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ABSTRACT. We study the quasi-static limit for the L^∞ entropy weak solution of the one-dimensional Burgers equation with boundary conditions. The quasi-stationary profile evolves with the quasi-static Burgers equation, whose entropy solution is determined by the stationary profile corresponding to the boundary data at a given time.

1. INTRODUCTION

Quasi-static evolution refers to dynamics driven by external boundary conditions or forces that change in a time scale much longer than the typical time scale of the convergence to stationary state of the dynamics. In the time scale of the changes of the exterior conditions the system is very close to the corresponding stationary state. This ideal evolutions are fundamental in thermodynamics and in many other situation. We are interested in studying dynamics where the corresponding quasi-stationary state is of *non-equilibrium*, i.e. it presents non-vanishing currents of conserved quantities.

In a companion article [4] we study the quasi-static limit for the one-dimensional open asymmetric simple exclusion process (ASEP). This is a dynamics where the stationary non-equilibrium states are well studied [5, 8, 9]. The macroscopic equation for the ASEP is given by Burgers equation on the one-dimensional finite interval $[0, 1]$:

$$\partial_t u + \partial_x \left(u(1-u)\right) = 0, \tag{1.1}$$

with boundary conditions, eventually time dependent, $u(t, 0) = \rho_-(t), u(t, 1) = \rho_+(t)$, resulting from the interaction with external reservoirs. For time independent boundary conditions and a special choice of the dynamics of the reservoirs for the open ASEP, equation (1.1) is obtained as hydrodynamic limit in [1]. More precisely the hydrodynamic limit generates the L^∞ entropy weak solution of (1.1) in the sense of [6].

Let us consider now the situation when the boundary conditions change in a slower time scale: for $\varepsilon > 0$ small, consider for (1.1) the boundary conditions $u(t, 0) = \rho_-(\varepsilon t), u(t, 1) = \rho_+(\varepsilon t)$. In order to see the effect of the changes in the boundaries, we need to look at the evolution in this time scale, i.e. defining

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$u^\varepsilon(t,x) = u(\varepsilon^{-1}t, x)$, it will satisfy the equation

$$\begin{cases} \varepsilon\partial_t u^\varepsilon + \partial_x \left(u^\varepsilon(1-u^\varepsilon)\right) = 0, \\ u^\varepsilon(t,0) = \rho_-(t), \quad u^\varepsilon(t,1) = \rho_+(t). \end{cases} \tag{1.2}$$

The main result in this article concerns the convergence of u^ε to the entropy weak solution of the quasi-static equation (see section 3.1 for the definition)

$$\partial_x J(u) = 0, \qquad u(t,0) = \rho_-(t), \quad u(t,1) = \rho_+(t). \tag{1.3}$$

It turns out that such solutions can only achieve two values with at most one upward discontinuity (shock) in the interior of the interval $[0,1]$, so they are necessarily of bounded variation (see Proposition 3.1). Outside the critical line $\{\rho_-(t) + \rho_+(t) = 1, \rho_-(t) < 1/2\}$ the solution is unique and constant in space (see Proposition 3.2). On the other hand on the critical line there are infinite many entropy solutions, corresponding to different position of the single shock, associated to the same value of the current. Consequently we can prove the convergence of u^ε to the unique quasi-static solution of the quasi-static equation only if $(\rho_-(t), \rho_+(t))$ remains outside the critical line for almost every t (see Theorem 3.5). On the critical line we can only prove the convergence to a *measure-valued* solution (cf. Remark 4.4). In all cases the quasi-stationary current $\mathcal{J}(t)$ is constant is space, and its value is determined by a variational problem (cf (3.11)): the entropy quasi-stationary solution minimize $J(\rho) = \rho(1-\rho)$ when $\rho_-(t) < \rho_+(t)$ (drift up-hill) and maximize it when $\rho_-(t) \geqslant \rho_+(t)$ (drift down-hill).

2. Burgers equation with boundary conditions

Consider the following initial–boundary problem of Burgers equation on the one-dimensional finite interval $[0,1]$

$$\begin{cases} \partial_t v(t,x) + \partial_x J(v(t,x)) = 0, & t > 0,\ x \in (0,1), \\ v(t,0) = \rho_-(t),\ v(t,1) = \rho_+(t), & t > 0, \\ v(0,x) = v_0(x), & x \in [0,1], \end{cases} \tag{2.1}$$

where $J(v) = v(1-v)$, $\rho_\pm \in C^1(\mathbb{R}_+)$, $v_0 \in L^\infty([0,1])$. Assume that the boundary and initial data are essentially bounded: $\|\rho_\pm\|_\infty \leqslant 1$ and $\|v_0\|_\infty \leqslant 1$, where $\|\cdot\|_\infty$ denotes the L^∞ norm. The solution $v \in L^\infty(\mathbb{R}_+ \times [0,1])$ is intended in the weak sense: for any $T > 0$ and $\varphi \in C_0^\infty([0,T] \times (0,1))$ such that $\varphi(T,x) = 0$,

$$\int_0^T \int_0^1 \left[v\partial_t\varphi + J(v)\partial_x\varphi\right] dx\, dt + \int_0^1 v_0(x)\varphi(0,x)dx = 0. \tag{2.2}$$

Furthermore, u satisfies the entropy inequality

$$\int_0^T \int_0^1 \left[S(v)\partial_t\varphi + Q(v)\partial_x\varphi\right] dx\, dt \geqslant 0 \tag{2.3}$$

for all pairs of functions (S,Q) such that

$$S, Q \in C^2(\mathbb{R}), \quad S'' \geqslant 0, \quad Q' = J'S' \tag{2.4}$$

and $\varphi \in C_0^\infty((0,T)\times(0,1))$ such that $\varphi \geqslant 0$. Functions (S,Q) satisfies (2.4) is called a *Lax entropy–entropy flux pair* associated to (2.1). Observe that (2.3) implies the Rankine–Hugoniot jump condition for (2.1): inside the interval eventual discontinuities must be *upwards shocks*.

Notice that discontinuities can appear in v at the boundaries. The boundary conditions in (2.1) are satisfied in the following sense. Assume for the moment that $v(t)$ are of bounded variation for each t, so that the limits

$$v_-(t) = \lim_{x\to 0+} v(t,x), \quad v_+(t) = \lim_{x\to 1-} v(t,x)$$

are well-defined. Then the Bardos–LeRoux–Nédélec boundary condition [2] of the entropy solution v reads for all $t>0$,

$$\operatorname{sign}(v_-(t)-\rho_-(t))\big[J(v_-(t))-J(k)\big] \leqslant 0 \tag{2.5}$$

for all $k \in I[v_-(t),\rho_-(t)]$ and

$$\operatorname{sign}(v_+(t)-\rho_+(t))\big[J(v_+(t))-J(k)\big] \geqslant 0 \tag{2.6}$$

for all $k \in I[v_+(t),\rho_+(t)]$, where $I[a,b]$ denotes the closed interval with extremes given by a and b.

The characterization of boundary conditions is extended to general entropy solutions $v \in L^\infty$ in [6]. A pair of two-variable functions (S,Q) is called a *boundary entropy–entropy flux pair* if S, $Q \in C^2(\mathbb{R}^2)$, $(S,Q)(\cdot,w)$ is a entropy–entropy flux pair for any $w\in\mathbb{R}$ and

$$S(w,w) = Q(w,w) = \partial_v S(v,w)|_{v=w} = 0, \quad \forall\, w\in\mathbb{R}. \tag{2.7}$$

The boundary conditions in (2.1) are then given by

$$\begin{aligned} &\operatorname*{esslim}_{r\to 0+} \int_0^T Q(v(t,r),\rho_-(t))\beta(t)dt \leqslant 0,\\ &\operatorname*{esslim}_{r\to 0+} \int_0^T Q(v(t,1-r),\rho_+(t))\beta(t)dt \geqslant 0, \end{aligned} \tag{2.8}$$

for any flux Q and $\beta \in C([0,T])$ such that $\beta \geqslant 0$.

The entropy solution v of (2.1) introduced above can be obtained through the *viscous approximation*. For $\delta>0$, let $v^\delta = v^\delta(t,x)$ be the classical solution of the viscous problem

$$\begin{cases} \partial_t v^\delta + \partial_x J(v^\delta) = \delta \partial_{xx} v^\delta, \quad t>0,\ x\in(0,1),\\ v^\delta(\cdot,0)=\rho_-, \quad v^\delta(\cdot,1)=\rho_+, \quad v^\delta(0,\cdot)=v_{0,\delta}, \end{cases} \tag{2.9}$$

where the mollified initial value $v_{0,\delta}\in C^\infty([0,1])$ satisfies that

$$\lim_{\delta\to 0+}\int_0^1 |v_{0,\delta}(x)-v_0(x)|dx = 0 \tag{2.10}$$

and the compatible conditions

$$v_{0,\delta}(0,0)=\rho_-(0), \quad v_{0,\delta}(0,1)=\rho_+(0). \tag{2.11}$$

By [7, Theorem 8.20], $v^\delta \to v$ in $C([0,T], L^1[0,1])$ for each $T > 0$.

3. Quasi-static evolution

3.1. **The quasi-static equation.** For $\varepsilon > 0$, let $u^\varepsilon \in L^\infty(\mathbb{R}_+ \times [0,1])$ be the entropy solution of

$$\begin{cases} \varepsilon \partial_t u^\varepsilon + \partial_x J(u^\varepsilon) = 0, \\ u^\varepsilon(t,0) = \rho_-(t), \quad u^\varepsilon(t,1) = \rho_+(t), \quad u^\varepsilon(0,x) = u_0(x), \end{cases} \tag{3.1}$$

in the sense of (2.2), (2.3) and (2.8).

Our aim is to prove that, as $\varepsilon \to 0$, the entropy solution u^ε of (3.1) converge to some $u \in L^\infty$ that is the entropy solution of the *quasi-static conservation law*

$$\partial_x J(u) = 0, \quad u(t,0) = \rho_-(t), \quad u(t,1) = \rho_+(t). \tag{3.2}$$

The L^∞ entropy solution of the quasi-static problem (3.2) is defined as a function $u \in L^\infty([0,T] \times [0,1])$ such that, for any function $\varphi \in C_0^\infty((0,T) \times (0,1))$,

$$\int_0^T \int_0^1 J(u) \partial_x \varphi \, dx \, dt = 0. \tag{3.3}$$

Furthermore for a flux function Q associated to some convex entropy S,

$$\int_0^T \int_0^1 Q(u) \partial_x \varphi \, dx \, dt \geqslant 0, \quad \forall \, \varphi \in C_0^\infty((0,T) \times (0,1)), \ \varphi \geqslant 0, \tag{3.4}$$

while the boundary conditions are satisfied in the same sense as in (2.8) with respect to a boundary entropy flux $Q(v,w)$.

Proposition 3.1. *Let $u(t,x)$ be L^∞ entropy solution of (3.2). Then there exists $a(t) \in [0,1/2]$ such that*

$$u(t,x) \in \{a(t), 1-a(t)\}, \quad (t,x) - a.s. \tag{3.5}$$

Furthermore, or $u(t,x)$ is a.s. constant in $x \in (0,1)$ for almost every t, or there is at most one upward jump from $a(t)$ to $1-a(t)$ inside $(0,1)$. In particular $u(t,x)$ is of bounded variation.

Proof. Since $u(t,x)$ solves $\partial_x J(u) = 0$ in the weak sense, there exists a bounded function $\mathcal{J}(t)$ such that $J(u(t,x)) = \mathcal{J}(t)$ almost surely in (t,x). Since $J(u) = u(1-u)$ let $a(t)$ such that $J(a(t)) = \mathcal{J}(t)$, and (3.5) follows.

The entropy condition (3.4) yields that $\partial_x Q(u(t,x))$ is negative in the sense of distribution. Observe now that for any $0 \leqslant a \leqslant 1/2$,

$$\begin{aligned} Q(a) - Q(1-a) &= \int_a^{1-a} Q'(u) du = \int_a^{\frac{1}{2}} \big[Q'(u) + Q'(1-u) \big] du \\ &= \int_a^{\frac{1}{2}} J'(u) \big[S'(u) - S'(1-u) \big] du \\ &= - \int_a^{\frac{1}{2}} J'(u) \left(\int_u^{1-u} S''(u^*) du^* \right) du \leqslant 0. \end{aligned} \tag{3.6}$$

Hence, only upward jumps from $a(t)$ to $1-a(t)$ can decrease the entropy flux $Q(u(t,x))$. This implies that we can have at most one such jump inside $(0,1)$. □

Since, by Proposition 3.1, entropy solution must be of bounded variation, then the boundary conditions are satified in the Bardos–LeRoux–Nédélec sense given by (2.5) and (2.6).

Define the critical segment

$$\Theta = \{(a,b) \in [0,1]^2; a < 1/2, a+b = 1\}. \tag{3.7}$$

The entropy solution of (3.2) is unique outside Θ and it can be calculated explicitly.

Proposition 3.2. *Suppose that $(\rho_-(t), \rho_+(t)) \notin \Theta$ for almost every $t \in [0,T]$. Then (3.2) has a unique entropy solution $u(t,x)$ given by*

$$u(t,x) = \begin{cases} \rho_-(t), & \text{if } \rho_-(t) < 1/2,\ \rho_-(t)+\rho_+(t) < 1, \\ \rho_+(t), & \text{if } \rho_+(t) > 1/2,\ \rho_-(t)+\rho_+(t) > 1, \\ 1/2, & \text{if } \rho_-(t) \geqslant 1/2,\ \rho_+(t) \leqslant 1/2. \end{cases} \tag{3.8}$$

Proof. We have to specify $a(t)$ through the boundary values $\rho_\pm$. From the argument above, $u(t,\cdot)$ has bounded total variation for each t, hence

$$u_-(t) = \lim_{x\to 0+} u(t,x), \quad u_+(t) = \lim_{x\to 0-} u(t,x) \tag{3.9}$$

are well-defined. Furthermore, $u_\pm(t) \in \{a(t), 1-a(t)\}$ and $u_-(t) \leqslant u_+(t)$. Rewrite (2.5) and (2.6) explicitly as

$$\begin{aligned} &\rho_- < 1/2 \Rightarrow u_- = \rho_- \text{ or } u_- \in [1-\rho_-, 1], \\ &\rho_- \geqslant 1/2 \Rightarrow u_- \geqslant 1/2, \\ &\rho_+ \leqslant 1/2 \Rightarrow u_+ \leqslant 1/2, \\ &\rho_+ > 1/2 \Rightarrow u_+ = \rho_+ \text{ or } u_+ \in [0, 1-\rho_+]. \end{aligned} \tag{3.10}$$

If $\rho_- \geqslant 1/2$, $\rho_+ \leqslant 1/2$, then $u_- = u_+ = 1/2$ so that $u(t,x) = 1/2$. If $\rho_- < 1/2$, $\rho_+ < 1-\rho_-$, then $u_- \leqslant u_+ \leqslant \max\{1/2, \rho_+\} < 1-\rho_-$, so that $u_- = \rho_-$. In view of (3.5), we have $a(t) = \rho_-$ and $u_+ = \rho_-$, hence $u(t,x) = \rho_-$. The case in which $\rho_+ > 1/2$, $\rho_- > 1-\rho_+$ is proved similarly. □

Remark 3.3. *If $(\rho_-(t), \rho_+(t)) \in \Theta$ for an interval of time of positive measure, then the entropy solution is not unique, but for any solution there exists one single shock with position $X(t)$ such that $u(t,x) = \rho_-(t)$ for $x < X(t)$ and $u(t,x) = \rho_+(t) = 1-\rho_-(t)$ for $x > X(t)$.*

Remark 3.4. *The entropy solution can also be characterized as the solution of the following variational problem:*

$$\mathcal{J}(t) = \begin{cases} \sup\{J(\rho); \rho \in [\rho_+(t), \rho_-(t)]\}, & \text{if } \rho_-(t) \geqslant \rho_+(t), \\ \inf\{J(\rho); \rho \in [\rho_-(t), \rho_+(t)]\}, & \text{if } \rho_-(t) < \rho_+(t). \end{cases} \tag{3.11}$$

This also includes the critical line $(\rho_-(t), \rho_+(t)) \in \Theta$, *where the current* $\mathcal{J} = J(\rho_-) = J(\rho_+)$ *is minimal in the interval* $[\rho_-(t), 1-\rho_-(t)]$.

3.2. **The quasi-static limit.**

Theorem 3.5. *Suppose that* $(\rho_-(t), \rho_+(t)) \notin \Theta$, *for almost every* t, *then the solution* u^ε *of* (3.1) *converges to* $u = u(t,x)$ *defined in* (3.8) *with respect to the weak-*$\star$ *topology of* $L^\infty([0,T]\times[0,1])$ *for all* $T > 0$.

Remark 3.6. *As* $\varepsilon \to 0$, $J(u^\varepsilon(t)) \overset{*}{\rightharpoonup} \mathcal{J}(t)$ *given by* (3.11). *Particularly, in the case* $(\rho_-(t), \rho_+(t)) \in \Theta$ *we can prove that* u^ε *converges weakly-*$\star$ *to a Young measure concentrated on* $\{\rho_\pm(t)\}$, *thus* $J(u^\varepsilon) \overset{*}{\rightharpoonup} \mathcal{J} = J(\rho_-) = J(\rho_+)$. *See Remark 4.4 at the end the the section.*

Remark 3.7. *Notice that the quasi-static limit in Theorem 3.5 does not depend on the initial condition* u_0 *for* u^ε.

Remark 3.8. *For* $\delta > 0$, *let* $v^\delta = v^\delta(t,x)$ *be the classical solution of the quasi-static problem associated to the viscous equation* (2.9):

$$\partial_x J(v^\delta) = \delta \partial_{xx} v^\delta, \quad v^\delta(t,0) = \rho_-(t), \quad v^\delta(t,1) = \rho_+(t). \tag{3.12}$$

When $(\rho_-(t), \rho_+(t)) \notin \Theta$, *it is not hard to see that* v^δ *also converges pointwisely to the function* u *given by* (3.8):

$$\lim_{\delta \to 0+} v^\delta(t,x) = u(t,x), \quad \forall x \in (0,1), \tag{3.13}$$

and the convergence is uniform on $[\gamma, 1-\gamma]$ *for any* $\gamma > 0$. *On the other hand, on the critical segment* $(\rho_-(t), \rho_+(t)) \in \Theta$, v^δ *is explicitly given by*

$$v^\delta(t,x) = \frac{1}{2} + \delta C(\delta,t) \tanh\left[C(\delta,t)\left(x - \frac{1}{2}\right)\right], \tag{3.14}$$

where $C = C(\delta,t)$ *is such that* $C\tanh(C/2) = \delta^{-1}(2\rho_+(t)-1)$. *Then* v^δ *converges pointwisely to the profile with an upward shock at* $1/2$:

$$\lim_{\delta \to 0+} v^\delta(t,x) = \rho_-(t)\mathbf{1}_{[0,\frac{1}{2})}(x) + \rho_+(t)\mathbf{1}_{(\frac{1}{2},1]}(x), \quad \forall x \in [0,1], \tag{3.15}$$

and the convergence is uniform on any closed interval does not contain $1/2$.

4. Proof of Theorem 3.5

For $\varepsilon > 0$, $\delta > 0$, consider viscous approximation of (3.1) given by

$$\begin{cases} \varepsilon \partial_t u^{\varepsilon,\delta} + \partial_x J(u^{\varepsilon,\delta}) = \delta \partial_{xx} u^{\varepsilon,\delta}, \quad t > 0,\ x \in (0,1), \\ u^{\varepsilon,\delta}(t,0) = \rho_-(t), \quad u^{\varepsilon,\delta}(t,1) = \rho_+(t), \quad u^{\varepsilon,\delta}(0,x) = u_{0,\delta}(x), \end{cases} \tag{4.1}$$

where $u_{0,\delta}$ is the mollified initial function satisfying (2.10) and the compatible conditions. Let $u^{\varepsilon,\delta} = u^{\varepsilon,\delta}(t,x)$ be the classical smooth solution of (4.1). We first present a priori estimate for $\|\partial_x u^{\varepsilon,\delta}\|_{L^2}$.

Proposition 4.1. *For any $t \geqslant 0$, there is a constant $C = C_t$ such that*

$$\varepsilon \int_0^1 u^{\varepsilon,\delta}(t,x)^2 dx + \delta \int_0^t \int_0^1 \big(\partial_x u^{\varepsilon,\delta}(s,x)\big)^2 dx\, ds \leqslant C. \tag{4.2}$$

Proof. Denote by $G(u)$ a primitive of $uJ'(u) = u - 2u^2$. Multiply (4.1) by $u^{\varepsilon,\delta}$ and integrate over $(0,t)\times(0,1)$ to obtain

$$\begin{aligned} &\frac{\varepsilon}{2}\int_0^1 u^{\varepsilon,\delta}(t,x)^2 dx - \frac{\varepsilon}{2}\int_0^1 u_{0,\delta}(x)^2 dx + \int_0^t \big[G(\rho_+(s)) - G(\rho_-(s))\big] ds \\ &= \delta \int_0^t \big[\rho_+(s)\partial_x u^{\varepsilon,\delta}(s,1) - \rho_-(s)\partial_x u^{\varepsilon,\delta}(s,0)\big] ds - \delta \iint (\partial_x u^{\varepsilon,\delta})^2 dx\, ds. \end{aligned} \tag{4.3}$$

In order to estimate the last line of (4.3) we test (4.1) against $\psi(s,x) := \rho_-(s) + x[\rho_+(s) - \rho_-(s)]$, obtaining that

$$\begin{aligned} &\varepsilon \int_0^1 \big[\psi(t,x)u^{\varepsilon,\delta}(t,x) - \psi(0,x)u_{0,\delta}(x)\big] dx - \varepsilon \iint u^{\varepsilon,\delta}\partial_s\psi\, dx\, ds \\ &+ \int_0^t \big[J(\rho_+(s))\rho_+(s) - J(\rho_-(s))\rho_-(s)\big] ds - \iint J(u^{\varepsilon,\delta})\partial_x\psi\, dx\, ds \\ &= \delta \int_0^t \big[\rho_+(s)\partial_x u^{\varepsilon,\delta}(s,1) - \rho_-(s)\partial_x u^{\varepsilon,\delta}(s,0)\big] ds - \delta \iint \partial_x u^{\varepsilon,\delta}\partial_x\psi\, dx\, ds. \end{aligned}$$

Then, Young inequality allows to estimate

$$\begin{aligned} &\left|\delta \int_0^t \big[\rho_+(s)\partial_x u^{\varepsilon,\delta}(s,1) - \rho_-(s)\partial_x u^{\varepsilon,\delta}(s,0)\big] ds\right| \\ &\leqslant C + \frac{\varepsilon}{4}\int_0^1 u^{\varepsilon,\delta}(t,x)^2 dx + \frac{\delta}{2}\int_0^t \int_0^1 (\partial_x u^{\varepsilon,\delta}(s,x))^2 dx ds, \end{aligned}$$

which, inserted into (4.3) gives the conclusion. □

In the following we denote $\Omega_T = [0,T]\times[0,1]$, $\Omega = \mathbb{R}_+ \times [0,1]$. As stated in §2, for each fixed $\varepsilon > 0$,

$$\lim_{\delta\to 0} \iint \varphi(t,x) F(t,x,u^{\varepsilon,\delta}(t,x)) dx\, dt = \iint \varphi(t,x) F(t,x,u^{\varepsilon}(t,x)) dx\, dt,$$

for all $F \in C(\Omega_T \times [0,1])$ and $\varphi \in L^1(\Omega_T)$, where $u^\varepsilon \in L^\infty(\Omega_T)$ is the entropy solution of (3.1). Observe that u^ε is uniformly bounded: $\|u^\varepsilon\|_{L^\infty(\Omega_T)} \leqslant 1$. Therefore, we can extract a weakly-$\star$ convergent subsequence:

$$\lim_{\varepsilon_n\to 0} \iint \varphi(t,x) F(t,x,u^{\varepsilon_n}(t,x)) dx\, dt = \iint \varphi(t,x) \int_0^1 F(t,x,\lambda)\nu_{t,x}(d\lambda) dx\, dt$$

where $\{\nu_{t,x}(d\lambda)\}_{(t,x)\in\Omega_T}$ is the limit Young measure.

It suffices to show that $\nu_{t,x}$ coincides with the delta measure concentrated on $u(t,x)$ given by (3.8). To this end, given boundary entropy–entropy flux pair

(S, Q), define the *boundary entropy production*

$$\overline{Q}_{\pm}(t,x) := \int Q(\lambda, \rho_{\pm}(t))\nu_{t,x}(d\lambda), \quad (t,x) \in \Omega_T. \tag{4.4}$$

The following proposition is the key argument.

Proposition 4.2. *For any boundary entropy flux Q,*

$$\overline{Q}_{-}(t,x) \leqslant 0, \quad \overline{Q}_{+}(t,x) \geqslant 0, \quad (t,x) - a.s. \tag{4.5}$$

Moreover, $\partial_x \overline{Q}_{\pm} \leqslant 0$ in the sense of distribution.

Proof. Recall that $u^{\varepsilon,\delta}$ is the classical solution of (4.1). For $w \in C^1([0,T])$ and boundary entropy–entropy flux (S, Q),

$$\begin{aligned}
&\varepsilon\partial_t S(u^{\varepsilon,\delta}, w) = \varepsilon\partial_u S(u^{\varepsilon,\delta}, w)\partial_t u^{\varepsilon,\delta} + \varepsilon\partial_w S(u^{\varepsilon,\delta}, w)w' \\
&= \delta\partial_x^2 S(u^{\varepsilon,\delta}, w) - \delta\partial_u^2 S(u^{\varepsilon,\delta}, w)(\partial_x u^{\varepsilon,\delta})^2 - \partial_x Q(u^{\varepsilon,\delta}, w) + \varepsilon\partial_w S(u^{\varepsilon,\delta}, w)w'.
\end{aligned}$$

Therefore, for $\varphi \in C^\infty(\Omega_T)$ such that $\varphi(0,x) = \varphi(T,x) = 0$,

$$\begin{aligned}
&\iint \big[\varepsilon S(u^{\varepsilon,\delta}, w)\partial_t\varphi + Q(u^{\varepsilon,\delta}, w)\partial_x\varphi + \varepsilon\partial_w S(u^{\varepsilon,\delta}, w)w'\varphi\big]dx\,dt \\
&= \delta \iint \big[\partial_x S(u^{\varepsilon,\delta}, w)\partial_x\varphi + \partial_u^2 S(u^{\varepsilon,\delta}, w)(\partial_x u^{\varepsilon,\delta})^2\varphi\big]dx\,dt \\
&\quad + \int_0^T \big[Q(u^{\varepsilon,\delta}(t,1), w(t)) - \delta\partial_x S(u^{\varepsilon,\delta}(t,1), w(t))\big]\varphi(t,1)dt \\
&\quad - \int_0^T \big[Q(u^{\varepsilon,\delta}(t,0), w(t)) - \delta\partial_x S(u^{\varepsilon,\delta}(t,0), w(t))\big]\varphi(t,0)dt.
\end{aligned}$$

Taking $w = \rho_-$, since $u^{\varepsilon,\delta}(\cdot,0) = \rho_-$ and $Q(w,w) = \partial_u S(w,w) = 0$ for all $w \in \mathbb{R}$, the last line above is 0. Hence, choosing $\varphi = \varphi_+$ such that

$$\varphi_+(t,1) = 0, \quad \varphi_+(0,x) = 0, \quad \varphi_+(T,x) = 0, \tag{4.6}$$

we obtain for any convex boundary entropy S that

$$\begin{aligned}
&\iint \big[\varepsilon S(u^{\varepsilon,\delta}, \rho_-)\partial_t\varphi_+ + Q(u^{\varepsilon,\delta}, \rho_-)\partial_x\varphi_+ + \varepsilon\partial_w S(u^{\varepsilon,\delta}, \rho_-)\rho_-'\varphi_+\big]dx\,dt \\
&\geqslant \delta \iint \partial_u S(u^{\varepsilon,\delta}, \rho_-)\partial_x u^{\varepsilon,\delta}\partial_x\varphi_+ dx\,dt.
\end{aligned}$$

Let $\delta \to 0+$ and apply the priori estimate in Proposition 4.1,

$$\iint \big[\varepsilon S(u^{\varepsilon}, \rho_-)\partial_t\varphi_+ + Q(u^{\varepsilon}, \rho_-)\partial_x\varphi_+ + \varepsilon\partial_w S(u^{\varepsilon}, \rho_-)\rho_-'\varphi_+\big]dx\,dt \geqslant 0.$$

Eventually, let $\varepsilon \to 0+$ along the convergent subsequence,

$$\iint \overline{Q}_{-}(t,x)\partial_x\varphi_+(t,x)dx\,dt \geqslant 0. \tag{4.7}$$

Since this holds for all nonnegative, smooth test function φ_+ satisfying (4.6), we conclude that $\overline{Q}_- \leqslant 0$ almost everywhere and $\partial_x \overline{Q}_- \leqslant 0$ as a distribution. For $\overline{Q}_+$, we replace (ρ_-, φ_+) with (ρ_+, φ_-) such that

$$\varphi_-(t,0) = 0, \quad \varphi_-(0,x) = 0, \quad \varphi_-(T,x) = 0, \tag{4.8}$$

and repeat the same argument. □

Theorem 3.5 follows directly from the following consequence.

Corollary 4.3. *The followings hold for a.e. (t,x):*

(1) If $\rho_-(t) < 1/2$, $\rho_-(t) + \rho_+(t) < 1$ then $\nu_{t,x} = \delta_{\rho_-(t)}$,
(2) If $\rho_+(t) > 1/2$, $\rho_-(t) + \rho_+(t) > 1$ then $\nu_{t,x} = \delta_{\rho_+(t)}$,
(3) If $\rho_-(t) \geqslant 1/2$, $\rho_+(t) \leqslant 1/2$ then $\nu_{t,x} = \delta_{1/2}$.

Proof. Consider the following boundary entropy

$$S(u,w) = \begin{cases} w \wedge \frac{1}{2} - u, & u \in [0, w \wedge \frac{1}{2}), \\ 0, & u \in [w \wedge \frac{1}{2}, 1]. \end{cases}$$

Note that S is not smooth, but it can be approximated by convex, smooth functions easily. For instance, let $s \in C^\infty(\mathbb{R})$ be such that

$$s(u) = -u, \ \forall\, u \leqslant -1, \ s(u) = 0, \ \forall\, u \geqslant 1, \ s'(0) = 0, \ s'' \geqslant 0.$$

Then $S_a(\cdot, w) \to S(\cdot, w)$ as $a \to 0+$, where

$$S_a(u,w) := a s\big(a^{-1}(u-w)\big), \quad a > 0.$$

The flux corresponding to S is

$$Q(u,w) = \begin{cases} J(w \wedge \frac{1}{2}) - J(u), & u \in [0, w \wedge \frac{1}{2}), \\ 0, & u \in [w \wedge \frac{1}{2}, 1]. \end{cases}$$

Since $Q(u, \rho_-) \geqslant 0$ for all $u \in [0,1]$ and $\overline{Q}_- \leqslant 0$, we know that $\nu_{t,x}$ concentrates on its zero set $[\rho_-(t) \wedge 1/2, 1]$. A similar argument yields that $\nu_{t,x}$ concentrates on $[0, \rho_+(t) \vee 1/2]$. Hence, $\nu_{t,x}$ concentrates on the interval

$$I_t = \left[\rho_-(t) \wedge \frac{1}{2}, \rho_+(t) \vee \frac{1}{2} \right].$$

Case 3 follows directly. In order to prove case 1 and 2, we choose

$$S_*(u,w) = |u-w|, \quad Q_*(u,w) = \operatorname{sign}(u-w)(J(u) - J(w)).$$

In case 1, $Q_*(u, \rho_-(t)) \geqslant 0$ on I_t and the only zero point is $\rho_-(t)$. As $\overline{Q}_- \leqslant 0$, we know that $\nu_{(t,x)} = \delta_{\rho_-}$. In Case 2, $Q_*(u, \rho_+(t)) \leqslant 0$ on I_t and the only zero point is $\rho_+(t)$, so the conclusion holds similarly. □

Remark 4.4. *Concerning the case $(\rho_-, \rho_+)(t) \in \Theta$, $Q_*(u, \rho_\pm(t))$ has opposite sign in I_t except two zero points $\rho_\pm(t)$, therefore $\nu_{t,x}$ concentrates on $\{\rho_-(t), \rho_+(t)\}$. Suppose $f(t,x) = \nu_{t,x}(\rho_+(t))$, then*

$$\nu_{t,x}(d\lambda) = [1 - f(t,x)]\delta_{\rho_-(t)}(d\lambda) + f(t,x)\delta_{\rho_+(t)}(d\lambda). \tag{4.9}$$

Observing that $J(\rho_+) = J(\rho_-)$*, so that*

$$J(u^\varepsilon(t,x)) \xrightarrow{*} \int_0^1 J(\lambda)\nu_{t,x}(d\lambda) = J(\rho_-(t)) = J(\rho_+(t)), \quad \varepsilon \to 0, \tag{4.10}$$

as stated in Remark 3.4.

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