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Pairs of Boolean elements based on a ‘concrete’ Rauszer Boolean algebra

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Abstract

With the development of practical applications in the domain of information systems, a wide variety of non-classical logics are part of current studies. Most of these systems have a natural algebraic model (namely their Lindenbaum algebra) which is a distributive lattice with additional operators. In order to find examples of these models, one way is to consider pairs of elements.

The purpose of this paper is to provide a general framework, built from a pair of Boolean operators -a quantifier and a co-quantifier- on a complete Heyting-Brouwer algebra, focusing on the lattice part, which is common to all these structures.

1. Introduction

It turns out that, for a variety of sound reasons, researchers are occasionally interested in topics related to pairs of sets. Below we will report some sources that have influenced our work.

According to ([13], p.88), the idea of considering pairs of elements in ordered sets in view of a generalization of the cardinal arithmetic (Hausdorff), comes back to Garrett Birkhoff ([1], p.312). Thus, given two ordered sets X and Y , “the product $X \cdot Y$ of X and Y is meant the system whose members are all couples $[x, y]$ with $x \in X$ and $y \in Y$, and in which $[x, y] \leq [x', y']$ means that $x \leq x'$ in X and $y \leq y'$ in Y . $X \cdot Y$ is an ordered set and if X and Y are lattices, then so is $X \cdot Y$. Moreover any laws such as the modular and distributive laws which hold in X and Y hold in $X \cdot Y$ ”.

Using ordered pairs of Boolean elements to represent algebraic structures related to logics comes back to Gr. C. Moisil. Indeed, in [12], this author was able to characterize -from an algebraic-algorithmic point of view- the Łukasiewicz three-valued logic. With the aim of giving an example as well as a representation of this structure, this author considered sets of pairs $[N\mu Nx, \mu x]$, where N is a Kleene negation and μ an endomorphism from the Łukasiewicz three-valued algebra to the Boolean subalgebra of Chrysippian elements. The construction of Moisil has been adapted numerous times to other propositional logics and has been generalized to other contexts.

In the case of lower and upper approximations of a set, Z. Pawlak (1981) proposed the notion of rough set, as the approximation of a set by other sets and T. B. Iwiński (1987) started their lattice-theoretical study considering rough sets as pairs of [lower approximation, upper approximation].

In order to approach a set of objects -by default and by excess- a very old idea is to consider the universe Ob provided with a partition P . From an algebraic point of view, this partition generates an equivalence relation R_P . Let R_P^* be the family of all equivalence classes $|x|$ of R_P , that is $R_P^* = \{|x| : x \in Ob\}$. It is well known that on the Boolean algebra $(\mathcal{P}(Ob), \cap, \cup, -, \emptyset, Ob)$, where $\mathcal{P}(Ob)$ denotes the powerset of Ob , and $-X$ the complementation of X , the relation R_P induces a **monadic closure operator** M_P and a **monadic interior operator** L_P in the following way, for $X \subseteq Ob$:

$$\begin{aligned} M_P X &= \bigcup \{|x| \in R_P^* : x \in X\} \\ L_P X &= \bigcup \{|x| \in R_P^* : |x| \subseteq X\}. \end{aligned}$$

Since $L_P X \subseteq X \subseteq M_P X$, ordered pairs of the form $[L_P X, M_P X]$ with $X \subseteq Ob$, are rough sets. The collection of these pairs, equipped with a suitable structure, allows us to give a ‘typical’ example of three-valued Łukasiewicz algebra. These algebraic structures are adapted to three-valued situations and, in particular, they are monadic Boolean algebras.

It is well known that the study of monadic Boolean algebras was initiated by Halmos around 1955 [3]. They are Boolean algebras with, in addition, a unary operator characterized by axioms analogous to those of an universal quantifier \forall or an existential quantifier \exists .

In the nineties, in particular in 1996 (the essays dedicated to the memory of Helena Rasiowa were published in 1999 [5]), I proved that the notion of monadic Boolean Algebra is the good abstract algebraic structure to manage rough sets. In order to be closer to Rasiowa’s lines of research I worked with Łukasiewicz algebras. Moreover, if the classical negation is not taken as an essential operation, the concept of monadic Tarski algebra, known in the literature, can replace the notion of monadic Boolean algebra [8].

Encouraged (around 2011) by the advise of an anonymous referee who has evaluated one of my papers, I have extended the study of lower and upper approximation by means of **preorders** in view of practical applications and representation theorems of some structures related to non-classical logics.

During years, a large number of non-classical logics and their algebraic semantic have been considered. In ([22], p.227 (or p.154)), we can read that most of the well-studied non-classical systems are logics based on classes of distributive lattices with operators (see for example, the T -structures in [6], [7] or T -rough algebras and their applications in [21], where T is a finite ordered set of co-operating agents).

In the case the approach of pairs is chosen, the main goal of this paper is to find a general pattern to be able to manage lattice operations of pairs.

In the particular case cited above, where the universe Ob is enriched with an equivalence relation R_P , the lower and upper approximation sets are related by means of the Boolean negation in $\mathcal{P}(Ob)$. In fact, $L_P X = -M_P - X$. The boundaries are empty. This is no longer the case if the relation R is only a preorder.

2. A ‘concrete’ Rauszer Boolean algebra

With the aim of taking into account the boundaries [10], we consider the ‘**concrete**’ **Rauszer Boolean algebra** introduced in ([17] and [18]) as an example of a bi-topological algebra (see also [9]).

Roughly speaking, they are Boolean algebras with, in addition, two unary operators, I_R an interior and C_R a closure, not related by the Boolean negation.

In this section we point out some basic notions related to a Rauszer Boolean algebra.

Let Ob be a nonempty set (set of objects) and R a **preorder relation** on Ob . For $x \in Ob$, let

$$R(x) = \{y \in Ob : xRy\}.$$

and $R^* = \{R(x) : x \in Ob\}$.

By the reflexivity of R we infer that $x \in R(x)$. Also, if $z \in R(x)$ and $u \in R(z)$ then xRz and zRu , so by transitivity xRu , that is $R(z) \subseteq R(x)$.

On the Boolean algebra $(\mathcal{P}(Ob), \cap, \cup, -, \emptyset, Ob)$, where $\mathcal{P}(Ob)$ denotes the powerset of Ob , and $-X$ the complementation of X , a preorder relation R induces two unary operators C_R and I_R in the following way, for $X \subseteq Ob$:

$$\begin{aligned} C_R X &= \bigcup \{R(x) \in R^* : x \in X\} \\ I_R X &= \bigcup \{R(x) \in R^* : R(x) \subseteq X\} \end{aligned}$$

C_R and I_R are respectively a **closure** and an **interior** operator over $\mathcal{P}(Ob)$. This ‘concrete’ Rauszer Boolean algebra $\mathcal{B} = (\mathcal{P}(Ob), I_R, C_R)$ has many interesting algebraic properties.

In fact:

$$I_R C_R X = C_R X \quad \text{and} \quad C_R I_R X = I_R X,$$

that is, they are **conjugate** over $\mathcal{P}(Ob)$ ([18], p.228).

Also, a set $X \in \mathcal{P}(Ob)$ is called **R -closed** in the case $C_R X = X$ and **R -open** in the case $I_R X = X$. As in the case of monadic Boolean algebras we have here that:

$$X \text{ is } R\text{-open} \quad \text{if and only if} \quad X \text{ is } R\text{-closed.}$$

Let \mathcal{O}_R be the family of all **R -open** (= **R -closed**) sets. Thus, $(\mathcal{O}_R, \cap, \cup, \emptyset, Ob)$ is a lattice with zero and unit.

For all $G, H \in \mathcal{O}_R$, where \supset is the classical implication, two operations \Rightarrow and $\dot{\div}$ on \mathcal{O}_R , are defined ([18], p.235):

$$G \Rightarrow H = I_R(G \supset H) \quad \text{and} \quad G \dot{\div} H = C_R(G \cap -H)$$

On \mathcal{O}_R , the operation \Rightarrow is the Heyting implication and the operation $\dot{\div}$ is a Brouwer implication (also called the pseudo-difference or the residual). In addition we have two negations: $\neg G = G \Rightarrow \emptyset$ and $\neg G = Ob \dot{\div} G$.

Therefore the system $(\mathcal{O}_R, \cap, \cup, \Rightarrow, \dot{\div}, \neg, \neg, \emptyset, Ob)$ is a Heyting-Brouwer algebra.

Let $I_R(\mathcal{P}(Ob))$ be the image of $\mathcal{P}(Ob)$ by I_R . We have the following equivalences:

$$Z \in I_R(\mathcal{P}(Ob)) \iff \text{there is } X \in \mathcal{P}(Ob) \text{ such that } I_RX = Z \iff \\ \text{there is } X \in \mathcal{P}(Ob) \text{ such that } I_RZ = I_RI_RX = I_RX = Z \iff Z \in \mathcal{O}_R$$

Thus, the image $I_R(\mathcal{P}(Ob)) = (\mathcal{O}_R, \cap, \cup, \emptyset, Ob)$ is a distributive lattice, with zero and unit.

In addition, \mathcal{O}_R satisfies the following property ([14], p.177):

$$\text{If for all } k \in K, G_k \in \mathcal{O}_R \text{ then the lower upper bound (l.u.b.) } \bigvee_{k \in K} G_k = \\ \bigcup_{k \in K} G_k \text{ is in } \mathcal{O}_R.$$

In other words, this means that $I_R(\bigcup_{k \in K} G_k) = \bigcup_{k \in K} G_k$. Hence \mathcal{O}_R is a **sup-complete distributive lattice**.

In the ‘concrete’ Rauszer Boolean algebra $\mathcal{B} = (\mathcal{P}(Ob), I_R, C_R)$, the lattice \mathcal{O}_R has another property which is, in general, not true in topological spaces.

Proposition 2.1 *If for all $k \in K$, $G_k \in \mathcal{O}_R$ then $I_R(\bigcap_{k \in K} G_k) = \bigcap_{k \in K} G_k$.*

Proof. Indeed, (i) $I_R(\bigcap_{k \in K} G_k) \subseteq \bigcap_{k \in K} G_k$ since I_R is an interior operator. On the other hand, we will prove that $\bigcap_{k \in K} G_k \subseteq I_R(\bigcap_{k \in K} G_k) = \bigcup \{R(u) : R(u) \subseteq \bigcap_{k \in K} G_k\}$.

Let $z \in \bigcap_{k \in K} G_k$. For all $k \in K$ we have $z \in G_k = I_RG_k$, that is, for all $k \in K$ there is $u_k \in G_k$ such that $z \in R(z) \subseteq R(u_k) \subseteq G_k$. Hence $z \in R(z) \subseteq \bigcap_{k \in K} G_k$. This leads to $z \in I_R(\bigcap_{k \in K} G_k)$, that is (ii) $\bigcap_{k \in K} G_k \subseteq I_R(\bigcap_{k \in K} G_k)$.

From (i) and (ii) we get the result.

Owing to the previous results, we infer that the ordered set $(\mathcal{O}_R, \cap, \cup, \emptyset, Ob)$ is a **complete distributive lattice with zero and unit**. The g.l.b. and l.u.b. being the intersection and union of sets respectively.

Let \mathcal{C}_R be the family of all **R -closed** elements under R and $C_R(Ob)$ the image of Ob by C_R . By a result above, we conclude that $\mathcal{C}_R = \mathcal{O}_R$.

This type of algebras were remarked by McKinsey and Tarski in ([11], p.129) and referred to as “double Brouwerian algebras”. According to these authors, this

notion seems to have been discussed for the first time in a paper by Skolem in 1919 (implicative and subtractive lattices). In the 1970's, they were extensively investigated by Rauszer in several papers, under the name of semi-Boolean algebras [17]. We remark that, in the literature, this latter name has also been used for other structures. They are an algebraic counterpart of an extension of the intuitionistic logic that she called Heyting-Brouwer ($H-B$)-logic [19]. For this reason, we preferred to call them **Heyting-Brouwer algebras** (**H-B**-algebras in brief). For more information see for example [4].

Under the name of co-Heyting algebras [10], or Bi-Heyting algebras [20], they occur in categorical logic (the Lawvere topos). Thus, Lawvere [10] wrote that “they occur in practice directly, not only as formal opposites of Heyting algebras”.

This structure, as in the case of toposes, allows us to define a boundary operator $\delta(G) = G \cap \neg(G)$ [10]. Otherwise, the idea of iterating the two negations can be found in ([18], p.222) and ([4], p.552). Using these facts, we are able to get two particular dual operators on \mathcal{O}_R needed to obtain our purpose.

For a later need we recall that, since \mathcal{O}_R is a **complete** distributive lattice, two “infinite distributive laws”: $G \cap \bigcup_t H_t = \bigcup_t (G \cap H_t)$ and $G \cup \bigcap_t H_t = \bigcap_t (G \cup H_t)$ hold in \mathcal{O}_R (cf. Birkhoff-Tarski or ([18], p.235)).

3. The complete H-B-subalgebra

$$(\mathcal{O}_R, \cap, \cup, \Rightarrow, \dot{-}, \neg, \neg\neg, \emptyset, Ob)$$

According to ([18], p.222), ([4], p.552), from negations $\neg G = G \Rightarrow \emptyset$ and $\neg\neg G = Ob \dot{-} G$ we can list a few fundamental properties, for all $G, H \in \mathcal{O}_R$:

- (1.) \neg and $\neg\neg$ are order reversing. Hence, $\neg\neg$ and $\neg\neg\neg$ are order preserving
- (2.) $\neg\emptyset = Ob$, $\neg Ob = \emptyset$; $\neg\neg\emptyset = Ob$, $\neg\neg Ob = \emptyset$
- (3.) $G \cap \neg G = \emptyset$, $G \cup \neg\neg G = Ob$; i.e. $\neg G$ is the largest element X such that $G \cap X = \emptyset$ and $\neg\neg G$ is the smallest element Y such that $G \cup Y = Ob$
- (4.) $\neg(G \cup H) = \neg G \cap \neg H$, $\neg\neg(G \cap H) = \neg\neg G \cup \neg\neg H$
- (5.) $\neg G \subseteq \neg\neg G$; $\neg\neg G \subseteq \neg\neg\neg G \subseteq G \subseteq \neg\neg\neg G \subseteq \neg\neg\neg\neg G$; $\neg\neg\neg\neg G = \neg\neg G$, $\neg\neg\neg\neg\neg G = \neg\neg\neg G$
- (6.) $\neg\neg(G \cap H) = \neg\neg(\neg\neg G \cup \neg\neg H) = (\neg\neg\neg G \cap \neg\neg\neg H)$
- (7.) $\neg\neg\neg(G \cup H) = \neg\neg\neg(\neg\neg G \cap \neg\neg H) = (\neg\neg\neg\neg G \cup \neg\neg\neg\neg H)$
- (8.) $\neg\neg \bigcap_t G_t = \bigcup_t \neg\neg G_t$ ([18], p.235)
- (9.) $\neg\neg \bigcup_t G_t = \bigcap_t \neg\neg G_t$ ([19], p.129).

Following [17], [18], [4], we can consider on \mathcal{O}_R the sequences S_n and T_n , for $n \geq 1$, in the following way, for all $G \in \mathcal{O}_R$:

$$\begin{aligned} S_1 G &= \neg \neg G & T_1 G &= \neg \neg G \\ S_n G &= \underbrace{\neg \neg \neg \neg \dots \neg \neg}_{2n} G & T_n G &= \underbrace{\neg \neg \neg \neg \dots \neg \neg}_{2n} G. \end{aligned}$$

In other words S_n , is obtained by iterating n times the composition $\neg \neg$, whereas T_n is obtained by iterating n times the dual composition $\neg \neg$.
By properties recalled above we deduce that:

$$S_1 G \subseteq G \subseteq T_1 G \quad \text{and} \quad S_n G \subseteq G \subseteq T_n G.$$

In addition, since the H-B-algebra \mathcal{O}_R is **complete** we can define operators S_R and T_R on \mathcal{O}_R as follows:

$$S_R G = \bigcap_n S_n G \quad \text{and} \quad T_R G = \bigcup_n T_n G$$

The notions of $T_R G$ and $S_R G$ are perfectly duals.

Theorem 3.1 *The operator T_R defined above is a **quantifier** on the bounded distributive lattice $(\mathcal{O}_R, \cap, \cup, \emptyset, Ob)$ and S_R is a **co-quantifier** on the same lattice. In other words, T_R satisfies the following conditions ([2], p.185):*

- | | |
|---|-----------------------------|
| (Q0) $T_R \emptyset = \emptyset$ | <i>normalized</i> |
| (Q1) $G \cap T_R G = G$ | <i>increasing</i> |
| (Q2) $T_R(G \cap T_R H) = T_R G \cap T_R H$ | <i>quasi-multiplicative</i> |
| (Q3) $T_R(G \cup H) = T_R G \cup T_R H$ | <i>additive</i> |

*The corresponding assertions for the **co-quantifier** S_R are obtained from (Q0)-(Q3) upon replacing T_R , \emptyset , \cap , and \cup by S_R , Ob , \cup , and \cap , respectively:*

- | |
|--|
| (Q'0) $S_R Ob = Ob$ |
| (Q'1) $S_R G \cup G = G$ |
| (Q'2) $S_R(G \cup S_R H) = S_R G \cup S_R H$ |
| (Q'3) $S_R(G \cap H) = S_R G \cap S_R H$ |

In order to shorten the proof of Theorem 3.1 we will first present some elementary consequences of the definitions given above. Also, in view of the perfect existing duality, some proofs can be omitted.

Lemma 3.2 *On account of definitions S_R and T_R we have:*

$$(p0) \quad S_R Ob = Ob ; T_R \emptyset = \emptyset ;$$

$$(p1) \quad S_R \text{ and } T_R \text{ are order preserving}$$

- (p2) $S_R G \subseteq G \subseteq T_R G$, more precisely
 $S_R G \subseteq S_n G \subseteq S_{n-1} G \subseteq \dots \subseteq G \subseteq \dots \subseteq T_{n-1} G \subseteq T_n G \subseteq T_R G$
- (p3) $\neg \sqcap S_R G = S_R G$ and $\sqcap \neg T_R G = T_R G$

Proof. These assertions follow immediately from properties (1.) – (9.) above.

In the past, authors have been interested in the study of modal operators of necessity and possibility on distributive lattices. In [20], the operators S_R and T_R have been considered to define two modal operators in complete bi-Heyting toposes. They have a very nice geometrical interpretation in the case of the bi-Heyting algebras of subgraphs of a given graph.

For the sake of proof, according to [20], we recall the following results. We note $(B(\mathcal{O}_R), \cap, \cup, -, \emptyset, \mathcal{O}_R)$ the subalgebra of Boolean elements in \mathcal{O}_R , that is the set of elements $G \in \mathcal{O}_R$ such that $\neg G = \sqcap G$. Here ‘-’ is the Boolean negation on $B(\mathcal{O}_R)$.

Lemma 3.3 *The operators S_R and T_R defined above, satisfy -in addition- the following properties:*

- (p4) S_R and T_R are applications from \mathcal{O}_R to $B(\mathcal{O}_R)$, that is:
 $\neg S_R G = \sqcap S_R G$ and $\neg T_R G = \sqcap T_R G$
- (p5) If G is Boolean, i.e. G in $B(\mathcal{O}_R)$ then, $S_R G = T_R G = G$
- (p6) $S_R(S_R G) = S_R G$; $T_R(T_R G) = T_R G$
- (p7) S_R and T_R are conjugate, i.e. $S_R(T_R G) = T_R G$ and $T_R(S_R G) = S_R G$
- (p8) $S_R G = \sqcap T_R \sqcap G$ and $T_R G = \neg S_R \neg G$
that is, S_R (resp. T_R) is definable in terms of T_R and \sqcap (resp. S_R and \neg)

Proof.

(p4) By (5.) we have (i) $\neg S_R G \subseteq \sqcap S_R G$. To prove the reverse inclusion, we see by (3.) that $\neg \sqcap S_R G \cap \sqcap S_R G = \emptyset$ and by (p3) we infer that $S_R G \cap \sqcap S_R G = \emptyset$. Thus (ii) $\sqcap S_R G \subseteq \neg S_R G$. From (i) and (ii) the result follows. Similar proof for $\neg T_R G = \sqcap T_R G$.

(p5) If G is Boolean (i.e. G in $B(\mathcal{O}_R)$) then, by (5.) we obtain $G \subseteq \neg \neg G = \neg \sqcap G = S_1 G \subseteq G$. By induction on n we obtain $S_R G = G$. Analogous proof for T_R .

(p6) Since $S_R G$ is Boolean then, from (p4) and (p5), we get $S_R(S_R G) = S_R G$. By duality $T_R(T_R G) = T_R G$.

(p7) By (p4), $T_R G$ is Boolean so, via (p6), we have $S_R(T_R G) = S_R G$.

(p8) In fact, by (p2) we get $\sqcap G \subseteq T_R \sqcap G$ then, by (1.) and (5.) we obtain $\sqcap T_R \sqcap G \subseteq \sqcap \sqcap G \subseteq G$. Consequently, $S_R(\sqcap T_R \sqcap G) \subseteq S_R G$ by (p1). Since $\sqcap T_R \sqcap G$ is Boolean, by (p5) we deduce (i) $\sqcap T_R \sqcap G \subseteq S_R G$.

In the opposite way, by (p2) we have $S_R G \subseteq G$. On account of (1.), (p1), (p4) and (p5) we deduce $\top G \subseteq \top S_R G$ and $T_R \top G \subseteq T_R \top S_R G = \top S_R G$. Therefore $T_R \top G \subseteq \top S_R G$ and this implies by (1.) that $\top \top S_R G \subseteq \top T_R \top G$. Thus (ii) $S_R G \subseteq \top T_R \top G$ because $S_R G$ is Boolean. From (i) and (ii) the result holds. By duality, $T_R G = \neg S_R \neg G$. Using "-" we can write: $S_R G = \neg T_R \top G$ and $T_R G = \neg S_R \neg G$

Finally, from the definitions and results already established, we can prove that operators T_R and S_R are respectively a quantifier and a co-quantifier on the bounded distributive lattice \mathcal{O}_R .

Proof. of Theorem 3.1.

(Q0) and (Q1) are (p0) and (p2) respectively.

(Q3) $T_R(G \cup H) = T_R G \cup T_R H$

In fact: $G \subseteq G \cup H$ and $H \subseteq G \cup H$ then by (p1) we get $T_R G \subseteq T_R(G \cup H)$ and $T_R H \subseteq T_R(G \cup H)$. Thus (i) $T_R G \cup T_R H \subseteq T_R(G \cup H)$.

In the other direction, by (p2), $G \subseteq T_R G$ and $H \subseteq T_R H$, then $G \cup H \subseteq T_R G \cup T_R H$. By (p1) we obtain $T_R(G \cup H) \subseteq T_R(T_R G \cup T_R H)$. Since, by (p4), $T_R G$ and $T_R H$ are in $B(\mathcal{O}_R)$ we infer $T_R G \cup T_R H$ is also in $B(\mathcal{O}_R)$. Therefore, by (p5), we conclude (ii) $T_R(G \cup H) \subseteq T_R G \cup T_R H$. From (i) and (ii) we obtain the result.

Interchanging the roles of $T_R H$ and $S_R H$ leads to (Q'3) $S_R(G \cap H) = S_R G \cap S_R H$.

(Q2) $T_R(G \cap T_R H) = T_R G \cap T_R H$

By (p2) we have $G \cap T_R H \subseteq T_R G \cap T_R H$ and as T_R is increasing we get (i) $T_R(G \cap T_R H) \subseteq T_R(T_R G \cap T_R H) = T_R G \cap T_R H$ by (p5) because $T_R G \cap T_R H$ is in $B(\mathcal{O}_R)$.

On the other side, by (p4), $G \cup \neg T_R H = (G \cap (T_R H \cup \neg T_R H)) \cup \neg T_R H = (G \cap T_R H) \cup (G \cap \neg T_R H) \cup \neg T_R H = (G \cap T_R H) \cup \neg T_R H$. By applying T_R of both sides of the equality we obtain: $T_R(G \cup \neg T_R H) = T_R((G \cap T_R H) \cup \neg T_R H)$. On account of (Q3) (twice), we deduce $T_R G \cup T_R \neg T_R H = T_R(G \cap T_R H) \cup T_R \neg T_R H$. Since $\neg T_R H$ is Boolean we infer by (p5) that $T_R G \cup \neg T_R H =$

$T_R(G \cap T_R H) \cup \neg T_R H$. Forming the intersection of both sides of this relation with $T_R H$ we obtain: $(T_R G \cup \neg T_R H) \cap T_R H = (T_R(G \cap T_R H) \cup \neg T_R H) \cap T_R H$. By distributivity $(T_R G \cap T_R H) \cup (\neg T_R H \cap T_R H) =$

$= (T_R(G \cap T_R H) \cap T_R H) \cup (\neg T_R H \cap T_R H)$. By (p4) and (p6) we deduce

(ii) $T_R G \cap T_R H = T_R(G \cap T_R H) \cap T_R H \subseteq T_R(G \cap T_R H)$. From (i) and (ii) we obtain (Q2).

Interchanging the roles of $T_R H$ and $S_R H$ leads to (Q'2) $S_R(G \cup S_R H) = S_R G \cup S_R H$.

This completes the proof of Theorem 3.1.

For every frame (Ob, R) , where R is a preorder on the set Ob , we can consider the system $(\mathcal{O}_R, S_R, T_R)$, where $(\mathcal{O}_R, \cap, \cup, \Rightarrow, \div, \neg, \top, \emptyset, Ob)$ is a complete **H-B**-algebra and S_R and T_R are respectively a co-quantifier and a quantifier on the

distributive lattice \mathcal{O}_R . This system provides a new algebraic structure. According to [15] we could call it a complete bi-monadic **H-B**-algebra and it deserves to be studied by its own interest.

To stay within the scope of our work, that is, to provide a general pattern of pairs of elements to model lattices, the fact of having obtained a quantifier T_R and a co-quantifier S_R leads us to consider pairs of the form $[S_R G, T_R G]$.

To supply a structure of lattice to the collection of these pairs, we are going to introduce, as suggested by ([16], p.160) two new operations \sqcap and \sqcup , for all $G, H \in \mathcal{O}_R$:

$$\begin{aligned} G \sqcap H &= T_R G \cap H \cap (G \cup \neg S_R H) \\ G \sqcup H &= S_R G \cup H \cup (G \cap \neg T_R H) \end{aligned}$$

We can point out that:

Lemma 3.4 *Operations \sqcap and \sqcup have two suitable proprieties:*

$$\begin{array}{l|l} (p9) S_R(G \sqcap H) = S_R G \cap S_R H & (p'9) S_R(G \sqcup H) = S_R G \cup S_R H \\ (p10) T_R(G \sqcap H) = T_R G \cap T_R H & (p'10) T_R(G \sqcup H) = T_R G \cup T_R H \end{array}$$

Proof. For technical convenience, we present (p9) and (p10) in two ways. Thus:

$$\begin{aligned} G \sqcap H &= T_R G \cap H \cap (G \cup \neg S_R H) = (G \cap H) \cup (T_R G \cap H \cap \neg S_R H) \\ G \sqcup H &= (G \cup H) \cap (S_R G \cup H \cup \neg T_R H) = S_R G \cup H \cup (G \cap \neg T_R H) \end{aligned}$$

(p9) Taking into account: ((Q'3) and (p5)), ((p7), (Q'2)), distributivity, ((p2), (p5)), (p4) we deduce, step by step, that:

$$\begin{aligned} S_R(G \sqcap H) &= S_R(T_R G \cap H \cap (G \cup \neg S_R H)) \\ &= S_R(T_R G) \cap S_R H \cap S_R(G \cup \neg S_R H) \\ &= T_R G \cap S_R H \cap (S_R G \cup \neg S_R H) \\ &= (T_R G \cap S_R H \cap S_R G) \cup (T_R G \cap S_R H \cap \neg S_R H) \\ &= (S_R H \cap S_R G) \cup (T_R G \cap S_R H \cap \neg S_R H) = (S_R H \cap S_R G) \cup (T_R G \cap \emptyset) \\ &= S_R G \cap S_R H \end{aligned}$$

(p10) It follows from ((Q3) and (p5)), (Q2), (distributivity and (p6)), ((p1) (twice) and (Q3)), ((p2) and (p4)), (p1) that the following equalities hold:

$$\begin{aligned} T_R(G \sqcap H) &= T_R((G \cap H) \cup (T_R G \cap H \cap \neg S_R H)) \\ &= T_R(G \cap H) \cup T_R(T_R G \cap H \cap \neg S_R H) \\ &= T_R(G \cap H) \cup (T_R T_R G \cap T_R H \cap \neg S_R H) \\ &= (T_R(G \cap H) \cup T_R G) \cap (T_R(G \cap H) \cup T_R H) \cap (T_R(G \cap H) \cup (T_R \neg S_R H)) \\ &= T_R G \cap T_R H \cap T_R((G \cap H) \cup \neg S_R H) \\ &= T_R G \cap T_R H \cap T_R(G \cup \neg S_R H) \cap (H \cup \neg S_R H) \\ &= T_R G \cap T_R H \cap T_R((G \cup \neg S_R H) \cap Ob) \\ &= T_R G \cap T_R H. \end{aligned}$$

In like manner we can establish that: (p'9) $S_R(G \uplus H) = S_R G \cup S_R H$ and (p'10) $T_R(G \uplus H) = T_R G \cup T_R H$

This completes the proof of Lemma 3.4.

Since $S_R G \subseteq G \subseteq T_R G$ we can consider the collection B^* of ordered pairs of the form $[S_R G, T_R G]$, for all $G \in \mathcal{O}_R$. Our next task will be to endow B^* with a lattice structure.

Theorem 3.5 *For every frame (Ob, R) , where R is a preorder on the set Ob , the system $(B^*, \cap, \cup, \emptyset, Ob)$ is a distributive lattice with zero and unit.*

Proof. Since $S_R G \subseteq G \subseteq T_R G$, we can consider -following the ideas above- the collection B^* of all pairs $[S_R G, T_R G]$. We will define on B^* the following pointwise operations \cap and \cup :

$$\begin{aligned} [S_R G, T_R G] \cap [S_R H, T_R H] &= [S_R(S_R G \cap S_R H), T_R(T_R G \cap T_R H)] \\ &= [(S_R G \cap S_R H), (T_R G \cap T_R H)] \\ &= [S_R(G \cap H), T_R(G \cap H)], \text{ on account of Lemma 3.4.} \\ [S_R G, T_R G] \cup [S_R H, T_R H] &= [S_R(S_R G \cup S_R H), T_R(T_R G \cup T_R H)] \\ &= [(S_R G \cup S_R H), (T_R G \cup T_R H)] \\ &= [S_R(G \uplus H), T_R(G \uplus H)], \text{ by Lemma 3.4.} \\ 0 &= [\emptyset, \emptyset] ; 1 = [Ob, Ob] \end{aligned}$$

This completes the proof of Theorem 3.5.

In this manner we have showed that the bounded distributive lattice $(B^*, \cap, \cup, \emptyset, Ob)$ is a sublattice of $\mathcal{P}(\mathcal{O}_R) \times \mathcal{P}(\mathcal{O}_R)$.

In the case that R is an equivalence relation, the H - B -algebra is Boolean, the quantificateurs S_R and T_R are related by the Boolean negation, the boundaries are empty, and we obtain classical results.

4. Conclusion

The suitable connection between pairs of subsets of a given ‘concrete’ universe Ob and the abstract structures to represent them has valuable advantages: to avoid duplications of questions and proofs, and knowing which property is general or depending on a particular situation.

Here, starting from an universe Ob enriched with a preorder R we have considered a ‘concrete’ Rauszer Boolean algebra $\mathcal{B} = (\mathcal{P}(Ob), I_R, C_R)$. On the complete **H-B**-subalgebra of closed elements $(\mathcal{O}_R, \cap, \cup, \Rightarrow, \div, \neg, \Gamma, \emptyset, Ob)$, the combination of two negations in two different ways, allows us by iteration and passing to the limit to obtain two Boolean operators: a quantifier T_R and a co-quantifier S_R .

Pairs of the form $[S_R G, T_R G]$, for $G \in \mathcal{O}_R$, provide a general framework for the distributive lattice part, which is common to many algebraic systems related to non-classical logics. In the search for algebraic models, this is the first common step.

The many questions asked in the context of information systems, in particular the approximation of a set by a pair of sets, lead readers to request about mathematical foundations of the specific applied results and help the advance of theoretical research.

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References

- [1] G. Birkhoff, *An extended arithmetic*, Duke Math. J. **3** (1937), 311–316.
- [2] R. Cignoli, *Quantifiers on distributive lattices*, Discrete Mathematics **96** (1991), 183–197.
- [3] P. R. Halmos, *Algebraic logic I. Monadic Boolean algebras*, Compositio Mathematica **12**, 217–249, 1955.
- [4] L. Iturrioz, *Sur les algèbres de Heyting-Brouwer*, Bull. Acad. Polon. Sci. **24** (1976), 551–558.
- [5] L. Iturrioz, *Rough sets and three-valued structures*, chapter 33, in: Logic at Work, Essays dedicated to the memory of Helena Rasiowa, ed. by E. Orłowska, Studies in Fuzziness and Soft Computing, Physica-Verlag, 1999, 596–603. arXiv:1905.09926v1 [math.LO] 19 May 2019.
- [6] L. Iturrioz, *Two representation theorems of three-valued structures by means of binary relations*, arXiv:0710.1007v1 [cs.DM] 4 Oct 2007, 11 pp.
- [7] L. Iturrioz, *Algebraic Structures Based on a Chain of Two Agents*, Essays dedicated to the memory of Professor Grigore C. Moisil, Multiple-Valued Logic. An International Journal **6** (2001), 137–155.
- [8] L. Iturrioz, *Two old representation theorems and Information Systems*, Logic Journal of IGPL, article dedicated to the memory of Cecilia Rauszer, Oxford University Press **20(6)** (2012), 1038-1045.
- [9] L. Iturrioz, *About a ‘concrete’ Rauszer Boolean algebra generated by a pre-order*, arXiv:1905.09928v1 [math.LO] 19 May 2019, 25 pp.
- [10] F.W. Lawvere, *Intrinsic Co-Heyting Boundaries and the Leibniz Rule in Certain Toposes*, Lectures Notes in Mathematics **1488** (1991), pp. 279–281.

- [11] J.C.C. McKinsey and A. Tarski: *On closed elements in closure algebras*, *Annals of Mathematics* **47** (1946), 122–162.
- [12] Gr. C. Moisil, *Recherches sur les logiques non-chrysiennes*, *Annales Scientifiques de l'Université de Jassy* **26** (1940), 431–466.
- [13] Gr. C. Moisil, *Notes sur les logiques non-chrysiennes*, *Annales Scientifiques de l'Université de Jassy* **27** (1941), 86–98.
- [14] A. Monteiro and H. Ribeiro, *L'opération de fermeture et ses invariants dans les systèmes partiellement ordonnés*, *Portugaliae Mathematica* **3(3)** (1942), 171–184.
- [15] A. Monteiro and O. Varsavsky, *Algebras de Heyting monádicas*, *Actas de la X Jornadas de la Unión Matemática Argentina*, Bahía Blanca, (1957), 52–62. A French translation is published as *Notas de Lógica Matemática n°1-2-3*, Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, 1974, 18 pp.
- [16] L. Monteiro and L. Gonzalez Coppola, *Sur une construction des Algèbres de Lukasiewicz Trivalentes*, *Portugaliae Mathematica* **23** (1964), 157–167. A preprint is published as *Notas de Lógica Matemática n°17*, Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina, 1964, 17 pages.
- [17] C. Rauszer, *Representation theorems for semi-Boolean algebras, I*, *Bull. Acad. Polon. Sci.* **19** (1971), 881–887.
- [18] C. Rauszer, *Semi-Boolean algebras and their applications to intuitionistic logic with dual operations*, *Fundamenta Mathematica* **83** (1974), 219–249.
- [19] C. Rauszer, *An algebraic approach to the Heyting-Brouwer predicate calculus*, *Fundamenta Mathematica* **96** (1977), 127–135.
- [20] G. E. Reyes and H. Zolfaghari, *Bi-Heyting algebras, toposes and modalities*, *J. Philos. Log.* **25** (1996), 25–43.
- [21] E. SanJuan, *Heyting algebras with Boolean operators for rough sets and information retrieval applications*, *Discrete Applied Mathematics* **156** (2008), 967–983.
- [22] V. Sofronie-Stokkermans, *Automated theorem proving by resolution in non-classical logics*, *Ann. Math. Artif. Intell* **49** (2007), 221–252. See also, JIM'2003, Fourth International Conference, Knowledge Discovery and Discrete Mathematics, eds. by M. Nadif, A. Napoli, E. SanJuan, A. Sigayret, Metz University, September, 2003, 151–167.