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Improving semi-groups bounds with resolvent estimates

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Abstract

The purpose of this paper is to revisit the proof of the Gearhardt-Prüss-Hwang-Greiner theorem for a semigroup $S(t)$, following the general idea of the proofs that we have seen in the literature and to get an explicit estimate on $\|S(t)\|$ in terms of bounds on the resolvent of the generator. A first version of this paper was presented by the two authors in ArXiv (2010) together with applications in semi-classical analysis and a part of these results has been published later in two books written by the authors. Our aim is to present new improvements, partially motivated by a paper of D. Wei. On the way we discuss optimization problems confirming the optimality of our results.

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1 Introduction

Let \mathcal{H} be a complex Hilbert space and let $[0, +\infty[\ni t \mapsto S(t) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ be a strongly continuous semigroup with $S(0) = I$. Recall that by the Banach-Steinhaus theorem, $\sup_J \|S(t)\| =: m(J)$ is bounded for every compact interval $J \subset [0, +\infty[$. Using the semigroup property it follows easily that there exist $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $S(t)$ has the property

$$P(M, \omega_0) : \quad \|S(t)\| \leq Me^{\omega_0 t}, \quad t \geq 0. \quad (1.1)$$

Let A be the generator of the semigroup (so that formally $S(t) = \exp tA$) and recall (cf. [4], Chapter II or [10]) that A is closed and densely defined. We also recall ([4], Theorem II.1.10) that

$$(z - A)^{-1} = \int_0^\infty S(t)e^{-tz} dt, \quad \|(z - A)^{-1}\| \leq \frac{M}{\Re z - \omega_0}, \quad (1.2)$$

when $P(M, \omega_0)$ holds and z belongs to the open half-plane $\Re z > \omega_0$.

According to the Hille-Yosida theorem ([4], Th. II.3.5), the following three statements are equivalent when $\omega \in \mathbb{R}$:

- $P(1, \omega)$ holds.
- $\|(z - A)^{-1}\| \leq (\Re z - \omega)^{-1}$, when $z \in \mathbb{C}$ and $\Re z > \omega$.
- $\|(\lambda - A)^{-1}\| \leq (\lambda - \omega)^{-1}$, when $\lambda \in]\omega, +\infty[$.

Here we may notice that we get from the special case $\omega = 0$ to general ω by passing from $S(t)$ to $\tilde{S}(t) = e^{-\omega t} S(t)$.

Also recall that there is a similar characterization of the property $P(M, \omega)$ when $M > 1$, in terms of the norms of all powers of the resolvent. This is the Feller-Miyadera-Phillips theorem ([4], Th. II.3.8). Since we need all powers of the resolvent, the practical usefulness of that result is less evident.

We next recall the Gearhardt-Prüss-Hwang-Greiner theorem, see [4], Theorem V.I.11, [15], Theorem 19.1:

Theorem 1.1

- (a) Assume that $\|(z - A)^{-1}\|$ is uniformly bounded in the half-plane $\Re z \geq \omega$. Then there exists a constant $M > 0$ such that $P(M, \omega)$ holds.
- (b) If $P(M, \omega)$ holds, then for every $\alpha > \omega$, $\|(z - A)^{-1}\|$ is uniformly bounded in the half-plane $\Re z \geq \alpha$.

The purpose of this paper is to revisit the proof of (a), following the general idea of the proofs that we have seen in the literature and to get an explicit t dependent estimate on $e^{-\omega t}\|S(t)\|$, implying explicit bounds on M .

This idea is essentially to use that the resolvent and the inhomogeneous equation $(\partial_t - A)u = w$ in exponentially weighted spaces are related via Fourier-Laplace transform and we can use Plancherel's formula. Variants of this simple idea have also been used in more concrete situations. See [1, 6, 9, 11] and a very complete overview of the possible applications in [2]. In this paper, we will obtain general results of the form:

If $\|S(t)\| \leq m(t)$ for some positive function m , and if we have a certain bound on the resolvent of A , then $\|S(t)\| \leq \tilde{m}(t)$ and hence $\|S(t)\| \leq \min(m(t), \tilde{m}(t))$ for a new function \tilde{m} that can be explicitly described.

Note that we can extend the conclusion of (a). If the property (a) is true for some ω then it is automatically true for some $\omega' < \omega$. We recall indeed the following

Lemma 1.2

If for some $r(\omega) > 0$, $\|(z - A)^{-1}\| \leq \frac{1}{r(\omega)}$ for $\Re z > \omega$, then for every $\omega' \in]\omega - r(\omega), \omega]$ we have

$$\|(z - A)^{-1}\| \leq \frac{1}{r(\omega) - (\omega - \omega')}, \quad \Re z > \omega'.$$

Let

$$\omega_1 = \inf\{\omega \in \mathbb{R}; \{z \in \mathbb{C}; \Re z > \omega\} \subset \rho(A) \text{ and } \sup_{\Re z > \omega} \|(z - A)^{-1}\| < \infty\}.$$

For $\omega > \omega_1$, we may define $r(\omega)$ by

$$\frac{1}{r(\omega)} = \sup_{\Re z > \omega} \|(z - A)^{-1}\|. \tag{1.3}$$

Then $r(\omega)$ is an increasing function of ω ; for every $\omega \in]\omega_1, \infty[$, we have $\omega - r(\omega) \geq \omega_1$ and for $\omega' \in [\omega - r(\omega), \omega]$ we have

$$r(\omega') \geq r(\omega) - (\omega - \omega').$$

Remark 1.3

Under the assumption $P(M, \omega_0)$ in (1.1), we already know from (1.2) that $\|(z - A)^{-1}\|$ is uniformly bounded in the half-plane $\Re z \geq \beta$, if $\beta > \omega_0$. If $\alpha \leq \omega_0$, we see that $\|(z - A)^{-1}\|$ is uniformly bounded in the half-plane $\Re z \geq \alpha$, provided that

- we have this uniform boundedness on the line $\Re z = \alpha$,
- A has no spectrum in the half-plane $\Re z \geq \alpha$,
- $\|(z - A)^{-1}\|$ does not grow too wildly in the strip $\alpha \leq \Re z \leq \beta$:

$$\|(z - A)^{-1}\| \leq \mathcal{O}(1) \exp(\mathcal{O}(1) \exp(k|\Im z|)),$$

where $k < \pi/(\beta - \alpha)$.

We then also have

$$\sup_{\Re z \geq \alpha} \|(z - A)^{-1}\| = \sup_{\Re z = \alpha} \|(z - A)^{-1}\|. \quad (1.4)$$

This follows from the subharmonicity of $\log \|(z - A)^{-1}\|$, basically Hadamard's theorem (or the one of Phragmén-Lindelöf in exponential coordinates).

The main result in [8] was:

Theorem 1.4

We make the assumptions of Theorem 1.1, (a) and let $r(\omega) > 0$ be as in (1.3). Let $m(t) \geq \|S(t)\|$ be a continuous positive function. Then for all $t, a, b > 0$, such that $t \geq a + b$, we have

$$\|S(t)\| \leq \frac{e^{\omega t}}{r(\omega) \left\| \frac{1}{m} \right\|_{e^{-\omega \cdot} L^2(0, a)} \left\| \frac{1}{m} \right\|_{e^{-\omega \cdot} L^2(0, b)}}. \quad (1.5)$$

Here the norms are always the natural ones obtained from \mathcal{H} , L^2 , thus for instance $\|S(t)\| = \|S(t)\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})}$, if u is a function on \mathbb{R} with values in \mathbb{C} or in \mathcal{H} , $\|u\|$ denotes the natural L^2 norm, when the norm is taken over a subset J of \mathbb{R} , this is indicated with a " $L^2(J)$ ". In (1.5) we also have the natural norm in the exponentially weighted space $e^{-\omega \cdot} L^2(0, a)$ and similarly with b instead of a ; $\|f\|_{e^{-\omega \cdot} L^2(0, a)} = \|e^{\omega \cdot} f(\cdot)\|_{L^2(0, a)}$.

The proof of these theorems was first presented in [8] and later published in the books of the authors. In [16], Dongyi Wei, motivated by our first version [8] has proved the following theorem:

Theorem 1.5 *Let $H = -A$ be an m -accretive operator in a Hilbert space \mathcal{H} . Then we have,*

$$\|S(t)\| \leq e^{-r(0)t + \frac{\pi}{2}}, \quad \forall t \geq 0. \quad (1.6)$$

Our aim is to deduce and improve these two theorems as a consequence of a unique basic estimate that we present now. Let Φ satisfy

$$0 \leq \Phi \in C^1([0, +\infty[) \text{ with } \Phi(0) = 0 \text{ and } \Phi(t) > 0 \text{ for } t > 0, \quad (1.7)$$

and assume that Ψ has the same properties. (By a density argument we can replace $C^1([0, +\infty[)$ in (1.7) by the space of locally Lipschitz functions on $[0, +\infty[$.) For $t > 0$, let ι_t be the reflection with respect to $t/2$: $\iota_t u(s) = u(t - s)$. With this notation, we have the following theorem.

Theorem 1.6 *Under the assumptions of Theorem 1.4, for any Φ and Ψ satisfying (1.7) and for any $\epsilon_1, \epsilon_2 \in \{-, +\}$, we have*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq e^{\omega t} \frac{\|(r(\omega)^2 \Phi^2 - \Phi'^2)_{\epsilon_1}^{\frac{1}{2}} m\|_{e^{\omega \cdot} L^2([0, t])} \|(r(\omega)^2 \Psi^2 - \Psi'^2)_{\epsilon_2}^{\frac{1}{2}} m\|_{e^{\omega \cdot} L^2([0, t])}}{\int_0^t (r(\omega)^2 \Phi^2 - \Phi'^2)_{\epsilon_1}^{\frac{1}{2}} (r(\omega)^2 \iota_t \Psi^2 - \iota_t \Psi'^2)_{\epsilon_2}^{\frac{1}{2}} ds}. \quad (1.8)$$

Here for $a \in \mathbb{R}$, $a_+ = \max(a, 0)$ and $a_- = \max(-a, 0)$.

We now discuss the consequences of this theorem that can be obtained with suitable choices of $\Phi, \Psi, \epsilon_1, \epsilon_2$.

The first one is a Wei like version of our previous Theorem 1.4.

Theorem 1.7 *For positive a and b , we have, for $t > a + b$,*

$$\|S(t)\| \leq \frac{e^{\omega t - r(\omega)(t-a-b)}}{r(\omega)} \frac{1}{\|\frac{1}{m}\|_{e^{-\omega \cdot} L^2(0, a)} \|\frac{1}{m}\|_{e^{-\omega \cdot} L^2(0, b)}}. \quad (1.9)$$

In the case of Wei's theorem we have $\omega = 0$, $m = 1$. With $b = a$ we first get

$$\|S(t)\| \leq \frac{1}{ar(0)} \exp -r(0)(t - 2a), \quad t > 2a.$$

Minimization with respect to a leads to $ar(0) = \frac{1}{2}$ and consequently to

$$\|S(t)\| \leq 2e \exp -r(0)t, \quad t > \frac{1}{r(0)},$$

which is not quite as sharp as (1.6), since $e^{\pi/2} \approx 4.81$, $2e \approx 5.44$.

We will show that a finer approach will permit to recover (1.6) and generalize it to more general m 's. We assume

$$0 < m \in C^1([0, +\infty[). \quad (1.10)$$

An important step will be to prove (we assume $\omega = 0$, $r(0) = 1$) as a consequence of Theorem 1.6 with $\epsilon_1 = -$ and $\epsilon_2 = +$, the following key proposition

Proposition 1.8 *Assume that $\omega = 0$, $r(\omega) = 1$. Let a, b positive. Then for $t \geq a + b$,*

$$\|S(t)\| \leq \exp -(t - a - b) \frac{(\inf_u \int_0^a m(s)^2 (u'(s)^2 - u^2(s))_+ ds)^{1/2}}{(\sup_\theta \int_0^b \frac{1}{m^2} (\theta(s)^2 - \theta'(s)^2) ds)^{1/2}}, \quad (1.11)$$

where

- $u \in H^1(]0, a[)$ satisfies $u(0) = 0$, $u(a) = 1$;
- $\theta \in H^1(]0, b[)$ satisfies $\theta(b) = 1$ and $|\theta'| \leq \theta$.

This proposition implies rather directly Theorem 1.7 in the following way. We first observe the trivial lower bound (take $\theta(s) = 1$)

$$\sup_\theta \int_0^b \frac{1}{m^2} (\theta(s)^2 - \theta'(s)^2) ds \geq \int_0^b \frac{1}{m^2} ds. \quad (1.12)$$

A more tricky argument based on the equality case in Cauchy-Schwarz' inequality (see Subsection 3.6 for details), gives

$$\inf_u \int_0^a m(s)^2 (u'(s)^2 - u(s)^2)_+ ds \leq \inf_u \int_0^a m(s)^2 u'(s)^2 ds \leq 1 / \int_0^a \frac{1}{m^2} ds, \quad (1.13)$$

Combining (1.8) with (1.12) and (1.13) gives directly (1.9) in the case $\omega = 0$, $r(\omega) = 1$. A rescaling argument (which will be detailed in Subsection 4.1) then gives (1.9) in general.

To refine the analysis of the right hand side of (1.11), we have to analyze for positive a and b the quantities

$$I_{\inf}(a) := \inf_u \int_0^a m(s)^2 (u'(s)^2 - u^2(s))_+ ds$$

and

$$J_{\max}(b) := \sup_\theta \int_0^b \frac{1}{m^2} (\theta(s)^2 - \theta'(s)^2) ds,$$

where u and θ satisfy the above conditions. This will be the main object of Section 3. To present some of the results in this introduction, we consider the Dirichlet-Robin realization $K_{m,a}^{DR}$ of the operator

$$K_m := -\frac{1}{m^2} \partial_s \circ m^2 \partial_s - 1, \quad (1.14)$$

in the interval $]0, a[$. The Dirichlet-Robin condition is

$$u(0) = 0, \quad u'(a) = u(a), \quad (1.15)$$

and we define the domain of $K_{m,a}^{DR}$ by

$$\mathcal{D}(K_{m,a}^{DR}) = \{u \in H^2(]0, a[); u \text{ satisfies (1.15)}\}.$$

We note that this realization is a self-adjoint operator on $L^2(]0, a[, m^2 ds)$, bounded from below and with purely discrete spectrum.

Let $\lambda^{DR}(a, m)$ denote the lowest eigenvalue of $K_{m,a}^{DR}$. Then $\lambda^{DR}(a, m) > 0$ when $a > 0$ is small enough. We define

$$a^* = a^*(m) = \sup\{\tilde{a} \in]0, \infty[; \lambda^{DR}(a, m) > 0 \text{ for } 0 < a < \tilde{a}\}, \quad (1.16)$$

so that $a^*(m) \in]0, +\infty[$. Since $\lambda^{DR}(a, m)$ is a continuous function of a , we have in the case $a^* < \infty$ that

$$\lambda^{DR}(a^*, m) = 0, \quad \lambda^{DR}(a, m) > 0 \text{ for } 0 < a < a^*.$$

We introduce the condition

$$\liminf_{s \rightarrow +\infty} \mu(s) > -1 \text{ with } \mu := m'/m. \quad (1.17)$$

Under this condition, we will show that $a^*(m) < +\infty$. We will show in Section 3 that if on $]0, a^*[$

$$\psi_0(s; m) = \psi_0 := u'_0(s)/u_0(s), \quad 0 < s < a^*, \quad (1.18)$$

where u_0 is the first eigenfunction of the DR -problem in $]0, a^*[$, then:

Theorem 1.9 *Let $\omega = 0$, $r(\omega) = 1$. When $a, b \in]0, +\infty[\cap]0, a^*$ and $t > a + b$, we have*

$$\|e^t S(t)\| \leq \exp(a + b) m(a) m(b) \psi_0(a)^{\frac{1}{2}} \psi_0(b)^{\frac{1}{2}}. \quad (1.19)$$

In particular, when $a^ < +\infty$, we have*

$$\|e^t S(t)\| \leq \exp(2a^*) m(a^*)^2, \quad t > 2a^*. \quad (1.20)$$

This theorem is the analog of Wei's theorem for general weights m .

By a general procedure described in Subsection 4.1, we have actually a more general statement. We consider \hat{A} with the same properties as A where the hat's are introduced to make easier the transition between the particular case above to the general case below. As before, we introduce $\hat{\omega}$ and $\hat{r} = \hat{r}(\hat{\omega})$.

Theorem 1.10 *Let $\hat{S}(\hat{t}) = e^{\hat{t}\hat{A}}$ satisfying*

$$\|\hat{S}(\hat{t})\| \leq \hat{m}(\hat{t}), \quad \forall \hat{t} > 0.$$

Then there exist uniquely defined¹ $\hat{a}^ := \hat{a}^*(\hat{m}, \hat{\omega}, \hat{r}) > 0$ and $\hat{\psi} := \hat{\psi}(\cdot; \hat{m}, \hat{\omega}, \hat{r})$ on $]0, \hat{a}^*[$ with the same general properties as above such that, if $\hat{a}, \hat{b} \in]0, +\infty[\cap]0, \hat{a}^*$ and $\hat{t} > \hat{a} + \hat{b}$, we have*

$$\|S(\hat{t})\| \leq \exp\left((\hat{\omega} - \hat{r}(\omega))(\hat{t} - (\hat{a} + \hat{b}))\right) \hat{m}(\hat{a}) \hat{m}(\hat{b}) \hat{\psi}(\hat{a})^{\frac{1}{2}} \hat{\psi}(\hat{b})^{\frac{1}{2}}. \quad (1.21)$$

¹The definition will be given in Subsection 4.1

Moreover, when $\hat{a}^* < +\infty$, the estimate is optimal for $\hat{a} = \hat{b} = \hat{a}^*$ and reads

$$\|\hat{S}(\hat{t})\| \leq \exp((\hat{\omega} - \hat{r})(\hat{t} - 2\hat{a}^*)) \hat{m}(\hat{a}^*)^2, \quad t > 2\hat{a}^*. \quad (1.22)$$

Moreover

$$\hat{a}^*(\hat{m}, \hat{\omega}) = \hat{r} a^*(e^{-\hat{\omega} \cdot \hat{m}}), \quad \hat{\psi}(\hat{s}; \hat{m}, \hat{\omega}, \hat{r}) = \psi_0(\hat{r} \hat{s}; e^{-\hat{\omega} \cdot \hat{m}}).$$

Theorem 1.7, Proposition 1.8 and Theorem 1.9 are based in Section 4 on Theorem 1.6, with the choice $(\epsilon_1, \epsilon_2) = (+, -)$ which is proved in Section 2. In the appendix we explore the consequences of the choice $(\epsilon_1, \epsilon_2) = (+, +)$. In this case it turned out to be more difficult to reach equally clear applications.

2 Proof of Theorem 1.6

2.1 Flux

Let $u(t) \in C^1([0, +\infty[; \mathcal{H}) \cap C^0([0, +\infty[; \mathcal{D}(A))$, $u^*(t) \in C^1(]-\infty, T]; \mathcal{H}) \cap C^0(]-\infty, T]; \mathcal{D}(A^*))$, solve $(A - \partial_t)u = 0$ and $(A^* + \partial_t)u^* = 0$ on $[0, +\infty[$ and $]-\infty, T]$ respectively. Then the flux (or Wronskian) $[u(t)|u^*(t)]$ is constant on $[0, T]$ as can be seen by computing the derivative with respect to t . Here we use the notations $[\cdot|\cdot]_{\mathcal{H}}$ and $|\cdot|_{\mathcal{H}}$ for the ‘‘point-wise’’ scalar product and norm in \mathcal{H} .

2.2 L^2 estimate

Write $L_{\phi}^2(I) = L^2(I; e^{-2\phi} dt) = e^{\phi} L^2(I)$, $\|u\|_{\phi} = \|u\|_{\phi, I} = \|u\|_{L_{\phi}^2(I)}$, where I is an interval and our functions take values in \mathcal{H} . (Our vector valued functions will be norm continuous, so we avoid the formal definition of these spaces with the Lebesgue integral and manage with the Riemann integral.) By Parseval-Plancherel, the Laplace transform

$$\mathcal{L}u(\tau) = \int e^{-t\tau} u(t) dt$$

gives a unitary map from $L_{\omega}^2(\mathbb{R})$ to $L^2(\Gamma_{\omega}; d\Im\tau/(2\pi))$, where $\Gamma_{\omega} \subset \mathbb{C}$ denotes the line given by $\Re\tau = \omega$ and ω is real. By applying \mathcal{L} we see that $(A - \partial_t)^{-1} : L_{\omega}^2(\mathbb{R}) \rightarrow L_{\omega}^2(\mathbb{R})$ is well-defined and bounded of norm $1/r(\omega)$.

Consider $(A - \partial_t)u = 0$ on $[0, +\infty[$ with $u \in L_{\omega}^2([0, +\infty[)$. Let Φ satisfy (1.7) and add temporarily the assumption that $\Phi(s)$ is constant for $s \gg 0$. Then Φu , $\Phi' u$ can be viewed as elements of $L_{\omega}^2(\mathbb{R})$ and from

$$(A - \partial_t)\Phi u = -\Phi' u,$$

we get, by the definition of $r(\omega)$,

$$\|\Phi u\|_{\omega} \leq \frac{1}{r(\omega)} \|\Phi' u\|_{\omega},$$

or, taking the square,

$$((r(\omega)^2\Phi^2 - \Phi'^2)u|u)_\omega \leq 0.$$

This can be rewritten as

$$((r(\omega)^2\Phi^2 - \Phi'^2)_+u|u)_\omega \leq ((r(\omega)^2\Phi^2 - \Phi'^2)_-u|u)_\omega, \quad (2.1)$$

or

$$\|(r(\omega)^2\Phi^2 - \Phi'^2)_+^{1/2}u\|_\omega \leq \|(r(\omega)^2\Phi^2 - \Phi'^2)_-^{1/2}u\|_\omega. \quad (2.2)$$

By a limiting procedure, we see that (2.1), (2.2) remain valid without the assumption that Φ be constant near $+\infty$.

Writing $\Phi = e^\phi$, $\phi \in C^1(]0, +\infty[)$, $\phi(t) \rightarrow -\infty$ when $t \rightarrow 0$, we have

$$r(\omega)^2\Phi^2 - \Phi'^2 = (r(\omega)^2 - \phi'^2)e^{2\phi},$$

and (2.1), (2.2) become

$$((r(\omega)^2 - \phi'^2)_+u|u)_{\omega \cdot -\phi} \leq ((r(\omega)^2 - \phi'^2)_-u|u)_{\omega \cdot -\phi}, \quad (2.3)$$

$$\|(r(\omega)^2 - \phi'^2)_+^{1/2}u\|_{\omega \cdot -\phi} \leq \|(r(\omega)^2 - \phi'^2)_-^{1/2}u\|_{\omega \cdot -\phi}. \quad (2.4)$$

We have in mind the case when $r(\omega)^2 - (\phi')^2 > 0$ away from a bounded neighborhood of $t = 0$.

Let $S(t) = e^{tA}$, $t \geq 0$ and let $m(t) > 0$ be a continuous function such that

$$\|S(t)\| \leq m(t), \quad t \geq 0. \quad (2.5)$$

Then we get

$$\|(r(\omega)^2 - \phi'^2)_+^{1/2}u\|_{\omega \cdot -\phi} \leq \|(r(\omega)^2 - \phi'^2)_-^{1/2}m\|_{\omega \cdot -\phi}|u(0)|_{\mathcal{H}}. \quad (2.6)$$

Note that we have also trivially

$$\|(r(\omega)^2 - \phi'^2)_-^{1/2}u\|_{\omega \cdot -\phi} \leq \|(r(\omega)^2 - \phi'^2)_-^{1/2}m\|_{\omega \cdot -\phi}|u(0)|_{\mathcal{H}}. \quad (2.7)$$

We get the same bound for the forward solution of $A^* - \partial_t$ and, after changing the orientation of time, for the backward solution of $A^* + \partial_t = (A - \partial_t)^*$. Then for $u^*(s)$, solving

$$(A^* + \partial_s)u^*(s) = 0, \quad s \leq t,$$

with $u^*(t)$ prescribed, we get

$$\|(r(\omega)^2 - \iota_t\phi'^2)_+^{1/2}u^*\|_{\omega(t-\cdot)-\iota_t\phi} \leq \|(r(\omega)^2 - \iota_t\phi'^2)_-^{1/2}\iota_t m\|_{\omega(t-\cdot)-\iota_t\phi}|u^*(t)|_{\mathcal{H}},$$

where $\iota_t\phi$ and $\iota_t m$ denote the compositions of ϕ and m respectively with the reflection ι_t in $t/2$ so that

$$\iota_t m(s) = m(t-s), \quad \iota_t\phi(s) = \phi(t-s).$$

More generally, we can replace ϕ by ψ with the same properties (see (1.7)) and consider $\Psi = \exp \psi$.

Note that we have

$$\|(r(\omega)^2 - \iota_t\psi'^2)_+^{1/2}u^*\|_{\omega(t-\cdot)-\iota_t\psi} \leq \|(r(\omega)^2 - \psi'^2)_-^{1/2}m\|_{\omega \cdot -\psi}|u^*(t)|_{\mathcal{H}}. \quad (2.8)$$

and also trivially

$$\|(r(\omega)^2 - \iota_t\psi'^2)_-^{1/2}u^*\|_{\omega(t-\cdot)-\iota_t\psi} \leq \|(r(\omega)^2 - \psi'^2)_-^{1/2}m\|_{\omega \cdot -\psi}|u^*(t)|_{\mathcal{H}}. \quad (2.9)$$

2.3 From L^2 to L^∞ bounds

In order to estimate $|u(t)|_{\mathcal{H}}$ for a given $u(0)$ it suffices to estimate $[u(t)|u^*(t)]_{\mathcal{H}}$ for arbitrary $u^*(t) \in \mathcal{H}$. Extend $u^*(t)$ to a backward solution $u^*(s)$ of $(A^* + \partial_s)u^*(s) = 0$, so that

$$[u(s)|u^*(s)]_{\mathcal{H}} = [u(t)|u^*(t)]_{\mathcal{H}}, \quad \forall s \in [0, t].$$

Let $M = M_t : [0, t] \rightarrow [0, +\infty[$ have mass 1:

$$\int_0^t M(s)ds = 1. \quad (2.10)$$

Then

$$|[u(t)|u^*(t)]_{\mathcal{H}}| = \left| \int_0^t M(s)[u(s)|u^*(s)]_{\mathcal{H}}ds \right| \leq \int_0^t M(s)|u(s)|_{\mathcal{H}}|u^*(s)|_{\mathcal{H}}ds. \quad (2.11)$$

Let $\epsilon_1, \epsilon_2 \in \{-, +\}$. Assume that

$$\text{supp } M \subset \{s; \epsilon_1(r(\omega)^2 - \phi'(s)^2) > 0, \epsilon_2(r(\omega)^2 - \iota_t \psi'(s)^2) > 0\}. \quad (2.12)$$

Then multiplying and dividing with suitable factors in the last member of (2.11), we get

$$\begin{aligned} |[u(t)|u^*(t)]_{\mathcal{H}}| &\leq e^{\omega t} \int_0^t \frac{M(s)e^{-\phi(s) - \iota_t \psi(s)}}{(r(\omega)^2 - \phi'(s)^2)_{\epsilon_1}^{\frac{1}{2}} (r(\omega)^2 - \iota_t \psi'(s)^2)_{\epsilon_2}^{\frac{1}{2}}} \times \\ &\quad \times e^{\phi(s) - \omega s} (r(\omega)^2 - \phi'(s)^2)_{\epsilon_1}^{\frac{1}{2}} |u(s)|_{\mathcal{H}} \times \\ &\quad \times e^{\iota_t \psi(s) - \omega(t-s)} (r(\omega)^2 - \iota_t \psi'(s)^2)_{\epsilon_2}^{\frac{1}{2}} |u^*(s)|_{\mathcal{H}} ds \\ &\leq e^{\omega t} \sup_{[0, t]} \frac{M e^{-\phi - \iota_t \psi}}{(r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} (r(\omega)^2 - \iota_t \psi'^2)_{\epsilon_2}^{\frac{1}{2}}} \times \\ &\quad \times \|(r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} u\|_{\omega, -\phi} \|(r(\omega)^2 - \iota_t \psi'^2)_{\epsilon_2}^{\frac{1}{2}} u^*\|_{\omega(t-), -\iota_t \psi}. \end{aligned}$$

Using (2.6), (2.8) when $\epsilon_j = +$ or (2.7), (2.9) when $\epsilon_j = -$, we get

$$\begin{aligned} |[u(t)|u^*(t)]_{\mathcal{H}}| &\leq e^{\omega t} \sup_{[0, t]} \frac{M e^{-\phi - \iota_t \psi}}{(r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} (r(\omega)^2 - \iota_t \psi'^2)_{\epsilon_2}^{\frac{1}{2}}} \times \\ &\quad \times \|(r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} m\|_{\omega, -\phi} \|(r(\omega)^2 - \psi'^2)_{\epsilon_2}^{\frac{1}{2}} m\|_{\omega, -\psi} |u(0)|_{\mathcal{H}} |u^*(t)|_{\mathcal{H}}. \end{aligned}$$

Choosing $u^*(t) = u(t)$, gives

$$\begin{aligned} |u(t)|_{\mathcal{H}} &\leq e^{\omega t} \sup_{[0, t]} \frac{M e^{-\phi - \iota_t \psi}}{(r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} (r(\omega)^2 - \iota_t \psi'^2)_{\epsilon_2}^{\frac{1}{2}}} \times \\ &\quad \times \|(r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} m\|_{\omega, -\phi} \|(r(\omega)^2 - \psi'^2)_{\epsilon_2}^{\frac{1}{2}} m\|_{\omega, -\psi} |u(0)|_{\mathcal{H}}. \end{aligned} \quad (2.13)$$

In order to optimize the choice of M , we let $0 \neq F \in C([0, t]; [0, +\infty[)$ and study

$$\inf_{\substack{0 \leq M \in C([0, t]), \\ \int M ds = 1}} \sup_s \frac{M(s)}{F(s)}. \quad (2.14)$$

We first notice that

$$1 = \int M ds = \int \frac{M}{F} F ds \leq \left(\sup_s \frac{M}{F} \right) \int F ds$$

and hence the quantity (2.14) is $\geq 1/\int F ds$. Choosing $M = \theta F$ with $\theta = 1/\int F(s) ds$, we get equality. ²

Lemma 2.1 *For any continuous function $F \geq 0$, non identically 0,*

$$\inf_{\substack{0 \leq M \in C([0, t]), \\ \int M(s) ds = 1}} \left(\sup_s \frac{M}{F} \right) = 1/\int F ds.$$

Applying the lemma to the supremum in (2.13) with

$$F = e^{\phi + \iota_t \psi} (r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} (r(\omega)^2 - \iota_t \psi'^2)_{\epsilon_2}^{\frac{1}{2}},$$

we get

$$|u(t)|_{\mathcal{H}} \leq e^{\omega t} \frac{\|(r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} m\|_{\omega \cdot -\phi} \|(r(\omega)^2 - \psi'^2)_{\epsilon_2}^{\frac{1}{2}} m\|_{\omega \cdot -\psi}}{\int_0^t e^{\phi + \iota_s \psi} (r(\omega)^2 - \phi'^2)_{\epsilon_1}^{\frac{1}{2}} (r(\omega)^2 - \iota_s \psi'^2)_{\epsilon_2}^{\frac{1}{2}} ds} |u(0)|_{\mathcal{H}}. \quad (2.15)$$

Since $u(0)$ is arbitrary, this is a rewriting of (1.8) and we get Theorem 1.6.

Remark 2.2 *If we do not impose any condition of the type (2.12), we get a variant of Theorem 1.6 which is easier to state, but probably less sharp: Adding the squares of (2.6), (2.7), leads to*

$$\| |r(\omega)^2 - \phi'^2|^{1/2} u \|_{\omega \cdot -\phi} \leq \sqrt{2} \| (r(\omega)^2 - \phi'^2)_{\epsilon_1}^{1/2} m \|_{\omega \cdot -\phi} |u(0)|_{\mathcal{H}}$$

Similarly, from (2.8), (2.9),

$$\| |r(\omega)^2 - \iota_t \psi'^2|^{1/2} u^* \|_{\omega(t \cdot) - \psi} \leq \sqrt{2} \| (r(\omega)^2 - \psi'^2)_{\epsilon_2}^{1/2} m \|_{\omega \cdot -\psi} |u^*(t)|_{\mathcal{H}}$$

² M does not necessarily satisfy condition (2.12) but we can proceed via a limiting argument.

We then follow a simplified variant of the estimates after (2.11):

$$\begin{aligned}
|[u(t)|u^*(t)]_{\mathcal{H}}| &\leq e^{\omega t} \int_0^t \frac{M(s)e^{-\phi(s)-\iota_t\psi(s)}}{|r(\omega)^2 - \phi'(s)^2|^{\frac{1}{2}}|r(\omega)^2 - \iota_t\psi'(s)^2|^{\frac{1}{2}}} \times \\
&\quad \times e^{\phi(s)-\omega s}|r(\omega)^2 - \phi'(s)^2|^{\frac{1}{2}}|u(s)|_{\mathcal{H}} \times \\
&\quad \times e^{\iota_t\psi(s)-\omega(t-s)}|r(\omega)^2 - \iota_t\psi'(s)^2|^{\frac{1}{2}}|u^*(s)|_{\mathcal{H}} ds \\
&\leq e^{\omega t} \sup_{[0,t]} \frac{Me^{-\phi-\iota_t\psi}}{|r(\omega)^2 - \phi'^2|^{\frac{1}{2}}|r(\omega)^2 - \iota_t\psi'^2|^{\frac{1}{2}}} \times \\
&\quad \times \||r(\omega)^2 - \phi'^2|^{\frac{1}{2}}u\|_{\omega,-\phi} \||r(\omega)^2 - \iota_t\psi'^2|^{\frac{1}{2}}u^*\|_{\omega(t-)-\iota_t\psi} \\
&\leq 2e^{\omega t} \sup_{[0,t]} \frac{Me^{-\phi-\iota_t\psi}}{|r(\omega)^2 - \phi'^2|^{\frac{1}{2}}|r(\omega)^2 - \iota_t\psi'^2|^{\frac{1}{2}}} \times \\
&\quad \times \|(r(\omega)^2 - \phi'^2)^{\frac{1}{2}}m\|_{\omega,-\phi} \|(r(\omega)^2 - \psi'^2)^{\frac{1}{2}}m\|_{\omega,-\psi} |u(0)|_{\mathcal{H}} |u^*(t)|_{\mathcal{H}}.
\end{aligned}$$

Choosing $u^*(t) = u(0)$ and applying Lemma 2.1 gives the following variant of (1.8),

$$\begin{aligned}
\|S(t)\|_{\mathcal{L}(\mathcal{H})} &\leq \\
2e^{\omega t} \frac{\|(r(\omega)^2\Phi^2 - \Phi'^2)^{\frac{1}{2}}m\|_{e^{\omega\cdot}L^2([0,t])} \|(r(\omega)^2\Psi^2 - \Psi'^2)^{\frac{1}{2}}m\|_{e^{\omega\cdot}L^2([0,t])}}{\int_0^t |r(\omega)^2\Phi^2 - \Phi'^2|^{\frac{1}{2}} |r(\omega)^2\iota_t\Psi^2 - \iota_t\Psi'^2|^{\frac{1}{2}} ds}. \quad (2.16)
\end{aligned}$$

Our goal is to show that starting from (1.8), (2.15) we can, by suitable choices of $\Phi, \phi, \Psi, \psi, \epsilon_1, \epsilon_2$, obtain and actually improve all the variants of the previously obtained statements [8, 16]. We will start by the analysis of two optimization problems which have their own independent interest.

3 Optimizers

3.1 Introduction

Motivated by Proposition 1.8, we study in this section the problem of minimizing an integral:

$$I_{\inf}(a) := \inf_{\{u \in H^1([0,a]); u(0)=0, u(a)=1\}} \int_0^a (u'^2 - u^2)_+ m^2 ds. \quad (3.1)$$

and of maximizing a similar integral:

$$J_{\sup}(b) := \sup_{\mathcal{G}} \int_0^b (\theta^2 - \theta'^2) m^{-2} ds, \quad (3.2)$$

where \mathcal{G} is defined by

$$\mathcal{G} = \{\theta \in H^1([0,b]); |\theta'| \leq \theta \text{ and } \theta(b) = 1\}. \quad (3.3)$$

The two problems are very similar, we devote most of the section to the minimization problem in the next four subsections and treat more shortly the maximization problem in the last Subsection 3.7.

3.2 Reduction

Let $0 < m \in C^1([0, +\infty[)$. If $0 \leq \sigma < \tau < +\infty$ and $S, T \in \mathbb{R}$ we put

$$H_{S,T}^1(] \sigma, \tau [) = \{u \in H^1(] \sigma, \tau [); u(\sigma) = S, u(\tau) = T\}. \quad (3.4)$$

Here and in the following all functions are assumed to be real-valued unless stated otherwise. In this section we let $a \in]0, +\infty[$ and study

$$\inf_{u \in H_{0,1}^1(]0, a [)} I(u), \text{ where } I(u) = I_{]0, a [}(u) = \int_0^a (u'^2 - u^2)_+ m^2 ds. \quad (3.5)$$

We shall show that we can here replace $H_{0,1}^1$ by a subspace that allows to avoid the use of positive parts. Put

$$\mathcal{H} = \mathcal{H}_{0,a}^{0,1}, \quad (3.6)$$

where for σ, τ, S, T as above,

$$\mathcal{H}_{\sigma,\tau}^{S,T} = \{u \in H_{S,T}^1(] \sigma, \tau [); 0 \leq u \leq u'\}. \quad (3.7)$$

Here the inequalities $0 \leq u, u \leq u'$ are valid in the sense of distributions, i. e. u and $u' - u$ are positive distributions on $]0, a[$. Notice that if $S, T > 0$ then for this space to be non-zero, it is necessary that

$$T \geq e^{\tau-\sigma} S. \quad (3.8)$$

Proposition 3.1

$$\inf_{H_{0,1}^1} I(u) = \inf_{\mathcal{H}} I(u).$$

Proof. Clearly

$$\inf_{H_{0,1}^1} I(u) \leq \inf_{\mathcal{H}} I(u). \quad (3.9)$$

We need to establish the opposite inequality.

Step 1. We first show

$$\inf_{u \in H_{0,1}^1} I(u) \geq \inf_{\{u \in H_{0,1}^1; u' \geq 0\}} I(u). \quad (3.10)$$

In the left hand side, we can replace $H_{0,1}^1$ by the dense subspace of Morse functions of class C^2 in $[0, a]$ where $0, 1$ are not critical values, $u(0) = 0, u(a) = 1$.

We shall see that we can replace u by a piecewise C^1 function³ v on $[0, a]$

³ We say that $u = [0, a] \mapsto \mathbb{R}$ is piecewise C^1 if $u \in C^0([0, a])$ and u' is piecewise continuous, i.e. with at most finitely many jump discontinuities. We denote by $C_{pw}^1([0, a])$ the space of piecewise C^1 functions.

with $v' \geq 0$, $v(0) = 0$, $v(a) = 1$, s.t. $I(v) \leq I(u)$.

Let $M(u) \geq 0$ be the number of critical points of u in $]0, a[$. If $M(u) = 0$, then u is increasing and we are done.

Assume that we can construct v as above⁴ when $M(u) \leq M$ for some $M \in \mathbb{N}$, and let us show that we can do the same when

$$M(u) = M + 1 \tag{3.11}$$

and we now consider that case.

Let $\sigma = \sup_{u(s)=0} s$. Then $u(\sigma) = 0$ and $u(s) > 0$ for $s > \sigma$. If $\sigma > 0$, then u has at least one critical point in $]0, \sigma[$ and hence u has at most M critical points in $]\sigma, a[$. Our induction hypothesis applies to $u|_{]0, \sigma[}$ so there is an increasing piecewise C^1 function \tilde{v} on $[\sigma, a]$ with $\tilde{v}(\sigma) = 0$, $\tilde{v}(a) = 1$ such that

$$I_{]0, \sigma[}(\tilde{v}) \leq I_{]0, \sigma[}(u).$$

We then get the desired conclusion with $v = 1_{]0, \sigma[}\tilde{v}$, and we have reduced the proof to the case when $u(s) > 0$ for $s > 0$.

Similarly we get a reduction to the case when $u(s) < 1$ for $s < a$, so we can now assume that $u(s) \in]0, 1[$ for $0 < s < a$ (and that (3.11) holds).

When s increases from 0 to a , u will first increase until it reaches a non-degenerate local maximum at some point $s_0 \in]0, a[$ with $u(s_0) \in]0, 1[$, then $u' < 0$ on some interval $]s_0, s_0 + \epsilon[$. Choose $\sigma \in]s_0, s_0 + \epsilon[$ and put $v_1(s) = \min(u(s), u(\sigma))$, $0 \leq s \leq \sigma$. Then $v_1 \in C_{pw}^1([0, \sigma])$, $v_1 \geq 0$, $v_1(\sigma) = u(\sigma)$ and $I_{]0, \sigma[}(v_1) \leq I_{]0, \sigma[}(u)$.

Clearly $u|_{]0, \sigma[}$ has M critical points and by the induction assumption (cf. Footnote 4) we have a piecewise C^1 function v_2 on $[\sigma, a]$ with $v_2' \geq 0$, $v_2(\sigma) = u(\sigma)$, $v_2(a) = 1$ s.t. $I_{]0, \sigma[}(v_2) \leq I_{]0, \sigma[}(u)$. We get the desired conclusion with $v = 1_{]0, \sigma[}v_1 + 1_{]0, \sigma[}v_2$.

Step 2. Let $u \in H_{0,1}^1$ with $u' \geq 0$. Then $u' \in L^2(]0, a[) \subset L^1(]0, a[)$ has mass 1 and we can find a sequence $v_j \in C^\infty([0, a];]0, \infty[)$, $j = 1, 2, \dots$ such that

$$v_j > 0, \quad \int_0^a v_j ds = 1, \quad v_j \rightarrow u' \text{ in } L^2.$$

If $u_j(s) = \int_0^s v_j(\sigma) d\sigma$, we have $u_j' > 0$, $u_j(0) = 0$, $u_j(a) = 1$ and $u_j \rightarrow u$ uniformly and hence in L^2 . Since $u_j' \rightarrow u'$ in L^2 we have that $u_j \rightarrow u$ in $H_{0,1}^1$. From (3.10) we then get

$$\inf_{u \in H_{0,1}^1} I(u) \geq \inf_{\{u \in H_{0,1}^1 \cap C^\infty([0, a]); u' > 0\}} I(u). \tag{3.12}$$

⁴ Notice that by affine dilations in s , u we have the seemingly more general statement that if \tilde{u} is a C^2 Morse function on $[\sigma, \tau]$, where $-\infty < \sigma < \tau < +\infty$, $\tilde{u}(\sigma) < \tilde{u}(\tau)$, and $\tilde{u}(\sigma)$, $\tilde{u}(\tau)$ are not critical values, then there is a piecewise C^1 function on $[\sigma, \tau]$, such that, $\tilde{v}' \geq 0$, $\tilde{v}(\sigma) = \tilde{u}(\sigma)$, $\tilde{v}(\tau) = \tilde{u}(\tau)$, and $I_{]0, \sigma[}(\tilde{v}) \leq I_{]0, \sigma[}(\tilde{u})$.

Step 3. Now, let $u \in H_{0,1}^1 \cap C^1([0, a])$ satisfy $u' > 0$ and let us construct $\tilde{u} \in \mathcal{H}$ such that $I(\tilde{u}) \leq I(u)$. Let $v \in C^1(]0, a[)$ satisfy

$$v'^2 - v^2 = (u'^2 - u^2)_+, \quad v(0) = 0, \quad v' \geq 0. \quad (3.13)$$

We can then apply the global Cauchy-Lipschitz theorem to

$$v' = \sqrt{v^2 + \phi}, \quad v(0) = 0,$$

with $\phi = (u'^2 - u^2)_+ \geq 0$.

The function $f(x, v) := \sqrt{v^2 + \phi(x)}$ is indeed Lipschitz in v along the graph of v , since $v^2 + \phi > 0$. Then $v' \geq v \geq 0$ and we now claim that $v \geq u$. From (3.13), we get indeed

$$v'^2 - v^2 \geq u'^2 - u^2,$$

which can be rewritten as

$$v'^2 - u'^2 \geq v^2 - u^2.$$

Factorizing both members in the last estimate and dividing with $v' + u' \geq u' > 0$, we get

$$(v - u)' \geq \frac{v + u}{v' + u'}(v - u). \quad (3.14)$$

Here $(v+u)/(v'+u') \geq 0$, so the differential inequality (3.14) and $v(0) - u(0) = 0$ imply that

$$v - u \geq 0. \quad (3.15)$$

In particular, $v(a) \geq u(a) = 1$. By (3.13) we have $I(v) = I(u)$.

Put $\tilde{u} = v(a)^{-1}v \in \mathcal{H}$. Then

$$I(\tilde{u}) = v(a)^{-2}I(v) = v(a)^{-2}I(u) \leq I(u).$$

End of the proof. Putting all the steps together, we can for any $u \in H_{0,1}^1$, construct a sequence \tilde{u}_n in $C^\infty([0, a]) \cap H_{0,1}^1(]0, a[)$ such that $\tilde{u}'_n > 0$ on $[0, a]$ and $I(\tilde{u}_n) \leq I(u) + \epsilon_n$ with $\epsilon_n \rightarrow 0$. Using Step 3 for \tilde{u}_n , we find $\hat{u}_n \in \mathcal{H}$ such that $I(\hat{u}_n) \leq I(\tilde{u}_n)$. This completes the proof of the lemma. \square

3.3 Existence of minimizers

As above, let $a \in]0, +\infty[$. We show that the infimum above is attained, i.e. that minimizers exist.

Proposition 3.2 *There exists $u_0 \in \mathcal{H}$ such that*

$$I_{\text{inf}}(a) := \inf_{u \in \mathcal{H}} I_{]0, a[}(u) = I_{]0, a[}(u_0). \quad (3.16)$$

Proof. The proof is standard. We recall it for completeness. Let $\|\cdot\|$ denote the norm in $L^2(]0, a[; m^2 ds)$ and define the norm in $H^1(]0, a[)$ by

$$\|u\|_1^2 = \|u\|^2 + \|\partial_s u\|^2.$$

Under our assumption on m , this norm is equivalent to the standard norm (corresponding to $m = 1$). Then

$$I_{]0, a[}(u) = \|u\|_1^2 - 2\|u\|^2.$$

For $u \in \mathcal{H}$ we have $0 \leq u \leq 1$, so $\|u\|^2 \leq C_m a$, $C_m = \int_0^a m^2 ds$. Hence, if $u \in \mathcal{H}$ and $I_{]0, a[}(u) \leq C$, we have

$$\|u\|_1^2 \leq C + 2\|u\|^2 \leq C + 2C_m a.$$

A closed ball in $H^1(]0, a[)$ of finite radius is compact for the weak topology in H^1 . It follows that every set $\{u \in \mathcal{H}; I_{]0, a[}(u) \leq C\}$ has the same property. Let $u_1, u_2, \dots \in \mathcal{H}$ be a sequence such that $I_{]0, a[}(u_\nu) \rightarrow \inf_{\mathcal{H}} I_{]0, a[}$ as $\nu \rightarrow +\infty$. After extracting a subsequence, we may assume that there exists $u_0 \in \mathcal{H}$ such that

$$u_\nu \rightharpoonup u_0 \text{ in } H^1(]0, a[), \quad u_\nu \rightarrow u_0 \text{ in } H^{\frac{3}{4}}(]0, a[).$$

We then deduce by continuity of the trace that $u_0(0) = 0$, $u_0(a) = 1$. Also $0 \leq u_0 \leq u'_0$ in the sense of distributions. Hence $u_0 \in \mathcal{H}$ and consequently

$$\inf_{u \in \mathcal{H}} I_{]0, a[}(u) \leq I_{]0, a[}(u_0).$$

Clearly $\|u_\nu\|^2 \rightarrow \|u_0\|^2$. From

$$\|u_0\|_1^2 = \lim_{\nu \rightarrow +\infty} (u_0, u_\nu)_1 \leq \|u_0\|_1 \limsup_{\nu \rightarrow +\infty} \|u_\nu\|_1,$$

we see that

$$\|u_0\|_1 \leq \limsup \|u_\nu\|_1.$$

Hence

$$\begin{aligned} I_{]0, a[}(u_0) &= \|u_0\|_1^2 - \|u_0\|^2 \\ &\leq \limsup (\|u_\nu\|_1^2 - \|u_\nu\|^2) = \limsup I_{]0, a[}(u_\nu) = \inf_{u \in \mathcal{H}} I_{]0, a[}(u). \end{aligned}$$

□

We have the following easy generalization.

Let $\sigma, \tau, S, T \in \mathbb{R}$, $\sigma < \tau$, $S, T \geq 0$, $T \geq e^{\tau-\sigma} S$. Let $0 < m \in C^1([\sigma, \tau])$ and define

$$\mathcal{H}_{\sigma, \tau}^{S, T} = \{u \in H^1(] \sigma, \tau[; \mathbb{R}); u(\sigma) = S, u(\tau) = T, 0 < u \leq u'\} \quad (3.17)$$

as in (3.7).

We then wish to study

$$\inf_{u \in \mathcal{H}_{\sigma, \tau}^{S, T}} I_{] \sigma, \tau[}(u),$$

where

$$I_{] \sigma, \tau[}(u) = \int_{\sigma}^{\tau} (u'^2 - u^2) m^2 ds.$$

The preceding proposition has a straight forward generalization:

Proposition 3.3 *There exists $u_0 \in \mathcal{H}_{\sigma, \tau}^{S, T}$ such that*

$$\inf_{u \in \mathcal{H}_{\sigma, \tau}^{S, T}} I_{] \sigma, \tau[}(u) = I_{] \sigma, \tau[}(u_0). \quad (3.18)$$

In the situation of the last proposition we call u_0 a minimizer in $\mathcal{H}_{\sigma, \tau}^{S, T}$.

3.4 On m -harmonic functions

3.4.1 Minimizers and m -harmonic functions

By (1.14), we have $K_m = m^{-2} P_m$ where

$$P_m = -\partial_s \circ m^2 \circ \partial_s - m^2.$$

If $0 \leq \sigma < \tau < +\infty$, we say that a function u on $] \sigma, \tau[$ is m -harmonic if $P_m u = 0$ on that interval.

The operator K_m is an unbounded self-adjoint operator in $L^2(] \sigma, \tau[; m^2 ds)$ when equipped with the domain $\mathcal{D} = (H_{0,0}^1 \cap H^2)(] \sigma, \tau[)$. It has discrete spectrum, contained in some interval $[-C, +\infty[$. If $\tau \leq a$ for some fixed $a \in]0, +\infty[$ and if $\tau - \sigma$ is small enough⁵ we have

$$m^{-2} P_m \geq 1/|\mathcal{O}(1)|. \quad (3.19)$$

Then $P_m : H_{0,0}^1 \cap H^2 \rightarrow H_0$ is a bijection and it is straight forward to see that for all $S, T \in \mathbb{R}$, the problem

$$\begin{cases} P_m u = 0 \text{ on }] \sigma, \tau[, \\ u(\sigma) = S, \quad u(\tau) = T, \end{cases} \quad (3.20)$$

has a unique solution $u \in H^2(] \sigma, \tau[)$.

Indeed, let $f \in C^2(] \sigma, \tau[)$ satisfy $f(\sigma) = S$, $f(\tau) = T$ and put $u = f + \tilde{u}$, where $\tilde{u} \in H_{0,0}^1 \cap H^2$ is the unique solution in $H_{0,0}^1 \cap H^2$ of $P_m \tilde{u} = -P_m f$. We denote by $u = f_{\sigma, \tau}^{S, T}$ the unique solution of (3.20).

The property (3.19) is equivalent to

$$I_{] \sigma, \tau[}(u) \geq \frac{1}{C} \|u\|_{H^1}^2, \quad \forall u \in H_{0,0}^1(] \sigma, \tau[). \quad (3.21)$$

Recall the definition of $H_{S, T}^1(] \sigma, \tau[)$ in 3.4. A general element $u \in H_{S, T}^1$ can be written

$$u = f + \tilde{u}, \quad \tilde{u} \in H_{0,0}^1(] \sigma, \tau[), \quad (3.22)$$

⁵ More precisely, there exist $C, \epsilon_0 > 0$ such that, for $|\sigma - \tau| < \epsilon_0$, the Dirichlet realization in $] \sigma, \tau[$ (also denoted by P_m) satisfies the lower bound $m^{-2} P_m \geq \frac{1}{C}$.

where $f = f_{\sigma,\tau}^{S,T}$.

We have with $I = I_{] \sigma, \tau [}$:

$$\begin{aligned} I(u) &= I(\tilde{u}) + 2 \int_{\sigma}^{\tau} (f' \tilde{u}' - f \tilde{u}) m^2 ds + I(f) \\ &\geq \frac{1}{C} \|\tilde{u}\|_{H^1}^2 - C_{S,T} \|\tilde{u}\|_{H^1} - C_{S,T} \\ &\geq \frac{1}{2C} \|\tilde{u}\|_{H^1}^2 - \tilde{C}_{S,T}. \end{aligned} \quad (3.23)$$

Thus

$$\|\tilde{u}\|_{H^1}^2 \leq \mathcal{O}(1)(I(u) + 1),$$

and combining this with the estimate

$$\|u\|_{H^1}^2 \leq 2(\|\tilde{u}\|_{H^1}^2 + \|f\|_{H^1}^2),$$

we get

$$\|u\|_{H^1}^2 \leq C_{S,T}(I(u) + 1), \quad (3.24)$$

with a new constant $C_{S,T}$.

Proposition 3.4 *Let σ, τ satisfy (3.19) or equivalently (3.21) and fix $S, T \in \mathbb{R}$. Then there exists $u_0 \in H_{S,T}^1(] \sigma, \tau [)$ such that*

$$I_{] \sigma, \tau [(u_0) = \inf_{u \in H_{S,T}^1(] \sigma, \tau [)} I_{] \sigma, \tau [(u). \quad (3.25)$$

The minimizer u_0 is equal to the unique solution $f_{\sigma,\tau}^{S,T}$ of (3.20) and hence belongs to $H^2(] \sigma, \tau [)$.

Proof. Thanks to (3.24) we can adapt the proof of Proposition 3.3 to see that there exists $u_0 \in H_{S,T}^1(] \sigma, \tau [)$, satisfying (3.25). The standard variational argument then shows that $u_0 = f_{\sigma,\tau}^{S,T}$ solves (3.20) and is therefore the unique minimizer. P_m being elliptic, we have $u_0 \in H^2(] \sigma, \tau [)$. \square

Remark 3.5 *Let $0 \leq \sigma < \tau$, $0 \leq S < T$, with $T \geq e^{\tau-\sigma} S$ as in (3.8) and assume that (3.19) holds on $] \sigma, \tau [$. If $u_0 := f_{\sigma,\tau}^{S,T}$ belongs to $\mathcal{H}_{\sigma,\tau}^{S,T}$ (i.e. if $0 \leq u_0 \leq u_0'$), then u_0 is equal to the unique minimizer in $H_{S,T}^1(] \sigma, \tau [)$ and hence it is a minimizer in the smaller space $\mathcal{H}_{\sigma,\tau}^{S,T}$. If u_1 is another minimizer in that space, then $I(u_1) = I(u_0)$, so it is also a minimizer in $H_{S,T}^1$ and by the uniqueness in that space, $u_1 = u_0$.*

Remark 3.6 *Let u_0 be a minimizer in $\mathcal{H}_{\sigma,\tau}^{S,T}$, let $\sigma \leq \tilde{\sigma} < \tilde{\tau} \leq \tau$ and set $\tilde{S} = u_0(\tilde{\sigma})$, $\tilde{T} = u_0(\tilde{\tau})$. Then $u_0|_{] \tilde{\sigma}, \tilde{\tau} [}$ is a minimizer in $\mathcal{H}_{\tilde{\sigma}, \tilde{\tau}}^{\tilde{S}, \tilde{T}}$. If $f_{\tilde{\sigma}, \tilde{\tau}}^{\tilde{S}, \tilde{T}}$ belongs to $\mathcal{H}_{\tilde{\sigma}, \tilde{\tau}}^{\tilde{S}, \tilde{T}}$ (assuming (3.19) holds on $] \tilde{\sigma}, \tilde{\tau} [$), then $u_0|_{] \tilde{\sigma}, \tilde{\tau} [} = f_{\tilde{\sigma}, \tilde{\tau}}^{\tilde{S}, \tilde{T}}$.*

3.4.2 Riccati equations and m -harmonic functions.

We next discuss m -harmonic functions from the point of view of first order non-linear ODE's, more specifically Riccati equations. Let f be an m -harmonic function on $]σ, τ[$ such that

$$0 < f \leq f'. \quad (3.26)$$

(For some arguments we relax this condition somewhat, still assuming that $f, f' > 0$.) Put

$$\mu = m'/m.$$

Then from

$$(\partial_s \circ m^2 \circ \partial_s + m^2)f = 0,$$

we get

$$(\partial_s^2 + 2\mu\partial_s + 1)f = 0. \quad (3.27)$$

Writing

$$\phi = \log f \text{ and } \psi = \phi' = f'/f,$$

we get

$$\psi \geq 1 \text{ and } \phi'' + \phi'^2 + 2\mu\phi' + 1 = 0. \quad (3.28)$$

We can rewrite the last equation in one of the two equivalent forms

$$\psi' = -(\psi^2 + 2\mu\psi + 1) \text{ or } \psi' = -2 \left(\mu + \frac{1}{2} \left(\psi + \frac{1}{\psi} \right) \right) \psi. \quad (3.29)$$

In the region $\psi > 1$ we can determine more explicitly when we have $\psi' > 0$, i.e. when

$$\psi^2 + 2\mu\psi + 1 < 0,$$

or equivalently when

$$\mu < -\frac{1}{2} \left(\psi + \frac{1}{\psi} \right).$$

Here the right hand side is ≤ -1 , so we have the necessary condition that

$$\mu < -1.$$

Assuming this to hold, we notice that $\psi^2 + 2\mu\psi + 1$ vanishes precisely for $\psi = -\mu \pm \sqrt{\mu^2 - 1}$. Clearly, $-\mu + \sqrt{\mu^2 - 1} > 1$ when $\mu < -1$. A small calculation (or using that the product of the two solutions is equal to 1) shows that $-\mu - \sqrt{\mu^2 - 1} < 1$ when $\mu < -1$.

In conclusion, we have proven

Lemma 3.7 *Consider a point s where (3.29) holds and $\psi(s) > 1$. Then:*

$$\psi'(s) > 0 \text{ if and only if } \mu(s) < -1 \text{ and } 1 < \psi(s) < -\mu(s) + \sqrt{\mu^2(s) - 1}.$$

We now put

$$f_+(s) = \begin{cases} 1, & \text{if } \mu(s) \geq -1, \\ -\mu(s) + \sqrt{\mu(s)^2 - 1}, & \text{if } \mu(s) < -1. \end{cases}$$

The last lemma tells us that

$$\psi'(s) \leq 0, \text{ when (3.29) holds and } \psi(s) \geq f_+(s). \quad (3.30)$$

This implies the following nice control of solutions of (3.29) in the direction of increasing “time” s :

If (3.29) holds for $\sigma < s < \tau$ and $\sigma < s_0 < \tau$, then

$$\psi(s) \leq \max(\psi(s_0), \max_{[s_0, s]} f_+), \quad s_0 \leq s < \tau. \quad (3.31)$$

Let $a \in]0, +\infty[$ be fixed and assume that $0 \leq \sigma < \tau \leq a$ with $\tau - \sigma$ small enough, so that the Dirichlet realization of $m^{-2}P_m$ is $\geq 1/|\mathcal{O}(1)|$ and let $f = f_{\sigma, \tau}^{S, T}$, so that $u = f$ satisfies (3.20). We restrict the attention to a region $\{s \in]0, a[; \psi(s) \in]1/2, 2C_0[\}$ where C_0 can be large but fixed. We have

$$\int_{\sigma}^{\tau} \psi(s) ds = \int_{\sigma}^{\tau} \partial_s \log f ds = \log \frac{f(\tau)}{f(\sigma)} = \log \frac{T}{S}. \quad (3.32)$$

Conversely, from (3.32), (3.29), we get $f(\tau)/f(\sigma) = T/S$ and after multiplying f with a suitable positive constant, we get (3.20).

Consider the differential equation (3.29) over an interval $]\sigma, \tau[$ with $1/2 \leq \psi \leq 2C_0$ with $C_0 > 1$ as above. If $\tau - \sigma$ is small enough, we have a unique such solution if we prescribe $\psi(\sigma)$ in the slightly smaller interval $]2/3, 3C_0/2[$ and we get $\psi(s) = \psi(\sigma) + \mathcal{O}(s - \sigma)$. Hence we have

$$m_{] \sigma, \tau [}(\psi) := \frac{1}{\tau - \sigma} \int_{\sigma}^{\tau} \psi(s) ds = \psi(\sigma) + \mathcal{O}(\tau - \sigma). \quad (3.33)$$

For $z \in]2/3, 3C_0/2[$, we define $\tilde{m}_{\sigma, \tau}(z) := m_{] \sigma, \tau [}(\psi)$ where ψ is the solution of (3.29) with $\psi(\sigma) = z$. $\tilde{m}_{\sigma, \tau}$ can be extended to a biholomorphic map from some fixed neighborhood of $[1, C_0]$ in \mathbb{C} onto a (σ, τ) -dependent neighborhood of the same type and 3.33 extends to the estimate:

$$\tilde{m}_{\sigma, \tau}(z) = z + \mathcal{O}(\tau - \sigma).$$

The inverse map $\tilde{m} \mapsto z$ satisfies trivially

$$z = \tilde{m}_{\sigma, \tau}(z) + \mathcal{O}(\tau - \sigma),$$

and this holds uniformly for $\sigma, \tau \in [0, a]$, $|\tau - \sigma| \ll 1$. (Once z has been determined from some \tilde{m} , we determine ψ from the differential equation (3.29) with initial condition $\psi(\sigma) = z$ and we have $\tilde{m} = m_{] \sigma, \tau [}(\psi)$.)

We can apply this to (3.32), that we write as

$$\tilde{m}_{\sigma,\tau}(z) = m_{]\sigma,\tau[}(\psi) = \frac{1}{\tau - \sigma} \log \frac{T}{S}.$$

If $(\tau - \sigma)^{-1} \log(T/S) \in \text{neigh}([1, C_0], \mathbb{R})$, we get⁶ a unique z and a real solution ψ of (3.32), (3.29) with $\psi(\sigma) = z$,

$$\psi(s) = \frac{1}{\tau - \sigma} \log \frac{T}{S} + \mathcal{O}(\tau - \sigma), \quad (3.34)$$

uniformly on $[\sigma, \tau]$. In particular, if

$$\frac{1}{\tau - \sigma} \log \frac{T}{S} \geq 1 + \frac{1}{|\mathcal{O}(1)|}, \quad (3.35)$$

we get

$$\psi(s) \geq 1 + \frac{1}{|\mathcal{O}(1)|} - \mathcal{O}(\tau - \sigma) > 1 \quad (3.36)$$

and we conclude that the corresponding solution $u = f_{\sigma,\tau}^{S,T}$ belongs to $\mathcal{H}_{\sigma,\tau}^{S,T}$.

In conclusion:

Proposition 3.8 *For every $C_0 > 1$, there exist $\epsilon_0 > 0$ and $C_1 > 0$ such that if $S, T > 0$, $0 \leq \sigma < \tau \leq a$, $\tau - \sigma < \epsilon_0$, $2/3 \leq \ln(T/S)/(\tau - \sigma) \leq 3C_0/2$, then $f = f_{\sigma,\tau}^{S,T}$ satisfies*

$$\left| \frac{f'}{f} - \frac{\ln(T/S)}{\tau - \sigma} \right| \leq C_1(\tau - \sigma) \text{ on }]\sigma, \tau[.$$

In particular, if

$$\frac{\ln(T/S)}{\tau - \sigma} \in [1 + 1/C_2, 3C_0/2],$$

where $C_2 > 0$, then

$$\frac{f'}{f} - 1 \geq \frac{1}{C_2} - C_1(\tau - \sigma) \text{ on }]\sigma, \tau[,$$

hence $f'/f - 1 \geq 1/(2C_2)$ on $]\sigma, \tau[$ and $f \in \mathcal{H}_{\sigma,\tau}^{S,T}$, if $\tau - \sigma$ is small enough.

3.5 Structure of minimizers

We will first discuss minimizers over a fixed interval $]0, a[$, $0 < a < \infty$. It may be useful to recall that if $u \in H^1(]0, a[)$, then u is Hölder continuous of order $1/2$, i.e. $u \in C^{1/2}$. In fact, if $0 \leq \sigma < \tau \leq a$,

$$|u(\tau) - u(\sigma)| \leq \int_{\sigma}^{\tau} |u'(s)| ds \leq \|u\|_{H^1}(\tau - \sigma)^{1/2}.$$

⁶Here we use "neigh(A, B)" as an abbreviation for "some neighborhood of A in B ".

Proposition 3.9 *Let $\phi \in H^1(]0, a[)$ be real-valued, $0 \leq \sigma < \tau \leq a$. Let*

$$\lambda = \frac{\phi(\tau) - \phi(\sigma)}{\tau - \sigma}.$$

Then there exist arbitrarily short intervals $[\tilde{\sigma}, \tilde{\tau}] \subset [\sigma, \tau]$ such that

$$\frac{\phi(\tilde{\tau}) - \phi(\tilde{\sigma})}{\tilde{\tau} - \tilde{\sigma}} = \lambda.$$

Proof.

If $I = [\tilde{\sigma}, \tilde{\tau}]$ is a subinterval of $[\sigma, \tau]$, then $m_I(\phi') := (\phi(\tilde{\tau}) - \phi(\tilde{\sigma})) / (\tilde{\tau} - \tilde{\sigma})$ is equal to the average over I of ϕ' . Let $N \in \mathbb{N}$, $N \geq 2$, and decompose $[\sigma, \tau]$ into the disjoint union of N intervals I_1, \dots, I_N of length $(\tau - \sigma)/N$. Then the mean value of the averages $m_{I_j}(\phi')$ is equal to λ . If no such average is equal to λ , there exist I_j, I_k , with $m_{I_j}(\phi') < \lambda$, $m_{I_k}(\phi') > \lambda$. Let $I^t = I_j + Ct$ with $C \in \mathbb{R}$ chosen so that $I^0 = I_j$, $I^1 = I_k$ or at least so that we have equality for the interiors. Then $m_{I^t}(\phi')$ varies continuously with t , so there exists $t \in]0, 1[$ such that $m_{I^t}(\phi') = \lambda$. \square

Let $u_0 \in \mathcal{H}(]0, a[) = \mathcal{H}_{0,a}^{0,1}$ be a minimizer for $I_{]0,a[}$. Let

$$\psi_0 = u'_0 / u_0 = \phi'_0 \text{ where } \phi_0 = \log u_0.$$

Then, we deduce from (3.17):

$$\psi_0 \geq 1, \tag{3.37}$$

$$\phi_{0|] \epsilon, a[} \in H^1(] \epsilon, a[), \tag{3.38}$$

for every $\epsilon > 0$.

Proposition 3.10 *Let $0 < \sigma < \tau \leq a$ and assume that $\lambda := m_{] \sigma, \tau[}(\psi_0) > 1$. Then there exists $s_0 \in] \sigma, \tau[$ and α, β with*

$$0 \leq \alpha < s_0 < \beta \leq a, \tag{3.39}$$

such that

$$u_0 \text{ is } m\text{-harmonic and } 0 < u_0 < u'_0 \text{ on }] \alpha, \beta[. \tag{3.40}$$

- If $\beta < a$, we have $\psi_0(s) \rightarrow 1$, $s \nearrow \beta$.
- If $\alpha > 0$, we have $\psi_0(s) \rightarrow 1$, $s \searrow \alpha$.

We recall that $\psi_0(s) \rightarrow -\infty$, when $s \searrow 0$. Moreover s_0 can be chosen so that $\psi_0(s_0)$ is arbitrarily close to λ .

Proof. By Proposition 3.9 there exist arbitrarily short intervals $I =] \tilde{\sigma}, \tilde{\tau}[\subset] \sigma, \tau[$ such that $m_I(\phi'_0) = \lambda$. For each such interval put $S = u_0(\tilde{\sigma})$, $T = u_0(\tilde{\tau})$, so that $\log(T/S) = \lambda(\tilde{\tau} - \tilde{\sigma})$. Let $f = f_{\tilde{\sigma}, \tilde{\tau}}^{S, T}$, $\psi = f'/f$, $\phi = \log f$. Then we have (3.29), (3.32) and we can apply (3.34) (or Proposition 3.8) with σ, τ there replaced by $\tilde{\sigma}, \tilde{\tau}$, to see that $\psi(s) = \lambda + \mathcal{O}(\tilde{\tau} - \tilde{\sigma})$, $\tilde{\sigma} \leq s \leq \tilde{\tau}$. In

particular $\psi > 1$ on $[\tilde{\sigma}, \tilde{\tau}]$ when $\tilde{\tau} - \tilde{\sigma}$ is small enough. Hence $f \in \mathcal{H}_{\tilde{\sigma}, \tilde{\tau}}^{S, T}$ and applying Remark 3.6, we conclude that

$$u_0 = f_{\tilde{\sigma}, \tilde{\tau}}^{u_0(\tilde{\sigma}), u_0(\tilde{\tau})} \text{ on }]\tilde{\sigma}, \tilde{\tau}[. \quad (3.41)$$

Choose $s_0 \in]\tilde{\sigma}, \tilde{\tau}[$ so that $\psi(s_0) = \psi_0(s_0)$ is as close to λ as we like.

Let $]\alpha, \beta[\subset]0, a[$ be the largest open interval containing s_0 on which u_0 is m -harmonic and $u'_0/u_0 > 1$.

Assume that $\alpha > 0$ and that $\psi_0(\alpha + 0) > 1$. Then we can find (new) arbitrarily short intervals $]\tilde{\sigma}, \tilde{\tau}[$, containing α , such that

$$m_{]\tilde{\sigma}, \tilde{\tau}[(\phi'_0) \geq 1 + \frac{1}{|\mathcal{O}(1)|}$$

and as above, we see that u_0 is m -harmonic and $u'_0/u_0 > 1$ on $]\tilde{\sigma}, \tilde{\tau}[$ which contradicts the maximality of $]\alpha, \beta[$. Hence, if $\alpha > 0$, we have $\psi_0(\alpha + 0) = 1$.

Similarly, if $\beta < a$, we have $\psi_0(\beta - 0) = 1$. \square

Let $J \subset]0, a[$ be the countable disjoint union of all open maximal intervals $I \subset]0, a[$, such that u_0 is m -harmonic with $u'_0/u_0 > 0$ on I .

Proposition 3.11 *ψ_0 is uniformly Lipschitz continuous on $]0, a[$, > 1 on J , and $= 1$ on $]0, a[\setminus J$.*

Proof. For $t \in]0, a[$, let $\partial_s^+ \psi_0(t)$ be the set of all limits $(\psi_0(t + \epsilon_j) - \psi_0(t))/\epsilon_j$ with $\epsilon_j \searrow 0$. Similarly let $\partial_s^- \psi_0(t)$ be the set of all limits $(\psi_0(t + \epsilon_j) - \psi_0(t))/\epsilon_j$ with $\epsilon_j \nearrow 0$. We can also define $\partial_s^- \psi_0(a)$.

When $t \in \mathcal{J}$, we have

$$\partial_s^+ \psi_0(t) = \partial_s^- \psi_0(t) = \{\psi'_0(t)\}.$$

When $t \in]0, a[\setminus \mathcal{J}$, we see using (3.29) that

$$\partial_s^+ \psi_0(t) \subset \begin{cases} \{0\} & \text{if } \mu \geq -1 \\ [0, -2(1 + \mu)] & \text{if } \mu < -1 \end{cases}, \quad (3.42)$$

$$\partial_s^- \psi_0(t) \subset \begin{cases} [-2(1 + \mu), 0] & \text{if } \mu \geq -1 \\ \{0\} & \text{if } \mu < -1 \end{cases}. \quad (3.43)$$

From this it follows that ψ_0 is Lipschitz. \square

We next discuss some consequences for the global structure of minimizers. As before, let $u_0 \in \mathcal{H}$ be a minimizer for $I_{]0, a[}$ and recall that u_0 is m -harmonic with $u'_0/u_0 > 1$ on a countable union J of maximal open subintervals of $]0, a[$. One of these subintervals is of the form $]0, \tilde{a}[$, for some $\tilde{a} \in]0, a[$, which is uniquely determined while $u_0|_{]0, \tilde{a}[}$ is unique up to a positive constant factor.

We have $\tilde{a} = a$ if $\mu \leq -1$ on $]0, a[$. In fact, by (3.29),

$$\begin{aligned} (\psi - 1)' &= -2(1 + \mu) - 2(1 + \mu)(\psi - 1) - (\psi - 1)^2 \\ &\geq -2(1 + \mu)(\psi - 1) - (\psi - 1)^2, \end{aligned}$$

so we cannot reach the region $\psi - 1 = 0$ in finite positive time from a point in the region $\psi - 1 > 0$.

When $\tilde{a} < a$, if $\mu(s) \geq -1$ for $\tilde{a} \leq s \leq a$ (and in particular if $\mu(s) \geq -1$ on $]0, a[$), it follows from Lemma 3.7 that $\psi_0(s) = 1$ for $s \geq \tilde{a}$. Indeed, otherwise there would be a maximal open subinterval $]b, c[\subset]\tilde{a}, a[$ on which ψ_0 is m -harmonic, in contradiction with the fact that $\psi_0' \leq 0$ there.

More generally, let $\tilde{a} < a$ and assume that $\psi_0 \neq 1$ on $]\tilde{a}, a[$. Let $I =]\sigma, \tau[$ be a maximal open subinterval of $]\tilde{a}, a[$ on which u_0 is m -harmonic with $u_0'/u_0 > 1$. Then $\psi_0 > 1$ on $]\sigma, \tau[$ and converges to 1 when $s \searrow \sigma$. When $\tau < a$ we also have that $\psi_0 \rightarrow 1$ when $\sigma \nearrow \tau$. Lemma 3.7 then tells us that there exist points $s > \sigma$ arbitrarily close to σ where $\mu(s) < -1$. Similarly, if $\tau < a$ there are points $s < \tau$ arbitrarily close to τ with $\mu(s) > -1$.

We get the following conclusion, where we represent J as a disjoint union of maximal subintervals I , where u_0 is m -harmonic with $u_0'/u_0 > 1$:

If $\mu \geq -1$ on $]0, a[$, then $J = I =]0, \tilde{a}[$ for some $0 < \tilde{a} \leq a$.

If $I =]0, \tilde{a}[$, $\tilde{a} < a$, then I contains a point s arbitrarily close to \tilde{a} where $\mu(s) > -1$.

If $I =]\sigma, \tau[$, $\tilde{a} < \sigma < \tau < a$, then I contains two points $\tilde{\sigma}$, $\tilde{\tau}$, arbitrarily close to σ and τ respectively, such that $\mu(\tilde{\sigma}) < -1$, $\mu(\tilde{\tau}) > -1$.

If $I =]\sigma, a[$, $\tilde{a} < \sigma < a$, then I contains a point $\tilde{\sigma}$, arbitrarily close to σ , such that $\mu(\tilde{\sigma}) < -1$.

We spell out the conclusion when $\mu \geq -1$:

Proposition 3.12 *Let u_0 be a minimizer for $I_{]0, a[}$ on $\mathcal{H} = \mathcal{H}_{0, a}^{0,1}$ and let \tilde{a} be the largest number in $]0, a[$ such that u_0 is m -harmonic with $u_0'/u_0 > 1$ on $]0, \tilde{a}[$. If $\tilde{a} < a$ and $\mu(s) \geq -1$ on $[\tilde{a}, a]$, then $u_0(s) = e^{s-a}$ on $[\tilde{a}, a[$ and u_0 is uniquely determined.*

Remark 3.13

- When $\tilde{a} < a$, we shall see that $\tilde{a} = a^*$ is independent of a . See Proposition 3.15.
- The proposition can be applied in the case m constant ($\mu = 0$) and more generally the case $m_\alpha(s) = \exp -\alpha s$ with $\alpha \leq 1$.

We end this subsection by studying global minimizers, more precisely minimizers defined on all of $]0, +\infty[$. Let

$$\mathcal{H}(]0, +\infty[) := \{u \in H_{\text{loc}}^1([0, +\infty[); 0 \leq u \leq u', u(0) = 0, u > 0 \text{ on }]0, +\infty[\}.$$

We say that $u_0 \in \mathcal{H}(]0, +\infty[)$ is a minimizer (or a global minimizer when emphasizing that we work on the whole half axis) if $u_0|_{]0, a[}$ is a minimizer in $\mathcal{H}_{0, a}^{0, u_0(a)}$ for every $a > 0$. Recall our assumption that $0 < m \in C^1([0, +\infty[)$.

Proposition 3.14 *A global minimizer $u_0 \in \mathcal{H}(]0, +\infty[)$ exists.*

Proof. Let $0 < a_1 < a_2 < \dots$ be a sequence such that $a_j \rightarrow +\infty$ when $j \rightarrow +\infty$. It suffices to find $u_0 \in \mathcal{H}(]0, +\infty[)$ such that $u_0|_{]0, a_j[}$ is a minimizer in $\mathcal{H}_{0, a_j}^{0, u_0(a_j)}$ for every j .

Let $u_1 \in \mathcal{H}_{0, a_1}^{0, 1}$ be a minimizer (and here we could replace 1 by any positive number). Let $\tilde{u}_2 \in \mathcal{H}_{0, a_2}^{0, 1}$ be a minimizer. Replacing \tilde{u}_2 by $\tilde{u}_2(a_1)^{-1}\tilde{u}_2$, we get a new minimizer $\tilde{u}_2 \in \mathcal{H}_{0, a_2}^{0, \tilde{u}_2(a_2)}$ with $\tilde{u}_2(a_1) = u_1(a_1) (= 1)$. Then both u_1 and $\tilde{u}_2|_{]0, a_1[}$ are minimizers in $\mathcal{H}_{0, a_1}^{0, 1}$, so

$$u_2 := 1_{]0, a_1]}u_1 + 1_{]a_1, a_2[}\tilde{u}_2$$

is also a minimizer in $\mathcal{H}_{0, a_2}^{0, u_2(a_2)}$ and has the property: $u_2|_{]0, a_1[} = u_1$.

Iterating this argument, we get a sequence of minimizers u_j in $\mathcal{H}_{0, a_j}^{0, u(a_j)}$, $j = 1, 2, \dots$ such that $u_{j+1}|_{]0, a_j[} = u_j$ for $j = 1, 2, \dots$ and it suffices to define u_0 on $]0, +\infty[$ by $u_0|_{]0, a_j[} = u_j$. \square

The discussion of the structure of minimizers in $\mathcal{H}_{0, a}^{0, 1}$ applies directly to global minimizers. In particular, we get:

Proposition 3.15 *If u_0 is a global minimizer, then u_0 is m -harmonic with $u'_0/u_0 > 1$ on a maximal interval of the form $]0, a^*[$, for some $a^* \in]0, +\infty[$. a^* is uniquely determined and (the m -harmonic function) $u_0|_{]0, a^*[}$ is unique up to a constant positive factor.*

This characterization of a^ is equivalent to the one in (1.16)*

Remark 3.16 *Note that we do not claim that we have uniqueness for u_0 up to multiplication with positive constants. However, we do get this uniqueness if we add the assumption that $\mu(s) \geq -1$ for $s \geq a^*$. Cf. Proposition 3.12.*

From the discussion with Riccati equations, we have also

Proposition 3.17 *If (1.17) holds, then a^* .*

Proof. Let $A > 0$ be such that $\mu(s) \geq -1 + \frac{1}{A}$ for $s \geq A$. It follows from (3.42)-(3.43) that there exists $B \geq A$ such that $\psi_0(A) \leq B$.

Then we get from (3.29)

$$\begin{cases} s \geq A \\ \psi(s) > 1 \end{cases} \text{ implies } \psi'_0 \leq -2/A,$$

so $\psi_0(s) = 1$ for $B - \frac{2}{A}(s - A) \leq 1$, i.e. for $s \geq \frac{A}{2}(B + 1)$. \square

3.6 Application to our minimization problem

Let $u_0 : [0, +\infty[\rightarrow \mathbb{R}$, satisfy $P_m u_0 = 0$, $u_0(0) = 0$, $u_0'(0) > 0$, so that u_0 is uniquely determined up to a constant positive factor. Then $u_0 > 0$ on $]0, a^*(m)[$ and when $a^*(m) < +\infty$, we have $u_0'(a^*(m)) = u_0(a^*(m))$ and u_0 is then the first eigenfunction of K_{m, a^*}^{DR} with eigenvalue 0.

Proposition 3.18 For $a \in]0, +\infty[\cap]0, a^*(m)[$,

$$I_{\inf}(a) = \psi_0(a) m^2(a), \quad (3.44)$$

where $\psi_0 = u_0'/u_0$, u_0 .

In particular, when $a = a^*(m) < +\infty$, we get

$$I_{\inf}(a^*(m)) = m^2(a^*(m)). \quad (3.45)$$

Proof. We have seen in Proposition 3.1 that

$$\inf_{\{u \in H^1(]0, a[), u(0)=0, u(a)=1\}} \int_0^a (u'^2 - u^2)_+ m^2 ds = \inf_{u \in \mathcal{H}} \int_0^a (u'^2 - u^2) m^2 ds.$$

Here the minimizer is $u = u_0(s)/u_0(a)$. Integration by parts and using that u is m -harmonic, gives

$$\int_0^a (u'^2 - u^2) m^2 ds = m^2(a) u(a) u'(a) = m^2(a) \psi_0(a).$$

We also recall that $\psi_0(a^*) = 1$. □

Remark 3.19 When $m = 1$, we obtain $a^* = \frac{\pi}{4}$ and $\psi_0(s) = \cot s$. More generally, we can consider $m_\alpha(s) = e^{-\alpha s}$ with $|\alpha| \leq 1$. Writing $\alpha = \cos \theta$ ($\theta \in [0, +\pi[$) we get

- for $\alpha = \cos \theta$ with $\theta \in]0, \pi[$

$$a^*(m_\alpha) = \frac{\pi - \theta}{2 \sin \theta},$$

- for $\alpha = \pm 1$,

$$a^*(m_{-1}) = \frac{1}{2} \text{ and } a^*(m_1) = +\infty.$$

The global minimizer restricted to $]0, a^*[$ is given by

$$u_\alpha(s) = \frac{1}{\sqrt{1 - \alpha^2}} \exp(\alpha s) \sin(\sqrt{1 - \alpha^2} s), \quad -1 < \alpha < 1 \quad (3.46)$$

and

$$u_{\pm 1}(s) = s \exp \pm s. \quad (3.47)$$

When $\alpha = \cos \theta$, we get the energy

$$\psi_\alpha(a) = \frac{\sin(\sin \theta a + \theta)}{\sin(\sin \theta a)}.$$

Another upper bound We start from the upper bound

$$\int_0^a (u'^2 - u^2)_+ m(s)^2 ds \leq \int_0^a u'^2(s) m(s)^2 ds$$

and minimize the right hand side.

Observing that

$$\begin{aligned} 1 = u(a) &= \int_0^a u'(s) ds = \int_0^a u'(s) m(s) m(s)^{-1} ds \\ &\leq \left(\int_0^a (u'(s) m(s))^2 ds \right)^{\frac{1}{2}} \left(\int_0^a \frac{1}{m(s)^2} ds \right)^{\frac{1}{2}}, \end{aligned}$$

we look for a u for which we have equality.

By the standard Cauchy-Schwarz criterion, this is the case if, for some constant $C > 0$,

$$u'(s) m(s) = \frac{C}{m(s)}.$$

Hence, we choose

$$u(s) = C \int_0^s \frac{1}{m(\tau)^2} d\tau,$$

where the choice of C is determined by imposing $u(a) = 1$. We obtain

Proposition 3.20 *For any $a > 0$,*

$$\inf_{\{u \in H^1(]0,a[), u(0)=0, u(a)=1\}} \int_0^a (u'^2 - u^2)_+ m^2 ds \leq \left(\int_0^a \frac{1}{m(s)^2} ds \right)^{-1}. \quad (3.48)$$

Note here that we have no condition on $a > 0$ and no condition on μ .

Minimization of $\exp(2a)I_{\inf}(a)$

In the application to the semi-group upper bound we will meet the natural question of minimizing over $]0, a^*]$ the quantity

$$a \mapsto \Theta(a) := \exp(2a) m^2(a) \psi_0(a). \quad (3.49)$$

The answer is given by the following proposition:

Proposition 3.21 *When $a^*(m) < +\infty$, we have*

$$\inf_{a \in]0, a^*]} \Theta(a) = \Theta(a^*) = \exp(2a^*) m^2(a^*). \quad (3.50)$$

Proof. We will simply show that $\Theta' < 0$ on $]0, a^*[$. Computing Θ' we get

$$\Theta'(a) = \exp(2a) m^2(a) \psi_0(a) (2 + 2\mu(a) + \psi_0'(a)/\psi_0(a)) \quad (3.51)$$

Using (3.29), we obtain $a < a^*$

$$\Theta'(a) = -\exp(2a) m^2(a) (\psi_0(a) - 1)^2. \quad (3.52)$$

Note also that $\Theta'(a^* - 0) = 0$. \square

3.7 Maximizers

As before, let $0 < m \in C^1([0, +\infty[)$ and let $0 < b < +\infty$. In the following, all functions are assumed to be real-valued if nothing else is specified. We recall that \mathcal{G} was introduced in (3.3) by

$$\mathcal{G} = \{\theta \in H^1(]0, b[); |\theta'| \leq \theta, \theta(b) = 1\}.$$

If $\theta \in \mathcal{G}$, we have

$$|\theta'/\theta| \leq 1, \text{ i.e. } |(\log \theta)'| \leq 1,$$

so $|\log \theta(s)| \leq b - s$,

$$e^{s-b} \leq \theta(s) \leq e^{b-s}, \quad 0 \leq s \leq b.$$

In this subsection we consider the problem of maximizing the functional on \mathcal{G}

$$J(\theta) = J_{]0, b[}(\theta) = \int_0^b (\theta^2 - \theta'^2) m^{-2} ds. \quad (3.53)$$

We recall from (1.12) that we have the easy lower bound

$$J_{\sup}(b) := \sup_{\theta \in \mathcal{G}} J(\theta) \geq \int_0^b m(s)^{-2} ds.$$

We notice that \mathcal{G} is a bounded subset of $H^1(]0, b[)$ and that $0 \leq J \leq \mathcal{O}(1)$ on that subset. As in Subsection 3.3 we can show the existence of a maximizer:

$$\text{There exists } \theta_0 \in \mathcal{G}, \text{ such that } J(\theta_0) = \sup_{\theta \in \mathcal{G}} J(\theta). \quad (3.54)$$

If $0 \leq \sigma < \tau \leq b$, $S, T > 0$, $|\log(T/S)| \leq \tau - \sigma$, put

$$\mathcal{G}_{\sigma, \tau}^{S, T} = \{u \in H_{S, T}^1(] \sigma, \tau [); |u'| \leq u\}. \quad (3.55)$$

We also define

$$\mathcal{G}_{\tau}^T = \{u \in H_T^1(]0, \tau [); |u'| \leq u\}, \quad (3.56)$$

where $H_T^1(]0, \tau [) := \{u \in H^1(]0, \tau [); u(\tau) = T\}$.

Finally, we introduce the functional

$$J_{] \sigma, \tau [}(u) = \int_{\sigma}^{\tau} (u^2 - u'^2) m^{-2} ds, \quad u \in H^1(] \sigma, \tau [). \quad (3.57)$$

Let θ_0 be a maximizer for J on \mathcal{G} . If $0 \leq \sigma < \tau \leq b$, we put $S = \theta_0(\sigma)$, $T = \theta_0(\tau)$. Then $\theta_0|_{] \sigma, \tau [}$ is a maximizer for $J_{] \sigma, \tau [}$ on $\mathcal{G}_{\sigma, \tau}^{S, T}$. Also $\theta_0|_{]0, \tau [}$ is a maximizer for $J_{]0, \tau [}$ on \mathcal{G}_{τ}^T .

For $0 < \sigma < \tau \leq b$, we assume that $u_0 \in H_{S,T}^1(] \sigma, \tau [)$ is a maximizer for $J_{] \sigma, \tau [}$ on $H_{S,T}^1(] \sigma, \tau [)$. Then by the same standard variational arguments as for minimizers (cf. Proposition 3.4), we see that u_0 is $1/m$ -harmonic on $] \sigma, \tau [$:

$$P_{1/m} u_0 := -(\partial_s \circ m^{-2} \partial_s + m^{-2}) u_0 = 0, \text{ on }] \sigma, \tau [, \quad (3.58)$$

so $u_0 \in H^2(] \sigma, \tau [)$.

When $\sigma = 0$, assume that $u_0 \in H_T^1(] 0, \tau [)$ is a maximizer for $J_{] 0, \tau [}$ on $H_T^1(] \sigma, \tau [)$. Then by variational calculations, we get

$$\begin{cases} P_{1/m} u_0 = 0 \text{ on }] 0, \tau [, \\ \partial_s u_0(0) = 0, \\ u_0(\tau) = T. \end{cases} \quad (3.59)$$

Also, if $\tau - \sigma > 0$ is small enough, we know that

$$m^2 P_{1/m} \geq 1/|\mathcal{O}(1)| \text{ on } (H^2 \cap H_{0,0}^1)(] \sigma, \tau [, \quad (3.60)$$

and consequently that

$$\begin{aligned} \forall S, T \in \mathbb{R}, \exists ! u =: g_{\sigma, \tau}^{S, T}, \text{ such that} \\ P_{1/m} u = 0 \text{ on }] \sigma, \tau [, \quad u(\sigma) = S, \quad u(\tau) = T. \end{aligned} \quad (3.61)$$

Similarly to what we have seen in Subsection 3.1, under this assumption, $g_{\sigma, \tau}^{S, T}$ is the unique maximizer for $J_{] \sigma, \tau [}$ on $H_{S,T}^1(] \sigma, \tau [)$.

When $0 < \tau \leq b$, $m^2 P_{1/m}$ is self-adjoint on $L^2(] 0, \tau [, m^{-2} ds)$ with domain $\mathcal{D} = \{u \in H_{T=0}^1(] 0, \tau [) \cap H^2(] 0, \tau [); \partial_s u(0) = 0\}$.

Moreover, $m^2 P_{1/m} \geq 1/|\mathcal{O}(1)|$ when $\tau > 0$ is small enough and for every $T \in \mathbb{R}$ we have a unique solution $u =: g_\tau^T$ of

$$P_{1/m} u = 0 \text{ on }] 0, \tau [, \quad \partial_s u(0) = 0, \quad u(\tau) = T. \quad (3.62)$$

Let θ_0 be a maximizer for $J_{] 0, b [}$ on $\mathcal{G}_{0,b}^1$. Let $0 < \sigma < \tau \leq b$ with $\tau - \sigma \ll 1$ and put $S = \theta_0(\sigma)$, $T = \theta_0(\tau)$. Then $g_{\sigma, \tau}^{S, T}$ is the unique maximizer for $J_{] \sigma, \tau [}$ in $H_{S,T}^1(] \sigma, \tau [)$. If $g_{\sigma, \tau}^{S, T}$ belongs to the smaller space $\mathcal{G}_{\sigma, \tau}^{S, T}$ then it is also the unique maximizer in that smaller space and we conclude that

$$\theta_0|_{] \sigma, \tau [} = g_{\sigma, \tau}^{S, T}. \quad (3.63)$$

Similarly, with $\sigma = 0$, if $0 < \tau$ is small enough, we see from (3.62) that $g_\tau^T \in \mathcal{G}_\tau^T$, when $T = \theta_0(\tau) > 0$. Now g_τ^T is the unique maximizer for $J_{] 0, \tau [}$ on $H_T^1(] 0, \tau [)$ and a fortiori on \mathcal{G}_τ^T , and we conclude that

$$\theta_0|_{] 0, \tau [} = g_\tau^T. \quad (3.64)$$

As above, let θ_0 be a maximizer for $J_{] 0, b [}$ on $\mathcal{G}_{0,b}^1$, put

$$\tilde{\phi}_0 = \log \theta_0, \quad \tilde{\psi}_0 = \tilde{\phi}'_0 = \theta'_0 / \theta_0, \quad (3.65)$$

and observe that $|\tilde{\psi}_0| \leq 1$. From (3.64) we deduce that this inequality is strict near $s = 0$.

Lemma 3.22 *We have $\theta'_0 \leq 0$, so $-\theta_0 \leq \theta'_0 \leq 0$, $-1 \leq \tilde{\psi}_0 \leq 0$.*

Proof. Assume that $\theta'_0 > 0$ on a set of positive measure and define $\theta_1 \in \mathcal{G}_\tau^1$ by

$$\theta_1(b) = 1, \quad \tilde{\psi}_1 := \theta'_1/\theta_1 = -|\tilde{\psi}_0|.$$

Then

$$\theta_1(t) = \exp \int_b^t \tilde{\psi}_1(s) ds, \quad \theta_0(t) = \exp \int_b^t \tilde{\psi}_0(s) ds$$

and

$$\theta_1(s) \geq \theta_0(s), \quad (3.66)$$

with strict inequality near $s = 0$.

Now, for $j = 0, 1$,

$$\theta_j(t)^2 - \theta'_j(t)^2 = \theta_j(t)^2(1 - \tilde{\psi}_j(t)^2)$$

where the last factor in the right hand side is independent of j . Hence by (3.66), we get

$$\theta_1(t)^2 - \theta'_1(t)^2 \geq \theta_0(t)^2 - \theta'_0(t)^2$$

and the inequality is strict near $t = 0$, so $J_{]0,b[}(\theta_1) > J_{]0,b[}(\theta_0)$, in contradiction with the maximality of θ_0 . \square

We now employ first order ODEs as in Subsubsection 3.4.2. Let f be an $1/m$ -harmonic function on some interval $] \sigma, \tau[\subset] 0, b[$ such that

$$-f < f' \leq 0. \quad (3.67)$$

Put

$$\mu = m'/m.$$

Then from

$$(\partial_s \circ m^{-2} \circ \partial_s + m^{-2})f = 0,$$

we get

$$(\partial_s^2 - 2\mu\partial_s + 1)f = 0. \quad (3.68)$$

Writing

$$\tilde{\phi} = \log f \text{ and } \tilde{\psi} = \tilde{\phi}' = f'/f,$$

we get

$$-1 < \tilde{\psi} \leq 0 \text{ and } \tilde{\phi}'' + \tilde{\phi}'^2 - 2\mu\tilde{\phi}' + 1 = 0. \quad (3.69)$$

We can rewrite the last equation in the form

$$\tilde{\psi}' = 2\mu\tilde{\psi} - \tilde{\psi}^2 - 1, \quad (3.70)$$

or equivalently,

$$\tilde{\psi}' = 2 \left(\mu - \frac{1}{2} \left(\tilde{\psi} + \frac{1}{\tilde{\psi}} \right) \right) \tilde{\psi}. \quad (3.71)$$

Notice that this is the same equation as (3.29), after replacing μ with $-\mu$.

In the region $-1 < \tilde{\psi} < 0$, we have $(-1/2)(\tilde{\psi} + 1/\tilde{\psi}) > 1$, hence

$$2 \left(\mu - \frac{1}{2} \left(\tilde{\psi} + \frac{1}{\tilde{\psi}} \right) \right) > 1 + \mu,$$

and we conclude that

$$\tilde{\psi}' < 0, \text{ when } -1 < \tilde{\psi} < 0 \text{ and } \mu \geq -1. \quad (3.72)$$

When, $\mu < -1$ and $-1 < \tilde{\psi} < 0$, we have the equivalences

$$\tilde{\psi}' < 0 \Leftrightarrow \mu - \frac{1}{2} \left(\tilde{\psi} + \frac{1}{\tilde{\psi}} \right) > 0 \Leftrightarrow g(\mu) < \tilde{\psi} < 0,$$

where $g = g(\mu)$ is the unique solution in $] -1, 0[$ of

$$\mu = \frac{1}{2} \left(g + \frac{1}{g} \right) \text{ or equivalently } g^2 - 2\mu g + 1 = 0,$$

i.e.

$$g(\mu) = \mu + \sqrt{\mu^2 - 1} = \frac{1}{\mu - \sqrt{\mu^2 - 1}}. \quad (3.73)$$

In other terms, when $\mu < -1$, $-1 < \tilde{\psi} < 0$, we have

$$\tilde{\psi}' \geq 0 \text{ if and only if } -1 < \tilde{\psi} \leq g(\mu). \quad (3.74)$$

In all cases, we see directly from (3.70) that

$$\tilde{\psi}'(s) < 0, \text{ when } |\tilde{\psi}(s)| \leq 1/|\mathcal{O}(1)|, \quad (3.75)$$

so integral curves of (3.71) cannot enter a neighborhood of $\tilde{\psi} = 0$ from a region where $\tilde{\psi} \leq -1/C$.

Remark 3.23 *We have seen that the equations (3.29) and (3.71) differ only by a change of sign of μ . There is a corresponding symmetry for the solutions: If $\psi \in C^1(] \sigma, \tau[;]0, +\infty[)$, $0 \leq \sigma < \tau \leq +\infty$, then*

$$\tilde{\psi}(s) := -1/\psi(s) \quad (3.76)$$

belongs to the same space and

1. ψ solves (3.29) if and only if $\tilde{\psi}$ solves (3.71).
2. Equivalently, if $u'/u = \psi$, $\theta'/\theta = \tilde{\psi}(= -u/u')$, with $u, \theta > 0$, then u is m -harmonic if and only if θ is $1/m$ -harmonic.
3. Pointwise: $\partial_s \psi(s) \geq 0 \iff \partial_s \tilde{\psi} \geq 0$.
4. Pointwise: $1 < \psi(s) < +\infty \iff -1 < \tilde{\psi}(s) < 0$.
5. We have $\psi(s) \rightarrow \infty$ when $s \rightarrow \sigma$ if and only if $\tilde{\psi}(s) \rightarrow 0$ when $s \rightarrow \sigma$.
6. Let $s_0 \in \{\sigma, \tau\}$. Then, $\psi(s) \rightarrow 1$ when $s \rightarrow s_0$ if and only if $\tilde{\psi}(s) \rightarrow -1$ when $s \rightarrow s_0$.

Structure of maximizers. Let us return to the maximizer $\theta_0 = e^{\tilde{\phi}_0}$ introduced before Lemma 3.22. We know that θ_0 is $1/m$ -harmonic on some interval $]0, \tau[$, $\tau > 0$ and that $\tilde{\psi}_0 := \tilde{\phi}'_0/\tilde{\phi}_0 \in [-1, 0[$. From (3.74), we see that $\tilde{\psi}'_0 < 0$ near 0 and

$$-1 \leq \tilde{\psi}_0(s) \leq -1/|\mathcal{O}(1)|,$$

on $] \epsilon, b[$, for every $\epsilon > 0$. Thus whenever θ_0 is $1/m$ -harmonic on a subinterval $\subset] \epsilon, b[$, we have the differential equation (3.71) (with $\tilde{\psi}$ replaced by $\tilde{\psi}_0$) with a nice uniform control (no blow up). Also

$$(\tilde{\phi}_0)|_{] \epsilon, b[} \in H^1(] \epsilon, b[), \quad (3.77)$$

for every $\epsilon > 0$.

As in Subsection 3.5 we have

Proposition 3.24 *Let $0 < \sigma < \tau \leq b$ and let us assume that $\lambda := m|_{\sigma, \tau}(\tilde{\psi}_0) > -1$. Then there exists $s_0 \in] \sigma, \tau[$ and α, β with*

$$0 \leq \alpha < s_0 < \beta \leq b, \quad (3.78)$$

such that

$$\theta_0 \text{ is } 1/m\text{-harmonic and } -1 < \tilde{\psi}_0 < 0 \text{ on }] \alpha, \beta[. \quad (3.79)$$

- If $\beta < b$, we have $\tilde{\psi}_0(s) \rightarrow -1$, $s \nearrow \beta$.
- If $\alpha > 0$, we have $\tilde{\psi}_0(s) \rightarrow -1$, $s \searrow \alpha$.

We recall that $\tilde{\psi}_0(s) \rightarrow 0$, when $s \searrow 0$.

Moreover s_0 can be chosen so that $\tilde{\psi}_0(s_0)$ is arbitrarily close to λ .

Let $J \subset]0, b[$ be the countable disjoint union of all open maximal intervals $I \subset]0, b[$, such that θ_0 is $1/m$ -harmonic and $-1 < \tilde{\psi}_0 < 1$ on I .

Proposition 3.25 *$\tilde{\psi}_0$ is uniformly Lipschitz continuous on $]0, b[$, > -1 on J , and $= -1$ on $]0, b[\setminus J$.*

Using Remark 3.23, we can carry over the results about minimizers u_0 on maximal subintervals where $u'_0/u_0 > 1$, to maximizers θ_0 on maximal subintervals where θ_0 is $1/m$ -harmonic with $-1 < \theta'_0/\theta_0 < 0$. Thus for instance we have

Proposition 3.26 *Assume that $\mu \geq -1$ on $[0, b]$ and let θ_0 be a maximizer for $J_{]0, b[}$ on $\mathcal{G} = \mathcal{G}_b^1$. Then there exists $\tilde{b} \in]0, b]$ such that*

$$\theta_0 \in C^2([0, \tilde{b}]), \theta'_0(0) = 0,$$

$$\theta_0 \text{ is } m\text{-harmonic and } -1 < \theta'_0/\theta_0 < 0 \text{ on }]0, \tilde{b}[,$$

$\theta_0(s) = e^{b-s}$ on $]s, b[$ (if this interval is $\neq \emptyset$).

We end this subsection with a discussion of global maximizers. Let

$$\mathcal{G}(]0, +\infty[) := \{u \in H_{\text{loc}}^1([0, +\infty[); 0 \leq u' \leq u, u > 0 \text{ on }]0, +\infty[\}.$$

We say that $\theta_0 \in \mathcal{G}(]0, +\infty[)$ is a maximizer (or a global maximizer when emphasizing that we work on the whole half axis) if $\theta_0|_{]0, b[}$ is a maximizer in $\mathcal{G}_b^{\theta_0(b)}$ for every $a > 0$.

Proposition 3.27 *A global maximizer $\theta_0 \in \mathcal{G}(]0, +\infty[)$ exists.*

Indeed, the proof of Proposition 3.14 applies with minor changes.

The discussion of the structure of maximizers in \mathcal{G}_b^1 carries over directly to that of global maximizers. In particular, if θ_0 is a global maximizer, then θ_0 is $1/m$ -harmonic with $-1 < \theta_0'/\theta_0 < 0$ on a maximal interval of the form $]0, b*[$ for some $b^* \in]0, +\infty[$. b^* is uniquely determined and (the $1/m$ -harmonic function) $\theta_0|_{]0, b^*[}$ is unique up to a constant positive factor.

By Remark 3.23, we have

$$b^* = a^*. \quad (3.80)$$

b^* is also characterized as the largest number in $]0, +\infty[$ such that the smallest eigenvalue of $K_{1/m, b}^{N\tilde{R}}$ is > 0 for $b < b^*$. Here $K_{1/m, b}^{N\tilde{R}}$ is defined as in the introduction, with m replaced by $1/m$ and with the domain

$$\mathcal{D}(K_{1/m, b}^{N\tilde{R}}) = \{u \in H^2(]0, b[); u'(0) = 0, u'(b) = -u(b)\}.$$

As for the minimization problem, we have

Proposition 3.28 *For $\mathbb{R} \ni b \in]0, a^*]$, we have*

$$\sup_{\mathcal{G}} \int_0^b (\theta^2 - \theta'^2) m^{-2} ds = -\frac{\tilde{\psi}_0(b)}{m(b)^2} = \frac{1}{m(b)^2} \frac{1}{\psi_0(b)}. \quad (3.81)$$

In particular, when $b = a^* < +\infty$:

$$\sup_{\mathcal{G}} \int_0^b (\theta^2 - \theta'^2) m^{-2} ds = \frac{1}{m(a^*)^2}. \quad (3.82)$$

Proof. Similarly to the proof of Proposition 3.18, we can this time start from the global maximizer θ_0 and compute for $b \leq b^*$ the integral $\int_0^b (\theta^2 - \theta'^2) m^{-2} ds$ with $\theta(s) = \theta_0(s)/\theta_0(b)$. We obtain

$$\int_0^b (\theta^2 - \theta'^2) m^{-2} ds = -\theta_0'(b) m(b)^{-2}. \quad (3.83)$$

We then use (3.80) and Remark 3.23. \square

Remark 3.29 In the case when $m = 1$. We have $b^* = \frac{\pi}{4}$.

$$\theta_0(s) = \sqrt{2} \cos s.$$

The corresponding energy is under the condition $0 < b \leq \frac{\pi}{4}$,

$$\int_0^b (\theta(s)^2 - \theta'(s)^2) ds = \tan b.$$

4 Optimization in Th. 1.6: case $\epsilon_1 = -\epsilon_2 = +$.

4.1 Reduction to $\omega = 0$ and $r(0) = 1$

Let $A, r = r(\omega), \omega$ be as in Theorem 1.6 and (1.3). Let $\hat{\omega} \in \mathbb{R}, \hat{r} = \hat{r}(\hat{\omega}) > 0$. Then $(\hat{A}, \hat{r}(\hat{\omega}), \hat{\omega})$ has the same properties, if we define \hat{A} by

$$\frac{1}{r}(A - \omega) = \frac{1}{\hat{r}}(\hat{A} - \hat{\omega}).$$

Notice here that (1.3) can be written

$$1 = \sup_{\Re w > 0} \|r(\omega)(A - \omega - w)^{-1}\|$$

and that

$$r(\omega)(A - \omega - w)^{-1} = \hat{r}(\hat{\omega})(\hat{A} - \hat{\omega} - \hat{w})^{-1},$$

if $\hat{w}/\hat{r} = w/r$.

Let $S(t) = \exp(tA), \hat{S}(\hat{t}) = \exp(\hat{t}\hat{A}), t, \hat{t} \geq 0$. If $\|S(t)\| \leq m(t)$ for some $t \geq 0$, then $\|\hat{S}(\hat{t})\| \leq \hat{m}(\hat{t})$ if

$$\frac{\hat{m}(\hat{t})}{e^{\hat{\omega}\hat{t}}} = \frac{m(t)}{e^{\omega t}}, \quad \hat{r}\hat{t} = rt.$$

This follows from,

$$e^{-\omega t} S(t) = \exp t(A - \omega) = \exp \hat{t}(\hat{A} - \hat{\omega}) = e^{-\hat{\omega}\hat{t}} \hat{S}(\hat{t}).$$

Theorem 1.6 tells us that if $\|S(t)\| \leq m(t), t \geq 0$, then $\|S(t)\| \leq m_{\text{new}}(t)$, for $t \geq 0$, where

$$\frac{m_{\text{new}}(t)}{e^{\omega t}} = \frac{\|(r(\omega)^2 \Phi^2 - \Phi'^2)^{\frac{1}{2}} m/e^{\omega \cdot}\|_{L^2([0,t])} \|(r(\omega)^2 \Psi^2 - \Psi'^2)^{\frac{1}{2}} m/e^{\omega \cdot}\|_{L^2([0,t])}}{\int_0^t (r(\omega)^2 \Phi^2 - \Phi'^2)^{\frac{1}{2}}_{\epsilon_1} (r(\omega)^2 \Psi^2 - \Psi'^2)^{\frac{1}{2}}_{\epsilon_2} ds}. \quad (4.1)$$

With $\hat{\Phi}(\hat{t}) = \Phi(t), \hat{\Psi}(\hat{t}) = \Psi(t)$, we have $\Phi'(t)/r(\omega) = \hat{\Phi}'(\hat{t})/\hat{r}(\hat{\omega})$ and similarly for $\Psi', \hat{\Psi}'$. If $\hat{m}_{\text{new}}(\hat{t})$ is defined by $\hat{m}_{\text{new}}(\hat{t})/e^{\hat{\omega}\hat{t}} = m_{\text{new}}(t)/e^{\omega t}$, then (4.1) implies the analogous relation for \hat{m}_{new} :

$$\frac{\hat{m}_{\text{new}}(\hat{t})}{e^{\hat{\omega}\hat{t}}} = \frac{\|(\hat{r}(\hat{\omega})^2 \hat{\Phi}^2 - \hat{\Phi}'^2)^{\frac{1}{2}} \hat{m}/e^{\hat{\omega} \cdot}\|_{L^2([0,\hat{t}])} \|(\hat{r}(\hat{\omega})^2 \hat{\Psi}^2 - \hat{\Psi}'^2)^{\frac{1}{2}} \hat{m}/e^{\hat{\omega} \cdot}\|_{L^2([0,\hat{t}])}}{\int_0^{\hat{t}} (\hat{r}(\hat{\omega})^2 \hat{\Phi}^2 - \hat{\Phi}'^2)^{\frac{1}{2}}_{\epsilon_1} (\hat{r}(\hat{\omega})^2 \hat{\Psi}^2 - \hat{\Psi}'^2)^{\frac{1}{2}}_{\epsilon_2} d\hat{s}}. \quad (4.2)$$

We also saw above that $\|\hat{S}(\hat{t})\| \leq \hat{m}_{\text{new}}(\hat{t})$. Thus if we have proved Theorem 1.6 for (A, ω, r, m) we get it also for $(\hat{A}, \hat{\omega}, \hat{r}, \hat{m})$, and vice versa. In particular we could reduce the proof of the theorem to the special case when $\omega = 0, r(\omega) = 1$.

We review the above scaling in a slightly special case, keeping an eye on the scaling of some optimizers from Section 3. Let $\hat{A}, \hat{r} = \hat{r}(\hat{\omega}), \hat{\omega}$ be as in Theorem 1.6 and (1.3), where we have added hats for notational convenience. Let

$$A = \frac{1}{\hat{r}(\hat{\omega})}(\hat{A} - \hat{\omega}).$$

As above, we check that A satisfies the general assumptions with $\omega = 0, r = r(\omega) = 1$. With $t = \hat{r}\hat{t} \geq 0$, we have

$$\|e^{tA}\| \leq m(t) \Leftrightarrow \|e^{\hat{t}\hat{A}}\| \leq \hat{m}(\hat{t}),$$

if $m(t) > 0, \hat{m}(\hat{t}) > 0$ are related by

$$m(t) = e^{-\hat{t}\hat{\omega}}\hat{m}(\hat{t}), \text{ or equivalently } \hat{m}(\hat{t}) = e^{t\hat{\omega}/\hat{r}}m(t).$$

Theorem 1.6 applies to $\hat{S}(\hat{t}) = e^{\hat{t}\hat{A}}$. It is a little more scale invariant to rewrite (1.8) as

$$e^{-\hat{\omega}\hat{t}}\|e^{\hat{t}\hat{A}}\| \leq \frac{\|(\hat{\Phi}^2 - (\hat{\Phi}'/\hat{r})^2)_-^{1/2} e^{-\hat{\omega}\cdot}\hat{m}\|_{[0,\hat{t}]}}{\int_0^{\hat{t}} (\hat{\Phi}^2 - (\hat{\Phi}'/\hat{r})^2)_{e_1}^{1/2} ((\iota_{\hat{t}}\hat{\Psi})^2 - (\iota_{\hat{t}}\hat{\Psi}')/\hat{r})_{e_2}^{1/2} d\hat{s}}, \quad (4.3)$$

where the subscript $[0, \hat{t}]$ indicates the interval over which we take the L^2 -norm.

Putting $s = \hat{r}\hat{s}, \Phi(s) = \hat{\Phi}(\hat{s}), \Psi(s) = \hat{\Psi}(\hat{s})$, we get $\hat{\Phi}'/\hat{r} = \Phi', \hat{\Psi}'/\hat{r} = \Psi'$,

$$e^{-\hat{\omega}\hat{t}}\|e^{\hat{t}\hat{A}}\| \leq \frac{\|(\Phi^2 - \Phi'^2)_-^{1/2} m\|_{[0,t]}\|(\Psi^2 - \Psi'^2)_- m\|_{[0,t]}}{\int_0^t (\Phi^2 - \Phi'^2)_{e_1}^{1/2} (\iota_t \Psi^2 - \iota_t \Psi'^2)_{e_2}^{1/2} ds}. \quad (4.4)$$

In (3.5) we studied the minimization of a factor in the numerator,

$$\inf_{u \in H_{0,1}^1([0,a])} I(u), \text{ where } I(u) = I_{[0,a]}(u) = \int_0^a (u'^2 - u^2)_+ m^2 ds. \quad (4.5)$$

The corresponding problem appearing in (4.3) is

$$\inf_{\hat{u} \in H_{0,1}^1([0,\hat{a}])} \hat{I}(\hat{u}), \text{ where } \hat{I}(\hat{u}) = \hat{I}_{[0,\hat{a}]}(\hat{u}) = \int_0^{\hat{a}} ((\hat{u}'/\hat{r})^2 - \hat{u}^2)_+ (e^{-\hat{\omega}\cdot}\hat{m})^2 d\hat{s}. \quad (4.6)$$

u is a minimizer for (4.5) iff \hat{u} is a minimizer for (4.6) when u, \hat{u} are related by

$$\hat{u}(\hat{s}) = u(s). \quad (4.7)$$

We have seen that a minimizer u for (4.5) belongs to the space

$$\mathcal{H}_{0,1}(]0, a[) = \{u \in H^1(]0, a[); 0 \leq u \leq u'\}.$$

The corresponding space for (4.6) is then

$$\hat{\mathcal{H}}_{0,1}(]0, \hat{a}[) = \{\hat{u} \in H^1(]0, \hat{a}[); 0 \leq \hat{u} \leq \hat{u}'/\hat{r}\}.$$

We have seen in Subsection 3.5 that I has an associated global minimizer u which is m -harmonic with $u' > u$ on $]0, a^*[$ and when $a^* < +\infty$ we have $u'(a^*) = u(a^*)$. Moreover a^* is uniquely determined, and up to multiplication with a positive constant, the same holds for $u|_{]0, a^*[}$. Similarly we have a global minimizer \hat{u} associated to \hat{I} , related to a global minimizer u via (4.7). The corresponding variational equation on any open interval where $0 \leq \hat{u} < \hat{u}'/\hat{r}$, is

$$\left(\frac{1}{\hat{r}} \partial_{\hat{s}} \circ (e^{-\hat{\omega}\hat{s}} \hat{m}(\hat{s}))^2 \frac{1}{\hat{r}} \partial_{\hat{s}} + (e^{-\hat{\omega}\hat{s}} \hat{m}(\hat{s}))^2 \right) \hat{u} = 0. \quad (4.8)$$

This holds on $]0, \hat{a}^*[$, where $a^* = \hat{r}\hat{a}^*$ and when $a^* < \infty$, we have

$$\hat{u}'(\hat{a}^*)/\hat{r} = \hat{u}(\hat{a}^*).$$

In Subsection 3.4.2 we studied a Riccati equation for an m -harmonic function u in terms of the logarithmic derivative $\psi = u'/u$. In the case of (4.8) with general \hat{r} , $\hat{\omega}$, the natural logarithmic derivative is $\hat{\psi} = (\hat{u}'/\hat{r})/\hat{u}$. In conclusion Theorem 1.10 is a direct consequence of Theorem 1.9.

4.2 Other preliminaries

We now assume $\omega = 0$ and $r(0) = 1$. In this case, (1.8) takes the form

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq \frac{\|(\Phi^2 - \Phi'^2)_-^{\frac{1}{2}} m\|_{L^2(]0, t])} \| \Psi^2 - (\Psi')^2_-^{\frac{1}{2}} m \|_{L^2(]0, t])}}{\int_0^t (\Phi^2 - (\Phi')^2)_+^{\frac{1}{2}} ((\iota_t \Psi)^2 - ((\iota_t \Psi)')^2_-^{\frac{1}{2}}) ds}. \quad (4.9)$$

Replacing (Φ, Ψ) by $(\lambda\Phi, \mu\Psi)$ give for any $(\lambda, \mu) \in (\mathbb{R} \setminus \{0\})^2$ does not change the right hand side. Hence we may choose a suitable normalization without loss of generality. We also choose Φ and Ψ to be piecewise $C^1(]0, t])$ (see Footnote 3 for the definition).

Given some $t > a + b$, we now give the conditions satisfied by Φ :

Property 4.1 ($P_{a,b}$)

1. $\Phi = e^a u$ on $]0, a[$ and $u \in \mathcal{H} := \mathcal{H}_{0,a}^{0,1}$ (cf (3.6))⁷.
2. On $[a, t - b]$, we take $\Phi(s) = e^s$, so $\Phi'^2(s) - \Phi(s)^2 = 0$.

⁷Here is our choice of normalization

3. On $[t - b, t]$ we take $\Phi(s) = e^{t-b}\theta(t - s)$ with $\theta \in \mathcal{G} = \mathcal{G}_b^1$.

Hence, we have

$$\text{Supp}(\Phi^2 - \Phi'^2)_+ \subset [t - b, t].$$

Similarly we assume that Ψ satisfies property $(P_{b,a})$ but with $\theta = 1$, hence

1. $\Psi(s) = e^b v(s)$ on $]0, b[$ with $v \in \mathcal{H}_b$, where $\mathcal{H}_b := \mathcal{H}_{0,b}^{0,1}$.
2. On $[b, t - a]$, we take $\Psi(s) = e^s$.
3. On $[t - a, t]$, $\Psi(s) = e^{t-a}$.

Recalling the definition of ι_t , we get for $\iota_t \Psi$:

1. On $[0, a]$, $\iota_t \Psi = e^{(t-a)}$, satisfying

$$(\iota_t \Psi)^{\prime 2} - (\iota_t \Psi)^2 = -e^{2(t-a)}.$$

2. On $[a, t - b]$, we have $\iota_t \Psi(s) = e^{t-s}$, hence

$$(\iota_t \Psi)'(s)^2 - \iota_t \Psi(s)^2 = 0.$$

3. On $]t - b, t[$, we have

$$(\iota_t \Psi)'(s)^2 - (\iota_t \Psi)^2(s) \geq 0.$$

Recall our choice of $\epsilon_1 = +$ and $\epsilon_2 = -$. Assuming that $t > a + b$, we have under these assumptions on Φ and Ψ

$$\{s; \Phi(s)^2 - (\Phi'(s))^2 > 0, \iota_t \Psi(s)^2 - (\iota_t \Psi'(s))^2 < 0\} \subset [t - b, b].$$

We now compute or estimate the various quantities appearing in (4.9).

We have

$$\|(\Phi^2 - \Phi'^2)_+^{\frac{1}{2}} m\| = e^a \left(\int_0^a (u'(s)^2 - u^2(s)) m(s)^2 ds \right)^{1/2}, \quad (4.10)$$

$$\|(\Psi^2 - (\Psi')^2)_-^{\frac{1}{2}} m\| = e^b \left(\int_0^b (v'(s)^2 - v(s)^2) m(s)^2 ds \right)^{1/2}, \quad (4.11)$$

and

$$\begin{aligned} & \int_0^t (\Phi^2 - \Phi'^2)_+^{\frac{1}{2}} ((\iota_t \Psi)^2 - (\iota_t \Psi')^2)_-^{\frac{1}{2}} ds \\ &= \int_{t-b}^t (\Phi^2 - \Phi'^2)_+^{\frac{1}{2}} ((\iota_t \Psi)^2 - (\iota_t \Psi')^2)_-^{\frac{1}{2}} ds \\ &= e^{t-b} \int_{t-b}^t (\theta(t-s)^2 - \theta'(t-s)^2)_+^{\frac{1}{2}} ((\iota_t \Psi)^2 - (\iota_t \Psi')^2)_-^{\frac{1}{2}} ds \\ &= e^t \int_0^b (\theta(s)^2 - \theta'(s)^2)_+^{\frac{1}{2}} (v'(s)^2 - v(s)^2)_-^{\frac{1}{2}} ds. \end{aligned} \quad (4.12)$$

So we get from (4.9)

$$\|e^t S(t)\|_{\mathcal{L}(\mathcal{H})} \leq e^{a+b} \left(\int_0^a (u'(s)^2 - u^2(s)) m(s)^2 ds \right)^{\frac{1}{2}} K(b, \theta, v), \quad (4.13)$$

where

$$K(b, \theta, v) := \frac{\left(\int_0^b (v'(s)^2 - v(s)^2) m(s)^2 ds \right)^{1/2}}{\int_0^b (\theta(s)^2 - \theta'(s)^2)^{\frac{1}{2}} (v'(s)^2 - v(s)^2)^{\frac{1}{2}} ds}. \quad (4.14)$$

We start by considering for a given $\theta \in \mathcal{G}$

$$K_{\inf}(b, \theta) := \inf_{v \in \mathcal{H}_b} K(b, \theta, v),$$

and get the following:

Lemma 4.2 *If $\theta \in \mathcal{G}$ and $\theta - \theta'$ is not identically 0 on $]0, b[$, we have*

$$K_{\inf}(b, \theta) = \frac{1}{\sqrt{\int_0^b (\theta(s)^2 - \theta'(s)^2) \frac{1}{m^2} ds}}. \quad (4.15)$$

Proof. Inspired by the proof in Subsection 3.6, we consider with

$$h(s) = (\theta(s)^2 - \theta'(s)^2)^{\frac{1}{2}} \geq 0$$

the denominator in (4.14),

$$\int_0^b h(s) (v'(s)^2 - v(s)^2)^{\frac{1}{2}} ds = \int_0^b (h(s)/m(s)) (m(v'(s)^2 - v(s)^2)^{\frac{1}{2}}) ds.$$

By the Cauchy-Schwarz inequality, we have

$$\int_0^b h(s) (v'(s)^2 - v(s)^2)^{\frac{1}{2}} ds \leq \left(\int_0^b (h(s)/m(s))^2 ds \right)^{\frac{1}{2}} \left(\int_0^b (m(s)^2 (v'(s)^2 - v(s)^2)) ds \right)^{\frac{1}{2}},$$

which implies that $K_{\inf}(b, \theta)$ is bounded from below by the right hand side of (4.15).

We have equality for some v in \mathcal{H}_b if and only if

$$m(s) (v'(s)^2 - v(s)^2)^{\frac{1}{2}} = c h(s)/m(s).$$

for some constant $c > 0$. In order to get such a v , we first consider $w \in H^1$ defined by

$$w' = \sqrt{w^2 + h^2 m^{-4}}, \quad w(0) = 0,$$

noticing that the right hand side of the differential equation is Lipschitz continuous in w , so that the Cauchy-Lipschitz theorem applies. According to our assumption on θ , we verify that $w(b) > 0$ and we choose

$$v = \frac{1}{w(b)}w, \quad c = \frac{1}{w(b)}.$$

For this pair (c, v) we get

$$\begin{aligned} & \left(\int_0^b (v'(s)^2 - v(s)^2)m(s)^2 ds \right)^{\frac{1}{2}} / \int_0^b (\theta(s)^2 - \theta'(s)^2)^{\frac{1}{2}} (v'(s)^2 - v(s)^2)^{\frac{1}{2}} ds \\ &= 1 / \left(\int_0^b h^2(s)m^{-2}(s) ds \right)^{\frac{1}{2}}. \end{aligned} \quad (4.16)$$

Returning to the definition of h shows that $K_{\inf}(b, \theta)$ is bounded from above by the right hand side of (4.15) and we get the announced result. \square

To conclude the proof of Proposition 1.8, we just combine Lemma 4.2 and (4.13).

A Appendix: Optimization with $\epsilon_1 = \epsilon_2 = +$

In this section we let $\epsilon_1 = \epsilon_2 = +$ in Theorem 1.6 and assume that $\text{supp}(r(\omega)^2 - \phi'^2)_- \subset [0, a]$, $\text{supp}(r(\omega)^2 - \psi'^2)_- \subset [0, b]$ for some $a, b > 0$, where $\Phi = e^\phi$, $\Psi = e^\psi$. The results we get in this case seem less decisive, but perhaps still of some interest. Assuming, to start with, that ϕ and ψ are given on $[0, a]$ and $[0, b]$ respectively, we shall discuss how to choose ϕ on $]a, +\infty[$ and ψ on $]b, +\infty[$, for every given $t > a + b$, in order to optimize the estimate on $\|S(t)\|$. A later problem will be to choose a, b with $a + b < t$ and the restrictions of ϕ, ψ to $[0, a]$ and $[0, b]$ respectively.

From (1.8) we get with $r = r(\omega)$,

$$\|S(t)\| \leq e^{\omega t} \frac{\|(r^2 - \phi'^2)_-^{1/2} m\|_{\phi-\omega} \|(r^2 - \psi'^2)_-^{1/2} m\|_{\psi-\omega}}{I(\phi, \psi)}, \quad (A.1)$$

where

$$I(\phi, \psi) = I_{a,b,t}(\phi, \psi) = \int_a^{t-b} e^{\phi + \iota\psi} (r^2 - \phi'^2)_+^{1/2} (r^2 - \iota\psi'^2)_+^{1/2} ds. \quad (A.2)$$

Here we put $\iota_t\psi(s) = \psi(t - s)$, and write simply ι when the choice of t is clear. We try to choose $\phi(s)$ for $s \geq a$ and $\psi(s)$ for $s \geq b$ so that $I(\phi, \psi)$ is as large as possible. Write

$$\phi(s) = \phi(a) + \tilde{\phi}(s - a), \quad \psi(s) = \psi(b) + \tilde{\psi}(s - b).$$

For $s \in [a, t - b]$, set $s = a + \tilde{s}$, $0 \leq \tilde{s} \leq t - a - b$. Then with $\tilde{t} = t - a - b$,

$$\begin{aligned}\phi(s) + \iota\psi(s) &= \phi(s) + \psi(t - s) = \phi(a + \tilde{s}) + \psi(t - a - \tilde{s}) \\ &= \phi(a) + \psi(b) + (\phi(a + \tilde{s}) - \phi(a)) + (\psi(b + (t - a - b) - \tilde{s}) - \psi(b)) \\ &= \phi(a) + \psi(b) + \tilde{\phi}(\tilde{s}) + \tilde{\psi}(t - a - b - \tilde{s}) \\ &= \phi(a) + \psi(b) + \tilde{\phi}(\tilde{s}) + \iota_{\tilde{t}}\tilde{\psi}(\tilde{s})\end{aligned}$$

and we get with $\tilde{t} = \iota_{\tilde{t}}$,

$$\begin{aligned}I(\phi, \psi) &= e^{\phi(a) + \psi(b)} \int_0^{\tilde{t}} e^{\tilde{\phi}(\tilde{s}) + \iota_{\tilde{t}}\tilde{\psi}(\tilde{s})} (r^2 - \tilde{\phi}'^2)_+^{1/2} (r^2 - \iota_{\tilde{t}}\tilde{\psi}'^2)_+^{1/2} d\tilde{s} \\ &= e^{\phi(a) + \psi(b)} I_{0,0,\tilde{t}}(\tilde{\phi}, \tilde{\psi}).\end{aligned}\tag{A.3}$$

We wish to choose $\tilde{\phi}, \tilde{\psi}$ with $\tilde{\phi}(0) = \tilde{\psi}(0) = 0$ such that $\tilde{I}(\tilde{\phi}, \tilde{\psi}) = I_{0,0,\tilde{t}}(\tilde{\phi}, \tilde{\psi})$ is as large as possible.

Drop the tildes for a while. The problem is then to choose ϕ, ψ with $\phi(0) = \psi(0) = 0$ such that

$$I(\phi, \psi) = \int_0^t e^{\phi + \iota\psi} (r^2 - \phi'^2)_+^{1/2} (r^2 - \iota\psi'^2)_+^{1/2} ds\tag{A.4}$$

is as large as possible.

At this moment we do not know how to solve this general problem, so we restrict the class of functions (satisfying $\phi(0) = \psi(0) = 0$) by requiring that

$$\phi + \iota\psi = \text{Const. on } [0, t].\tag{A.5}$$

In other words, we require that $(\iota\psi)' = -\phi'$. The constant in (A.5) is then equal to $\phi(t) = \int_0^t \phi'(s) ds$. With $\|\cdot\|$ denoting the $L^2([0, t])$ -norm, we get

$$I(\phi, \psi) = \exp\left(\int_0^t \phi'(s) ds\right) \int_0^t (r^2 - \phi'(s)^2) ds \leq \exp(\sqrt{t}\|\phi'\|)(tr^2 - \|\phi'\|^2),\tag{A.6}$$

requiring also the $|\phi'| \leq r$. Here we have equality precisely when ϕ' is equal to some constant $\alpha \in [0, r]$, so for any given value $\beta \in [0, r\sqrt{t}]$ of $\|\phi'\|$, we should choose

$$\phi' = \alpha, \quad \phi(s) = \psi(s) = \alpha s, \quad \text{with } \sqrt{t}\alpha = \beta.\tag{A.7}$$

The corresponding maximal value of $I(\phi, \psi)$ is given by

$$J(\alpha, t) = te^{\alpha t}(r^2 - \alpha^2)\tag{A.8}$$

We look for the maximum of this function of α :

$$\partial_\alpha J(\alpha, t) = -t^2 e^{\alpha t} \left(\alpha^2 + \frac{2}{t}\alpha - r^2 \right)$$

The two critical points are given by a local maximum at

$$\alpha_+ = \alpha_+(t) = \frac{1}{t}(\sqrt{1+(rt)^2} - 1) \in]0, r[\quad (\text{A.9})$$

and a local minimum at

$$\alpha_- = \alpha_-(t) = -\frac{1}{t}(\sqrt{1+(rt)^2} + 1) < 0.$$

We see that α_+ is a global maximum. The corresponding maximal value is given by

$$J_{\max}(t) = J(\alpha_+, t) = e^{\sqrt{1+(rt)^2}-1} \frac{2}{t}(\sqrt{1+(rt)^2} - 1) \quad (\text{A.10})$$

Let us compute the asymptotic behaviour of $J_{\max}(t)$ when $t \rightarrow +\infty$: We get

$$\begin{aligned} \sqrt{1+(rt)^2} &= rt + \mathcal{O}\left(\frac{1}{rt}\right), \quad \alpha_+ = r - \frac{1}{t} + \mathcal{O}\left(\frac{1}{trt}\right), \\ J_{\max}(t) &= \left(1 + \mathcal{O}\left(\frac{1}{rt}\right)\right) \frac{2}{e} e^{rt} r, \quad rt \rightarrow +\infty. \end{aligned} \quad (\text{A.11})$$

Here we recall that the new $t = \tilde{t}$ is equal to $t - a - b$ for the original t . Returning to the original $I(\phi, \psi) = I_{a,b,t}(\phi, \psi)$ (cf. (A.3)) with $\phi|_{[0,a]}$, $\psi|_{[0,b]}$ prescribed, we get with the choice

$$\begin{cases} \phi(s) - \phi(a) = \alpha_+(s-a), & s \geq a, \\ \psi(s) - \psi(b) = \alpha_+(s-b), & s \geq b, \end{cases} \quad (\text{A.12})$$

$$\alpha_+ = \alpha_+(\tilde{t}), \quad \tilde{t} = t - a - b, \quad (\text{A.13})$$

that

$$\begin{aligned} I(\phi, \psi) &= J_{\max}(t-a-b) e^{\phi(a)+\psi(b)} \\ &= \left(1 + \mathcal{O}\left(\frac{1}{t-a-b}\right)\right) \frac{2}{e} e^{\phi(a)+\psi(b)} r e^{r(t-a-b)}, \quad t-a-b \rightarrow +\infty, \end{aligned} \quad (\text{A.14})$$

when $r = r(\omega) > 0$ is fixed.

Summing up the discussion so far, we get from (A.1), (A.14):

Proposition A.1 *Let $a, b > 0$ and let $\Phi \in C([0, a]; \mathbb{R})$, $\Psi \in C([0, b])$ be increasing, piecewise C^1 with $\Phi(0) = \Psi(0) = 0$,*

$$\begin{cases} r(\omega)^2 \Phi^2 - \Phi'^2 \leq 0, & \text{on } [0, a], \\ r(\omega)^2 \Psi^2 - \Psi'^2 \leq 0, & \text{on } [0, b]. \end{cases}$$

Write $\Phi = e^\phi$, $\Psi = e^\psi$, with ϕ, ψ real. Then for $t > a + b$, with $r = r(\omega)$,

$$e^{-\omega t} \|S(t)\| \leq \frac{\|(r^2 \Phi^2 - \Phi'^2)^{\frac{1}{2}} m\|_{e^\omega \cdot L^2([0,a])} \|(r^2 \Psi^2 - \Psi'^2)^{\frac{1}{2}} m\|_{e^\omega \cdot L^2([0,b])}}{J_{\max}(t-a-b) \Phi(a) \Psi(b)}, \quad (\text{A.15})$$

where $J_{\max}(\tilde{t})$ is given in (A.10) and has the asymptotics (A.11). In particular for large values of $t - a - b$,

$$e^{-\omega t} \|S(t)\| \leq \left(1 + \mathcal{O}\left(\frac{1}{r(t-a-b)}\right)\right) \frac{e}{2r} e^{-r(t-a-b)} \times \frac{\|(r^2\Phi^2 - \Phi'^2)_{-}^{\frac{1}{2}} m\|_{e^{\omega \cdot L^2}([0,a])} \|(r^2\Psi^2 - \Psi'^2)_{-}^{\frac{1}{2}} m\|_{e^{\omega \cdot L^2}([0,b])}}{\Phi(a)\Psi(b)}. \quad (\text{A.16})$$

Here we meet the same quantities as in the previous section. Hence we obtain (cf Theorem 1.9), if $a^*(m)$ is bounded, $r = 1$, $\omega = 0$ and $a, b \leq a^*(m)$, $t > a + b$,

$$\|e^t S(t)\| \leq \left(1 + \mathcal{O}\left(\frac{1}{(t-a-b)}\right)\right) \frac{e}{2} m(a)m(b) e^{a+b} \psi_0(a)^{\frac{1}{2}} \psi_0(b)^{\frac{1}{2}}. \quad (\text{A.17})$$

As $t \rightarrow +\infty$, we have lost a factor $(e/2)$ in comparison with the statement of Theorem 1.9. However it is not excluded that for some t the estimate obtained by this approach is better.

Non optimality. Possible improvements? We have solved the optimization problem for $I(\phi, \psi)$ in (A.4) for (ϕ, ψ) varying in a restricted class. The purpose of this remark is to show that the solution (ϕ, ψ) in (A.7) with $\alpha = \alpha_+$ is not a critical point for $I(\phi, \psi)$ when (ϕ, ψ) varies more freely and hence we can perturb our special solution slightly (leaving the restricted class) and find an even larger value of $I(\phi, \psi)$.

Write $f = \iota\psi$ for simplicity. We then want to find $\phi, f \in C^2([0, 1])$ with

$$\phi(0) = f(t) = 0, \quad (\text{A.18})$$

ϕ increasing, f decreasing (i.e. $\phi' \geq 0$, $f' \leq 0$) with

$$r^2 - \phi'^2 > 0, \quad r^2 - f'^2 > 0, \quad (\text{A.19})$$

such that $I(\phi, \iota f)$ is as large as possible and in particular such that (ϕ, f) is a critical point for I . We make a variational calculation considering infinitesimal variations $(\phi + \delta\phi, f + \delta f)$ with $\delta\phi(0) = \delta f(t) = 0$. Then

$$\delta I(\phi, \iota f) = \text{I} + \text{II} + \text{III},$$

where with $K(\phi, f)(s) := e^{\phi+f} (r^2 - \phi'^2)^{\frac{1}{2}} (r^2 - f'^2)^{\frac{1}{2}}$:

$$\begin{aligned} \text{I} &= \int_0^t K(\phi, f)(s) (\delta\phi(s) + \delta f(s)) ds, \\ \text{II} &= \int_0^t K(\phi, f)(s) \frac{\delta((r^2 - \phi'^2)^{1/2})}{(r^2 - \phi'^2)^{1/2}} ds, \end{aligned}$$

$$\text{III} = \int_0^t K(\phi, f)(s) \frac{\delta((r^2 - f'^2)^{1/2})}{(r^2 - f'^2)^{1/2}} ds.$$

Here,

$$\delta((r^2 - \phi'^2)^{1/2}) = -(r^2 - \phi'^2)^{-1/2} \phi' \delta \phi'$$

and similarly for f , so

$$\text{II} = - \int_0^t K(\phi, f)(s) (r^2 - \phi'^2)^{-1} \phi' \delta \phi' ds,$$

$$\text{III} = - \int_0^t K(\phi, f)(s) (r^2 - f'^2)^{-1} f' \delta f' ds.$$

Here we integrate by parts, using that $\delta \phi(0) = \delta f(t) = 0$:

$$\text{II} = \int_0^t (\partial_s \circ K(\phi, f) (r^2 - \phi'^2)^{-1} \circ \partial_s \phi) \delta \phi ds - K(\phi, f) (r^2 - \phi'^2) \phi' \delta \phi(t),$$

$$\text{III} = \int_0^t (\partial_s \circ K(\phi, f) (r^2 - f'^2)^{-1} \circ \partial_s f) \delta f ds + K(\phi, f) (r^2 - f'^2) f' \delta f(0).$$

This gives,

$$\begin{aligned} \delta(\phi, \iota f) &= \int_0^t (K(\phi, f) + \partial_s \circ K(\phi, f) (r^2 - \phi'^2)^{-1} \circ \partial_s \phi) \delta \phi ds \\ &\quad - K(\phi, f) (r^2 - \phi'^2) \phi' \delta \phi(t) + \\ &\quad \int_0^t (K(\phi, f) + \partial_s \circ K(\phi, f) (r^2 - f'^2)^{-1} \circ \partial_s f) \delta f ds \\ &\quad + K(\phi, f) (r^2 - f'^2) f' \delta f(0). \end{aligned}$$

The Assumption (A.19) implies that $K(\phi, f) (r^2 - \phi'^2) > 0$, $K(\phi, f) (r^2 - f'^2) > 0$ and we see that (ϕ, f) is a critical point precisely when

$$\phi'(t) = 0, \quad f'(0) = 0, \tag{A.20}$$

(in addition to (A.18)) and

$$\begin{cases} K(\phi, f) + \partial_s \circ K(\phi, f) (r^2 - \phi'^2)^{-1} \circ \partial_s \phi = 0 \\ K(\phi, f) + \partial_s \circ K(\phi, f) (r^2 - f'^2)^{-1} \circ \partial_s f = 0 \end{cases} \quad \text{on } [0, t]. \tag{A.21}$$

We conclude that (ϕ, ψ) in (A.7) with $\alpha = \alpha_+$, is not a critical point for $I(\cdot, \cdot)$, since it does not satisfy (A.19). Hence by modifying ϕ slightly near $s = t$, we can increase $I(\phi, \psi)$ further.

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