QUASI-STATIC LIMIT FOR THE ASYMMETRIC SIMPLE EXCLUSION

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Abstract. We study the one-dimensional asymmetric simple exclusion process on the lattice $\{1,\dots,N\}$ with creation/annihilation at the boundaries. The boundary rates are time dependent and change on a slow time scale N^{-a} with $a>0$. We prove that at the time scale N^{1+a} the system evolves quasi-statically with a macroscopic density profile given by the entropy solution of the stationary Burgers equation with boundary densities changing in time, determined by the corresponding microscopic boundary rates. We consider two different types of boundary rates: the "Liggett boundaries" that correspond to the projection of the infinite dynamics, and the reversible boundaries, that correspond to the contact with particle reservoirs in equilibrium. The proof is based on the control of the Lax boundary entropy–entropy flux pairs and a coupling argument.

1. Introduction

The one-dimensional open asymmetric simple exclusion process (ASEP) is one of the most interesting models in non-equilibrium statistical mechanics, in particular because its stationary state can be explicitly computed (cf. [9, 13, 14, 3]). Particles perform asymmetric random walks on the finite lattice $\{1,\dots,N\}$ with the exclusion rule (the jump is suppressed if the site is occupied), and at the boundaries particles are created and absorbed with given rates. The dynamics is then characterized by 5 parameters (the asymmetry of the random walks and the 4 boundary rates).

In a seminal article [10], Liggett introduced special boundary rates such that the corresponding dynamics approximates optimally the dynamics of the infinite system with different densities at $\pm\infty$. These boundary conditions correspond to the projection of the infinite dynamics with respect to the Bernoulli measures with different densities on the left and on the right of the system. Under this choice of boundary conditions Bahadoran [1] proved the hydrodynamic limit for the density profile: under the hyperbolic space-time scaling the density profile converges to the unique L^∞ entropy solution of the Burgers equation satisfying the Bardos–Leroux–Nédélec boundary conditions [2] in the sense of Otto [11]. In [1] Bahadoran also proves the hydrostatic limit, i.e. the macroscopic limit of the stationary profile satisfies the stationary Burgers equation with same boundary conditions. This is the solution of the variational problem maximizing the stationary flux in case of a density gradient with opposite sign to the drift generated by the asymmetry, or minimizing the stationary flux in the other case. This is consistent with the phase diagram proved in [9, 13]. The proof in [1] relies on an extension of the coupling argument used by Rezakhanlou in [12] in the infinite dynamics, and on the particular boundary conditions which

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are such that at equal density (balanced case) the stationary measure is known explicitly (given by the Bernoulli measure at the boundary density). As far as we know there is no proof of the hydrodynamic limit for general boundary conditions.

In this article we study the quasi-static hydrodynamic limit for the open ASEP. This limit is taken in a time scale that is larger than the typical one where the system converges to equilibrium. Changing the boundary condition at this time scale, the system is globally close to the corresponding stationary state. Quasi-static evolutions are usually presented as idealization of real thermodynamic transformations among equilibrium states. They are necessary concepts in order to construct thermodynamic potentials, for example to define the thermodynamic entropy from Carnot cycles. Here we are interested in the quasi-static evolution among *non-equilibrium stationary states*. These quasi-static hydrodynamic limits have been already studied in the symmetric simple exclusion as well as in other diffusive systems [4]. We are here interested in the asymmetric case where currents of density do not vanish in the limit. Since in ASEP the typical time scale of convergence to stationarity is hyperbolic, we look at larger time scales changing the boundary rates in this time scale. Consequently, at each instant of time the system is *close* to the corresponding stationary state determined by the varying boundary conditions. We prove that the density profile converges to the entropy solution of the *quasi-static Burgers equation* with the corresponding boundary condition, now time dependent. We prove this quasi-static evolution for two types of boundary rates:

(1) Liggett boundaries: if $p > \frac{1}{2}$ is the probability of jumping to the right in the bulk of the system, at a macroscopic time t we choose $[p\bar\rho_-(t), (1-p)(1-\bar\rho_-(t))]$ as rates of creation and destruction on the left side, $[(1-p)(1-\bar\rho_+(t)), p\bar\rho_+(t)]$ on the right side.
(2) Reversible boundaries: we choose $[\rho_-(t), 1-\rho_-(t)]$ as rates of creation and destruction on the left side, $[\rho_+(t), 1-\rho_+(t)]$ on the right side, by accelerating this boundary rates and the symmetric part of the exclusion process in the bulk. These rates correspond to contact with reversible reservoirs of particles at the corresponding densities $\rho_\pm(t)$, and are independent from the asymmetry bulk parameter p. Even when $\rho_- = \rho_+$ and time independent, the stationary probability distribution is not a product measure in general.

The proof of the result for both cases will proceed as follows.

We first prove it in the balanced but time dependent cases ($\bar\rho_-(t) = \bar\rho_+(t)$ or $\rho_-(t) = \rho_+(t)$). Surprisingly this is the most difficult part, and it is proven by controlling the time average of the *microscopic boundary entropy flux*. The unbalanced situation is then proven by a coupling argument.

The use of the microscopic entropy production associated to a Lax entropy–entropy flux pair is already present in the seminal article of Rezakhanlou [12]. J. Fritz and collaborators combined this idea with a stochastic version of compensated compactness in order to deal with non-attractive dynamics [6, 7]. Otto [11] introduced the *boundary entropy–entropy flux pairs* in order to characterize the boundary conditions in the scalar hyperbolic equations. The main point of this article is to prove that, in the balanced case when we take the same

density on the boundaries, the time average of the microscopic boundary entropy flux is negligeable in the quasi-static time scale, even when boundary conditions change in time (see Propositions 8.1 and 8.2).

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2. ASEP with open boundaries

For $N \geq 2$, the asymmetric simple exclusion process (ASEP) with open boundary conditions is the Markov process on the configuration space

$$\Omega_N := \big\{\eta = (\eta_1, \eta_2, \dots, \eta_N),\ \eta_i \in \{0,1\}\big\}, \tag{2.1}$$

with generator

$$L_N f = \lambda_0 L_{\text{exc}} f + \lambda_- L_- f + \lambda_+ L_+ f, \tag{2.2}$$

where λ_0, $\lambda_\pm > 0$, f is any function on Ω_N, L_{exc} is the generator of the simple exclusion:

$$\begin{aligned} L_{\text{exc}} f &:= \sum_{i=1}^{N-1} c_{i,i+1} \left[f\left(\eta^{i,i+1}\right) - f(\eta) \right], \\ c_{i,j} &:= p\eta_i(1-\eta_j) + (1-p)\eta_j(1-\eta_i), \end{aligned} \tag{2.3}$$

where $1/2 < p \leq 1$, $\eta^{i,i+1}$ is the configuration obtained from η upon exchanging η_i and η_{i+1}. $L_\pm$ are the generators of creates/annihilates processes at the boundaries $i = 1$ and $i = N$:

$$\begin{aligned} L_- f &:= [\rho_-(1-\eta_1) + (1-\rho_-)\eta_1] \left[f(\eta^1) - f(\eta) \right], \\ L_+ f &:= [\rho_+(1-\eta_N) + (1-\rho_+)\eta_N] \left[f(\eta^N) - f(\eta) \right], \end{aligned} \tag{2.4}$$

where $\rho_\pm \in [0,1]$, η^i is the configuration obtained from η by shifting the status at site i from η_i to $1-\eta_i$. Notice that $L_\pm$ are reversible for the Bernoulli probability with density $\rho_\pm$, respectively.

Remark 1 (Boundary rates)**.** *To relate the boundary parameters to the ones used in* [14, 3]*, define*

$$\begin{aligned} \alpha &:= \lambda_- \rho_- \qquad \textit{entry rate from the left,} \\ \gamma &:= \lambda_-(1-\rho_-) \qquad \textit{exit rate to the left,} \\ \beta &:= \lambda_+(1-\rho_+) \qquad \textit{exit rate to the right,} \\ \delta &:= \lambda_+ \rho_+ \qquad \textit{entry rate from the right.} \end{aligned} \tag{2.5}$$

Note that $\lambda_- = \alpha + \gamma$ and $\lambda_+ = \gamma + \delta$ give the total rates at left and right boundaries.

Remark 2 (Stationary states)**.** *Following* [3]*, defining*

$$\begin{aligned} \tau_- &:= \frac{1}{2\alpha}\left(2p - 1 - \alpha + \gamma + \sqrt{(2p-1-\alpha+\gamma)^2 + 4\alpha\gamma}\right), \\ \tau_+ &:= \frac{1}{2\beta}\left(2p - 1 - \beta + \delta + \sqrt{(2p-1-\beta+\delta)^2 + 4\beta\delta}\right), \end{aligned} \tag{2.6}$$

then if the following conditions are satisfied (see [3]*)*

$$\min(\alpha,\beta)>0,\quad \tau_-=\tau_+^{-1}, \tag{2.7}$$

the stationary state of L_N is given by the Bernoulli product measure with homogeneous density

$$\tilde\rho:=\frac{1}{\tau_-+1}=\frac{\tau_+}{\tau_++1}. \tag{2.8}$$

2.1. **Liggett boundaries.** A special choice of the boundary rates has been introduced by Liggett [10]: given $\bar\rho_\pm\in[0,1]$, choose the parameters in (2.2) as

$$\begin{aligned}&\lambda_0=1,\quad \lambda_-=p\bar\rho_-+(1-p)(1-\bar\rho_-),\quad \lambda_+=p(1-\bar\rho_+)+(1-p)\bar\rho_+,\\ &\rho_-=\frac{p\bar\rho_-}{p\bar\rho_-+(1-p)(1-\bar\rho_-)},\quad \rho_+=\frac{(1-p)\bar\rho_+}{p(1-\bar\rho_+)+(1-p)\bar\rho_+}.\end{aligned} \tag{2.9}$$

This means to choose the boundary rates in (2.5) as

$$(\alpha,\beta,\gamma,\delta)=\big(p\bar\rho_-,p(1-\bar\rho_+),(1-p)(1-\bar\rho_-),(1-p)\bar\rho_+\big). \tag{2.10}$$

This choice of boundaries correspond to the projection on the finite interval $[[1,N]]$ of the infinite ASEP dynamics with Bernoulli distribution with density ρ_- on the left of 1, and with density ρ_+ on the right of N (see formulas (5) and (6) in [1]). Liggett's motivation for this choice was to give the best approximation to the infinite dynamics of the ASEP [10].

With the above choice of boundary rates, for $\bar\rho_-=\bar\rho_+=\bar\rho$, the corresponding stationary measure is

$$\nu_{\bar\rho}^N:=\prod_{i=1}^N\nu_{\bar\rho}(\eta_i),\quad \nu_{\bar\rho}(\eta):=\bar\rho^\eta(1-\bar\rho)^{1-\eta}. \tag{2.11}$$

Observe that in this case generally $\rho_-\neq\rho_+$, but for $p\to1/2$ we have $\rho_\pm\to\bar\rho$.

These are the boundary conditions chosen in [1] for the hydrodynamic limit.

3. Quasi-static evolution

3.1. **Quasi-static Burgers' equation.** Let $\bar p\in(0,1]$ and consider two $\mathcal{C}^1$ functions $\tilde\rho_\pm:[0,\infty)\to[0,1]$. For some technical reason (see Section 4), we assume the following condition: for any $T>0$,

$$[0,T]=\cup_{i\geq1}I_i\cup A_0, \tag{3.1}$$

where A_0 is a Lebesgue null set and I_i's are closed intervals such that either $\tilde\rho_+(t)\geq\tilde\rho_-(t)$ on I_i or $\tilde\rho_+(t)\leq\tilde\rho_-(t)$ on I_i.

The L^∞ entropy solution of the quasi-static conservation law

$$\partial_xJ(\rho(x,t))=0,\quad x\in(0,1),\quad J(\rho)=\bar p\rho(1-\rho), \tag{3.2}$$

with boundary data

$$\rho(0,t)=\tilde\rho_-(t),\quad \rho(1,t)=\tilde\rho_+(t), \tag{3.3}$$

is defined as the unique limit, for $\varepsilon \to 0+$ of the L^∞ entropy solution of

$$\begin{aligned} &\varepsilon \partial_t \rho^\varepsilon(x,t) + \partial_x J(\rho^\varepsilon(x,t)) = 0, \\ &\rho^\varepsilon(0,t) = \tilde\rho_-(t), \quad \rho^\varepsilon(1,t) = \tilde\rho_+(t). \end{aligned} \tag{3.4}$$

The definition, existence and uniqueness of the entropy solution of (3.4), given an initial condition, follow from the work of Otto [11], by the characterization of the boundary entropy fluxes:

Definition 3.1. *A boundary Lax entropy–entropy flux pair for* (3.4) *is a couple of* $\mathcal{C}^2$ *functions* $(F,Q) : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ *such that*

$$J'(u)\partial_u F(u,w) = \partial_u Q(u,w), \quad F(w,w) = Q(w,w) = \partial_u F(w,w) = 0 \tag{3.5}$$

for all $u \in [0,1]$ *and* $w \in \mathbb{R}$. *Moreover, we say that the pair* (F,Q) *is convex if* $F(u,w)$ *is convex in* u *for all* $w \in \mathbb{R}$.

Otto's boundary conditions reads in this case as

$$\begin{aligned} &\operatorname*{esslim}_{r\to 0^+} \int_0^T Q(\rho^\varepsilon(t,r), \tilde\rho_-(t))\beta(t)dt \le 0, \\ &\operatorname*{esslim}_{r\to 0^+} \int_0^T Q(\rho^\varepsilon(t,1-r), \tilde\rho_+(t))\beta(t)dt \ge 0, \end{aligned} \tag{3.6}$$

for any boundary entropy flux Q and $\beta \in \mathcal{C}(0,T)$ such that $\beta(t) \ge 0$. In the case of bounded variation solutions, this coincides with the Bardos–Leroux–Nédélec boundary conditions [2].

The existence and uniqueness of the *quasi-static* limit $\rho^\varepsilon \to \rho$ as $\varepsilon \to 0$ is proven in [8]. Moreover, the $\rho = \rho(x,t)$ satisfies the variational conditions ([9, 13, 1, 8]):

$$J(\rho(x,t)) = \begin{cases} \sup\{J(\rho); \rho \in [\tilde\rho_+(t), \tilde\rho_-(t)]\}, & \text{if } \tilde\rho_-(t) > \tilde\rho_+(t), \\ \inf\{J(\rho); \rho \in [\tilde\rho_-(t), \tilde\rho_+(t)]\}, & \text{if } \tilde\rho_-(t) \le \tilde\rho_+(t). \end{cases} \tag{3.7}$$

Observe that on the line

$$\Theta := \{(a,b) \in [0,1]^2; a < 1/2, a+b = 1\}, \tag{3.8}$$

$\rho(x,t)$ may attain two values and it may be not constant, while if $(\tilde\rho_-(t), \tilde\rho_+(t)) \notin \Theta$, then $\rho(x,t) = \rho(t)$, i.e. is constant in x and it is explicitly given by

$$\rho(t) = \begin{cases} \tilde\rho_-(t), & \text{if } \tilde\rho_-(t) < 1/2,\ \tilde\rho_-(t) + \tilde\rho_+(t) < 1 \text{ (low density)}, \\ \tilde\rho_+(t), & \text{if } \tilde\rho_-(t) > 1/2,\ \tilde\rho_-(t) + \tilde\rho_+(t) > 1 \text{ (high density)}, \\ 1/2, & \text{if } \tilde\rho_-(t) \ge 1/2,\ \tilde\rho_+(t) \le 1/2 \text{ (max current)}. \end{cases} \tag{3.9}$$

In particular if $(\tilde\rho_-(t), \tilde\rho_+(t)) \in \Theta$, the solution may not be unique.

3.2. **Quasi-static hydrodynamic limit.** In this article, we consider time dependent parameters $\rho_\pm(t) \in [0,1]$ and $\lambda_\pm(t) > 0$. As in (2.2), define the Markov generator

$$L_{N,t} = \lambda_0 L_{\text{exc}} + \lambda_-(t) L_{-,t} + \lambda_+(t) L_{+,t}, \quad t \geq 0, \tag{3.10}$$

where $L_{\pm,t}$ are operators defined by (2.4) with $\rho_\pm$ replaced by $\rho_\pm(t)$. We multiply $L_{N,t}$ by N^{1+a} for some $a > 0$ and study the macroscopic limit of the corresponding dynamics. We now distinguish two cases: the Liggett boundaries, where we only have speed up the generator by N^{1+a} to the quasi-static time scale, and the general reversible boundaries where there is a further speeding of the symmetric part of the generator and of the boundary rates.

3.2.1. *Liggett boundaries.* In this case, we fix $\lambda_0 = 1$ and take two $\mathcal{C}^1$ functions $\bar\rho_\pm : [0,\infty) \to [0,1]$ satisfying the condition expressed in (3.1) for $\tilde\rho_\pm(t)$.

Define $\bar p = 2p - 1 > 0$, then the exclusion part of $L_{N,t}$ can be rewritten as

$$\lambda_0 L_{\text{exc}} = \bar p L_{\text{tasep}} + \frac{1-\bar p}{2} L_{\text{ssep}}, \tag{3.11}$$

where L_{ssep} and L_{tasep} are respectively given by

$$\begin{aligned} L_{\text{ssep}} f &:= \sum_{i=1}^{N-1} \left[f\left(\eta^{i,i+1}\right) - f(\eta) \right], \\ L_{\text{tasep}} f &:= \sum_{i=1}^{N-1} \eta_i (1-\eta_{i+1}) \left[f\left(\eta^{i,i+1}\right) - f(\eta) \right]. \end{aligned} \tag{3.12}$$

Define also $L_{\pm,t}$ by (2.4) with $(\lambda_\pm, \rho_\pm) = (\lambda_\pm, \rho_\pm)(t)$ given by (2.9).

For some $a > 0$, denote by $\eta(t) = (\eta_1(t), \dots, \eta_N(t)) \in \Omega_N$ the process generated by $N^{1+a} L_{N,t}$. Let $\chi_{i,N}$ be the indicator function

$$\chi_{i,N}(x) := \mathbf{1}_{\left\{ \left[\frac{i}{N} - \frac{1}{2N}, \frac{i}{N} + \frac{1}{2N}\right) \cap [0,1] \right\}}(x), \qquad \forall\, x \in [0,1]. \tag{3.13}$$

For each N, define the empirical density $\zeta_N = \zeta_N(x,t)$ as

$$\zeta_N(x,t) := \sum_{i=1}^{N} \chi_{i,N}(x) \eta_i(t), \quad (x,t) \in [0,1] \times \mathbb{R}_+. \tag{3.14}$$

Our aim is to show that, as $N \to \infty$, ζ_N converges to the L^∞ entropy solution of the quasi-static conservation law (3.2)–(3.3) with $\tilde\rho_\pm(t) = \bar\rho_\pm(t)$.

Denote by $\mathbb{P}_N$ the distribution on the path space of $\eta(\cdot)$. Let $\mathbb{E}_N$ be the corresponding expectation. For $\rho \in [0,1]$, recall that ν_ρ is the product Bernoulli measure with density ρ. Given a local function f on Ω_N, denote

$$\langle f \rangle(\rho) := \int f d\nu_\rho. \tag{3.15}$$

Theorem 3.2 (Liggett boundaries)**.** *Suppose that $\bar\rho_\pm$ satisfies* (3.1) *and $(\bar\rho_-,\bar\rho_+)\notin\Theta$ for almost all $t\geq 0$, where Θ is given by* (3.8)*. Assume further that $a>1/2$, then for local function f and $\varphi\in\mathcal{C}([0,1]\times[0,\infty))$ we have the following convergence in probability: for all $T>0$,*

$$\lim_{N\to\infty}\int_0^T\frac{1}{N}\sum_{i=1}^{N-k_f}\varphi\left(\frac{i}{N},t\right)f\big(\tau_i\eta(t)\big)dt=\int_0^T\left[\int_0^1\varphi(x,t)dx\right]\langle f\rangle(\rho(t))dt, \tag{3.16}$$

where τ_i is the shift operator, $[[1,k_f]]$ is the support of the local function f, and $\rho(t)$ is given by (3.9) *with $\tilde\rho_\pm(t)=\bar\rho_\pm(t)$.*

3.2.2. *Reversible boundaries.* In Liggett boundary rates, $\lambda_\pm(t)$ and $\rho_\pm(t)$ are chosen in accordance with p. To deal with more general case in which the boundary rates are independent of p, we need to speed up the boundary operators. We also need to apply a speed change at symmetric exclusion. Let σ_N and $\widetilde\sigma_N$ be two sequences satisfying

$$\lim_{N\to\infty}\frac{\sigma_N}{N}=0,\quad \lim_{N\to\infty}\widetilde\sigma_N=\infty,\quad \lim_{N\to\infty}\frac{\sigma_N\widetilde\sigma_N}{\sqrt{N}}=\infty. \tag{3.17}$$

Fix some $\bar p\in[0,1]$ and $\mathcal{C}^1$ functions

$$\bar\lambda_\pm:[0,\infty)\to(0,\infty),\quad \rho_\pm:[0,\infty)\to[0,1].$$

Define the generator $L_{N,t}$ with the following choices of parameters

$$\lambda_0=\sigma_N,\quad p=\frac{1}{2}+\frac{\bar p}{2\sigma_N},\quad \lambda_\pm(t)=\widetilde\sigma_N\bar\lambda_\pm(t). \tag{3.18}$$

Notice that with the generators defined in (3.12),

$$\lambda_0L_{\text{exc}}=\bar pL_{\text{tasep}}+\frac{\sigma_N-\bar p}{2}L_{\text{ssep}}. \tag{3.19}$$

Hence, σ_N and $\widetilde\sigma_N$ correspond to the speed of the symmetric part of the exclusion dynamics and boundaries, respectively. These are necessary in order to obtain the macroscopic quasi-static law. Observe that this dynamic is not the so called *weakly asymmetric exclusion*, since with our choice of the parameters the asymmetry is always strong.

For $a>0$, let $\eta(t)=(\eta_1(t),\dots,\eta_N(t))\in\Omega_N$ be the Markov process generated by $N^{1+a}L_{N,t}$ and some initial distribution $\mu_{N,0}$. We show that ζ_N defined by (3.14) converges, as $N\to\infty$, to the L^∞ entropy solution of the quasi-static conservation law (3.2)–(3.3) with $\tilde\rho_\pm(t)=\rho_\pm(t)$. Observe that the macroscopic limit equation, and in particular the boundary conditions, do not depend on the choice of $\bar\lambda_\pm(t)$.

Theorem 3.3 (Reversible boundaries)**.** *Assume that $\rho_\pm$ satisfies* (3.1) *and $(\rho_-,\rho_+)\notin\Theta$ for almost all $t\geq 0$. In additional, assume* (3.17) *and*

$$\lim_{N\to\infty}N^{a-\frac{1}{2}}\sigma_N=\infty. \tag{3.20}$$

Then, the convergence in (3.16) *holds for local function f, $\varphi\in\mathcal{C}([0,1]\times[0,\infty))$ and $\rho=\rho(t)$ given by* (3.9) *with $\tilde\rho_\pm(t)=\rho_\pm(t)$.*

Remark 3. *Observe from* (3.17) *and* (3.20) *that, if a is large enough, the sequence σ_N does not have to grow with N.*

4. Proof of the main theorem

In this section we give the proof of Theorem 3.2 and 3.3 as a consequence of three main steps that will be proven in the following sections. Fix some $T > 0$ and denote $\Sigma_T = [0,1] \times [0,T]$.

By a Young measure ν we mean a family $\{\nu_{x,t}; (x,t) \in \Sigma_T\}$ of probability measures on $[0,1]$ such that the mapping

$$(x,t) \mapsto \int f(x,t,y)\nu_{x,t}(dy)$$

is measurable for all $f \in \mathcal{C}(\Sigma_T \times [0,1])$. Denote by $\mathcal{Y}$ the set of all Young measures, endowed with the vague topology, i.e. a sequence $\nu^n \to \nu$ if for all $f \in \mathcal{C}(\Sigma_T \times [0,1])$,

$$\lim_{n\to\infty} \iint_{\Sigma_T} dx\,dt \int_0^1 f(x,t,y)\nu^n_{x,t}(dy) = \iint_{\Sigma_T} dx\,dt \int_0^1 f(x,t,y)\nu_{x,t}(dy). \tag{4.1}$$

Under the vague topology, $\mathcal{Y}$ is metrizable, separable and compact.

Recall the empirical distribution function $\zeta_N = \zeta_N(x,t)$ defined in (3.14). Since $\zeta_N \in [0,1]$ for any N, the corresponding Young measures

$$\nu^N := \big\{\nu^N_{x,t} = \delta_{\zeta_N(x,t)}; (x,t) \in \Sigma_T\big\} \in \mathcal{Y}. \tag{4.2}$$

Denote by $\mathfrak{Q}_N$ the distribution of ν^N on $\mathcal{Y}$.

Since $\mathcal{Y}$ is compact, the sequence $\mathfrak{Q}_N$ is tight, thus we can extract a subsequence $\mathfrak{Q}_{N_h}$ that converges weakly to some measure $\mathfrak{Q}$ on $\mathcal{Y}$, which means that for all $f \in \mathcal{C}(\Sigma_T \times [0,1])$ we have

$$\begin{aligned} \mathbb{E}_{N_h}\left[\iint_{\Sigma_T} f\big(x,t,\zeta_{N_h}(x,t)\big)dx\,dt\right] =\ & E^{\mathfrak{Q}_{N_h}}\left[\iint_{\Sigma_T} dx\,dt \int_0^1 f(x,t,y)\nu_{x,t}(dy)\right] \\ \underset{h\to\infty}{\longrightarrow}\ & E^{\mathfrak{Q}}\left[\iint_{\Sigma_T} dx\,dt \int_0^1 f(x,t,y)\nu_{x,t}(dy)\right] \end{aligned} \tag{4.3}$$

In both Liggett and reversible cases, it suffices to show that any limit point $\mathfrak{Q}$ of $\mathfrak{Q}_N$ is concentrated on $\delta_{\rho(x,t)}$ with $\rho(x,t) = \rho(t)$ the entropy solution of (3.2)–(3.3) characterized by (3.9)[1], i.e.

$$\mathfrak{Q}\big\{\nu_{x,t} = \delta_{\rho(t)},\ (x,t)\text{-a.s. in } \Sigma_T\big\} = 1. \tag{4.4}$$

The uniqueness of the entropy solution implies the uniqueness of the limit of $\mathfrak{Q}_N$ and thus the convergence in probability.

The proof of (4.4) is divided into several steps.

[1]Hereafter $\tilde\rho_\pm(t) = \bar\rho_\pm(t)$ for the Liggett case and $\tilde\rho_\pm(t) = \rho_\pm(t)$ for the reversible case.

First step: macroscopic current. The *microscopic currents* are defined by the conservation law $L_{N,t}[\eta_i] = j_{i-1,i} - j_{i,i+1}$ for $i = 1, 2, ..., N$. In Section 5 we show that for a.e. $t \in [0,T]$, there exists $\mathcal{J}(t) < \infty$ such that

$$\mathcal{J}(t) = \lim_{N\to\infty} \mathbb{E}_N[j_{i,i+1}(t)], \quad \forall\, i = 0, \dots, N. \tag{4.5}$$

$\mathcal{J}(t)$ relates to the current $J(\rho) = \bar{p}\rho(1-\rho)$ in (3.2) in the following way:

$$\mathcal{J}(t) = E^{\mathfrak{Q}}\left[\int_0^1 dx \int_0^1 J(\rho)\nu_{x,t}(d\rho)\right], \quad t\text{-a.s. in } [0,T]. \tag{4.6}$$

Formula (4.6) is obtained in Section 7.1. The proof relies on the estimates on the Dirichlet forms and the mesoscopic block averages. These results are established in Section 6 and 7, respectively.
Second step: the balanced case. Next, we show the result for the *balanced case*, i.e. when $\tilde{\rho}_-(t) = \tilde{\rho}_+(t)$. The following lemma is proved in Section 8:

Lemma 4.1. *If $\tilde{\rho}_-(t) = \tilde{\rho}_+(t)$ for all $t \in [0,T]$, (4.4) holds.*

The proof exploits the boundary entropy–entropy flux pair defined in 3.1. In fact in the balanced case we expect equality in (3.6), and we prove in section 8.1 that the time average of the boundary entropy flux on the microscopic scale is negligible, see Proposition 8.1 and 8.2.
Third step: coupling. With Lemma 4.1 and a coupling argument with the balanced dynamics, we show in Section 9 that the limit Young measures concentrate in the interval between $\tilde{\rho}_-(t)$ and $\tilde{\rho}_+(t)$:

Lemma 4.2. *The following holds $\mathfrak{Q}$-almost surely:*

$$\nu_{x,t}\big\{I[\tilde{\rho}_-(t), \tilde{\rho}_+(t)]\big\} = 1, \quad (x,t)\text{-a.s. in } \Sigma_T. \tag{4.7}$$

where $I[a,b]$ is the closed interval with extremes a and b.

Meanwhile, the same coupling argument applied to the current $\mathcal{J}(t)$ yields the following proposition:

Proposition 4.3. *If $\tilde{\rho}_-(t) \geq \tilde{\rho}_+(t)$ for all $t \in [0,T]$,*

$$\int_0^T \mathcal{J}(t)dt = \int_0^T \sup\big\{J(\rho); \tilde{\rho}_+(t) \leq \rho \leq \tilde{\rho}_-(t)\big\}dt. \tag{4.8}$$

If $\tilde{\rho}_-(t) \leq \tilde{\rho}_+(t)$ for all $t \in [0,T]$,

$$\int_0^T \mathcal{J}(t)dt = \int_0^T \inf\big\{J(\rho); \tilde{\rho}_-(t) \leq \rho \leq \tilde{\rho}_+(t)\big\}dt. \tag{4.9}$$

Proof of Theorem 3.2 and 3.3. From the third step we have all the information to conclude the proof of the main theorems.

Proofs of Theorem 3.2 and 3.3. First observe that if $\tilde\rho_-(s)-\tilde\rho_+(s)$ does not change sign in $[0,t]$, (4.4) holds up to time t. Indeed, suppose that $\tilde\rho_-\geq\tilde\rho_+$ for $s\in[0,t]$, then $\rho=\rho(s)$ given by (3.9) is the unique solution such that

$$J(\rho(s))=\sup\big\{J(\rho),\tilde\rho_+(s)\leq\rho\leq\tilde\rho_-(s)\big\}. \tag{4.10}$$

Therefore, by (4.8), $\int_0^t\mathcal{J}(s)ds=\int_0^t J(\rho(s))ds$. This together with (4.6) yields that

$$\int_0^t E^{\mathfrak{Q}}\left[\int_0^1 dx\int_{\tilde\rho_+(s)}^{\tilde\rho_-(s)}\big[J(\rho)-J(\rho(s))\big]\nu_{x,s}(d\rho)\right]ds=0. \tag{4.11}$$

Since $J(\rho)-J(\rho(s))\leq 0$ for all $\rho\in[\tilde\rho_+(s),\tilde\rho_-(s)]$, the Young measure can only be concentrated in its zero set, more precisely: with $\mathfrak{Q}$-probability 1, $\nu_{x,s}=\delta_{\rho(s)}$ a.s. in Σ_t. For the case $\tilde\rho_-(s)\leq\tilde\rho_+(s)$, since $(\tilde\rho_-,\tilde\rho_+)\notin\Theta$, $\rho=\rho(s)$ is the unique solution such that

$$J(\rho(s))=\inf\big\{J(\rho),\tilde\rho_-(s)\leq\rho\leq\tilde\rho_+(s)\big\}, \tag{4.12}$$

and the argument is similar.

For general $\tilde\rho_\pm(t)$, by (3.1), $\tilde\rho_+-\tilde\rho_-$ keeps its sign in each $I_i=[t_i,t_i']$. Denote by μ_{N,t_i} the distribution of $\eta(t_i)$ on Ω_N. Consider the process generated by $N^{1+a}L_{N,s}$, $0\leq s\leq t_i'-t_i$ and initial distribution μ_{N,t_i}. As the arguments above are valid for any initial distribution, the result holds almost surely in each I_i, and hence in the whole interval $[0,T]$. □

5. Microscopic currents

The *microscopic currents* $j_{i,i+1}$ associated to the generator $L_{N,t}$ are defined by the conservation law $L_{N,t}[\eta_i]=j_{i-1,i}-j_{i,i+1}$ and they are equal to

$$j_{i,i+1}=\begin{cases}p\bar\rho_-(t)-\big[p\bar\rho_-(t)+(1-p)(1-\bar\rho_-(t))\big]\eta_1, & i=0,\\ \bar p\eta_i(1-\eta_{i+1})+\frac{1-\bar p}{2}(\eta_i-\eta_{i+1}), & 1\leq i\leq N-1,\\ \big[p(1-\bar\rho_+(t))+(1-p)\bar\rho_+(t)\big]\eta_N-(1-p)\bar\rho_+(t), & i=N.\end{cases} \tag{5.1}$$

in the case of Liggett boundary rates and

$$j_{i,i+1}=\begin{cases}\widetilde\sigma_N\bar\lambda_-(t)(\rho_-(t)-\eta_1), & i=0,\\ \bar p\eta_i(1-\eta_{i+1})+\frac{\sigma_N-\bar p}{2}(\eta_i-\eta_{i+1}), & 1\leq i\leq N-1,\\ \widetilde\sigma_N\bar\lambda_+(t)(\eta_N-\rho_+(t)), & i=N.\end{cases} \tag{5.2}$$

in the case of reversible boundary rates.

Follow the argument as in [5, Section 2], for $i=1,\dots N-1$ define the counting processes associated to the process $\{\eta(t)\}_{t\geq0}$ generated by $N^{1+a}L_{N,t}$ by

$$\begin{aligned}&h_+(i,t):=\text{number of jumps } i\to i+1 \text{ in } [0,t].\\ &h_-(i,t):=\text{number of jumps } i+1\to i \text{ in } [0,t].\\ &h(i,t):=h_+(i,t)-h_-(i,t).\end{aligned} \tag{5.3}$$

These definitions extend to the boundaries $i = 0$ and $i = N$ as

$$\begin{aligned} &h_+(0,t) := \text{number of particles created at 1 in } [0,t],\\ &h_-(0,t) := \text{number of particles annihilated at 1 in } [0,t],\\ &h_+(N,t) := \text{number of particles annihilated at } N \text{ in } [0,t],\\ &h_-(N,t) := \text{number of particles created at } N \text{ in } [0,t]. \end{aligned} \tag{5.4}$$

The conservation law is microscopically given by

$$\eta_i(t) - \eta_i(0) = h(i-1,t) - h(i,t), \quad \forall\, x = 1, \dots, N. \tag{5.5}$$

Furthermore, for $i = 0, \dots, N$ there is a martingale $M_i(t)$ such that

$$h(i,t) = N^{1+a} \int_0^t j_{i,i+1}(s)ds + M_i(t).$$

As $|\eta_i(t)| \le 1$, (5.5) yields that $|h(i,t) - h(i',t)| \le |i - i'|$. Therefore,

$$\mathbb{E}_N \left[\int_t^{t'} \big(j_{i,i+1}(s) - j_{i',i'+1}(s) \big) ds \right] = O(N^{-a}), \quad \forall\, t < t'.$$

Hence, there exists $\mathcal{J}(t) < \infty$ such that (4.5) holds for all $0 \le i \le N$.

In particular for the reversible boundary rates,

$$\begin{aligned} &\widetilde{\sigma}_N \bar{\lambda}_-(t) \mathbb{E}_N^{\eta(0)}[\eta_1(t) - \rho_-(t)] = -\mathcal{J}(t) + O(N^{-a}),\\ &\widetilde{\sigma}_N \bar{\lambda}_+(t) \mathbb{E}_N^{\eta(0)}[\eta_N(t) - \rho_+(t)] = \mathcal{J}(t) + O(N^{-a}). \end{aligned} \tag{5.6}$$

Since generally $\mathcal{J}(t) \neq 0$, we obtain that for the reversible boundaries case that

$$\Big| \mathbb{E}_N^{\eta(0)} \big[\eta_1(t) \big] - \rho_-(t) \Big| + \Big| \mathbb{E}_N^{\eta(0)} \big[\eta_N(t) \big] - \rho_+(t) \Big| \le \frac{C}{\tilde{\sigma}_N}$$

This means the acceleration of the boundary rates ($\tilde{\sigma}_N \to \infty$) obliges the boundary conditions $\rho_\pm(t)$ in this microscopic sense. Observe that this is not true in the Liggett boundary case.

6. Microscopic entropy production: bounds on the Dirichlet forms

For any $\rho \in [0,1]$, let ν_ρ be the product Bernoulli measure on $\Omega_N = \{0,1\}^N$ with rate ρ. In particular, for the time dependent parameters $\tilde{\rho}_\pm = \tilde{\rho}_\pm(t)$ denote

$$\nu_{\pm,t}(\eta) := \nu_{\tilde{\rho}_\pm(t)}(\eta) = \prod_{i=1}^N \tilde{\rho}_\pm(t)^{\eta_i} \big[1 - \tilde{\rho}_\pm(t) \big]^{1-\eta_i}, \quad \forall\, \eta \in \Omega_N. \tag{6.1}$$

Recall that $\eta(t)$ is the process generated by $N^{1+a} L_{N,t}$ and denote by $\mu_{N,t}$ the distribution of $\eta(t)$ in Ω_N. For $N \ge 2$, define the Dirichlet form

$$\mathfrak{D}_{\mathrm{exc},N}(t) := \frac{1}{2} \sum_{\eta \in \Omega_N} \sum_{i=1}^{N-1} \left(\sqrt{\mu_{N,t}(\eta^{i,i+1})} - \sqrt{\mu_{N,t}(\eta)} \right)^2. \tag{6.2}$$

Let $f^{\pm}_{N,t}$ be the density of $\mu_{N,t}$ with respect to $\nu_{\pm,t}$ and define the boundary Dirichlet forms as

$$\begin{aligned}\mathfrak{D}_{-,N}(t) &:= \frac{1}{2}\sum_{\eta} \tilde{\rho}_-^{1-\eta_1}(1-\tilde{\rho}_-)^{\eta_1}\left(\sqrt{f^-_{N,t}(\eta^1)}-\sqrt{f^-_{N,t}(\eta)}\right)^2 \nu_{-,t}(\eta),\\ \mathfrak{D}_{+,N}(t) &:= \frac{1}{2}\sum_{\eta} \tilde{\rho}_+^{1-\eta_N}(1-\tilde{\rho}_+)^{\eta_N}\left(\sqrt{f^+_{N,t}(\eta^N)}-\sqrt{f^+_{N,t}(\eta)}\right)^2 \nu_{+,t}(\eta).\end{aligned} \tag{6.3}$$

In this section we establish some useful bounds for these Dirichlet forms. We start from the Liggett boundaries in which $\tilde{\rho}_\pm(t) = \bar{\rho}_\pm(t)$ and prove the next result.

Proposition 6.1 (Liggett boundary)**.** *For all $t < t'$, there exists C such that*

$$\int_t^{t'} \big[\mathfrak{D}_{-,N}(s) + \mathfrak{D}_{\mathrm{exc},N}(s) + \mathfrak{D}_{+,N}(s)\big] ds \le C \tag{6.4}$$

for all N. Moreover if $\bar{\rho}_-(s) = \bar{\rho}_+(s)$ for all $s \in [t,t']$, then

$$\int_t^{t'} \big[\mathfrak{D}_{-,N}(s) + \mathfrak{D}_{\mathrm{exc},N}(s) + \mathfrak{D}_{+,N}(s)\big] ds \le \frac{C}{N^a}. \tag{6.5}$$

Proof. In this proof, any function f on Ω_N is viewed as a local function on $\{0,1\}^{\mathbb{Z}}$ in the standard way.

Given any probability measure μ on Ω_N, for a given $t \ge 0$ we extend it as a measure $\bar{\mu}$ on $\{0,1\}^{\mathbb{Z}} \sim \Omega_N \times \prod_{i\notin\{1,..,N\}}\{0,1\}$ where outside $\{1,..,N\}$ is a product measure with $\bar{\mu}\{\eta_i = 1\} = \bar{\rho}_-(t)$ for $i \le 0$ and $\bar{\mu}\{\eta_i = 1\} = \bar{\rho}_+(t)$ for $i > N$.

Recall that $L_{N,t}$ is the generator with Liggett boundary rates in (2.9). Also define for all local function f on $\{0,1\}^{\mathbb{Z}}$

$$L^{\mathbb{Z}}_{\mathrm{exc}} f := \sum_{i\in\mathbb{Z}} c_{i,i+1}\left[f\left(\eta^{i,i+1}\right) - f(\eta)\right], \quad c_{i,j} = p\eta_i(1-\eta_j) + (1-p)\eta_j(1-\eta_i).$$

$L^{\mathbb{Z}}_{\mathrm{exc}}$ generates the asymmetric simple exclusion process on $\{0,1\}^{\mathbb{Z}}$. As obtained in [10, p. 244] in the proof of Theorem 2.4, for a function $g = g(\eta_1,\dots,\eta_N)$,

$$\begin{aligned} &L_{N,t}g(\eta) - L^{\mathbb{Z}}_{\mathrm{exc}}g(\eta)\\ &= \big[p(\bar{\rho}_- - \eta_0)(1-\eta_1) + (1-p)(\eta_0 - \bar{\rho}_-)\eta_1\big]\big[g(\eta^1) - g(\eta)\big]\\ &\quad + \big[(1-p)(\bar{\rho}_+ - \eta_{N+1})(1-\eta_N) + p(\eta_{N+1} - \bar{\rho}_+)\eta_N\big]\big[g(\eta^N) - g(\eta)\big].\end{aligned} \tag{6.6}$$

Observe that the integral of the right hand side is 0 with respect to a measure $\bar{\mu}$ on $\{0,1\}^{\mathbb{Z}}$ obtained by extending any measure μ on Ω_N as above.

Recall the Bernoulli measure $\nu_{\pm,t}$ defined in (6.1) and the corresponding density function $f^{\pm}_{N,t}$: $\mu_{N,t} = f^{\pm}_{N,t}\nu_{\pm,t}$. Consider the relative entropy $H_{\pm,N}(t)$ given by

$$H_{\pm,N}(t) := \sum_{\eta\in\Omega_N} f^{\pm}_{N,t}(\eta)\log f^{\pm}_{N,t}(\eta)\nu_{\pm,t}(\eta) = \sum_{\eta\in\Omega_N} \log f^{\pm}_{N,t}(\eta)\mu_{N,t}(\eta).$$

Using Kolmogorov equation and (6.6), we obtain for the entropy production that

$$\begin{aligned}\frac{d}{dt}H_{-,N}(t) &= N^{1+a}\sum_{\eta\in\Omega_N} L_{N,t}[\log f^-_{N,t}]\mu_{N,t} - \sum_{\eta\in\Omega_N} f^-_{N,t}\frac{d}{dt}\nu_{-,t}\\ &= N^{1+a}\sum_{\eta\in\{0,1\}^{\mathbb{Z}}} L^{\mathbb{Z}}_{\mathrm{exc}}[\log f^-_{N,t}]\bar\mu_{N,t} - \sum_{\eta\in\Omega_N} f^-_{N,t}\frac{d}{dt}\nu_{-,t} \qquad (6.7)\\ &= N^{1+a}\sum_{\eta\in\{0,1\}^{\mathbb{Z}}} f^-_{N,t}L^{\mathbb{Z}}_{\mathrm{exc}}\big[\log f^-_{N,t}\big]\bar\nu_{-,t} - \sum_{\eta\in\Omega_N} f^-_{N,t}\frac{d}{dt}\nu_{-,t},\end{aligned}$$

where $\bar\nu_{-,t} = \prod_{i\le N}\nu_{\bar\rho_-(t)}(\eta_i)\prod_{i>N}\nu_{\bar\rho_+(t)}(\eta_i)$. Observe that the last term in (6.7) is bounded by CN.

Exploiting the inequality $x(\log y - \log x)\le 2\sqrt{x}(\sqrt{y}-\sqrt{x})$ for all x, $y>0$, and denoting that $\bar p = 2p-1$, $g^-_{N,t} = (f^-_{N,t})^{1/2}$,

$$\begin{aligned}&\sum_{\eta\in\{0,1\}^{\mathbb{Z}}} f^-_{N,t}L^{\mathbb{Z}}_{\mathrm{exc}}\big[\log f^-_{N,t}\big]\bar\nu_{-,t} \le 2\sum_{\eta\in\{0,1\}^{\mathbb{Z}}} g^-_{N,t}L^{\mathbb{Z}}_{\mathrm{exc}}\big[g^-_{N,t}\big]\bar\nu_{-,t}\\ &= \sum_{\eta\in\{0,1\}^{\mathbb{Z}}}\sum_{i\in\mathbb{Z}}\big[1+\bar p(\eta_i-\eta_{i+1})\big]\left[g^-_{N,t}(\eta^{i,i+1})-g^-_{N,t}\right]g^-_{N,t}\bar\nu_{-,t} \qquad (6.8)\\ &= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4,\end{aligned}$$

where the right-hand side reads

$$\mathcal{I}_1 = -\frac{1}{2}\sum_{\eta\in\{0,1\}^{\mathbb{Z}}}\sum_{i\in\mathbb{Z}}\big[g^-_{N,t}(\eta^{i,i+1})-g^-_{N,t}(\eta)\big]^2\bar\nu_{-,t}(\eta),$$

$$\begin{aligned}\mathcal{I}_2 &= -\frac{1}{2}\sum_{\eta\in\{0,1\}^{\mathbb{Z}}}\sum_{i\in\mathbb{Z}}(g^-_{N,t})^2(\eta)\big[\bar\nu_{-,t}(\eta)-\bar\nu_{-,t}(\eta^{i,i+1})\big]\\ &= \frac{1}{2}\sum_{\eta\in\{0,1\}^{\mathbb{Z}}} f^-_{N,t}(\eta)\big[\bar\nu_{-,t}(\eta^{N,N+1})-\bar\nu_{-,t}(\eta)\big],\end{aligned}$$

$$\begin{aligned}\mathcal{I}_3 &= \bar p\sum_{\eta\in\{0,1\}^{\mathbb{Z}}}\sum_{i\in\mathbb{Z}}(\eta_i-\eta_{i+1})g^-_{N,t}(\eta^{i,i+1})g^-_{N,t}\bar\nu_{-,t}\\ &= \frac{\bar p}{2}\sum_{\eta\in\{0,1\}^{\mathbb{Z}}}(\eta_{N+1}-\eta_N)g^-_{N,t}(\eta^{N,N+1})g^-_{N,t}\big[\bar\nu_{-,t}(\eta^{N,N+1})-\bar\nu_{-,t}\big],\end{aligned}$$

$$\mathcal{I}_4 = \bar p\sum_{\eta\in\{0,1\}^{\mathbb{Z}}}\sum_{i\in\mathbb{Z}}(\eta_i-\eta_{i+1})\left[-(g^-_{N,t})^2(\eta)\right]\bar\nu_{-,t}(\eta) = \bar p\big[\bar\rho_+(t)-\bar\rho_-(t)\big].$$

Notice that $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ are uniformly bounded in N and they vanish when $\bar\rho_- = \bar\rho_+$. On the other hand, since $f^-_{N,t}$ depends only on $\{\eta_1,\dots,\eta_N\}$,

$$\mathcal{I}_1 = -\mathfrak{D}_{\mathrm{exc},N}(t) - \mathcal{I}_{1,l} - \mathcal{I}_{1,r},$$

where $\mathcal{I}_{1,l}$ and $\mathcal{I}_{1,r}$ are computed respectively as

$$\begin{aligned}\mathcal{I}_{1,l} &= \frac{1}{2}\sum_{\eta\in\Omega_N}\left[g^-_{N,t}(\eta^1)-g^-_{N,t}\right]^2\nu_{-,t}(\eta)\sum_{\eta_0}\big(\eta_0(1-\eta_1)+\eta_1(1-\eta_0)\big)\nu_{\bar\rho_-(t)}(\eta_0)\\ &= \frac{1}{2}\sum_{\eta\in\Omega_N}\big(\bar\rho_-(t)(1-\eta_1)+\eta_1(1-\bar\rho_-(t))\big)\left[g^-_{N,t}(\eta^1)-g^-_{N,t}\right]^2\nu_{-,t}=\mathfrak{D}_{-,N}(t),\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{1,r} = \frac{1}{2}\sum_{\eta\in\Omega_N}&\left[g^-_{N,t}(\eta^N)-g^-_{N,t}\right]^2\nu_{-,t}(\eta)\\ &\sum_{\eta_{N+1}}\big(\eta_N(1-\eta_{N+1})+\eta_{N+1}(1-\eta_N)\big)\nu_{\bar\rho_+(t)}(\eta_{N+1})\ge 0.\end{aligned}$$

Hence, from (6.8) we obtain a constant $C>0$ such that

$$\sum_{\eta\in\{0,1\}^{\mathbb{Z}}} f^-_{N,t}L^{\mathbb{Z}}_{\mathrm{exc}}\big[\log f^-_{N,t}\big]\nu_{-,t}\le -\mathfrak{D}_{-,N}(t)-\mathfrak{D}_{\mathrm{exc},N}(t)+C.$$

Plugging this into (6.7) and integrating in time,

$$\int_0^t\big[\mathfrak{D}_{-,N}(s)+\mathfrak{D}_{\mathrm{exc},N}(s)\big]ds\le C.$$

The proof of (6.4) is completed by repeating the argument with $H^+_{N,t}$.

Now suppose that $\bar\rho_-(t)=\bar\rho_+(t)$, then $\nu_{-,t}=\nu_{+,t}$ and we have $\mathcal{I}_{1,r}=\mathfrak{D}_{+,N}(t)$. Therefore, (6.8) yields that

$$\sum_{\eta\in\mathbb{Z}} f_{N,t}L^{\mathbb{Z}}_{\mathrm{exc}}\big[\log f_{N,t}\big]\nu_t\le -\mathfrak{D}_{-,N}(t)-\mathfrak{D}_{\mathrm{exc},N}(t)-\mathfrak{D}_{+,N}(t).$$

Since $H_N(0)=O(N)$, (6.5) follows similarly. □

For reversible boundaries $\tilde\rho_\pm(t)=\rho_\pm(t)$, the following estimate holds.

Proposition 6.2 (Reversible boundary)**.** *For all $t<t'$, there exists C such that*

$$\int_t^{t'}\big[\widetilde\sigma_N\bar\lambda_-(s)\mathfrak{D}_{-,N}(s)+\sigma_N\mathfrak{D}_{\mathrm{exc},N}(s)+\widetilde\sigma_N\bar\lambda_+(s)\mathfrak{D}_{+,N}(s)\big]ds\le C \tag{6.9}$$

for all N. Moreover if $\rho_-(s)=\rho_+(s)$ for all $s\in[t,t']$, then

$$\begin{aligned}&\int_t^{t'}\big[\widetilde\sigma_N\bar\lambda_-(s)\mathfrak{D}_{-,N}(s)+\sigma_N\mathfrak{D}_{\mathrm{exc},N}(s)+\widetilde\sigma_N\bar\lambda_+(s)\mathfrak{D}_{+,N}(s)\big]ds\\ &\le\frac{\bar p}{\widetilde\sigma_N}\int_0^t\left[\frac{\mathcal{J}(s)}{\bar\lambda_-(s)}+\frac{\mathcal{J}(s)}{\bar\lambda_+(s)}\right]ds+\frac{C}{N^a}\le C'\left(\frac{1}{\widetilde\sigma_N}+\frac{1}{N^a}\right).\end{aligned} \tag{6.10}$$

Proof. Similarly to (6.7), we obtain for the entropy production that

$$\frac{d}{dt}H_{-,N}(t)\le N^{1+a}\sum_{\eta\in\Omega_N} f^-_{N,t}L_{N,t}[\log f^-_{N,t}]\nu_{-,t}+CN. \tag{6.11}$$

Applying the argument used in (6.8),

$$\begin{aligned}&\sum_{\eta\in\Omega_N} f^-_{N,t}\big(L_{N,t}-\widetilde{\sigma}_N\bar{\lambda}_+(t)L_{+,t}\big)[\log f^-_{N,t}]\nu_{-,t}\\ &\le -\widetilde{\sigma}_N\bar{\lambda}_-(t)\mathfrak{D}_{-,N}(t)-\sigma_N\mathfrak{D}_{\mathrm{exc},N}(t)+\bar{p}\sum_{\eta\in\Omega_N}(\eta_N-\eta_1)f^-_{N,t}\nu_{-,t}.\end{aligned}$$

In view of (5.6), the last term can be bounded as

$$\bar{p}\sum_{\eta}(\eta_N-\eta_1)f^-_{N,t}\nu_{-,t}=\bar{p}\left[\rho_+(t)-\rho_-(t)+\frac{\mathcal{J}(t)}{\widetilde{\sigma}_N\bar{\lambda}_-(t)}+\frac{\mathcal{J}(t)}{\widetilde{\sigma}_N\bar{\lambda}_+(t)}\right]+\frac{C}{N^a}.$$

For $L_{+,t}$, since $f^-_{N,t}\nu_{-,t}=f^+_{N,t}\nu_{+,t}$,

$$\sum_{\eta} f^-_{N,t}L_{+,t}[\log f^-_{N,t}]\nu_{-,t}\le-\mathfrak{D}_{+,N}(t)+\sum_{\eta} f^-_{N,t}L_{+,t}\left[\log\left(\frac{\nu_{+,t}}{\nu_{-,t}}\right)\right]\nu_{-,t}.$$

Standard manipulation shows that

$$L_{+,t}\left[\log\left(\frac{\nu_{+,t}}{\nu_{-,t}}\right)\right]=-\left\{\log\left[\frac{\rho_+(t)}{1-\rho_+(t)}\right]-\log\left[\frac{\rho_-(t)}{1-\rho_-(t)}\right]\right\}(\eta_N-\rho_+(t)).$$

Let $F(\rho)=\rho\log\rho+(1-\rho)\log(1-\rho)$, so that $F'(\rho)=\log(\rho/(1-\rho))$. By (5.6),

$$\sum_{\eta} f^-_{N,t}L_{+,t}\left[\log\left(\frac{\nu_{+,t}}{\nu_{-,t}}\right)\right]\nu_{-,t}\le-\big[F'(\rho_+(t))-F'(\rho_-(t))\big]\frac{\mathcal{J}(t)+O(N^{-a})}{\widetilde{\sigma}_N\bar{\lambda}_+(t)}.$$

Putting all these estimates together, we obtain from (6.11) that

$$\begin{aligned}\frac{d}{dt}H_{-,N}(t)\le&-N^{1+a}\big[2\widetilde{\sigma}_N\bar{\lambda}_-(t)\mathfrak{D}_{-,N}(t)+\sigma_N\mathfrak{D}_{\mathrm{exc},N}(t)+2\widetilde{\sigma}_N\bar{\lambda}_+(t)\mathfrak{D}_{+,N}(t)\big]\\&+N^{1+a}\big[C(\rho_\pm,t)+\widetilde{\sigma}_N^{-1}C(\bar{\lambda}_\pm,t)\big]+CN,\end{aligned}$$

where $C(\rho_\pm,t)$ and $C(\bar{\lambda}_\pm,t)$ are constants given by

$$\begin{aligned}C(\rho_\pm,t)&=\bar{p}\big[\rho_+(t)-\rho_-(t)\big]-\mathcal{J}(t)\big[F'(\rho_+(t))-F'(\rho_-(t))\big],\\C(\bar{\lambda}_\pm,t)&=\bar{p}\mathcal{J}(t)\big[\bar{\lambda}_+^{-1}(t)+\bar{\lambda}_-^{-1}(t)\big].\end{aligned}$$

We can then conclude (6.9) by integrating in time. For (6.10), it suffices to observe that $C(\rho_\pm,t)=0$ when $\rho_-(t)=\rho_+(t)$. □

Remark 4. *The condition $\rho_-(s)=\rho_+(s)$ is necessary for $C(\rho_\pm,s)=0$. Indeed, in view of Proposition 4.3, if $\rho_+(s)<\rho_-(s)$, $\mathcal{J}(s)=\sup\{J(\rho);\rho_+(s)\le\rho\le\rho_-(s)\}$, so*

$$C(\rho_\pm,s)=\int_{\rho_+(s)}^{\rho_-(s)}\left(\frac{\mathcal{J}(s)}{\rho(1-\rho)}-\bar{p}\right)d\rho=\int_{\rho_+(s)}^{\rho_-(s)}\frac{\mathcal{J}(s)-J(\rho)}{\rho(1-\rho)}d\rho>0.$$

Meanwhile if $\rho_+(s)\ge\rho_-(s)$, $\mathcal{J}(s)=\inf\{J(\rho);\rho_-(s)\le\rho\le\rho_+(s)\}$, so

$$C(\rho_\pm,s)=\int_{\rho_-(s)}^{\rho_+(s)}\left(\bar{p}-\frac{\mathcal{J}(s)}{\rho(1-\rho)}\right)d\rho=\int_{\rho_-(s)}^{\rho_+(s)}\frac{J(\rho)-\mathcal{J}(s)}{\rho(1-\rho)}d\rho>0.$$

Hence, better bounds are available only when $\rho_- = \rho_+$.

7. Block average estimates

For $1 \le k \le N$, define the left-sided uniform block averages by

$$\bar\eta_{i,k} := \frac{1}{k}\sum_{i'=0}^{k-1} \eta_{i-i'}, \quad \forall\, i = k, k+1, \dots, N. \tag{7.1}$$

Recall that $\mu_{N,t}$ is the distribution of $\eta(t)$ on Ω_N and ν_ρ is the Bernoulli measure with rate ρ. In this section we show that, in both Liggett and reversible cases, $\mu_{N,t}$ on the k-block $\{\eta_{i-k+1}, \dots, \eta_i\}$ can be estimated by $\nu_{\bar\eta_{i,k}}$ with errors bounded by the corresponding Dirichlet forms in Section 6. The main results are stated below.

Proposition 7.1. *Given a local function $f = f(\eta_0, \dots, \eta_\ell)$ and a vector $\mathbf{w}_k = (w_{k,0}, \dots, w_{k,k-1})$, define*

$$\mathcal{F}_{i,k} = \sum_{i'=0}^{k-1} w_{k,i'}\big(f\circ\tau_{i-i'} - \langle f\rangle(\bar\eta_{i+\ell,k+\ell})\big).$$

Then there is a constant C independent from N, t, k or ℓ, such that

$$\int \sum_{i=k}^{N-\ell} \big|\mathcal{F}_{i,k}\big|^2 d\mu_{N,t} \le C\|\mathbf{w}_k\|^2\big[\ell(k+\ell)^3\mathfrak{D}_{\mathrm{exc},N}(t) + N\big], \tag{7.2}$$

where $\|\cdot\|$ denotes the Euclidean norm.

Typically we will apply Proposition 7.1 for choices of $\mathbf{w}_k$ such that $\|\mathbf{w}_k\|^2 \sim k^{-1}$.

The next proposition deals with the blocks located at the boundaries.

Proposition 7.2. *There is a constant C independent from N, t or k, such that*

$$\int \big|\mathcal{G}_{\pm.k}\big|^2 d\mu_{N,t} \le C\|\mathbf{w}_k\|^2\big[k^2\mathfrak{D}_{\mathrm{exc},N}(t) + k\mathfrak{D}_{\pm,N}(t) + 1\big] \tag{7.3}$$

for any $\mathbf{w}_k = (w_{k,1}, \dots, w_{k,k}) \in \mathbb{R}^k$, where

$$\mathcal{G}_{-,k} = \sum_{i=1}^{k} w_{k,i}\big(\eta_i - \tilde\rho_-(t)\big), \quad \mathcal{G}_{+,k} = \sum_{i=1}^{k} w_{k,i}\big(\eta_{N-k+i} - \tilde\rho_+(t)\big).$$

Proof of Proposition 7.1. For $\rho_* \in \{i/k; i = 0, 1, \dots, k\}$, let $\nu^k(\cdot|\rho_*)$ be the uniform measure on

$$\Omega_{k,\rho_*} = \left\{\eta = (\eta_1, \dots, \eta_k) \in \Omega_k \;\middle|\; \frac{1}{k}\sum_{i=1}^{k}\eta_i = \rho_*\right\}. \tag{7.4}$$

For $i = k,\, k+1,\, ...,\, N$, define the measures

$$\begin{aligned} \bar{\mu}^{i,k}_{N,t}(\rho_*) &:= \mu_{N,t}\left\{(\eta_{i-k+1},\dots,\eta_i) \in \Omega_{k,\rho_*}\right\}, \\ \mu^{i,k}_{N,t}(\eta_{i-k+1},\dots,\eta_i|\rho_*) &:= \frac{\mu_{N,t}(\eta_{i-k+1},\dots,\eta_i)}{\bar{\mu}^{i,k}_{N,t}(\rho_*)}. \end{aligned} \tag{7.5}$$

By the relative entropy inequality, for each $i = k,\, k+1,...,\, N-\ell$,

$$\begin{aligned} \int \left|\mathcal{F}_{i,k}\right|^2 d\mu_{N,t} &= \sum_{\rho_*} \bar{\mu}^{i+\ell,k+\ell}_{N,t}(\rho_*) \int \left|\mathcal{F}_{i,k}\right|^2 d\mu^{i+\ell,k+\ell}_{N,t}(\,\cdot\,|\rho_*) \\ &\le \frac{1}{a}\sum_{\rho_*} \bar{\mu}^{i+\ell,k+\ell}_{N,t}(\rho_*)\Big\{ H\left(\mu^{i+\ell,k+\ell}_{N,t}(\,\cdot\,|\rho_*); \nu^{k+\ell}(\,\cdot\,|\rho_*)\right) \\ &\quad + \log \int \exp\left\{a|\mathcal{F}_{i,k}|^2\right\} d\nu^{k+\ell}(\,\cdot\,|\rho_*)\Big\}, \quad \forall\, a > 0, \end{aligned} \tag{7.6}$$

where H is the relative entropy: for two measures μ, ν on Ω_{k,ρ_*},

$$H(\mu;\nu) := \int (\log\mu - \log\nu) d\mu. \tag{7.7}$$

The logarithmic Sobolev inequality (A.1) yields that there is an universal constant C_{LS}, such that for each i, k and ρ_*,

$$H\left(\mu^{i+\ell,k+\ell}_{N,t}(\,\cdot\,|\rho_*); \nu^{k+\ell}(\,\cdot\,|\rho_*)\right) \le C_{\mathrm{LS}}(k+\ell)^2 \mathfrak{D}^{i+\ell,k+\ell}_{N,\rho_*}(t), \tag{7.8}$$

where the Dirichlet form in the right-hand side is defined as

$$\mathfrak{D}^{i,k}_{N,\rho_*}(t) := \frac{1}{2}\sum_{\eta\in\Omega_{k,\rho_*}} \sum_{i'=1}^{k-1} \left(\sqrt{\mu^{i,k}_{N,t}(\eta^{i',i'+1}|\rho_*)} - \sqrt{\mu^{i,k}_{N,t}(\eta|\rho_*)}\right)^2. \tag{7.9}$$

Plugging (7.8) into (7.6) and summing up for $i \ge k$, and using Schwarz inequality, we obtain

$$\begin{aligned} &\int \sum_{i=k}^{N-\ell} \left|\mathcal{F}_{i,k}\right|^2 d\mu_{N,t} \le \frac{C_{\mathrm{LS}}(k+\ell)^3}{2a} \mathfrak{D}_{\mathrm{exc},N}(t) \\ &+ \frac{1}{a}\sum_{i=k}^{N-\ell} \int \bar{\mu}^{i+\ell,k+\ell}_{N,t}(d\rho_*) \log \int \exp\left\{a|\mathcal{F}_{i,k}|^2\right\} d\nu^{k+\ell}(\,\cdot\,|\rho_*). \end{aligned}$$

The desired estimate then follows if we can show that

$$\log \int \exp\left\{a|\mathcal{F}_{x,k}|^2\right\} d\nu^{k+\ell}(\,\cdot\,|\rho_*) \le C, \quad \forall\, a < \frac{1}{(\ell+1)\|\mathbf{w}_k\|^2}, \tag{7.10}$$

where C is a universal constant C.

We are left with the proof of (7.10). Without loss of generality, we assume that the local function $f \in [0,1]$. By Hoeffding's lemma, for all $\rho \in [0,1]$,

$$\log \int e^{a[f-\nu_\rho(f)]} d\nu_\rho \le \frac{a^2}{8}, \quad \forall\, a \in \mathbb{R}. \tag{7.11}$$

As $f \circ \tau_i$ is independent from f if $|i| \ge \ell + 1$, by splitting the family $\{f \circ \tau_{i-i'}\}$ into independent groups and applying Cauchy–Schwarz inequality,

$$\log \int \exp\left\{ a \sum_{i'=0}^{k-1} w_{k,i'} \big(f \circ \tau_{i-i'} - \langle f \rangle(\rho) \big) \right\} d\nu_\rho \le \frac{(\ell+1)\|\mathbf{w}_k\|^2 a^2}{8}. \tag{7.12}$$

Hence, if $a^{-1} \ge (\ell+1)\|\mathbf{w}_k\|^2$,

$$\log \int \exp\left\{ a \Big| \sum_{i'=0}^{k-1} w_{k,i'} \big(f \circ \tau_{i-i'} - \langle f \rangle(\rho) \big) \Big|^2 \right\} d\nu_\rho \le 3. \tag{7.13}$$

In order to get (7.10) it suffices to replace ν_ρ with its conditional measure $\nu^{k+\ell}(\,\cdot\,|\rho_*)$. It follows from an elementary estimate that

$$\nu^{k+\ell}\big(\eta|_\Gamma = \tilde\eta \,\big|\, \rho_*\big) \le C \nu_{\rho_*}(\eta|_\Gamma = \tilde\eta) \tag{7.14}$$

for all $\Gamma \subset \{x-k+1, \dots, x+\ell\}$ such that $|\Gamma| \le 2k/3$. □

Proof of Proposition 7.2. Without loss of generality, we show the result for $\mathcal{G}_{-,k}$. The proof goes similarly to the previous one. Denote by $\mu^k_{N,t}$ the distribution of $\{\eta_1, \dots, \eta_k\}$ at time t and let $f^k_{N,t} = \mu^k_{N,t}/\nu_{\tilde\rho_-}$. By the relative entropy inequality,

$$E_{\mu_{N,t}}\big[\mathcal{G}^2_{-,k}\big] \le \frac{1}{a}\left[H\big(\mu^k_{N,t}; \nu_{\tilde\rho_-}\big) + \log \int e^{a\mathcal{G}^2_{-,k}} d\nu_{\tilde\rho_-} \right], \quad \forall\, a > 0.$$

Applying Proposition A.1 proved in Appendix A,

$$\begin{aligned} H(\mu^k_{N,t}; \nu_{\tilde\rho_-}) \le{}& \frac{Ck^2}{2} \sum_{\eta\in\Omega_k} \sum_{j=1}^{k-1} \Big(\sqrt{f^k_{N,t}(\eta^{j,j+1})} - \sqrt{f^k_{N,t}(\eta)} \Big)^2 \nu_{\tilde\rho_-}(\eta) \\ &+ \frac{Ck}{2} \sum_{\eta\in\Omega_k} \tilde\rho_-^{1-\eta_1} (1-\tilde\rho_-)^{\eta_1} \Big(\sqrt{f^k_{N,t}(\eta^1)} - \sqrt{f^k_{N,t}(\eta)} \Big)^2 \nu_{\tilde\rho_-}(\eta), \end{aligned}$$

with some constant independent of N, t or k. Therefore,

$$H(\mu^k_{N,t}; \nu_{\tilde\rho_-}) \le Ck^2 \mathfrak{D}_{\mathrm{exc},N}(t) + Ck \mathfrak{D}_{-,N}(t).$$

Similarly to (7.10), we can bound the exponential moment as

$$\log \int \exp\big\{ a|\mathcal{G}_{-,k}|^2 \big\} d\nu_{\tilde\rho_-} \le 3, \quad \forall\, a < \frac{1}{\|\mathbf{w}_k\|^2}.$$

We only need to put these estimate together and integrate in time. □

7.1. **Macroscopic current.** A direct consequence of Proposition 7.1 and the bounds on the Dirichlet forms is the explicit formula (4.6) for the macroscopic current function $\mathcal{J}(t)$ defined in (4.5). Define empirical density corresponding to $\bar{\eta}_{i,k}$ by

$$\zeta_{N,k}(x,t) := \sum_{i=k}^{N} \chi_{i,N}(x)\bar{\eta}_{i,k}(t), \quad (x,t) \in [0,1] \times \mathbb{R}_+. \tag{7.15}$$

Observe that for any $T > 0$ and $\varphi \in \mathcal{C}^1([0,1] \times \mathbb{R}_+)$,

$$\int_0^T \int_0^1 \varphi(x,t)\big(\zeta_N(x,t) - \zeta_{N,k}(x,t)\big)dx\,dt \le \frac{Ck}{N}, \tag{7.16}$$

where ζ_N is the density in (3.14). Recall the distribution $\mathfrak{Q}$ on the space of Young measures defined in (4.3). From the observation above, for fixed k or $k = k(N) \ll N$, one can view $\mathfrak{Q}$ as the weak limit of the same subsequence $\zeta_{N',k}$. To prove (4.6), we distinguish the two cases.

Liggett boundary. Recall the (microscopic) currents $j_{i,i+1}$ given by (5.1). For $1 \le i \le N-1$, we can furthermore write that

$$j_{i,i+1} = J_{i,i+1} - \frac{1-\bar{p}}{2}\nabla\eta_i, \quad J_{i,i+1} := \bar{p}\eta_i(1-\eta_{i+1}). \tag{7.17}$$

Direct computation then shows that for fixed k,

$$\begin{aligned} \mathcal{J}(t) &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N-1} E_{\mu_{N,t}}[j_{i,i+1}] = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N-1} E_{\mu_{N,t}}[J_{i,i+1}] \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=k}^{N-1} \frac{1}{k}\sum_{i'=0}^{k-1} E_{\mu_{N,t}}[J_{i-i',i-i'+1}]. \end{aligned} \tag{7.18}$$

Choose $\ell = 1$, $f = \bar{p}\eta_0(1-\eta_1)$ and note that

$$f \circ \tau_i = J_{i,i+1}, \quad \langle f \rangle(\rho) = J(\rho) = \bar{p}\rho(1-\rho). \tag{7.19}$$

Now take arbitrarily t, $t' \in [0,T]$ such that $t < t'$. Applying Proposition 7.1 with $\mathbf{w}_k = (k^{-1}, \dots, k^{-1})$ and using the estimate in Proposition 6.1,

$$\int_t^{t'} ds \int \sum_{i=k}^{N-1} \left| \frac{1}{k}\sum_{i'=0}^{k-1} J_{i-i',i-i'+1} - J(\bar{\eta}_{i+1,k+1}) \right|^2 d\mu_{N,s} \le C\left(k^2 + \frac{N}{k}\right). \tag{7.20}$$

Dividing the above estimate by N and let $N \to \infty$, we get

$$\left| \int_t^{t'} \mathcal{J}(s)ds - \lim_{N\to\infty} \frac{1}{N}\int_t^{t'} \sum_{i=k}^{N-1} E_{\mu_{N,s}}[J(\bar{\eta}_{i+1,k+1})]ds \right| \le \frac{C}{k}, \tag{7.21}$$

for any $k \ge 1$. From (7.15) and the discussion below it,

$$\left| \int_t^{t'} \mathcal{J}(s)ds - E^{\mathfrak{Q}}\left[\int_t^{t'} ds \int_0^1 dx \int_0^1 J(\rho)\nu_{x,s}(d\rho) \right] \right| \le \frac{C}{k}. \tag{7.22}$$

As t, t' and k are arbitrary we conclude (4.6).
Reversible boundary. In this case, $j_{i,i+1}$ is given by (5.2) and

$$j_{i,i+1} = J_{i,i+1} - \frac{\sigma_N - \bar{p}}{2}\nabla\eta_i, \quad \forall\, i = 1, \dots, N-1. \tag{7.23}$$

Note that from (3.17), $\sigma_N = o(N)$ so that (7.18) remains valid. By applying Proposition 7.1 in the same way as before and Proposition 6.2,

$$\int_t^{t'} ds \int \sum_{i=k}^{N-1} \left| \frac{1}{k} \sum_{i'=0}^{k-1} J_{i-i',i-i'+1} - J(\bar{\eta}_{i+1,k+1}) \right|^2 d\mu_{N,s} \le C \left(\frac{k^2}{\sigma_N} + \frac{N}{k} \right). \tag{7.24}$$

Hence, (4.6) can be proved similarly to the previous case.

8. The boundary entropy

This section is devoted to the proof of Lemma 4.1 under both Liggett and reversible boundaries. Throughout this section, $K \ge 1$ represents some mesoscopic scale that is growing with N such that $K = o(N)$.

Define the *smoother weighted averages* by

$$\hat{\eta}_{i,K} := \sum_{|j|<K} w_j \eta_{i-j}, \quad w_j = \frac{K - |j|}{K^2}. \tag{8.1}$$

Recall the non-gradient current $J_{i,i+1}$ in (7.17) and let

$$\hat{J}_{i,K} := \sum_{|j|<K} w_j J_{i-j,i-j+1}. \tag{8.2}$$

The empirical process $\rho_N = \rho_{N,K}$ is defined by

$$\rho_N(x,t) := \sum_{i=K+1}^{N-K} \chi_{i,N}(x) \hat{\eta}_{i,K}(t), \quad (x,t) \in [0,1] \times \mathbb{R}_+, \tag{8.3}$$

with $\chi_{i,N}$ in (3.13). Observe that η_1 and η_N do not appear in ρ_N, so that the boundary generator does not contribute to the time evolution.

Remark 5. *Recall the function ζ_N defined in* (3.14) *and observe that*

$$\int_0^T \int_0^1 \varphi(x,t)\big(\zeta_N(x,t) - \rho_N(x,t)\big) dx\, dt \le \frac{C_\varphi T K}{N}, \tag{8.4}$$

for any $T > 0$ and $\varphi \in \mathcal{C}^1(\Sigma_T)$. As $K = o(N)$, ρ_N can be identified with ζ_N in the limit $N \to \infty$, in the sense that (4.3) *holds with ζ_N replaced by ρ_N.*

8.1. **The boundary entropy production.** Recall the boundary entropy–entropy flux pair (F, Q) defined in Definition 3.1. For $\psi \in \mathcal{C}^1(\Sigma_T)$ and $w \in \mathcal{C}^1([0,T])$, define the *boundary entropy production* as

$$\begin{aligned} X_N^F(\psi, w) = N^{-a} \iint_{\Sigma_T} \big[F(\rho_N, w)\partial_t\psi + \partial_w F(\rho_N, w) w' \psi\big] dx\, dt \\ + \iint_{\Sigma_T} Q(\rho_N, w)\partial_x\psi\, dx\, dt, \end{aligned} \tag{8.5}$$

where ρ_N is defined in (8.3).

From now on we fix an arbitrary $\mathcal{C}^1$ smooth function $\rho = \rho(t)$ on $[0,T]$ such that $\rho(t) \in [0,1]$. Our aim is to prove the following results.

Proposition 8.1 (Liggett boundary)**.** *Assume that $\bar\rho_-(t) = \bar\rho_+(t) = \rho(t)$ and $a > 1/2$. With $K = K(N)$ satisfying that*

$$\sqrt{N} \ll K \ll \min\{N^a, N\}, \tag{8.6}$$

we have

$$\lim_{N\to\infty} \mathbb{E}_N^{\mu_{N,0}} \left[\left|X_N^F(\psi, \rho)\right|\right] = 0, \tag{8.7}$$

for any initial distribution $\mu_{N,0}$, boundary entropy–entropy flux pair (F, Q) and $\psi \in \mathcal{C}^2(\Sigma_T)$ such that $\psi(\cdot, 0) = \psi(\cdot, T) = 0$.

Proposition 8.2 (Reversible boundary)**.** *Assume that $\rho_-(t) = \rho_+(t) = \rho(t)$ and* (3.17), (3.20)*. With $K = K(N)$ satisfying that*

$$\max\left\{\sqrt{N}, \sigma_N\right\} \ll K \ll \min\{N^a\sigma_N, \widetilde\sigma_N\sigma_N, N\}, \tag{8.8}$$

the result in Proposition (8.1) *still holds. Observe that* (3.17) *and* (3.20) *assure the existence of such $K(N)$.*

The proofs of Proposition 8.1 and 8.2 are similar, and are postponed to Sections 8.2 and 8.3. From (8.7) we can conclude the proof of Lemma 4.1 as follows.

Proof of Lemma 4.1. Given any boundary entropy–entropy flux pair (F, Q), define the boundary entropy flux of a Young measure $\nu \in \mathcal{Y}$ with respect to boundary data $w \in \mathcal{C}([0,T])$ as the functional

$$\widetilde Q(\psi; \nu, w) := \iint_{\Sigma_T} \psi(x,t) dx\, dt \int_0^1 Q(y, w(t)) \nu_{x,t}(dy), \quad \forall\, \psi \in \mathcal{C}(\Sigma_T). \tag{8.9}$$

Let $\hat\nu^N$ be the Young measure associated to ρ_N, i.e. $\hat\nu^N_{x,t} = \delta_{\rho_N(x,t)}$. Since F and $\partial_w F$ are bounded and $a > 0$, for all $\psi \in \mathcal{C}^1(\Sigma_T)$,

$$\lim_{N\to\infty} \mathbb{E}_N^{\mu_{N,0}} \left[\left|\widetilde Q(\partial_x\psi; \hat\nu_N, \rho) - X_N^F(\psi, \rho)\right|\right] = 0, \tag{8.10}$$

where $\rho(t) = \tilde{\rho}_{\pm}(t)$. Since the map $\nu \mapsto \widetilde{Q}(\partial_x\psi; \nu, \rho)$ is a bounded linear functional on $\mathcal{Y}$, it is continuous and consequently the set $\{\nu; |\widetilde{Q}(\partial_x\psi; \nu, \rho)| < \varepsilon\}$ is open. Recall that the distribution of $\hat{\nu}^N$ converges (along a subsequence) to $\mathfrak{Q}$. By (8.7) we have

$$\mathfrak{Q}\left(\left|\widetilde{Q}(\partial_x\psi; \nu, \rho)\right| > \varepsilon\right) \leq \liminf_{N\to\infty} \mathbb{P}_N\left(\left|X_N^F(\psi, \rho)\right| > \varepsilon\right) = 0 \tag{8.11}$$

for any $\varepsilon > 0$ and $\psi \in \mathcal{C}^2(\Sigma_T)$ such that $\psi(\cdot, 0) = \psi(\cdot, T) = 0$. Hence, the following holds with $\mathfrak{Q}$-probability 1:

$$\bar{Q}(x,t) := \int_0^1 Q\big(y, \rho(t)\big)\nu_{x,t}(dy) = 0, \quad (x,t)\text{-a.s. in } \Sigma_T. \tag{8.12}$$

To prove Lemma 4.1, it suffices to show that $\nu_{x,t} = \delta_{\rho(t)}$ if (8.12) holds for all boundary entropy flux Q. We make use of the boundary entropy

$$F(u,w) = \begin{cases} w \wedge \frac{1}{2} - u, & u \in [0, w \wedge \frac{1}{2}), \\ 0, & u \in [w \wedge \frac{1}{2}, 1], \end{cases} \tag{8.13}$$

The corresponding boundary entropy flux is

$$Q(u,w) = \begin{cases} J(w \wedge \frac{1}{2}) - J(u), & u \in [0, w \wedge \frac{1}{2}), \\ 0, & u \in [w \wedge \frac{1}{2}, 1]. \end{cases} \tag{8.14}$$

As $Q(u,w) \geq 0$ for all (u,w) but $\bar{Q}(x,t) = 0$, we conclude that $\nu_{x,t}$ concentrates on the zero set of Q, which is $[\rho(t) \wedge 1/2, 1]$. Similarly, choose

$$F(u,w) = \begin{cases} 0, & u \in [0, w \vee \frac{1}{2}], \\ u - w \vee \frac{1}{2}, & u \in (w \vee \frac{1}{2}, 1], \end{cases}$$

$$Q(u,w) = \begin{cases} 0, & u \in [0, w \vee \frac{1}{2}], \\ J(u) - J(w \vee \frac{1}{2}), & u \in (w \vee \frac{1}{2}, 1]. \end{cases}$$

As $Q(u,w) \leq 0$. the condition $\bar{Q}(x,t) = 0$ then implies that $\nu_{x,t}$ concentrates on $[0, \rho_(t) \vee 1/2]$. Hence, $\nu_{x,t}(\Lambda_t) = 1$ almost surely on Σ_T, where

$$\Lambda_t = \left[\rho(t) \wedge \frac{1}{2}, \rho(t) \vee \frac{1}{2}\right]. \tag{8.15}$$

Finally, to close the proof we choose

$$F(u,w) = |u - w|, \quad Q(u,w) = \text{sign}(u-w)(J(u) - J(w)). \tag{8.16}$$

If $\rho(t) < 1/2$, $Q(u, \rho(t)) \geq 0$ on $\Lambda_t = [\rho(t), 1/2]$ and the only zero point is $\rho(t)$, so that $\bar{Q} = 0$ implies $\nu_{(x,t)} = \delta_{\rho(t)}$. If $\rho(t) \geq 1/2$ the argument is similar. □

Before proceeding to prove Proposition 8.1 and 8.2, we prepare some notations. For each $N \geq 1$,

$$B_N := \left[0, \frac{2K+1}{2N}\right) \cup \left[1 - \frac{2K-1}{2N}, 1\right]. \tag{8.17}$$

Observe that for $G : \Sigma_T \to \mathbb{R}$ and $w : [0,T] \to \mathbb{R}$,

$$G(\rho_N(x,t), w(t)) = \sum_{i=K+1}^{N-K} \chi_{i,N}(x) G(\hat{\eta}_{i,K}, w(t)) + G(0, w(t)) \mathbf{1}_{B_N}(x).$$

For $\psi : \Sigma_T \to \mathbb{R}$ and each $i = 1, ..., N$,

$$\bar{\psi}_i(t) := N \int_0^1 \psi(x,t) \chi_{i,N}(x) dx, \quad \tilde{\psi}_i(t) := \psi\left(\frac{i}{N} - \frac{1}{2N}, t\right). \tag{8.18}$$

We shall fix some boundary Lax entropy–entropy flux pair (F, Q) and write X_N instead of X_N^F for short. We also omit the arbitrary initial measure $\mu_{N,0}$ and denote the expectation with respect to $\{\eta(t); t \in [0,T]\}$ by $\mathbb{E}_N$.

8.2. **Proof of Proposition 8.1.** Throughout this part we assume the conditions in Proposition 8.1.

Lemma 8.3. *X_N satisfies the following decomposition:*

$$X_N(\psi, w) = M_N(\psi, w) - \sum_{i=1}^{4} A_N^{(i)}(\psi, w), \tag{8.19}$$

where M_N is a square integrable martingale and $A_N^{(i)}$ are given by

$$\begin{aligned}
A_N^{(1)} &:= \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i \partial_u F(\hat{\eta}_{i,K}, w) \nabla^* \left[\hat{J}_{i,K} - J(\hat{\eta}_{i,K}) \right] dt, \\
A_N^{(2)} &:= \frac{1-\bar{p}}{2} \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i \partial_u F(\hat{\eta}_{i,K}, w) \Delta \hat{\eta}_{i,K} \, dt, \\
A_N^{(3)} &:= \frac{1}{N} \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i \left(\varepsilon_{i,K}^{(1)} + \varepsilon_{i,K}^{(2)} \right) dt - \int_0^T \int_{B_N} Q(0,w) \partial_x \psi \, dx \, dt, \\
A_N^{(4)} &:= \int_0^T \sum_{i=K+1}^{N-K} \left[\bar{\psi}_i \partial_u Q(\hat{\eta}_{i,K}, w) \nabla^* \hat{\eta}_{i,K} - \nabla \tilde{\psi}_i Q(\hat{\eta}_{i,K}, w) \right] dt.
\end{aligned} \tag{8.20}$$

Here $\varepsilon_{i,K}^{(1)}$ and $\varepsilon_{i,K}^{(2)}$ are respectively given by

$$\begin{aligned}
\varepsilon_{i,K}^{(1)} &:= \frac{N}{2} \sum_{j=i-K}^{i+K-1} \left(\bar{p}\eta_i + \frac{1-\bar{p}}{2} \right) \partial_u^2 F(\tilde{\eta}_{i,j,K}, w) \left(\hat{\eta}_{i,K}^{j,j+1} - \hat{\eta}_{i,K} \right)^2, \\
\varepsilon_{i,K}^{(2)} &:= N \left[\nabla^* J(\hat{\eta}_{i,K}) - J'(\hat{\eta}_{i,K}) \nabla^* \hat{\eta}_{i,K} \right].
\end{aligned} \tag{8.21}$$

where $\tilde{\eta}_{i,j,K}$ is some intermediate value between $\hat{\eta}_{i,K}$ and $\hat{\eta}_{i,K}^{j,j+1}$.

Proof. We omit in this proof the dependence on w in (F, Q). By applying an integration by parts in time and recalling the fact that ρ_N is independent of η_1 and η_N, (8.5) can be rewritten as

$$X_N(\psi, w) = X_{L,N}(\psi, w) + Q_N(\psi, w) + M_N(\psi, w), \tag{8.22}$$

where $X_{L,N}$, Q_N are respectively given by

$$\begin{aligned} X_{L,N}(\psi, w) &:= -\int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i L_{\mathrm{exc}}[F(\hat{\eta}_{i,K})]\, dt, \\ Q_N(\psi, w) &:= \int_0^T \sum_{i=K+1}^{N-K} Q(\hat{\eta}_{i,K}) \nabla \tilde{\psi}_i \, dt + \int_0^T \int_{B_N} Q(0) \partial_x \psi \, dx \, dt, \end{aligned}$$

and M_N is a square integrable martingale. For each i,

$$\begin{aligned} L_{\mathrm{exc}}[F(\hat{\eta}_{i,K})] &= \sum_{j=i-K}^{i+K-1} \left(\bar{p}\eta_i + \frac{1-\bar{p}}{2} \right) \left[F\left(\hat{\eta}_{i,K}^{j,j+1}\right) - F(\hat{\eta}_{i,K}) \right] \\ &= \partial_u F(\hat{\eta}_{i,K}) L_{\mathrm{exc}}[\hat{\eta}_{i,K}] + N^{-1} \varepsilon_{i,K}^{(1)} \\ &= \partial_u F(\hat{\eta}_{i,K}) \left(\nabla^* \hat{J}_{i,K} + \frac{1-\bar{p}}{2} \Delta \hat{\eta}_{i,K} \right) + N^{-1} \varepsilon_{i,K}^{(1)}, \end{aligned}$$

where the last equality follows from (7.17). Moreover,

$$\nabla^* \hat{J}_{i,K} = \nabla^* \left[\hat{J}_{i,K} - J(\hat{\eta}_{i,K}) \right] + J'(\hat{\eta}_{i,K}) \nabla^* \hat{\eta}_{i,K} + N^{-1} \varepsilon_{i,K}^{(2)}. \tag{8.23}$$

Therefore, we can rewrite $X_{L,N}$ as

$$\begin{aligned} X_{L,N}(\psi, w) = &- A_N^{(1)} - A_N^{(2)} - \frac{1}{N} \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i \left(\varepsilon_{i,K}^{(1)} + \varepsilon_{i,K}^{(2)} \right) dt \\ &- \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i \partial_u F(\hat{\eta}_{i,K}) J'(\hat{\eta}_{i.K}) \nabla^* \hat{\eta}_{i,K} \, dt. \end{aligned}$$

The conclusion follows from (8.22) and that $J' \partial_u F = \partial_u Q$. □

To show Proposition 8.1, we evaluate each term in the right-hand side of (8.19). We begin with the martingale M_N.

Lemma 8.4. *If $a > 0$, $\lim_{N\to\infty} \mathbb{E}_N[|M_N(\psi, \rho)|] = 0$.*

Proof. By Dynkin's formula, the quadratic variance of M_N satisfies that

$$\begin{aligned}\langle M_N\rangle &= \int_0^T \sum_{j=1}^{N-1} \frac{c_{j,j+1}}{N^{1+a}} \left[\sum_{i=K+1}^{N-K} \bar{\psi}_i\big(F(\hat{\eta}_{i,K}^{j,j+1}) - F(\hat{\eta}_{i,K})\big)\right]^2 dt \\ &\le \frac{1}{N^{1+a}} \int_0^T \sum_{j=1}^{N-1} \left[\sum_{i=K+1}^{N-K} \bar{\psi}_i \partial_u F(\tilde{\eta}_{i,j,K})\big(\hat{\eta}_{i,K}^{j,j+1} - \hat{\eta}_{i,K}\big)\right]^2 dt,\end{aligned}$$

where $\tilde{\eta}_{i,j,K}$ is some intermediate value between $\hat{\eta}_{i,K}$ and $\hat{\eta}_{i,K}^{j,j+1}$. Direct computation shows that

$$\hat{\eta}_{i,K}^{j,j+1} = \hat{\eta}_{i,K} - \operatorname{sgn}\left(i - j - \frac{1}{2}\right) \frac{\nabla \eta_j}{K^2}, \quad j - K + 1 \le i \le j + K \tag{8.24}$$

and otherwise $\hat{\eta}_{i,K}^{j,j+1} - \hat{\eta}_{i,K} = 0$. Hence, define the block

$$\Lambda_j := \{K + 1 \le i \le N - K\} \cap \{j - K + 1 \le i \le j + K\}. \tag{8.25}$$

Since $|\Lambda_j| \le 2K$, we obtain from (8.24) the estimate

$$\begin{aligned}\langle M_N\rangle &\le \frac{|\partial_u F|_\infty^2}{N^{1+a}} \int_0^T \sum_{j=1}^{N-1} \sum_{i\in\Lambda_j} \bar{\psi}_i^2 \sum_{i\in\Lambda_j} \big(\hat{\eta}_{i,K}^{j,j+1} - \hat{\eta}_{i,K}\big)^2 dt \\ &\le \frac{C|\partial_u F|_\infty^2}{N^{1+a}K^3} \int_0^T \sum_{j=1}^{N-1} \sum_{i\in\Lambda_z} \bar{\psi}_i^2 dt = \frac{C|\partial_u F|_\infty^2}{N^a K^2} \|\psi\|_{L^2(\Sigma_T)}^2.\end{aligned} \tag{8.26}$$

The conclusion then follows from Doob's inequality. □

To estimate the other terms appeared in the right-hand side of (8.19), we make use of the following block estimates.

Proposition 8.5 (One-block estimate)**.** *If $\bar{\rho}_- = \bar{\rho}_+$ for $t \in [0, T]$,*

$$\mathbb{E}_N \left[\int_0^T \sum_{i=K}^{N-K} \left[\hat{J}_{i,K} - J(\hat{\eta}_{i,K})\right]^2 dt\right] \le C\left(\frac{K^2}{N^a} + \frac{N}{K}\right), \tag{8.27}$$

with some constant C independent of K or N.

Proposition 8.6 (Two-block estimate)**.** *If $\bar{\rho}_- = \bar{\rho}_+$ for $t \in [0, T]$,*

$$\mathbb{E}_N \left[\int_0^T \sum_{i=K}^{N-K} (\nabla \hat{\eta}_{i,K})^2 dt\right] \le C\left(\frac{1}{N^a} + \frac{N}{K^3}\right), \tag{8.28}$$

with some constant C independent of K or N.

Proposition 8.7 (Boundary estimates)**.** *If* $\bar\rho_-(t)=\bar\rho_+(t)=\rho(t)$,

$$\mathbb{E}_N\left[\int_0^T\left|\hat\eta_{K+1,K}-\rho(t)\right|^2dt\right]\le C\left(\frac{K}{N^a}+\frac{1}{K}\right),$$
$$\mathbb{E}_N\left[\int_0^T\left|\hat\eta_{N-K+1,K}-\rho(t)\right|^2dt\right]\le C\left(\frac{K}{N^a}+\frac{1}{K}\right),$$
$$\mathbb{E}_N\left[\int_0^T\left|\nabla\hat\eta_{K+1,K}\right|^2dt\right]\le C\left(\frac{1}{N^aK}+\frac{1}{K^3}\right),$$
$$\mathbb{E}_N\left[\int_0^T\left|\nabla\hat\eta_{N-K+1,K}\right|^2dt\right]\le C\left(\frac{1}{N^aK}+\frac{1}{K^3}\right),$$

with some constant C *independent of* K *or* N.

Remark 6. *From Proposition 6.1 and the proofs below, the factor* N^{-a} *in the upper bounds in Proposition 8.5–8.7 is available only when* $\bar\rho_-=\bar\rho_+$. *Without this condition we obtain these estimates with* N^{-a} *replaced by* 1.

Proof of Proposition 8.5. In Proposition 7.1 take $f=\eta_0$, $k=2K-1$ and

$$w_{2K-1,j}=\frac{K-|j-K+1|}{K^2},\quad j=0,1,\dots,2K-2.$$

Then $\langle f\rangle(\rho)=\rho$ and for $2K-1\le i\le N$,

$$\mathcal{F}_{i,k}=\sum_{j=0}^{2K-2}w_{2K-1,j}(\eta_{i-j}-\bar\eta_{i,2K-1})=\hat\eta_{i-K+1,K}-\bar\eta_{i,2K-1}.$$

Therefore, we obtain some constant C such that

$$\int\sum_{i=K}^{N-K+1}\left[\hat\eta_{i,K}-\bar\eta_{i+K-1,2K-1}\right)^2d\mu_{N,t}\le C\left[K^2\mathfrak{D}_{\mathrm{exc},N}(t)+\frac{N}{K}\right].\tag{8.29}$$

Similarly, take $f=\bar p\eta_0(1-\eta_1)$, $k=2K-1$ and the same vector $\mathbf{w}_k$ as above, we obtain that $\langle f\rangle(\rho)=J(\rho)$ and

$$\int\sum_{i=K}^{N-K}\left(\hat J_{i,K}-J(\bar\eta_{i+K,2K})\right)^2d\mu_{N,t}\le C'\left[K^2\mathfrak{D}_{\mathrm{exc},N}(t)+\frac{N}{K}\right].\tag{8.30}$$

Note that for each i, $|\bar\eta_{i+K,2K}-\bar\eta_{i+K-1,2K-1}|\le K^{-1}$, so the conclusion follows from (8.29), (8.30) and Proposition 6.1. □

Proof of Proposition 8.6. Observe that for all $i=K$, $K+1$, ..., $N-K$,

$$\nabla\hat\eta_{i,K}=\frac{\bar\eta_{x+K,K}-\bar\eta_{x,K}}{K}.\tag{8.31}$$

In Proposition 7.1 take $f=\eta_0$, $k=2K$ and

$$w_{2K,j}=\frac{1}{K^2}\mathrm{sgn}\left(K-j-\frac12\right),\quad j=0,1,\dots,2K-1,$$

we have $\mathcal{F}_{i,k} = \nabla\hat{\eta}_{i,K}$. The result follows again from Proposition 6.1. □

Proof of Proposition 8.7. In Proposition 7.2 take $k = 2K+1$ and

$$\mathbf{w}_{2K+1} = \left(0, \frac{1}{K^2}, \dots, \frac{K}{K^2} = \frac{1}{K}, \frac{K-1}{K^2}, \dots, 0\right).$$

The first inequality then follows from Proposition 6.1. The other estimates can be proved in the same way. □

Now we bound each term in (8.20) under the condition $\bar{\rho}_- = \bar{\rho}_+$.

Lemma 8.8. *Assume $a > 1/2$ and (8.6), then*

$$\lim_{N\to\infty} \mathbb{E}_N\left[\left|A_N^{(1)}(\psi,\rho)\right|^2\right] = 0. \tag{8.32}$$

Proof. By summation by parts and the intermediate value theorem,

$$\begin{aligned}
A_N^{(1)}(\psi,\rho) &= A_N^{(1,1)} + A_N^{(1,2)} + A_N^{(1,-)} - A_N^{(1,+)},\\
A_N^{(1,1)} &= \int_0^T \sum_{i=K+1}^{N-K} \nabla\bar{\psi}_i \partial_u F\big(\hat{\eta}_{i+1,K}, \rho(t)\big)\left[\hat{J}_{i,K} - J(\hat{\eta}_{i,K})\right] dt,\\
A_N^{(1,2)} &= \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i \partial_u^2 F\big(\xi_{x,K}, \rho(t)\big)\nabla\hat{\eta}_{i,K}\left[\hat{J}_{i,K} - J(\hat{\eta}_{i,K})\right] dt,\\
A_N^{(1,-)} &= \int_0^T \bar{\psi}_{K+1} \partial_u F\big(\hat{\eta}_{K+1,K}, \rho(t)\big)\left[\hat{J}_{K,K} - J(\hat{\eta}_{K,K})\right] dt,\\
A_N^{(1,+)} &= \int_0^T \bar{\psi}_{N-K+1} \partial_u F\big(\hat{\eta}_{N-K+1,K}, \rho(t)\big)\left[\hat{J}_{N-K,K} - J(\hat{\eta}_{N-K,K})\right] dt,
\end{aligned}$$

where $\xi_{i,K}$ is some intermediate value between $\hat{\eta}_{i,K}$ and $\hat{\eta}_{x+1,K}$. By the one-block estimate in Proposition 8.5,

$$\mathbb{E}_N\left[\left|A_N^{(1,1)}\right|^2\right] \le C|\partial_x\psi|_\infty^2 |\partial_u F|_\infty^2 \left(\frac{K^2}{N^{1+a}} + \frac{1}{K}\right). \tag{8.33}$$

For $A_N^{(1,2)}$, using Cauchy–Schwarz inequality, Proposition 8.5 and 8.6,

$$\mathbb{E}_N\left[\left|A_N^{(1,2)}\right|^2\right] \le C|\psi|_\infty^2 |\partial_u^2 F|_\infty^2 \left(\frac{K^2}{N^{2a}} + \frac{N^2}{K^4}\right). \tag{8.34}$$

For the boundary terms, recall that $\partial_u F(u,w)|_{u=w} \equiv 0$, then

$$\partial_u F(\hat{\eta}_{K+1,K}, \rho(t)) = \partial_u^2 F(\eta_-, \rho(t))\big(\hat{\eta}_{K+1,K} - \rho(t)\big), \tag{8.35}$$

for some intermediate value η_- between $\hat{\eta}_{K+1,K}$ and $\rho(t)$. Hence,

$$\mathbb{E}_N\left[\left|A_N^{(1,-)}\right|^2\right] \le C|\psi|_\infty^2 |\partial_u^2 F|_\infty^2 \left(\frac{K}{N^a} + \frac{1}{K}\right), \tag{8.36}$$

thanks to Proposition 8.7 and the boundedness of J. The right boundary term $A_N^{(1,+)}$ can be estimated similarly. When $N \to \infty$, all the upper bounds vanish since K is chosen to satisfy (8.6). □

Lemma 8.9. *Assume $a > 0$ and $K \gg N^{1/3}$, then*

$$\lim_{N\to\infty} \mathbb{E}_N \left[\left| A_N^{(2)}(\psi, \rho) \right| \right] = 0. \tag{8.37}$$

Proof. Similarly to $A_N^{(1)}$, with some $\xi_{i,K}$ between $\hat{\eta}_{i,K}$ and $\hat{\eta}_{i+1,K}$,

$$\begin{aligned}
A_N^{(2)}(\psi,\rho) &= A_N^{(2,1)} + A_N^{(2,2)} + A_N^{(2,-)} + A_N^{(2,+)},\\
A_N^{(2,1)} &= -\frac{1-\bar{p}}{2} \int_0^T \sum_{i=K}^{N-K} \nabla\bar{\psi}_i \partial_u F\big(\hat{\eta}_{i+1,K}, \rho(t)\big) \nabla\hat{\eta}_{i,K}\, dt,\\
A_N^{(2,2)} &= -\frac{1-\bar{p}}{2} \int_0^T \sum_{i=K}^{N-K} \bar{\psi}_i \partial_u^2 F\big(\xi_{x,K}, \rho(t)\big) \big(\nabla\hat{\eta}_{i,K}\big)^2 dt,\\
A_N^{(2,-)} &= -\frac{1-\bar{p}}{2} \int_0^T \bar{\psi}_K \partial_u F\big(\hat{\eta}_{K,K}, \rho(t)\big) \nabla\hat{\eta}_{K+1,K}\, dt,\\
A_N^{(2,+)} &= \frac{1-\bar{p}}{2} \int_0^T \bar{\psi}_{N-K+1} \partial_u F\big(\hat{\eta}_{N-K+1,K}, \rho(t)\big) \nabla\hat{\eta}_{N-K,K}\, dt.
\end{aligned}$$

Due to the two-block estimate in Proposition 8.6,

$$\mathbb{E}_N \left[\left| A_N^{(2,1)} \right|^2 + \left| A_N^{(2,2)} \right| \right] \le C(\psi, F) \left(\frac{1}{N} + 1 \right) \left(\frac{1}{N^a} + \frac{N}{K^3} \right). \tag{8.38}$$

For the boundary terms, similarly to (8.35),

$$\begin{aligned}
\mathbb{E}_N \left[\left| A_N^{(2,-)} \right|^2 \right] &\le C|\psi|_\infty^2 |\partial_u^2 F|_\infty^2 \times \mathbb{E}_N \left[\int_0^T \big(\nabla\hat{\eta}_{K+1,K}\big)^2 dt \right]\\
&\qquad \times \mathbb{E}_N \left[\int_0^T \big(\hat{\eta}_{K,K} - \rho(t)\big)^2 dt \right]\\
&\le C'|\psi|_\infty^2 |\partial_u^2 F|_\infty^2 \left(\frac{1}{N^{2a}} + \frac{1}{K^4} \right),
\end{aligned}$$

where the last line follows from Proposition 8.7. The last term is bounded similarly. Observe that all bounds vanish under our conditions. □

Lemma 8.10. *Assume $a > 1/2$ and (8.6), then $A_N^{(3)}(\psi, \rho) \to 0$ uniformly.*

Proof. Observe from (8.21) and (8.24) that for any i,

$$\lim_{N\to\infty} \left| \varepsilon_{i,K}^{(1)} \right| \le \lim_{N\to\infty} \frac{CN|\partial_u^2 F|_\infty}{K^3} = 0. \tag{8.39}$$

Meanwhile, noting that $J = \bar{p}\rho(1-\rho)$ and $J'' = -2\bar{p}$, we obtain that

$$\begin{aligned}\left|\varepsilon^{(2)}_{i,K}\right| &= N\left|J'(c\hat{\eta}_{i-1,K} + (1-c)\hat{\eta}_{i,K}) - J'(\hat{\eta}_{i,K})\right|\left|\nabla^*\hat{\eta}_{i,K}\right| \\ &= 2N\bar{p}(1-c)\left|\nabla^*\hat{\eta}_{i,K}\right|^2 \le \frac{CN}{K^2},\end{aligned}$$

with some $\xi \in [0,1]$. Therefore, they vanish uniformly as $N \to \infty$.

We are left with the integral with respect to B_N. Recall the definition of B_N in (8.17) and note that it has Lebesgue measure $2K/N$, so that

$$\left|\int_0^T \int_{B_N} Q(0,\rho)\partial_x\psi \, dx \, dt\right| \le \frac{C|\partial_x\psi|_\infty |Q|_\infty K}{N}. \tag{8.40}$$

Thus, this term also vanishes uniformly as $N \to \infty$. □

Lemma 8.11. *Assume $a > 1/2$ and (8.6), then*

$$\lim_{N\to\infty} \mathbb{E}_N\left[\left|A^{(4)}_N(\psi,\rho)\right|\right] = 0. \tag{8.41}$$

Proof. Similarly to $A^{(2)}_N$, with some $\xi_{i,K}$ between $\hat{\eta}_{i,K}$ and $\hat{\eta}_{i+1,K}$,

$$\begin{aligned}
A^{(4)}_N(\psi,\rho) &= A^{(4,1)}_N + A^{(4,2)}_N + A^{(4,\mathrm{bd})}_N, \\
A^{(4,1)}_N &= \int_0^T \sum_{i=K+1}^{N-K} \big(\bar{\psi}_i - \tilde{\psi}_i\big)\partial_u Q\big(\hat{\eta}_{i,K},\rho(t)\big)\nabla^*\hat{\eta}_{i,K}\, dt, \\
A^{(4,2)}_N &= -\int_0^T \sum_{i=K+1}^{N-K} \tilde{\psi}_i\big[\partial_u Q(\hat{\eta}_{i,K},\rho)\nabla^*\hat{\eta}_{i,K} - \nabla^* Q(\hat{\eta}_{i,K},\rho)\big]dt, \\
A^{(4,\mathrm{bd})}_N &= \int_0^T \tilde{\psi}_{K+1} Q\big(\hat{\eta}_{K,K},\rho(t)\big)dt - \int_0^T \tilde{\psi}_{N-K+1} Q\big(\hat{\eta}_{N-K,K},\rho(t)\big)dt.
\end{aligned}$$

For $A^{(4,1)}_N$, direct calculation shows that $|\bar{\psi}_i - \tilde{\psi}_i| \le C|\partial_x\psi|_\infty N^{-1}$, so that $|A^{(4,1)}_N| \le C(\psi,Q)K^{-1}$. Meanwhile, $|A^{(4,2)}_N| \le C(\psi,Q)NK^{-2}$ because

$$\left|\partial_u Q(\hat{\eta}_{i,K},\rho)\nabla^*\hat{\eta}_{i,K} - \nabla^* Q(\hat{\eta}_{i,K},\rho)\right| \le |\partial_u^2 Q|_\infty \left|\nabla^*\hat{\eta}_{i,K}\right|^2. \tag{8.42}$$

Therefore, these two terms vanish uniformly if $K^2 \gg N$.

We are left with the boundary term. Recalling that $Q(w,w) \equiv 0$ for all $w \in \mathbb{R}$, we have $|Q(\hat{\eta}_{K,K},\rho(t))| \le |\partial_u Q|_\infty |\hat{\eta}_{K,K} - \rho(t)|$. Since similar estimate holds for $Q(\hat{\eta}_{N-K,K},\rho(t))$, in view of Proposition 8.7,

$$\mathbb{E}_N\left[\left|A^{(4,\mathrm{bd})}_N\right|^2\right] \le C|\psi|^2_\infty |\partial_u Q|^2_\infty \left(\frac{K}{N^a} + \frac{1}{K}\right).$$

The desired estimate then follows from (8.6). □

8.3. **Proof of Proposition 8.2.** Throughout this part we assume the conditions in Proposition 8.2. The proof goes parallel to the previous case. We here emphasize the difference.

By the same computation as in Lemma 8.3, X_N satisfies the decomposition formula (8.19), where $A_N^{(i)}$, $i = 1, 3, 4$ and $\varepsilon_{i,K}^{(2)}$ are given in (8.20), (8.21),

$$A_N^{(2)} := \frac{\sigma_N - \bar{p}}{2} \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i \partial_u F(\hat{\eta}_{i,K}, w) \Delta \hat{\eta}_{i,K}\, dt,$$

$$\varepsilon_{i,K}^{(1)} := \frac{N}{2} \sum_{j=i-K}^{i+K-1} \left(\bar{p}\eta_i + \frac{\sigma_N - \bar{p}}{2} \right) \partial_u^2 F(\tilde{\eta}_{i,j,K}, w) \left(\hat{\eta}_{i,K}^{j,j+1} - \hat{\eta}_{i,K} \right)^2,$$

with proper intermediate value $\tilde{\eta}_{i,j,K}$ between $\hat{\eta}_{i,K}$ and $\hat{\eta}_{i,K}^{j,j+1}$.

To continue, we make use of the following block estimates. Observe that they differ from those obtained for Liggett boundaries in Proposition 8.5–8.7, since the Dirichlet forms possess different upper bounds here (Proposition 6.2).

Proposition 8.12 (One-block estimate)**.** *If $\rho_- = \rho_+$ for $t \in [0, T]$,*

$$\mathbb{E}_N \left[\int_0^T \sum_{i=K}^{N-K} \left[\hat{J}_{i,K} - J(\hat{\eta}_{i,K}) \right]^2 dt \right] \le C \left[\frac{K^2}{\sigma_N} \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{N}{K} \right], \tag{8.43}$$

with some constant C independent of K or N.

Proposition 8.13 (Two-block estimate)**.** *If $\rho_- = \rho_+$ for $t \in [0, T]$,*

$$\mathbb{E}_N \left[\int_0^T \sum_{i=K}^{N-K} (\nabla \hat{\eta}_{i,K})^2 dt \right] \le C \left[\frac{1}{\sigma_N} \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{N}{K^3} \right], \tag{8.44}$$

with some constant C independent of K or N.

Proposition 8.14 (Boundary estimates)**.** *If $\rho_-(t) = \rho_+(t) = \rho(t)$,*

$$\mathbb{E}_N \left[\int_0^T \left| \hat{\eta}_{K+1,K} - \rho(t) \right|^2 dt \right] \le C \left(\frac{K}{\sigma_N} + \frac{1}{\widetilde{\sigma}_N} \right) \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{C}{K},$$

$$\mathbb{E}_N \left[\int_0^T \left| \hat{\eta}_{N-K+1,K} - \rho(t) \right|^2 dt \right] \le C \left(\frac{K}{\sigma_N} + \frac{1}{\widetilde{\sigma}_N} \right) \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{C}{K},$$

$$\mathbb{E}_N \left[\int_0^T \left| \nabla \hat{\eta}_{K+1,K} \right|^2 dt \right] \le \frac{C}{K^2} \left(\frac{K}{\sigma_N} + \frac{1}{\widetilde{\sigma}_N} \right) \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{C}{K^3},$$

$$\mathbb{E}_N \left[\int_0^T \left| \nabla \hat{\eta}_{N-K+1,K} \right|^2 dt \right] \le \frac{C}{K^2} \left(\frac{K}{\sigma_N} + \frac{1}{\widetilde{\sigma}_N} \right) \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{C}{K^3},$$

with some constant C independent of K or N.

The proofs are omitted as they are same as the Liggett case. The only modification is to estimate the Dirichlet forms by Proposition 6.2 instead of Proposition 6.1.

To show Proposition 8.2, it suffices to evaluate each term in the decomposition (8.19). We sketch the proof in two lemmas.

Lemma 8.15. *Assume* (3.17), (3.20) *and* (8.8), *then*

$$\lim_{N\to\infty} \mathbb{E}_N \left[|M_N| + \left|A_N^{(1)}\right|^2 + \left|A_N^{(3)}\right|^2 + \left|A_N^{(4)}\right|^2 \right] = 0. \tag{8.45}$$

Proof. For the martingale M_N, its quadratic variance $\langle M_N \rangle$ reads

$$\frac{1}{N^{1+a}} \int_0^T \sum_{j=1}^{N-1} \left(\bar{p}\eta_i + \frac{\sigma_N - \bar{p}}{2} \right) \left[\sum_{i=K+1}^{N-K} \bar{\psi}_i \big(F(\hat{\eta}_{i,K}^{j,j+1}) - F(\hat{\eta}_{i,K}) \big) \right]^2 dt. \tag{8.46}$$

Using (8.24) and the same argument as in proving Lemma 8.4,

$$\langle M_N \rangle \le \frac{C(\sigma_N + \bar{p})}{N^{1+a}K} \int_0^T \sum_{j=1}^{N-1} \bar{\psi}_i^2 dt = \frac{C(\sigma_N + \bar{p})}{N^a K^2} \|\psi\|^2_{L^2(\Sigma_T)}. \tag{8.47}$$

The other terms are estimated by Proposition 8.12–8.14. For $A_N^{(1)}$, applying the argument we have used in proving Lemma 8.8,

$$\begin{aligned} \mathbb{E}_N \left[\left|A_N^{(1)}\right|^2 \right] \le\, & C_1(\psi, F) \left[\frac{K^2}{N\sigma_N} \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{1}{K} \right] \\ & + C_2(\psi, F) \left[\frac{K}{\sigma_N} \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{N}{K^2} \right]^2 \\ & + C_3(\psi, F) \left[\left(\frac{K}{\sigma_N} + \frac{1}{\widetilde{\sigma}_N} \right) \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{1}{K} \right]. \end{aligned}$$

For $A_N^{(3)}$, it vanishes uniformly as

$$\left| \varepsilon_{i,K}^{(1)} \right| \le \frac{CN\sigma_N}{K^3}, \quad \left| \varepsilon_{i,K}^{(2)} \right| \le \frac{CN}{K^2}, \quad \left| \int_{B_N} dx \right| \le \frac{CK}{N}. \tag{8.48}$$

For $A_N^{(4)}$, we can argue similarly to Lemma 8.11 to obtain that

$$\begin{aligned} \mathbb{E}_N \left[\left|A_N^{(4)}\right|^2 \right] \le\, & C(\psi, Q) \left(\frac{1}{K} + \frac{N}{K^2} \right) \\ & + C'(\psi, Q) \left[\left(\frac{K}{\sigma_N} + \frac{1}{\widetilde{\sigma}_N} \right) \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) + \frac{1}{K} \right]. \end{aligned}$$

Thanks to (3.17) and (8.8), we have as $N \to \infty$,

$$\frac{K}{N} = o(1), \quad \frac{K}{\sigma_N} \left(\frac{1}{N^a} + \frac{1}{\widetilde{\sigma}_N} \right) = o(1), \quad \frac{N}{K^2} = o(1), \quad \frac{N\sigma_N}{K^3} = o(1), \tag{8.49}$$

which assures the vanishing of all the bounds above. □

Lemma 8.16. *Assume* (3.17), (3.20) *and* (8.8), *then*

$$\lim_{N\to\infty} \mathbb{E}_N \left[\left| A_N^{(2)}(\psi, \rho) \right| \right] = 0. \tag{8.50}$$

Proof. With some $\xi_{i,K}$ between $\hat\eta_{i,K}$ and $\hat\eta_{i+1,K}$ we have

$$
\begin{aligned}
A_N^{(2)}(\psi,\rho) &= A_N^{(2,1)} + A_N^{(2,2)} + A_N^{(2,-)} + A_N^{(2,+)},\\
A_N^{(2,1)} &= -\frac{\sigma_N-\bar p}{2}\int_0^T \sum_{i=K}^{N-K} \nabla\bar\psi_i \partial_u F\big(\hat\eta_{i+1,K},\rho(t)\big)\nabla\hat\eta_{i,K}\,dt,\\
A_N^{(2,2)} &= -\frac{\sigma_N-\bar p}{2}\int_0^T \sum_{i=K}^{N-K} \bar\psi_i \partial_u^2 F\big(\xi_{x,K},\rho(t)\big)\big(\nabla\hat\eta_{i,K}\big)^2 dt,\\
A_N^{(2,-)} &= -\frac{\sigma_N-\bar p}{2}\int_0^T \bar\psi_K \partial_u F\big(\hat\eta_{K,K},\rho(t)\big)\nabla\hat\eta_{K+1,K}\,dt,\\
A_N^{(2,+)} &= \frac{\sigma_N-\bar p}{2}\int_0^T \bar\psi_{N-K+1} \partial_u F\big(\hat\eta_{N-K+1,K},\rho(t)\big)\nabla\hat\eta_{N-K,K}\,dt.
\end{aligned}
$$

Due to Proposition 8.13 and the assumption $\sigma_N \ll N$,

$$
\begin{aligned}
\mathbb{E}_N\Big[\big|A_N^{(2,1)}\big|^2 + \big|A_N^{(2,2)}\big|\Big] &\le C(\psi,F)\left(\frac{\sigma_N^2}{N}+\sigma_N\right)\left(\frac{1}{N^a\sigma_N}+\frac{1}{\widetilde\sigma_N\sigma_N}+\frac{N}{K^3}\right)\\
&\le C'(\psi,F)\left(\frac{1}{N^a}+\frac{1}{\widetilde\sigma_N}+\frac{N\sigma_N}{K^3}\right).
\end{aligned}
$$

We are left with the boundary terms. Similarly to (8.35),

$$
\begin{aligned}
\mathbb{E}_N\Big[\big|A_N^{(2,-)}\big|^2\Big] &\le C(\psi,F)\times\sigma_N^2\times\mathbb{E}_N\left[\int_0^T\big(\nabla\hat\eta_{K,K}\big)^2dt\right]\\
&\qquad\times\mathbb{E}_N\left[\int_0^T\big(\hat\eta_{K,K}-\rho_-(t)\big)^2dt\right]\\
&\le C'(\psi,F)\left[\left(1+\frac{\sigma_N}{K\widetilde\sigma_N}\right)\left(\frac{1}{N^a}+\frac{1}{\widetilde\sigma_N}\right)+\frac{\sigma_N}{K^2}\right]^2,
\end{aligned}
$$

where the last line follows from Proposition 8.14. As we choose $\sigma_N \ll K \ll N$ and $\widetilde\sigma_N \gg 1$, this term is bounded from above by

$$
\mathbb{E}_N\Big[\big|A_N^{(2,-)}\big|^2\Big] \le C''(\psi,F)\left(\frac{1}{N^{2a}}+\frac{1}{\widetilde\sigma_N^2}+\frac{N^2}{K^4}\right). \tag{8.51}
$$

The right boundary term is estimated similarly. Finally, the proof is completed by noting that all the bounds vanish as $N\to\infty$ under our conditions. □

Remark 7. *From the proof above we see that the expectation of $A_N^{(2)}$ does not vanish if $\rho_- \neq \rho_+$. Hence, it is responsible for the non-zero entropy production associated to the solution of* (3.2) *and* (3.3) *in this case.*

9. Coupling

In this section we prove Lemma 4.2 and Proposition 4.3 by a coupling argument. To establish the result for both Liggett and reversible boundaries, we use the time dependent entry/exit rate functions $(\alpha,\beta,\gamma,\delta)=(\alpha,\beta,\gamma,\delta)(t)$. For Liggett boundaries, these are given by

$$(\alpha,\beta,\gamma,\delta)=\left(\frac{1+\bar p}{2}\bar\rho_-,\frac{1+\bar p}{2}(1-\bar\rho_+),\frac{1-\bar p}{2}(1-\bar\rho_-),\frac{1-\bar p}{2}\bar\rho_+\right), \tag{9.1}$$

while for reversible boundaries,

$$(\alpha,\beta,\gamma,\delta)=\widetilde\sigma_N\big(\bar\lambda_-\rho_-,\bar\lambda_+(1-\rho_+),\bar\lambda_-(1-\rho_-),\bar\lambda_+\rho_+\big). \tag{9.2}$$

Recall that we have defined in (4.3) the limit distribution $\mathfrak{Q}$ and in (4.5) the current $\mathcal{J}(t)$, both associated to the boundary rates $(\alpha,\beta,\gamma,\delta)$.

Lemma 9.1. *If $\alpha\le\alpha_*$, $\gamma\ge\gamma_*$ on $[0,t]$, then for each $y\in[0,1]$,*

$$E^{\mathfrak{Q}}\big[\nu_{x,s}([y,1])\big]\le E^{\mathfrak{Q}_*}\big[\nu_{x,s}([y,1])\big],\quad (x,s)\text{ -a.s. in }\Sigma_t, \tag{9.3}$$

where $\mathfrak{Q}_*$ is the Young measure corresponding to $(\alpha_*,\beta,\gamma_*,\delta)$. The same result holds for $(\alpha,\beta_*,\gamma,\delta_*)$ such that $\beta\ge\beta_*$ and $\delta\le\delta_*$ on $[0,t]$.

Lemma 9.2. *Suppose that $\alpha\le\alpha_*$, $\gamma\ge\gamma_*$ on $[0,t]$. By $\mathcal{J}_*(s)$ we denote the current associated to the boundary rates $(\alpha_*,\beta,\gamma_*,\delta)$, then*

$$\int_0^t\mathcal{J}(s)ds\le\int_0^t\mathcal{J}_*(s)ds. \tag{9.4}$$

On the other hand, if $\beta\ge\beta_*$, $\delta\le\delta_*$ on $[0,t]$, then

$$\int_0^t\mathcal{J}(s)ds\ge\int_0^t\mathcal{J}_*(s)ds, \tag{9.5}$$

where $\mathcal{J}_*(s)$ is the current associated to $(\alpha,\beta_*,\gamma,\delta_*)$.

The proofs of Lemma 9.1 and 9.2 are postponed to the end of this section. We here first show Lemma 4.2 and Proposition 4.3 based on them.

Proof of Lemma 4.2. In Liggett case, let $\rho'(t)=\max\{\bar\rho_-(t),\bar\rho_+(t)\}$ for $t\in[0,T]$. Define $(\alpha_*,\beta_*,\gamma_*,\delta_*)$ through (9.1) with $\bar\rho_-=\bar\rho_+=\rho'$, then $\alpha\le\alpha_*$, $\gamma\ge\gamma_*$, $\beta\ge\beta_*$, $\delta\le\delta_*$. Denote by $\mathfrak{Q}_*$ the limit point associated to $(\alpha_*,\beta_*,\gamma_*,\delta_*)$. Lemma 4.1 yields that $\mathfrak{Q}_*$ concentrates on the Young measure $\nu_{x,t}=\delta_{\rho_*(t)}$. Applying Lemma 9.1,

$$E^{\mathfrak{Q}}\big[\nu_{x,t}([y,1])\big]\le E^{\mathfrak{Q}_*}\big[\nu_{x,t}([y,1])\big]=\mathbf{1}\{y\le\rho'(t)\}. \tag{9.6}$$

With similar argument, $E^{\mathfrak{Q}}[\nu_{x,t}([y,1])]\ge\mathbf{1}\{y\le\min(\bar\rho_-,\bar\rho_+)\}$. Hence, the proof of Lemma 4.2 is completed for Liggett boundaries. The reversible case can be proved in exactly the same way. □

Proof of Proposition 4.3. As before we prove only the Liggett case. Suppose that $\bar\rho_-(t) \le \bar\rho_+(t)$ for $t \in [0,T]$ and let $\rho_*(t) \in [\bar\rho_-(t), \bar\rho_+(t)]$ such that

$$J(\rho_*(t)) = \inf\big\{J(t); \rho \in [\bar\rho_-(t), \bar\rho_+(t)]\big\}. \tag{9.7}$$

Observe that ρ_* is not unique when $\bar\rho_- + \bar\rho_+ = 1$. In view of (4.6) and Lemma 4.2, $\mathcal{J}(t) \ge J(\rho_*(t))$. Meanwhile, define $(\alpha_*, \beta_*, \gamma_*, \delta_*)$ through (9.1) with $\bar\rho_- = \bar\rho_+ = \rho_*$. Since $\alpha \le \alpha_*$, $\gamma \ge \gamma_*$, $\beta \le \beta_*$, $\delta \ge \delta_*$, thanks to Lemma 9.2 and 4.1,

$$\int_0^T \mathcal{J}(t)dt \le \int_0^T \mathcal{J}_*(t)dt = \int_0^T J(\rho_*(t))dt. \tag{9.8}$$

Therefore, $\int \mathcal{J}(t)dt = \int J(\rho_*(t))dt$ and (4.9) then follows. The other criteria (4.8) is proved similarly. □

Both Lemma 9.1 and Lemma 9.2 are consequences of the so-called standard coupling for simple exclusion process. To construct the coupling, define $\bar\Omega_N := \{\xi = \eta \oplus \eta'; \eta_i \le \eta'_i, \forall\, i = 1, \dots, N\}$. For $\xi \in \bar\Omega_N$, let

$$\begin{aligned}
&\xi^{1,+} := \eta^{1,+} \oplus (\eta')^{1,+}, && \xi^{1,-} := \eta^{1,-} \oplus (\eta')^{1,-},\\
&\xi^{N,+} := \eta^{N,+} \oplus (\eta')^{N,+}, && \xi^{N,-} := \eta^{N,-} \oplus (\eta')^{N,-},\\
&\xi^{N,*} := \eta^{N,-} \oplus (\eta')^{N,+}, && \xi^{x,x+1} := \eta^{i,i+1} \oplus (\eta')^{i,i+1},
\end{aligned}$$

where for $\eta \in \Omega_N$, $\eta^{1,\pm}$ are $\eta^{N,\pm}$ are obtained through

$$\begin{aligned}
&\eta^{1,+} := (1, \eta_2, \dots, \eta_N), && \eta^{1,-} := (0, \eta_2, \dots, \eta_N),\\
&\eta^{N,+} := (\eta_1, \dots, \eta_{N-1}, 1), && \eta^{N,-} := (\eta_1, \dots, \eta_{N-1}, 0).
\end{aligned}$$

Note $\xi^{1,\pm}$, $\xi^{N,\pm}$, $\xi^{N,*}$ and $\xi^{i,i+1}$ all belong to $\bar\Omega_N$.

Fix some $N \ge 2$ and without loss of generality take $\lambda_0 = 1$. Let $(\alpha, \beta, \gamma, \delta)$ and $(\alpha, \beta, \gamma, \delta_*)$ be two groups of boundary rates, such that $\delta(s) \le \delta_*(s)$ for $0 \le s \le t$. Define the Markov generator $\bar L_{N,s}$ on $\bar\Omega_N$ as

$$\bar L_{N,s} := \bar L^{(1)}_{N,s} + \bar L^{(2)}_{N,s} + \bar L^{(3)}_{N,s} + \bar L^{(4)}_{N,s}, \tag{9.9}$$

where for any f defined on $\bar\Omega_N$,

$$\begin{aligned}
\bar L^{(1)}_{N,s} f &= \sum_{i=1}^{N-1} \big(p\eta'_i(1-\eta_{i+1}) + (1-p)\eta'_{i+1}(1-\eta_i)\big)\big(f(\xi^{i,i+1}) - f(\xi)\big)\\
\bar L^{(2)}_{N,s} f &= \alpha(s)(1-\eta_1)\big(f(\xi^{1,+}) - f(\xi)\big) + \gamma(s)\eta'_1\big(f(\xi^{1,-}) - f(\xi)\big)\\
\bar L^{(3)}_{N,s} f &= \delta(s)(1-\eta_N)\big(f(\xi^{N,+}) - f(\xi)\big) + \beta(s)\eta'_N\big(f(\xi^{N,-}) - f(\xi)\big)\\
\bar L^{(4)}_{N,s} f &= \big(\delta_*(s) - \delta(s)\big)(1 + \eta_N - \eta'_N)\big(f(\xi^{N,*}) - f(\xi)\big).
\end{aligned}$$

Denote by $\xi = \xi(s)$ the Markov process generated by $\bar L_{N,s}$. Observe that ξ couples the processes associated respectively to $(\alpha, \beta, \gamma, \delta)$ and $(\alpha, \beta, \gamma, \delta_*)$. Indeed, if f is a function on $\bar\Omega_N$ such that $f(\eta \oplus \eta') = g(\eta)$, it is not hard to verify that $\bar L_{N,t} f(\eta \oplus \eta') = L_{N,t} g(\eta)$. Similarly, $\bar L_{N,t} f(\eta \oplus \eta') = L'_{N,t} g(\eta')$ if $f(\eta \oplus \eta') = g(\eta')$.

Proof of Lemma 9.1. We prove here for $\beta = \beta_*$, $\delta \le \delta_*$. The other cases are similar. In the coupled process $\xi = \eta \oplus \eta'$, $\eta_i(t) \le \eta'_i(t)$, so that pointwisely,

$$\iint_{\Sigma_T} f(x,t) g(\zeta_N(x,t)) dx\, dt \le \iint_{\Sigma_T} f(x,t) g(\zeta'_N(x,t)) dx\, dt \tag{9.10}$$

for positive function $f \in \mathcal{C}(\Sigma_T)$ and increasing function $g \in \mathcal{C}([0,1])$. By subtracting subsequence and taking the limit $N \to \infty$,

$$E^{\mathfrak{Q}} \left[\iint_{\Sigma_T} f(x,t) dx\, dt \int_0^1 g d\nu_{x,t} \right] \le E^{\mathfrak{Q}_*} \left[\iint_{\Sigma_T} f(x,t) dx\, dt \int_0^1 g d\nu_{x,t} \right].$$

As f is an arbitrary continuous positive function,

$$E^{\mathfrak{Q}} \left[\int_0^1 g d\nu_{x,t} \right] \le E^{\mathfrak{Q}_*} \left[\int_0^1 g d\nu_{x,t} \right], \quad (x,t)\text{ -a.s.} \tag{9.11}$$

The conclusion follows since we can approximate the indicator function $\mathbf{1}_{[y,1]}$ by a sequence of continuous increasing functions. □

Proof of Lemma 9.2. We prove here (4.9) with $\beta = \beta_*$, $\delta \le \delta_*$. For the coupled process $\xi = \eta \oplus \eta'$, let $\eta_i^\Delta = \eta'_i - \eta_i$ be the *second class particle process*. Recall the counting process $h = h_+ - h_-$ defined for $\eta(\cdot)$ in (5.3)–(5.4). We define similar counting process h', $h'_\pm$ and h^Δ, $h^\Delta_\pm$ for $\eta'(\cdot)$ and $\eta^\Delta(\cdot)$, respectively. Observe that $h^\Delta_+(0,s) \equiv 0$ for $0 \le s \le t$ and

$$h'(i,t) - h(i,t) = h^\Delta(i,t), \quad 0 \le i \le N. \tag{9.12}$$

For any $\eta \in \Omega_N$, let $\xi(0) = \eta \oplus \eta$, which means that $\theta_i(0) = 0$ for all i. By summing up (9.12) and noting that in θ particles can enter only from the right,

$$\sum_{i=0}^N h'(i,s) - \sum_{i=0}^N h(i,s) = \sum_{i=0}^N h^\Delta(i,s) \le 0, \quad \forall s \in [0,t]. \tag{9.13}$$

From the definition of $\mathcal{J}(s)$ in (4.5),

$$\begin{aligned}
\int_0^t \mathcal{J}(s) ds &= \lim_{N\to\infty} \int_0^t \frac{1}{N} \sum_{i=0}^N \mathbb{E}_N^{\eta(0)}[j_{i,i+1}(s)] ds \\
&= \lim_{N\to\infty} \frac{1}{N^{2+a}} \sum_{i=0}^N \mathbb{E}_N^{\eta(0)}[h(i,t)] \\
&\ge \lim_{N\to\infty} \frac{1}{N^{2+a}} \sum_{i=0}^N \mathbb{E}_N^{\eta(0)}[h'(i,t)] = \int_0^t \mathcal{J}_*(s) ds.
\end{aligned}$$

The other cases follow from similar arguments. □

Appendix A. Appendix: Logarithmic Sobolev inequalities

In this appendix we fix a box of length k. For $\rho \in (0,1)$, let ν_ρ be the product Bernoulli measure on $\Omega_k = \{0,1\}^K$ with density ρ. For $h = 0, 1, ..., k$, let $\nu_\rho(\eta|h) = \tilde{\nu}(\eta|h)$ be the uniform distribution on

$$\Omega_{k,h} := \left\{ \eta \in \Omega_k \;\middle|\; \sum_{i=1}^{k} = h \right\},$$

and $\bar{\nu}_\rho(h)$ be the Binomial distribution $\mathcal{B}(k,\rho)$.

The log-Sobolev inequality for the simple exclusion yields that ([15]) there exists a universal constant C_{LS} such that

$$\sum_{\eta\in\Omega_{k,h}} f(\eta)\log f(\eta)\tilde{\nu}(\eta|h) \le \frac{C_{\mathrm{LS}}k^2}{2}\sum_{\eta\in\Omega_{k,h}}\sum_{i=1}^{k-1}\left(\sqrt{f}(\eta^{i,i+1})-\sqrt{f}(\eta)\right)^2\tilde{\nu}(\eta|h). \tag{A.1}$$

for any $f \ge 0$ on $\Omega_{k,h}$ such that $\sum_{\eta\in\Omega_{k,h}} f\tilde{\nu}(\eta|h) = 1$..

In the following we expand (A.1) to a log-Sobolev inequality associated to the product measure ν_ρ with boundaries. The result is necessary for the boundary block estimates in Section 8.2 and 8.3.

Proposition A.1. *There exists constants C_{LS} and C_ρ such that*

$$\begin{aligned}\sum_{\eta\in\Omega_k} f(\eta)\log f(\eta)\nu_\rho(\eta) \le{}& \frac{C_{\mathrm{LS}}k^2}{2}\sum_{\eta\in\Omega_k}\sum_{i=1}^{k-1}\left(\sqrt{f}(\eta^{i,i+1})-\sqrt{f}(\eta)\right)^2\nu_\rho(\eta)\\ &+\frac{C_\rho k}{2}\sum_{\eta\in\Omega_k}\rho^{1-\eta_1}(1-\rho)^{\eta_1}\left(\sqrt{f}(\eta^1)-\sqrt{f}(\eta)\right)^2\nu_\rho(\eta).\end{aligned} \tag{A.2}$$

for any $f \ge 0$ on Ω_k such that $\sum_{\eta\in\Omega_k} f\nu_\rho = 1$.

Proof. In the following proof we define

$$\bar{f}(h) := \sum_{\eta\in\Omega_{k,h}} f\tilde{\nu}(\eta|h), \quad f(\eta|h) := \frac{f(\eta)}{\bar{f}(h)}$$

for $\eta \in \Omega_{k,h}$ and $g = \sqrt{f}$, $\bar{g} = \sqrt{\bar{f}}$, $g(\eta|h) = \sqrt{f(\eta|h)}$. Observe that

$$\begin{aligned}\sum_\eta f\log f\nu_\rho = {}&\sum_{h=0}^{k}\bar{f}(h)\bar{\nu}_\rho(h)\sum_{\eta\in\Omega_{k,h}} f(\eta|h)\log f(\eta|h)\tilde{\nu}(\eta|h)\\ &+\sum_{h=0}^{k}\bar{f}(h)\log\bar{f}(h)\bar{\nu}_\rho(h).\end{aligned} \tag{A.3}$$

By (A.1) the first term on the RHS of (A.3) is bounded by

$$\begin{aligned}&\frac{C_{\mathrm{LS}}k^2}{2}\sum_{h=0}^{k}\bar f(h)\bar\nu_\rho(h)\sum_{\eta\in\Omega_{k,h}}\sum_{i=1}^{k-1}\big(g(\eta^{i,i+1}|h)-g(\eta|h)\big)^2\,\tilde\nu(\eta|h)\\ =&\frac{C_{\mathrm{LS}}k^2}{2}\sum_{\eta\in\Omega_k}\sum_{i=1}^{k-1}\big(g(\eta^{i,i+1})-g(\eta)\big)^2\,\nu_\rho(\eta).\end{aligned}\tag{A.4}$$

The second term on the RHS of (A.3) can be written and bounded by the log-Sobolev inequality for the dynamics where in each site particles are created and destroyed with intensities ρ and $1-\rho$. Since this is a product dynamics, the log-Sobolev constant is uniform in k. This gives

$$\begin{aligned}&\sum_{\eta\in\Omega_k}\bar f(\xi(\eta))\log f(\xi(\eta))\nu_\rho(\eta)\\ \le\, &C_\rho\sum_{\eta\in\Omega_k}\sum_{i=1}^{k-1}\rho^{1-\eta_i}(1-\rho)^{\eta_i}\left[\bar g(\xi(\eta^i))-\bar g(\xi(\eta))\right]^2\nu_\rho(\eta),\end{aligned}\tag{A.5}$$

where $\xi(\eta)=\sum_{i=1}^k\eta_i$. Notice that, denoting $T_{1,i}f(\eta)=f(\eta^{1,i})$ and recalling that $\nu_\rho(\eta)=\prod_i\nu_\rho(\eta_i)=\prod_i(1-\rho)^{1-\eta_i}\rho^{\eta_i}$,

$$T_{1,i}\left[\nu_\rho((\eta^i)_i)\left[\bar g(\xi(\eta^i)-\bar g(\xi(\eta))\right]^2\right]=\nu_\rho((\eta^1)_1)\left[\bar g(\xi(\eta^1)-\bar g(\xi(\eta))\right]^2.\tag{A.6}$$

This implies, using Jensen inequality,

$$\begin{aligned}&\sum_{\eta\in\Omega_k}\sum_{i=1}^{k-1}\rho^{1-\eta_i}(1-\rho)^{\eta_i}\left[\bar g(\xi(\eta^i))-\bar g(\xi(\eta))\right]^2\nu_\rho(\eta)\\ =\,&k\sum_{\eta\in\Omega_k}\rho^{1-\eta_1}(1-\rho)^{\eta_1}\left[\bar g(\xi(\eta^1)-\bar g(\xi(\eta))\right]^2\nu_\rho(\eta)\\ \le\,&k\sum_{\eta\in\Omega_k}\rho^{1-\eta_1}(1-\rho)^{\eta_1}\big(g(\eta^1)-g(\eta)\big)^2\,\nu_\rho(\eta),\end{aligned}\tag{A.7}$$

which completes the proof. □

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