) \\ &= i\left(-\frac{2}{\pi}\right) \log\left(\sin \frac{\theta}{2}\right) (T((-2N+1)^-) - T((-2N+1+2m)^-) - T((2N-1-2m)^+) + T((2N-1)^+)) \end{aligned}$$

For  $N=1$  we recover the vector  $V$  in Theorem 1.1, as

$$V_1 = i\left(-\frac{2}{\pi}\right) \log\left(\sin \frac{\theta}{2}\right) (T((-2N+1)^-) - T((-2N+1)^+) - T((2N-1)^-) + T((2N-1)^+)).$$

We note that for at least one  $m \in \{1, \dots, N\}$  we have  $V_m \neq 0$ .

Continuing similarly as in §2.3-2.4 we obtain the following result.

**Theorem 3.1.** *Consider a polygonal line  $\chi_0(x)$  with  $2N$  corners of angle  $\theta$  located at  $x \in \{-2N+1, \dots, 2N-1\}$ . Let  $\chi(t, x)$  be its evolution by the binormal flow by the Hasimoto method and denote  $T(t, x)$  the tangent vector.*

*There exists  $t_{\theta, N}, \tilde{t}_{\theta, N} \in (0, 1)$  and  $n_{\theta, N} \in \mathbb{N}$  such that for  $n \in \mathbb{N}, n \geq n_{\theta, N}$ , for  $t \in (\frac{\tilde{t}_{\theta, N}}{n^2 \log^2 n}, \frac{t_{\theta, N}}{n^2})$  satisfying  $\frac{1}{t} \in 8\pi\mathbb{Z}$ , for  $m \in \{1, \dots, N\}$  and for  $\xi$  satisfying either  $|\xi - \frac{m}{2\pi t}| \leq \frac{1}{n}$  or  $|\xi + \frac{m}{2\pi t}| \leq \frac{1}{n}$ , the following holds*

$$(38) \quad |\widehat{T}_x(t, \xi) - V_m \log(n)| \leq \frac{1}{2} |V_m| \log n,$$

where  $V_m$  is defined in (37).

As a consequence, for  $t \in (0, \frac{t_{\theta, N}}{n_{\theta, N}^2})$  there exists  $C_{\theta, N} > 0$  such that

$$(39) \quad \sup_{\xi} |\widehat{T}_x(t, \xi)| \geq C_{\theta, N} \log t.$$

Finally for  $\xi$  such that  $|\xi - \frac{m}{2\pi t}| \geq \frac{3}{8\pi t}$  and  $|\xi + \frac{m}{2\pi t}| \geq \frac{3}{8\pi t}$  for all  $m \in \{1, \dots, N\}$ , and  $t \in (0, \frac{1}{4\pi})$  we have an upper-bound of  $|\widehat{T}_x(t, \xi)|$  depending only on  $\theta$  and  $N$ .

**Acknowledgements:** This research is partially supported by the Institut Universitaire de France, by the French ANR project SingFlows, by ERCEA Advanced Grant 2014 669689 - HADE, by MEIC (Spain) projects Severo Ochoa SEV-2017-0718, and PGC2018-1228 094522-B-I00, and by Eusko Jaurlaritza project IT1247-19 and BERC program.

#### REFERENCES

- [1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems. *Studies in Appl. Math.* **53** (1974), 249–315.
- [2] R.J. Arms and F.R. Hama, Localized-induction concept on a curved vortex and motion of an elliptic vortex ring, *Phys. Fluids* **8** (1965), 553–560.
- [3] V. Banica and L. Vega, *The initial value problem for the binormal flow with rough data*, Ann. Sci. Éc. Norm. Supér. **48** (2015), 1421–1453.
- [4] V. Banica and L. Vega, Evolution of polygonal lines by the binormal flow, *Ann. PDE* **6** (2020), Paper No. 6, 53 pp.
- [5] V. Banica and L. Vega, On the energy of critical solutions of the binormal flow, *Comm. PDE* **45** (2020), 820–845.
- [6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I. Schrödinger equations, *Geom. Funct. Anal.* **3** (1993), 107–156.
- [7] T. F. Buttke, A numerical study of superfluid turbulence in the self-induction approximation, *J. Comput. Phys.* **76** (1988), 301–326.
- [8] R. Carles and T. Kappeler, Norm-inflation with infinite loss of regularity for periodic NLS equations in negative Sobolev spaces, *Bull. Soc. Math. France* **145** (2017), 623–642.
- [9] A.J. Callegari and L. Ting, Motion of a curved vortex filament with decaying vertical core and axial velocity, *SIAM. J. Appl. Math.* **35** (1978), 148–175.
- [10] T. Cazenave and F.B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in  $H^s$ , *Nonlinear Anal., Theory Methods Appl.* **14** (1990), 807–836.
- [11] M. Christ, Power series solution of a nonlinear Schrödinger equation, Mathematical aspects of nonlinear dispersive equations, 131-155, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

- [12] M. Christ, J. Colliander and T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, *Am. J. Math.* **125** (2003), 1235–1293.
- [13] J. Colliander, M. Keel, G. Staffilani, H. Takoaka, and T. Tao, Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation, *Invent. Math.* **181** (2010), 39–113.
- [14] P. Constantin, C. Fefferman and A. J. Majda, Geometric constraints on potentially singular solutions for the 3-D Euler equations, *Comm. Partial Differential Equations* **21** (1996), 559–571.
- [15] L.S. Da Rios, On the motion of an unbounded fluid with a vortex filament of any shape, *Rend. Circ. Mat. Palermo* **22** (1906), 117–135.
- [16] F. de la Hoz and L. Vega, On the relationship between the one-corner problem and the  $M$ -corner problem for the vortex filament equation, *J. Nonlinear Sci.* **28** (2018), 2275–2327.
- [17] P. Gérard and S. Grellier, The cubic Szegő equation, *Astérisque* **248** (2017), 126 p.
- [18] M. Guardia and V. Kaloshin. Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation, *J. Eur. Math. Soc.* **17** (2015), 71–149.
- [19] P. Gérard, E. Lenzmann, O. Pocovnicu and P. Raphaël A two-soliton with transient turbulent regime for the cubic half-wave equation on the real line. *Ann. PDE* **4** (2018), no. 1, Paper No. 7, 166 pp.
- [20] S. Gutiérrez, J. Rivas and L. Vega, Formation of singularities and self-similar vortex motion under the localized induction approximation, *Commun. PDE* **28** (2003) 927–968.
- [21] F.F. Grinstein, E. Gutmark and T. Parr, Near field dynamics of subsonic free square jets. A computational and experimental study, *Phys. Fluids* **7** (1995), 1483–1497.
- [22] A. Grünrock, Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, *Int. Math. Res. Not.* **41** (2005), 2525–2558.
- [23] A. Grünrock, and S. Herr. Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data, *SIAM J. Math. Anal.* **39** (2008)1890–1920.
- [24] S. Gutierrez and L. Vega, Self-similar solutions of the localized induction approximation: singularity formation, *Nonlinearity* **17** (2004), 2091–2136.
- [25] S. Gutierrez and L. Vega, On the stability of self-similar solutions of 1D cubic Schrödinger equations, *Math. Ann.* **356** (2013), 259–300.
- [26] Z. Hani, Long-time instability and unbounded Sobolev orbits for some periodic nonlinear Schrödinger equations, *Arch. Ration. Mech. Anal.* **211** (2014), 929–964.
- [27] Z. Hani, B. Pausader, N. Tzvetkov and N. Visciglia, Modified scattering for the cubic Schrödinger equation on product spaces and applications, *Forum Math. Pi* **3** (2015), 63p.
- [28] B. Harrop-Griffiths, R. Killip and M. Visan, Sharp well-posedness for the cubic NLS and mKdV in  $H^s(\mathbb{R})$ , it ArXiv 2003.05011.
- [29] H. Hasimoto, A soliton in a vortex filament, *J. Fluid Mech.* **51** (1972), 477–485.
- [30] R. L. Jerrard et C. Seis, *On the vortex filament conjecture for Euler flows*, Arch. Ration. Mech. Anal. **224** (2017), 135–172, pdf.
- [31] R. L. Jerrard and D. Smets, On the motion of a curve by its binormal curvature, *J. Eur. Math. Soc.* **17** (2015), 1148–1515.
- [32] C. Kenig, G. Ponce and L. Vega, On the ill-posedness of some canonical non-linear dispersive equations, *Duke Math. J.* **106** (2001), 716–633.
- [33] R. Killip, M. Visan and X. Zhang, Low regularity conservation laws for integrable PDE, *Geom. Funct. Anal.* **28** (2018), 1062–1090.
- [34] N. Kishimoto, Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity, *Differential Integral Equations* **22** (2009), 447–464.
- [35] N. Kita, Mode generating property of solutions to the nonlinear Schrödinger equations in one space dimension, Nonlinear dispersive equations, *GAKUTO Internat. Ser. Math. Sci. Appl., Gakkotosho, Tokyo* **26** (2006), 111–128.
- [36] H. Koch and D. Tataru, Conserved energies for the cubic NLS in 1-d, *Duke Math. J.* **167** (2018), 3207–3313.

- [37] M. Lakshmanan and M. Daniel, On the evolution of higher dimensional Heisenberg continuum spin systems. *Physica A* **107** (1981), 533–552.
- [38] M. Lakshmanan, T. W. Ruijgrok, and C. J. Thompson, On the the dynamics of a continuum spin system, *Physica A* **84** (1976), 577–590.
- [39] T. Lipniacki, Quasi-static solutions for quantum vortex motion under the localized induction approximation, *J. Fluid Mech.* **477** (2003), 321–337.
- [40] T. Lipniacki, Shape-preserving solutions for quantum vortex motion under localized induction approximation, *Phys. Fluids* **15** (2003) 1381–1395.
- [41] Y. Murakami, H. Takahashi, Y. Ukita and S. Fujiwara, On the vibration of a vortex filament, *Appl. Phys. Colloquium* (1937), 1–5.
- [42] T. Oh, A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces, *Funkcial. Ekvac.* **60** (2017), 259–277.
- [43] T. Oh and Y. Wang, Global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces, *J. Differential Equations* **269** (2020), 612–640.
- [44] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension. *Comm. Math. Phys.* **139** (1991), no. 3, 479–493.
- [45] L.A. Takhtadzhyan, Integration of the continuous Heisenberg spin chain through the inverse scattering method, *Phys. Lett. A* **64** (1977), 235–237.
- [46] Y. Tsutsumi,  $L^2$ -solutions for nonlinear Schrödinger equations and nonlinear groups, *Funkcial. Ekvac.*, **30** (1987), 115–125.
- [47] A. Vargas and L. Vega, Global wellposedness of 1D cubic nonlinear Schrödinger equation for data with infinity  $L^2$  norm, *J. Math. Pures Appl.* **80** (2001), 1029–1044.
- [48] V.E. Zakharov and S.V. Manakov, On the complete integrability of a nonlinear Schrödinger equation, *Theoret. and Math. Phys.* **19** (1974), 551–559.
- [49] V.E. Zakharov and A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Ž. Èksper. Teoret. Fiz.* **61** (1971), 118–134.
- [50] V.E. Zakharov and L.A. Takhtadzhyan, Equivalence of the nonlinear Schrödinger equation and the equation of a Heisenberg ferromagnet, *Theor. Math. Phys.* **38** (1979), 17–23.

(V. Banica) SORBONNE UNIVERSITÉ, CNRS, UNIVERSITÉ DE PARIS, LABORATOIRE JACQUES-LOUIS LIONS (LJLL), F-75005 PARIS, FRANCE, AND INSTITUT UNIVERSITAIRE DE FRANCE (IUF)

*E-mail address:* Valeria.Banica@ljll.math.upmc.fr

(L. Vega) BCAM-UPV/EHU BILBAO, SPAIN, LUIS.VEGA@EHU.ES

*E-mail address:* lvega@bcamath.org