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L^1 -THEORY FOR HELE-SHAW FLOW WITH LINEAR DRIFT

NOUREDDINE IGBIDA[†]

ABSTRACT. The main goal of this paper is to prove L^1 -comparison and contraction principles for weak solutions of PDE system corresponding to a phase transition diffusion model of Hele-Shaw type with addition of a linear drift. The flow is considered with a source term and subject to mixed homogeneous boundary conditions : Dirichlet and Neumann. The PDE can be focused to model for instance biological applications including multi-species diffusion-aggregation models and pedestrian dynamics with congestion. Our approach combines DiPerna-Lions renormalization type with Kruzhkov device of doubling and de-doubling variables. The L^1 -contraction principle allows afterwards to handle the problem in a general framework of nonlinear semigroup theory in L^1 , taking thus advantage of this strong theory to study existence, uniqueness, comparison of weak solutions, L^1 -stability as well as many further questions.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction and main contributions. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. We are interested in the existence and uniqueness as well as the L^1 -comparison principle for the weak solution concerning the PDE of the type

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = f \\ u \in \text{Sign}(p) \end{cases} \quad \text{in } Q := (0, T) \times \Omega.$$

Here, Sign is the maximal monotone graph defined in \mathbb{R} by

$$\text{Sign}(r) = \begin{cases} 1 & \text{for any } r > 0 \\ [-1, 1] & \text{for } r = 0 \\ -1 & \text{for any } r < 0, \end{cases}$$

V and f are the given velocity field and source term respectively, satisfying besides assumptions we precise next. In the case of nonnegative solution (one phase problem), the problem may be written in the widespread form

$$(1.2) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = f \\ 0 \leq u \leq 1, p \geq 0, p(u - 1) = 0 \end{cases} \quad \text{in } Q.$$

The linear version of the problem which corresponds to the case where Sign 's graph is replaced by the identity ; that is to take $p = u$, the problem matches with Fokker-Planck equation. Existence, uniqueness and stability of weak solutions for this case is studied in [34] with possibly linear degenerate

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diffusion, additive noise and BV -vector fields V in \mathbb{R}^N . One can see moreover the work [41] where the case of irregular coefficient and the associated stochastic differential equation are treated.

Nonlinear versions of Fokker-Planck equation cover several mathematical models in biological applications and pedestrian flow. The system (1.1) is a density dependent flow model which may be obtained following hydrodynamical approach for collective motion. Examples may be found in [12]. In the PDE system (1.1), the density constraint $|\rho| \leq 1$ is strongly connected to the microscopic non-overlapping constraint between the agents, and its coupling in a second order equation with linear drift is natural in many settings. This becomes widespread in the description of dynamics with congestion, particularly in pedestrian flow (cf. [42]) and in biological applications including multi-species diffusion-aggregation models (cf. [20] and the references therein). Indeed, even if the transport equation (linear or non-linear) remains to be the master equation for these type of phenomena, one needs to go with second order terms to perform some local behavior likely connected to the ‘thinking’ concept of the agents (cf. [36]). In general, the paths followed by the agents in transport equation are chosen a priori independently from their local dynamics. This may lead to greater accumulation of agents obstructing the non overlapping constraint $|\rho| \leq 1$. The introduction of the second variable p with the complementary condition $\rho \in \text{Sign}(p)$ typically allows to avoid this obstruction and describes the motion of congested zones. Roughly speaking, while the drift manages the dynamic of the crowd with an overview vector field V , the second order term gives rise to patch the dynamic in the congested zones with a local view looking out to allowable neighbors positions. A typical example of this point of view remains to be the constrained diffusion-transport equation which was performed in the pioneering work by B. Maury and al. (cf. [42]) using a gradient flow in the Wasserstein space of probability measures. In [42], this coupling was performed for one population leading to one phase Hele-Shaw problem ; i.e. the main actor for the dynamic remains to be the density of lonely population $0 \leq \rho \leq 1$. However, the case of two-species in inter-actions and occupying the same habitat, like diffusion-aggregation models, may be described by two-phase Hele-Shaw problem of the type (1.1), where the unknown ρ represents through its positive and negative parts the densities of each specie respectively. The source term f can model reaction phenomena connected to agent supply in biological models. This happens in particular when one deals with reaction diffusion system coupling the equation (1.1) with other PDE. As to the boundary condition, we’ll focus on homogeneous (null) mixed one. Neumann boundary condition is connected to the absence of crossing boundary possibilities ; i.e. no mass goes through the boundary. And, Dirichlet one for p is connected to the possibility of crossing other part of the boundary (exits) without any charge. One can see [33] for other possibilities of boundary conditions and their interpretations.

In general, the problem is of elliptic-hyperbolic-parabolic type. Despite the broad results on this class of nonlinear PDE, the structure of (1.1) and (1.2) excludes them definitely out the scope of the current literature, at least concerning uniqueness (cf. [2, 7, 8, 21, 39, 47], see also the expository paper [9] for a complete list of refs). In spite of the “hard” non-linearity connected to the sign graph, the Fokker-Planck look of the equation with linear drift seems to be very fruitful for the analysis based on gradient flow in the space of probability measures equipped with Wasserstein distance (the distance arising in the Monge-Kantorovich optimal transport problem) in the one phase case (1.2) with no-flux Neumann boundary condition and no-source ; i.e. $f \equiv 0$. Numerous results on the existence, properties and estimates on the weak solution were elaborated in the last decade (one can see for instance [45] and the references therein). Other interesting progress on qualitative properties of a solution was obtained using the notion of viscosity solution (one can see for instance [1] and the references therein). The key utility of the viscosity concept is the skill to describe the pointwise behavior of the free boundary evolution. Nonetheless, the uniqueness is still an open problem and seems to be hard in general.

In the case of monotone velocity field V , the uniqueness of absolute continuous solution in the set of probability equipped with Wasserstein distance has been obtained craftily in [29] for (1.2), in the framework of gradient flow in euclidean Wasserstein space again in the case of Neumann boundary condition and $g \equiv 0$.

Our main results in this paper concern the L^1 -comparison and contraction principles for the diffusion-transport equation of the type (1.1) subject to mixed homogeneous boundary conditions : Dirichlet and Neumann. This allows to handle the problem in the context of nonlinear semigroup theory in $L^1(\Omega)$ to prove thus existence, uniqueness, comparison of weak solutions, L^1 -stability as well as many other questions. We'll not cover all the possibilities with this framework here, but let us mention at least its prospect to tackle the continuous dependence with respect to the Sign graph, including for instance the connection between this problem and the so called incompressible limit of the porous medium equation even in the singular case. This subject would be likely treated in forthcoming works. We tackle the problem using a renormalization approach of DiPerna-Lions type (cf. [30]) combined with Kruzhkov device of doubling variables. We treat the problem in the case of outgoing vector field velocity on the boundary which remains practically useful for many applications like in crowd motion. On the hand, this condition seems to be optimal in the case of Dirichlet boundary condition (see the counter-example in Remark 2).

Among the propose of this paper, let us mention the treatment of the one phase problem (1.2) which will be concerned as well with sufficient condition on V and f . For the application in crowd motion, the conditions may be heuristically connected to the reaction of the concerned population at the position $x \in \Omega$ and time $t > 0$ within "congested" circumstance. Performing its value regarding to the divergence of the velocity vector field V may avoid the congestion.

1.2. Motivation and related works. The nonlinear equation of the type (1.1) is usually called Hele-Shaw Flow. Indeed, in the case where $V \equiv 0$, the equation is a free-boundary problem modeling the evolution of a slow incompressible viscous fluid moving between slightly separated plates (cf. [32, 48, 49, 31, 28, 52] for physical and mathematical formulation). The equation appears also as the viscous incompressible limit of the porous medium equation (see for instance [37, 35, 14] and the references therein).

The study of the Hele-Shaw flow with a reaction term (without drift) goes back to [15] and [16] in the study of the limit of the m -porous medium equation as $m \rightarrow \infty$. Concrete models with a dissimilar typical reaction term appeared after in the study of the tumor growth (cf. [50]). The emergence of a drift in the Hele-Shaw flow appeared for the first time in the study of congested crowd motion (cf. [42, 43, 44]). In this case, the function u is assumed to be nonnegative to model the density of population and the constraints on u prevents an evolution beyond a given threshold.

Completed with boundary condition (Dirichlet, Neumann or mixed Dirichlet-Neumann) and initial data, the questions of existence and uniqueness of a weak solution is well understood by now in the case where $V \equiv 0$ (one can see for instance the papers [15, 16] for the case where $f = f(., u)$, and [50] for a different structure of a reaction term). For general V , though the existence of a solution seems to be more or less well understood, the uniqueness question seems to be very delicate and challenging. The main difficulties comes from the combination between the "hard" non-linearity Sign (so called hard-congestion) and the arbitrary pointing of velocity field for the drift. Although the first order term is linear, one notices the hyperbolic character of the equation (outside the congested region). This motivates the questions of existence and uniqueness of a weak solution, and challenges new as well as standard techniques for first order hyperbolic equations and second order hyperbolic-parabolic

structures as well. In particular, one sees that setting

$$b = (I + \text{Sign}^{-1})^{-1} \quad \text{and} \quad \beta = (I + \text{Sign})^{-1},$$

$v := u - p$, the problem (1.1) falls into the scope of diffusion-transport equation of the type

$$(1.3) \quad \frac{\partial b(v)}{\partial t} - \Delta \beta(v) + \nabla \cdot (V(x)\Phi(v)) = f \quad \text{in } Q := (0, \infty) \times \Omega.$$

The study of this class of degenerate parabolic problems has under-gone a considerable progress in the last twenty years, thanks to the fundamental paper of J. Carrillo [21] in which the Kruzhkov device of doubling of variables was extended to this class of hyperbolic-parabolic-elliptic problems. In [21], the appropriate notion of solution was established : the so called weak-entropy solution. This definition (or, sometimes, parts of the uniqueness techniques of [21]) led to many developments, we refer to the expository paper [9] for a survey and complete references on this subject. As much as restrictive it is, the approach of [21] serves out primarily and ingeniously the case where V does not depend on space and homogeneous Dirichlet boundary condition. Strengthened with L^1 -nonlinear semi-group techniques (cf. [9]), the approach of [21] enables to achieve successful as well the uniqueness of weak solution in the "weakly degenerate" convection-diffusion problems of parabolic-elliptic type ; that is $[\beta = 0] \subseteq [\Phi = 0]$. As far as Dirichlet boundary condition is concerned (cf. [21, 39, 7, 7, 9]), the case of homogeneous Neumann was treated in [6]. Yet, the notion of entropic solution à la Carrillo is definitively the suitable notion for general problem of the type (1.3). Nevertheless, following the theory of hyperbolic equation, the case of linear drift is a particular case for which one expects the uniqueness of weak solution. As far as we are aware of, this case remains to be out of the scope of the current literature on this subject.

In spite of the serving of the second order term, we can not pass over the forcefulness of the first order term, which is here linear and carrying away along a vector field V with an arbitrary pointing in Ω . Heuristically, in the likely case $p \equiv 0$, the PDE turns into a linear initial-boundary value problem for continuity equation. In general, one guess the uniqueness of weak solution under reasonable assumption on V , like bounded total variation coefficients. As far as the regularity of V is involved, the way how to handle the boundary condition is hooked to the proofs of uniqueness and adduce different difficulties to the problem. Also, the treatment of boundary conditions is a crucial step in the proof of uniqueness. This is closely connected in some sense to the regularity of the solution as well as to the pointing direction of V on the boundary. This affects the manner to treat the weak/strong trace of the flux on the boundary (see the papers [3], [5], [10], [22], [25] and also [5], [8], [6]). The approach based on the concept of renormalized solution introduced in 1989 by Lions and DiPerna [30] for tangential velocity field seems to be unmissable and powerful in general for the proof of uniqueness of weak solution for this kind of problem. This concept was extended to bounded domains in \mathbb{R}^N with inflow boundary conditions and velocity field with a kind of Sobolev regularity in [17], [18] and [46]. These results were generalized to velocity fields with BV regularity in [3, 4, 5, 26]. The concept was used also in [34] and [41] to tackle the existence and uniqueness of weak solutions for Fokker-Planck equation and its associated stochastic differential equation in some extreme cases like linear degenerate diffusion with irregular coefficient, additive noise and BV -vector fields V .

In this paper, we show how to use this approach in the presence of second order term of Hele-Shaw type to prove the uniqueness of weak solution in the contest of mixed Dirichlet-Neuman boundary condition with an outward pointing velocity fields V on the boundary. Otherwise, one needs to be more careful with the treatment of flux trace on the boundary. For instance, in the case of purely

Dirichlet boundary condition we'll see that we can loose the uniqueness in the absence of outward pointing boundary condition on V .

1.3. Plan of the paper. In the following subsection, we give the main assumptions we'll use throughout the paper and the definition of weak solution we are dealing with. Section 2, is devoted to the L^1 -comparison principle for the weak solution. We introduce renormalized like formulations and use them with doubling and de-doubling variable techniques à la Kruzhkov inside Ω , to prove first that weak solutions satisfies some kind of local Kato's inequalities. Then, linking the outgoing assumption on the velocity field V on the boundary with the distance-to-boundary function, we prove that we can go with our Kato's inequalities up to the boundary proving thus L^1 -comparison principle, and then deduce the uniqueness. In section 3, we prove the existence of a weak solution by using non-linear semigroup theory governed by L^1 -accretive operator. Here the main ingredient is to use the L^1 -contraction principle for weak solution of stationary problem associated with the ϵ -Euler implicit time discretization of the evolution problem. Then, we pass to the limit in the so called ϵ -approximate solution and prove that the limit is the weak solution of the evolution problem. Section 4 is devoted to some remarks, comments and possible extensions. At last, in Section 5, for completeness we give a complement for the proof of de-doubling variables process to handle space dependent vector field with arbitrary pointing out.

1.4. Preliminaries, remarks and main assumptions. We assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set, with regular boundary of class \mathcal{C}^2 , splitted into regular partition $\partial\Omega = \Gamma_D \cup \Gamma_N$, such that $\Gamma_D \cap \Gamma_N = \emptyset$ and

$$\mathcal{L}^{N-1}(\Gamma_D) > 0.$$

We consider $H_0^1(\Omega)$ (resp. $H_D^1(\Omega)$) the usual space of functions in the Sobolev space $H^1(\Omega)$, with null trace on the boundary $\partial\Omega$ (resp. Γ_D). For any $h > 0$, we denote by

$$(1.4) \quad \xi_h(t, x) = \frac{1}{h} \min \left\{ h, d(x, \partial\Omega) \right\} \quad \text{and} \quad \nu_h(x) = -\nabla \xi_h, \quad \text{for any } x \in \Omega,$$

where $d(\cdot, \partial\Omega)$ names the euclidean distance-to-the-boundary function. We see that $\xi_h \in H_0^1(\Omega)$ is concave, $0 \leq \xi_h \leq 1$ and

$$(1.5) \quad \nu_h(x) = -\frac{1}{h} \nabla d(\cdot, \partial\Omega), \quad \text{for any } x \in \Omega \text{ s.t. } d(x, \partial\Omega) < h \leq h_0 \text{ (small enough).}$$

In particular, for such x , we have $h\nu_h(x) = \nu(\pi(x))$, where $\pi(x)$ design the projection of x on the boundary $\partial\Omega$, and $\nu(y)$ represents the outward unitary normal to the boundary $\partial\Omega$ at y .

We denote by Sign^+ the maximal monotone graph given by

$$\text{Sign}^+(r) = \begin{cases} 1 & \text{for } r > 0 \\ [0, 1] & \text{for } r = 0 \\ 0 & \text{for } r < 0. \end{cases}$$

Moreover, we define Sign_0 and Sign_0^\pm , the discontinuous applications defined from \mathbb{R} to \mathbb{R} by

$$\text{Sign}_0(r) = \begin{cases} 1 & \text{for } r > 0 \\ 0 & \text{for } r = 0 \\ -1 & \text{for } r < 0, \end{cases} \quad \text{Sign}_0^+(r) = \begin{cases} 1 & \text{for } r > 0 \\ 0 & \text{for } r \leq 0 \end{cases} \quad \text{Sign}_0^-(r) = \begin{cases} 0 & \text{for } r \geq 0 \\ 1 & \text{for } r < 0 \end{cases}$$

Throughout the paper, we assume that u_0 and the velocity vector filed V satisfy the following assumptions :

- $u_0 \in L^\infty(\Omega)$ and $0 \leq |u_0| \leq 1$ a.e. in Ω .
- $V \in W^{1,2}(\Omega)^N$, $\nabla \cdot V \in L^\infty(\Omega)$ and satisfies (outward pointing velocity vector field condition on the boundary)

$$(1.6) \quad V \cdot \nu \geq 0 \quad \text{on } \Gamma_D \quad \text{and} \quad V \cdot \nu = 0 \quad \text{on } \Gamma_N.$$

Remark 1. (1) *Since $V \in W^{1,2}(\Omega)^N$ and $\nabla \cdot V \in L^\infty(\Omega)$, then $V \cdot \nu \in H^{-\frac{1}{2}}(\partial\Omega)$. So, a priori (1.6) needs to be understood in a weak sense ; i.e.*

$$\int_{\Omega} V \cdot \nabla \xi \, dx + \int_{\Omega} \nabla \cdot V \, \xi \, dx \geq 0, \quad \text{for any } 0 \leq \xi \in H^1(\Omega)$$

and

$$(1.7) \quad \int_{\Omega} V \cdot \nabla \xi \, dx + \int_{\Omega} \nabla \cdot V \, \xi \, dx = 0, \quad \text{for any } 0 \leq \xi \in H_D^1(\Omega).$$

In particular, this condition implies that

$$(1.8) \quad \liminf_{h \rightarrow 0} \frac{1}{h} \int_{D_h} \xi V(x) \cdot \nu(\pi(x)) \, dx \geq 0, \quad \text{for any } 0 \leq \xi \in H^1(\Omega),$$

where, for any $0 < h \leq h_0$, D_h is a neighbourhood of $\partial\Omega$. Nevertheless, in order to prove uniqueness we'll assume that (1.8) is fulfilled for any $0 \leq \xi \in L^\infty(\Omega)$ (see Lemma 2.5). We do not know if this is a consequence of the assumption (1.6) even if it remains to be true for a large class of practical situations (see Remark 3 and Remark 7).

- (2) *In some models of congested crowd motion, the vector field velocity may be given by*

$$(1.9) \quad V = -\nabla d(\cdot, \Gamma_D), \quad \text{in } \Omega.$$

This means that the pedestrian choose the euclidean geodesic trajectory to the escape door to get away from the environment Ω . See that in this case the velocity vector field is outward pointing on Γ_D as we need, but strictly inward pointing on Γ_N . To fall into the scope of (1.6) one needs for instance to consider (1.9) outside a small neighbourhood of Γ_N and set V to be tangential in this neighbor. This example constitute a typical practical situation (among many others) for which all the results of this paper may be applied. By the way, one sees that in this case (1.8) is fulfilled for any $0 \leq \xi \in L^\infty(\Omega)$ (thanks to (1.5))

To begin with, we consider first the problem

$$(1.10) \quad \left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = f \\ u \in \text{Sign}(p) \end{array} \right\} \quad \text{in } Q$$

$$\left\{ \begin{array}{l} p = 0 \\ (\nabla p + u V) \cdot \nu = 0 \\ u(0) = u_0 \end{array} \right\} \quad \begin{array}{l} \text{on } \Sigma_D := (0, T) \times \Gamma_D \\ \text{on } \Sigma_N := (0, T) \times \Gamma_N \\ \text{in } \Omega, \end{array}$$

where $f \in L^1(Q)$ is given.

Definition 1.1 (Notion of solution). *A couple (u, p) is said to be a weak solution of (1.10) if $(u, p) \in L^\infty(Q) \times L^2(0, T; H_D^1(\Omega))$, $u \in \text{Sign}(p)$ a.e. in Q and*

$$\frac{d}{dt} \int_{\Omega} u \xi + \int_{\Omega} (\nabla p - u V) \cdot \nabla \xi = \int_{\Omega} f \xi, \quad \text{in } \mathcal{D}'(0, T), \quad \forall \xi \in H_D^1(\Omega).$$

We'll say plainly that u is a solution of (1.10) if $u \in C([0, T], L^1(\Omega))$, $u(0) = u_0$ and there exists $p \in L^2(0, T; H_D^1(\Omega))$ such that the couple (u, p) is a weak solution of (1.10).

Remark 2. *The assumptions we suppose on V look alike to be stronger regard to the literature on (linear) continuity equation. We think it is possible to extend our results for weak solution to more general V . But, in the presence of the second order term, the problem seems to be heavy and much more technical. Let us give here at least a short reminder on the uniqueness of weak solution for continuity equation, which could comprise some possible extensions (open questions !) for the Hele-Shaw flow with a linear Drift.*

- (1) *The analysis of the continuity equation (without a second order term) in the case when V has low regularity has drawn considerable attention. For an overview of some of the main contributions, we refer to the lecture notes by Ambrosio and Crippa [4] (see also [26] for more references on this subject). Counter-example for the uniqueness when V is not enough regular (up to the boundary) may be found in [26]. Indeed, regardless the orientation of V at the boundary, uniqueness may be violated as soon as V enjoys BV regularity in every open set $\omega \subset\subset \Omega$, but not at the boundary of Ω (cf. [26] for more details and discussions on this subject).*
- (2) *The outward pointing condition on Γ_D seems to be an optimal condition for uniqueness when one deals with purely Dirichlet boundary condition. Indeed, one sees easily that the 1-dimensional example in (0, 2) :*

$$(1.11) \quad \left\{ \begin{array}{l} \partial_t u - \partial_{xx} p + \partial_x u = 0 \\ u \in \text{Sign}(p) \end{array} \right\} \quad \text{in } (0, \infty) \times (0, 2)$$

$$\left\{ \begin{array}{l} p(0) = p(2) = 0 \\ u(0) \equiv 1 \end{array} \right\} \quad \begin{array}{l} \text{in } (0, \infty) \\ \text{in } (0, 2), \end{array}$$

where $V \equiv 1$ is inward pointing on 0 and outward pointing on 2, has many solutions. Indeed, one can take any smooth function $F : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $F(r) = 1$ for any $r \leq 0$ and $F(r) = 0$ for any $r \geq 1$, the function $u(t, x) = F(t - x)$ for any $(t, x) \in [0, \infty) \times [0, 2]$ is a solution of (1.11). Here $p \equiv 0$ in $[0, \infty) \times [0, 2]$, and the structure of Dirichlet boundary condition in (1.10) leaves u free on the boundary. To handle/avoid this kind of situation one needs to change the selection criteria on the boundary.

- (3) *When the vector field is outward pointing, heuristically the solution would not be worst at/near the boundary. Indeed, at least in the smooth case the solution may be simply "carried out" of the domain along the characteristics and, consequently, the behavior of the solution inside the domain is not substantially affected by what happens close to the boundary. However, let us remind the reader that examples are given in [26] discussing that even if V is outward pointing at $\partial\Omega$, then uniqueness may be violated as soon as the regularity (BV regularity) deteriorates at the boundary.*

2. L^1 -COMPARISON PRINCIPLE

In this section, we focus first on the uniqueness and L^1 -Comparison principle of weak solution. Our first main result is the following.

Theorem 2.1. *Under the assumptions of Section 1.4, we assume moreover that*

$$(2.1) \quad \liminf_{h \rightarrow 0} \frac{1}{h} \int_{\Omega \setminus \Omega_h} \xi V(x) \cdot \nu(\pi(x)) dx \geq 0, \quad \text{for any } 0 \leq \xi \in L^2(\Omega).$$

If (u_1, p_1) and (u_2, p_2) are two weak solutions of (1.10) associated with $f_1, f_2 \in L^1(Q)$ respectively, then there exists $\kappa \in L^\infty(Q)$, such that $\kappa \in \text{Sign}^+(u_1 - u_2)$ a.e. in Q and

$$(2.2) \quad \frac{d}{dt} \int_{\Omega} (u_1 - u_2)^+ dx \leq \int_{\Omega} \kappa (f_1 - f_2) dx, \quad \text{in } \mathcal{D}'(0, T).$$

In particular, we have

$$\frac{d}{dt} \|u_1 - u_2\|_1 \leq \|f_1 - f_2\|_1, \quad \text{in } \mathcal{D}'(0, T).$$

Moreover, if $f_1 \leq f_2$, a.e. in Q , and u_1, u_2 are two corresponding solutions satisfying $u_1(0) \leq u_2(0)$ a.e. in Ω , then

$$u_1 \leq u_2, \quad \text{a.e. in } Q.$$

Remark 3. *The assumption (2.1) is technical for the the proof of Theorem 2.1. It is fulfilled for a large class of vector field V , like for instance the case V is outward pointing in the neighborhood of the boundary. We postpone the technicality of this assumption in Remark 3 after the proof of Theorem 2.1.*

As an immediate and primary consequence of Theorem 2.1, we have the following results. Further consequences, will be given in the following sections.

Corollary 2.1. *Let $u_0 \in L^\infty(\Omega)$ and $f \in L^1(Q)$.*

- (1) *There exists at most one $u \in L^\infty(Q)$ such that, there exists $p \in L^2(0, T; H_D^1(\Omega))$, $u \in \text{Sign}(p)$ a.e. in Q and the couple (u, p) satisfies*

$$-\int_Q u \xi \partial_t \psi dx dt + \int_Q (\nabla p - u V) \cdot \nabla \xi \psi dx dt = \int_Q f \xi \psi dx dt + \int_{\Omega} u_0 \psi(0) \xi dx,$$

for any $\psi \in \mathcal{D}([0, T])$ and $\xi \in H_D^1(\Omega)$.

- (2) *The problem (1.10) has at most one solution, in the sense that $u \in \mathcal{C}([0, T], L^1(\Omega))$, $u(0) = u_0$ and there exists $p \in L^2(0, T; H_D^1(\Omega))$ such that the couple (u, p) satisfies (1.10).*
- (3) *If $u_0 \geq 0$ and $f \geq 0$, then any solution is nonnegative.*

Remark 4. *The main difference between part 1 and part 2 in Corollary 2.1 concerns the way one handle initial data u_0 . While the first assertion handle it in a weak sense, the second one is doing the job in a strong sense (since u is assumed there to be in $\mathcal{C}([0, T], L^1(\Omega))$). Of course the second notion implies the first one. In this paper, we favor the notion of solution in $\mathcal{C}([0, T], L^1(\Omega))$, since this is automatically obtained through nonlinear semigroup theory. After all, one sees that both solutions will exist and are equivalent.*

The main tool to prove this result is doubling and de-doubling variables. Since the degeneracy of the problem, we prove first that a weak solution satisfies some kind of (new) renormalized formulation, and then carry out trickily doubling and de-doubling variables device to prove (2.2).

Recall that, the merely transport equation case corresponds to the situation where $p \equiv 0$. In this case, the renormalized formulation reads

$$\partial_t \beta(u) + V \cdot \nabla \beta(u) + u \nabla \cdot V \beta'(u) = f \beta'(u) \quad \text{in } \mathcal{D}'(Q),$$

for any $\beta \in C^1(\mathbb{R})$, where from now on, for any $z \in L^1(\Omega)$ the notation $V \cdot \nabla z$ needs to be understood in the sense of distributions as follows

$$(2.3) \quad V \cdot \nabla z = \nabla \cdot (z V) - z \nabla \cdot V.$$

Proposition 2.1. *Under the assumptions of Section 1.4, if (u, p) is a weak solution of (1.10), then*

$$(2.4) \quad \begin{aligned} & \partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + u \nabla \cdot V(\beta'(u) \chi_{[p=0]} + \text{Sign}_0^+(p)) \\ & \leq f(\beta'(u) \chi_{[p=0]} + \text{Sign}_0^+(p)) \quad \text{in } \mathcal{D}'(Q) \end{aligned}$$

and

$$(2.5) \quad \begin{aligned} & \partial_t \beta(u) - \Delta p^- + V \cdot \nabla \beta(u) + u \nabla \cdot V(\beta'(u) \chi_{[p=0]} - \text{Sign}_0^-(p)) \\ & \leq f(\beta'(u) \chi_{[p=0]} - \text{Sign}_0^-(p)) \quad \text{in } \mathcal{D}'(Q), \end{aligned}$$

for any $\beta \in C^1(\mathbb{R})$ such that $0 \leq \beta' \leq 1$.

Remark 5. *The formulations (2.4) and (2.5) describe some kind of renormalized formulation for the solution u . For the one phase Hele-Shaw problem this formulation reads simply as follows*

$$\begin{aligned} & \partial_t \beta(u) - \Delta p + V \cdot \nabla \beta(u) + u \beta'(u) \nabla \cdot V \chi_{[p=0]} + \nabla \cdot V \chi_{[p \neq 0]} \\ & \leq f(\beta'(u) \chi_{[p=0]} + \chi_{[p \neq 0]}) \quad \text{in } \mathcal{D}'(Q), \end{aligned}$$

for any $\beta \in C^1(\mathbb{R})$ such that $0 \leq \beta' \leq 1$.

We prove this results in two steps, we begin with the case where $\beta \equiv 0$. Then proceeding skillfully with the positive and negative part of p , and using a smoothing procedure with a commutator à la DiPerna-Lions (cf. [30], see also [3]), we prove that any weak solution u satisfies renormalized formulations (2.4) and (2.5).

Lemma 2.1. *Under the assumptions of Section 1.4, if (u, p) is a weak solution of (1.10), then the renormalized formulations (2.4) and (2.5) are fulfilled with $\beta \equiv 0$ in $\mathcal{D}'((0, T) \times \overline{\Omega})$. That is*

$$(2.6) \quad -\Delta p^+ + (\nabla \cdot V - f) \text{Sign}_0^+(p) \leq 0 \quad \text{in } \mathcal{D}'((0, T) \times \overline{\Omega}),$$

and

$$(2.7) \quad -\Delta p^- + (\nabla \cdot V + f) \text{Sign}_0^-(p) \leq 0 \quad \text{in } \mathcal{D}'((0, T) \times \overline{\Omega}).$$

Moreover, we have

$$(2.8) \quad \partial_t u - \Delta p^+ + \nabla \cdot (u V) + \nabla \cdot V \text{Sign}_0^-(p) \leq f(1 - \text{Sign}_0^-(p)) \quad \text{in } \mathcal{D}'(Q),$$

Proof. By density, it is enough to prove (2.6), (2.7) and (2.8) with any test function of the type $\psi \xi$, with non-negative $\psi \in \mathcal{D}(0, T)$ and $\xi \in H^1(\Omega)$. To this aim, we extend p onto $\mathbb{R} \times \Omega$ by 0 for any $t \notin (0, T)$, and for any $h > 0$, we consider

$$\Phi^h(t, x) = \xi(x) \psi(t) \frac{1}{h} \int_t^{t+h} \mathcal{H}_\varepsilon^+(p(s, x)) ds, \quad \text{for a.e. } (t, x) \in Q,$$

where ψ is extended in turn onto \mathbb{R} by 0, and $\mathcal{H}_\varepsilon^+$ is given by

$$\mathcal{H}_\varepsilon^+(r) = \min\left(\frac{r^+}{\varepsilon}, 1\right), \quad \text{for any } r \in \mathbb{R},$$

for arbitrary $\varepsilon > 0$. It is clear that $\Phi_h \in W^{1,2}(0, T; H_D^1(\Omega)) \cap L^\infty(Q)$ is an admissible test function for the weak formulation, so that

$$(2.9) \quad - \iint_Q u \partial_t \Phi^h + \iint_Q (\nabla p - V u) \cdot \nabla \Phi^h = \iint_Q f \Phi^h.$$

See that

$$(2.10) \quad \iint_Q u \partial_t \Phi^h = \iint_Q u \partial_t \psi \frac{1}{h} \int_t^{t+h} \mathcal{H}_\varepsilon^+(p(s)) ds + \iint_Q u(t) \frac{\mathcal{H}_\varepsilon^+(p(t+h)) - \mathcal{H}_\varepsilon^+(p(t))}{h} \psi(t) \xi.$$

Moreover, using the fact that for a.e. $t \in (0, T)$, $-1 \leq u(t) \leq 1$, $\mathcal{H}_\varepsilon^+ \geq 0$ and $\mathcal{H}_\varepsilon^+(0) = 0$, we have $u(t, x) \mathcal{H}_\varepsilon^+(p(t, x)) = \mathcal{H}_\varepsilon^+(p(t, x))$ and $u(t, x) \mathcal{H}_\varepsilon^+(p(t+h, x)) \leq \mathcal{H}_\varepsilon^+(p(t+h, x))$ a.e. $(t, x) \in Q$. So, for any $h > 0$ (small enough), we have

$$\begin{aligned} \iint_Q u(t) \frac{\mathcal{H}_\varepsilon^+(p(t+h)) - \mathcal{H}_\varepsilon^+(p(t))}{h} \psi(t) \xi &\leq \iint_Q \frac{\mathcal{H}_\varepsilon^+(p(t+h)) - \mathcal{H}_\varepsilon^+(p(t))}{h} \psi(t) \xi \\ &\leq \iint_Q \frac{\psi(t-h) - \psi(t)}{h} \mathcal{H}_\varepsilon^+(p(t)) \xi. \end{aligned}$$

This implies that

$$\limsup_{h \rightarrow 0} \iint_Q u(t) \frac{\mathcal{H}_\varepsilon^+(p(t+h)) - \mathcal{H}_\varepsilon^+(p(t))}{h} \psi(t) \xi \leq - \iint_Q \partial_t \psi \mathcal{H}_\varepsilon^+(p(t)) \xi,$$

so that, by letting $h \rightarrow 0$ in (2.10), we get

$$\lim_{h \rightarrow 0} \iint_Q u \partial_t \Phi^h \leq 0.$$

Then, by letting $h \rightarrow 0$ in (2.9)

$$(2.13) \quad \iint_Q (\nabla p - V u) \cdot \nabla (\mathcal{H}_\varepsilon^+(p(t)) \xi) - \iint_Q f \mathcal{H}_\varepsilon^+(p(t)) \xi \leq \iint_Q f \mathcal{H}_\varepsilon^+(p(t)) \xi.$$

On the other hand, using again the fact that $u \mathcal{H}_\varepsilon^+(p) = \mathcal{H}_\varepsilon^+(p)$, a.e. in Q , we have

$$\begin{aligned} \iint_Q (\nabla p - u V) \cdot \nabla (\mathcal{H}_\varepsilon^+(p) \xi) \psi &= \iint_Q \mathcal{H}_\varepsilon^+(p) \nabla p \cdot \nabla \xi \psi + \iint_Q |\nabla p|^2 (\mathcal{H}_\varepsilon^+)'(p) \xi \psi \\ &\quad - \iint_Q V \cdot \nabla (\xi \mathcal{H}_\varepsilon^+(p)) \psi \\ &\geq \iint_Q \mathcal{H}_\varepsilon^+(p) \nabla p \cdot \nabla \xi \psi + \iint_Q \nabla \cdot V (\xi \mathcal{H}_\varepsilon^+(p)) \psi, \end{aligned}$$

where we use (1.7) and the fact that $|\nabla p|^2 (\mathcal{H}_\varepsilon^+)'(p) \geq 0$. Thanks to (2.13), this implies that

$$\iint_Q \nabla p \cdot \nabla \xi \mathcal{H}_\varepsilon^+(p) \xi \psi + \iint_Q \nabla \cdot V \xi \mathcal{H}_\varepsilon^+(p) \psi \leq \iint_Q f \mathcal{H}_\varepsilon^+(p(t)) \xi.$$

Letting now $\varepsilon \rightarrow 0$, we get (2.6). As to (2.7), it follows by using the that $(-u, -p)$ is also a solution of (1.10) with f replaced by $-f$, and applying (2.6) to $(-u, -p)$. At last, recall that

$$(2.14) \quad \partial_t u - \Delta p + \nabla \cdot (u V) = f \quad \text{in } \mathcal{D}'(Q).$$

So, summing (2.7) restrained to $\mathcal{D}'(Q)$ and (2.14), we get (2.8). \square

Now, in order to prove the proposition by using (2.6) and (2.8), we prove the following technical lemma.

Lemma 2.2. *Let $u \in L^1_{loc}(Q)$, $F \in L^1_{loc}(Q)^N$ and $J_1 \in L^1_{loc}(Q)$ be such that*

$$(2.15) \quad \partial_t u + V \cdot \nabla u - \nabla \cdot F \leq J_1 \quad \text{in } \mathcal{D}'(Q)$$

where $V \cdot \nabla u$ is taken in the sense $V \cdot \nabla u = \nabla \cdot (u V) - u \nabla \cdot V$, in $\mathcal{D}'(Q)$. If

$$(2.16) \quad -\nabla \cdot F \leq J_2 \quad \text{in } \mathcal{D}'(Q),$$

for some $J_2 \in L^1_{loc}(Q)$, then

$$(2.17) \quad \partial_t \beta(u) + V \cdot \nabla \beta(u) - \nabla \cdot F \leq J_1 \beta'(u) + J_2(1 - \beta'(u)) \quad \text{in } \mathcal{D}'(\Omega),$$

for any $\beta \in C^1(\mathbb{R})$ such that $0 \leq \beta' \leq 1$.

Proof. We set $Q_\varepsilon := \{(t, x) \in Q : d((t, x), \partial Q) > \varepsilon\}$. Moreover, for any $z \in L^1_{loc}(Q)$, we denote by z_ε the usual regularization of z by convolution given and denoted by

$$z_\varepsilon := z \star \rho_\varepsilon, \quad \text{in } Q_\varepsilon,$$

where ρ_ε is the usual mollifiers sequence defined here in $\mathbb{R} \times \mathbb{R}^N$. It is not difficult to see that (2.15) and (2.16) implies respectively

$$(2.18) \quad \partial_t u_\varepsilon + V \cdot \nabla u_\varepsilon - \nabla \cdot F_\varepsilon \leq J_{1\varepsilon} + \mathcal{C}_\varepsilon \quad \text{in } Q_\varepsilon$$

and

$$(2.19) \quad -\nabla \cdot F_\varepsilon \leq J_{2\varepsilon} \quad \text{in } Q_\varepsilon,$$

where \mathcal{C}_ε is the usual commutator given by

$$\mathcal{C}_\varepsilon := V \cdot \nabla u_\varepsilon - (V \cdot \nabla u)_\varepsilon.$$

Using (2.3), here $(V \cdot \nabla u)_\varepsilon$ needs to be understood in the sense

$$(V \cdot \nabla u)_\varepsilon = (u V) \star \nabla \rho_\varepsilon - (u \nabla \cdot V) \star \rho_\varepsilon, \quad \text{in } Q_\varepsilon.$$

Multiplying (2.18) by $\beta'(u_\varepsilon)$ and (2.19) by $1 - \beta'(u_\varepsilon)$ and adding the resulting equations, we obtain

$$\beta'(u_\varepsilon) \partial_t u_\varepsilon + \beta'(u_\varepsilon) V \cdot \nabla u_\varepsilon - \nabla \cdot F_\varepsilon \leq \mathcal{C}_\varepsilon \beta'(u_\varepsilon) + J_{1\varepsilon} \beta'(u_\varepsilon) + J_{2\varepsilon}(1 - \beta'(u_\varepsilon)) \quad \text{in } Q_\varepsilon$$

and then

$$(2.20) \quad \partial_t \beta(u_\varepsilon) + V \cdot \nabla \beta(u_\varepsilon) - \nabla \cdot F_\varepsilon \leq \mathcal{C}_\varepsilon \beta'(u_\varepsilon) + J_{1\varepsilon} \beta'(u_\varepsilon) + J_{2\varepsilon}(1 - \beta'(u_\varepsilon)) \quad \text{in } Q_\varepsilon.$$

Since $V \in W^{1,1}_{loc}(\Omega)$ and $\nabla \cdot V \in L^\infty(\Omega)$, it is well known by now that, by taking a subsequence if necessary, the commutator converges to 0 in $L^1_{loc}(Q)$, as $\varepsilon \rightarrow 0$ (see for instance Ambrosio [3]). So, letting $\varepsilon \rightarrow 0$ in (2.20), we get (2.17). \square

Now, let us prove how to get the renormalized formulations (2.4) and (2.5).

Proof of Proposition 2.1. Thanks to Lemma 2.1, by using the fact that

$$\nabla \cdot V \text{Sign}_0^-(p) = -u \nabla \cdot V \text{Sign}_0^-(p) \text{ and } \nabla \cdot V \text{Sign}_0^+(p) = u \nabla \cdot V \text{Sign}_0^+(p),$$

we see that (2.15) and (2.16) are fulfilled with

$$F := \nabla p^+, \quad J_1 := (f - u \nabla \cdot V)(1 - \text{Sign}_0^-(p))$$

and

$$J_2 := (f - u \nabla \cdot V) \text{Sign}_0^+(p).$$

Thanks to Lemma 2.2, for any $\beta \in \mathcal{C}^1(\mathbb{R})$ such that $0 \leq \beta' \leq 1$, we deduce that

$$\begin{aligned} \partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + (u \nabla \cdot V - f)(1 - \text{Sign}_0^-(p))\beta'(u) \\ + (f - u \nabla \cdot V) \text{Sign}_0^+(p)(\beta'(u) - 1) \leq 0 \quad \text{in } \mathcal{D}'(Q), \end{aligned}$$

Using again the fact that $\nabla \cdot V \text{Sign}_0^+(p) = u \nabla \cdot V \text{Sign}_0^+(p)$ this implies that

$$\begin{aligned} \partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + u \nabla \cdot V (1 - \text{Sign}_0^-(p))\beta'(u) + u \nabla \cdot V \text{Sign}_0^+(p)(1 - \beta'(u)) \\ \leq f(\text{Sign}_0^+(p)(1 - \beta'(u)) + (1 - \text{Sign}_0^-(p))\beta'(u)) \quad \text{in } \mathcal{D}'(Q). \end{aligned}$$

That is

$$\begin{aligned} \partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + u \nabla \cdot V(\beta'(u)(1 - \text{Sign}_0^-(p) - \text{Sign}_0^+(p)) + \text{Sign}_0^+(p)) \\ \leq f(\beta'(u)(1 - \text{Sign}_0^-(p) - \text{Sign}_0^+(p)) + \text{Sign}_0^+(p)) \quad \text{in } \mathcal{D}'(Q), \end{aligned}$$

and then

$$\begin{aligned} \partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + u \nabla \cdot V(\beta'(u)\chi_{[p=0]} + \text{Sign}_0^+(p)) \\ \leq f(\beta'(u)\chi_{[p=0]} + \text{Sign}_0^+(p)) \quad \text{in } \mathcal{D}'(Q). \end{aligned}$$

Thus (2.4). At last, using the fact that the couple $(-u, -p)$ is a solution of (1.10) with f replaced by $-f$, and applying (2.4) to $(-u, -p)$, we deduce (2.5). \square

See here that the attendance of a nonlinear second order term creates an obstruction to use standard approaches for the uniqueness of weak solutions based effectively on the linearity of the first order term. In order to prove uniqueness in our case, we proceed by doubling and de-doubling variable techniques. To this aim, we'll use the fact that the renormalized formulation implies some kind of local entropic inequalities. This is the aim of the following Lemma.

Lemma 2.3. *Under the assumptions of Section 1.4, if (u, p) is a weak solution of (1.10), then we have*

$$\begin{aligned} \partial_t(u - k)^+ - \Delta p^+ + \nabla \cdot ((u - k)^+ V) + k \nabla \cdot V \text{Sign}_0^+(u - k) \\ (2.21) \quad \leq f \text{Sign}_0^+(u - k) \quad \text{in } \mathcal{D}'(Q), \quad \text{for any } k < 1 \end{aligned}$$

and

$$\begin{aligned} \partial_t(u - k)^- - \Delta p^- + \nabla \cdot ((k - u)^+ V) - k \nabla \cdot V \text{Sign}_0^+(k - u) \\ (2.22) \quad \leq (u - f) \text{Sign}_0^+(k - u) \quad \text{in } \mathcal{D}'(Q) \quad \text{for any } k > -1 \end{aligned}$$

Remark 6. *See here that the entropic inequalities (2.21) and (2.22) are local and do not proceed up to the boundary like in [21]. Indeed, in contrast of Carillo's approach, we are intending here to handle the boundary condition separately by working with the test functions ξ_h combined with the outward pointing condition (2.1). In particular, as we'll see this would reduce the technicality of de-doubling variables process (see the proof Proposition 2.2).*

Proof of Lemma 2.3. Let us consider

$$\beta_\epsilon(r) = \tilde{\mathcal{H}}_\epsilon(r - k), \quad \text{for any } r \in \mathbb{R},$$

where

$$\tilde{\mathcal{H}}_\epsilon(r) = \begin{cases} \frac{1}{2\epsilon} r^{+2} & \text{for } r \leq \epsilon \\ r - \frac{\epsilon}{2} & \text{elsewhere .} \end{cases}$$

In this case, $\beta'_\epsilon(1) = \mathcal{H}_\epsilon^+(1) = 1$, for any $0 < \epsilon \leq \epsilon_0$ (small enough), and we have

$$\beta'_\epsilon(u) = \mathcal{H}_\epsilon^+(u - k) \rightarrow \text{Sign}_0^+(u - k), \quad \text{as } \epsilon \rightarrow 0.$$

For any $k < 1$, since

$$\text{Sign}_0^+(u - k) \chi_{[p=0]} + \text{Sign}_0^+(p) = \text{Sign}_0^+(u - k),$$

letting $\epsilon \rightarrow 0$, in (2.4) where we replace β by $\tilde{\mathcal{H}}_\epsilon$, we get

$$\begin{aligned} \partial_t(u - k)^+ - \Delta p^+ + V \cdot \nabla(u - k)^+ + u \nabla \cdot V \text{Sign}_0^+(u - k) \\ \leq f \text{Sign}_0^+(u - k) \quad \text{in } \mathcal{D}'(Q), \end{aligned}$$

which implies (2.21).

For the second part of the Lemma, we use again the fact that see that $(\tilde{u} := -u, \tilde{p} := -p)$ is a weak solution of (1.10) with f replaced by $\tilde{f} := -f$. So, using (2.21) for (\tilde{u}, \tilde{p}) with \tilde{f} , for any $k < 1$, we have

$$\begin{aligned} -\Delta \tilde{p}^+ + \nabla \cdot ((\tilde{u} - k)^+ V) + k \nabla \cdot V \text{Sign}_0^+(\tilde{u} - k) \\ \leq (\tilde{f} - \tilde{u}) \text{Sign}_0^+(\tilde{u} - k) \quad \text{in } \mathcal{D}'(Q). \end{aligned}$$

This implies that, for any $-1 < s$, we have

$$\begin{aligned} \partial_t(s - u)^- - \Delta p^- + \nabla \cdot ((s - u)^+ V) - s \nabla \cdot V \text{Sign}_0^+(s - u) \\ \leq (u - f) \text{Sign}_0^+(s - u) \quad \text{in } \mathcal{D}'(Q). \end{aligned}$$

Thus the result of the lemma. □

Now, we are able to prove some kind of "local" Kato's inequality which generates our L^1 -approach for the uniqueness.

Proposition 2.2 (Kato's inequality). *Under the assumptions of Section 1.4, if (u_1, p_1) and (u_2, p_2) are two couples of $L^\infty(Q) \times L^2(0, T; H_D^1(\Omega))$ satisfying (2.21) and (2.22) corresponding to $f_1 \in L^1(Q)$ and $f_2 \in L^1(Q)$ respectively, then there exists $\kappa \in L^\infty(Q)$, such that $\kappa \in \text{Sign}^+(u_1 - u_2)$ a.e. in Q and*

$$(2.23) \quad \partial_t(u_1 - u_2)^+ - \Delta(p_1^+ + p_2^-) + \nabla \cdot ((u_1 - u_2)^+ V) \leq \kappa(f_1 - f_2) \quad \text{in } \mathcal{D}'(Q).$$

Proof. The proof of this lemma is based on doubling and de-doubling variable techniques. Let us give here briefly the arguments. To double the variables, we fix $\tau > 0$, and since $u_1(s, y) - \tau < 1$, we use the fact that (u_1, p_1) satisfies (2.21) with $k = u_1(s, y) - \tau$, we have

$$\begin{aligned} & \frac{d}{dt} \int (u_1(t, x) - u_2(s, y) + \tau)^+ \zeta + \int (\nabla_x p_1^+(t, x) - (u_1(t, x) - u_2(s, y) + \tau)^+ V(x) \cdot \nabla_x \zeta \\ & + \int_{\Omega} \nabla_x \cdot V u_2(s, y) \zeta \text{Sign}_0^+(u_1(t, x) - u_2(s, y) + \tau) \leq \int f_1(t, x) \text{Sign}_0^+(u_1(t, x) - u_2(s, y) + \tau) \zeta, \end{aligned}$$

for any $0 \leq \eta \in \mathcal{D}(\Omega \times \Omega)$. See that $\int \nabla_y p_2^-(s, y) \cdot \nabla_x \zeta dx = 0$, so that

$$\begin{aligned} & \frac{d}{dt} \int (u_1(t, x) - u_2(s, y) + \tau)^+ \zeta + \int (\nabla_x p_1^+(t, x) + \nabla_y p_2^-(t, x)) \cdot \nabla_x \zeta - \int (u_1(t, x) - u_2(s, y) + \tau)^+ V(x) \cdot \nabla_x \zeta \\ & + \int \nabla_x \cdot V u_2(s, y) \zeta \text{Sign}_0^+(u_1(t, x) - u_2(s, y) + \tau) \leq \int f_1(t, x) \text{Sign}_0^+(u_1(t, x) - u_2(s, y) + \tau) \zeta. \end{aligned}$$

Denoting by

$$u(t, s, x, y) = u_1(t, x) - u_2(s, y) + \tau, \quad \text{and} \quad p(t, s, x, y) = p_1^+(t, x) + p_2^-(s, y),$$

and integrating with respect to y , we obtain

$$\begin{aligned} & \frac{d}{dt} \int u(t, s, x, y)^+ \zeta + \iint (\nabla_x + \nabla_y) p(t, s, x, y) \cdot \nabla_x \zeta - \iint u(t, s, x, y)^+ V(x) \cdot \nabla_x \zeta \\ & + \iint \nabla_x \cdot V u_2(s, y) \zeta \text{Sign}_0^+ u(t, s, x, y) \leq \iint f_1(t, x) \text{Sign}_0^+ u(t, s, x, y) \zeta. \end{aligned}$$

On the other hand, since $u_1(t, x) + \tau > -1$, using the fact that (u_2, p_2) satisfies (2.22) with $k = u_1(t, x) + \tau$, we have

$$\begin{aligned} & \frac{d}{ds} \int u(t, s, x, y)^+ \zeta + \int (\nabla_y p_2^-(s, y) - u(t, s, x, y)^+ V(y) \cdot \nabla_y \zeta \\ & - \int_{\Omega} \nabla_y \cdot V u_1(t, x) \zeta \text{Sign}_0^+(u(t, s, x, y)) \leq - \int f_2(s, y) \text{Sign}_0^+(u(t, s, x, y)) \zeta. \end{aligned}$$

Working in the same way, we get

$$\begin{aligned} & \frac{d}{ds} \int u(t, s, x, y)^+ \zeta + \iint (\nabla_x + \nabla_y) p(t, s, x, y) \cdot \nabla_y \zeta - \iint u(t, s, x, y)^+ V(y) \cdot \nabla_y \zeta \\ & - \iint \nabla_y \cdot V(y) u_1(t, x) \zeta \text{Sign}_0^+(u(t, s, x, y)) \leq - \iint f_2(s, y) \text{Sign}_0^+(u(t, s, x, y)) \zeta. \end{aligned}$$

Adding both inequalities, we obtain

$$\begin{aligned} (2.24) \quad & \left(\frac{d}{dt} + \frac{d}{ds} \right) \iint u(t, s, x, y)^+ \zeta + \iint (\nabla_x + \nabla_y) p(t, s, x, y) \cdot (\nabla_x + \nabla_y) \zeta \\ & - \iint u(t, s, x, y)^+ (V(x) \cdot \nabla_x \zeta + V(y) \cdot \nabla_y \zeta) \\ & + \iint (\nabla_x \cdot V(x) u_2(s, y) - \nabla_y \cdot V(y) u_1(t, x)) \zeta \text{Sign}_0^+(u(t, s, x, y)) \\ & \leq \iint (f_1(t, x) - f_2(s, y)) \text{Sign}_0^+(u(t, s, x, y)) \zeta. \end{aligned}$$

Now, we can de-double the variables t and s , as well as x and y , by taking as usual the sequence of test functions

$$\psi_\varepsilon(t, s) = \psi\left(\frac{t+s}{2}\right) \rho_\varepsilon\left(\frac{t-s}{2}\right) \text{ and } \zeta_\lambda(x, y) = \xi\left(\frac{x+y}{2}\right) \delta_\lambda\left(\frac{x-y}{2}\right),$$

for any $t, s \in (0, T)$ and $x, y \in \Omega$. Here $\xi \in \mathcal{D}(\Omega)$, $\psi \in \mathcal{D}(0, T)$, ρ_ε is a sequence of usual mollifiers in \mathbb{R} , and δ_λ is the sequence of mollifiers in \mathbb{R}^N given for instance by (5.1). See that

$$\left(\frac{d}{dt} + \frac{d}{ds}\right) \psi_\varepsilon(t, s) = \rho_\varepsilon\left(\frac{t-s}{2}\right) \dot{\psi}\left(\frac{t+s}{2}\right)$$

and

$$(\nabla_x + \nabla_y) \zeta_\lambda(x, y) = \delta_\lambda\left(\frac{x-y}{2}\right) \nabla \xi\left(\frac{x+y}{2}\right)$$

Moreover, for any $h \in L^1((0, T)^2 \times \Omega^2)$ and $\Phi \in L^1((0, T)^2 \times \Omega^2)^N$, it is not difficult to prove that

- $\lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^T \int_\Omega \int_\Omega h(t, s, x, y) \zeta_\lambda(x, y) \psi_\varepsilon(t, s) = \int_0^T \int_\Omega h(t, t, x, x) \xi(x) \psi(t).$
- $\lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^T \int_\Omega \int_\Omega h(t, s, x, y) \zeta_\lambda(x, y) \left(\frac{d}{dt} + \frac{d}{ds}\right) \psi_\varepsilon(t, s) = \int_0^T \int_\Omega h(t, t, x, x) \xi(x) \dot{\psi}(t).$
- $\lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \int_\Omega \Phi(t, s, x, y) \cdot (\nabla_x + \nabla_y) \zeta_\lambda(x, y) \psi_\varepsilon(t, s) = \int_0^T \int_\Omega \Phi(t, t, x, x) \cdot \nabla \xi(x) \psi(t) dt dx.$

So replacing ζ in (2.24) by ζ_λ , testing with ψ_ε and, letting $\varepsilon \rightarrow 0$ and $\lambda \rightarrow 0$, we get

$$\begin{aligned} & - \int_0^T \int_\Omega \left\{ (u_1 - u_2)^+ \xi \dot{\psi} + \nabla(p_1^+ + p_2^-) \cdot \nabla \xi \psi - (u_1 - u_2 + \tau)^+ (V \cdot \nabla \xi + \nabla \cdot V \xi) \psi \right\} \\ & \leq \int_0^T \int_\Omega \kappa_\tau(x) (f_1 - f_2) \xi \psi + \lim_{\lambda \rightarrow 0} \int_0^T \int_\Omega \int_\Omega u(t, t, x, y)^+ (V(x) - V(y)) \cdot \nabla_y \zeta \psi, \end{aligned}$$

where $\kappa_\tau \in L^\infty(Q)$ is such that $\kappa_\tau \in \text{Sign}^+(u_1 - u_2 + \tau)$ a.e. in Q . Thus

$$\begin{aligned} (2.25) \quad & \frac{d}{dt} \int (u_1 - u_2 + \tau)^+ \xi + \int \nabla(p_1^+ + p_2^-) \cdot \nabla \xi - \int (u_1 - u_2 + \tau)^+ (V \cdot \nabla \xi + \nabla \cdot V \xi) \\ & \leq \int \kappa(x) (f_1 - f_2) \xi + \lim_{\lambda \rightarrow 0} \int_\Omega \int_\Omega u(t, t, x, y)^+ (V(x) - V(y)) \cdot \nabla_y \zeta. \end{aligned}$$

To pass to the limit in the last term corresponding to the vector field V , we use moreover the technical result of Lemma 5.7 (cf. Appendix) which is more or less well known. We put back its proof in the Appendix. This implies that

$$\begin{aligned} & \frac{d}{dt} \int (u_1(t, x) - u_2(t, x) + \tau)^+ \xi dx + \int \nabla(p_1^+ + p_2^-) \cdot \nabla \xi dx \\ & - \int (u_1 - u_2 + \tau)^+ V \cdot \nabla \xi dx \leq \int \kappa_\tau (f_1 - f_2) \xi dx. \end{aligned}$$

Letting then $\tau \rightarrow 0$, and using again the fact that $\kappa_\tau \rightarrow \kappa$ weakly to $L^\infty(Q)$, with $\kappa \in \text{Sign}^+(u_1 - u_2)$, a.e. in Q , the result of the proposition follows. \square

The aim now is to proceed with the sequence of test function ξ_h given by (1.4) in Kato's inequality and let $h \rightarrow 0$, to cover (2.2).

Lemma 2.4. *Under the assumptions of Section 1.4, if (u, p) is a weak solution of (1.10), then for any $0 \leq \psi \in D(0, T)$, we have*

$$\liminf_{h \rightarrow 0} \int_0^T \int_{\Omega} \nabla p^{\pm} \cdot \nabla \xi_h \psi(t) dt dx \geq 0.$$

Proof : Using (2.6), we see that, for any $0 \leq \psi \in D(0, T)$ we have

$$\begin{aligned} \int_0^T \int_{\Omega} \nabla p^+ \cdot \nabla \xi_h \psi dt dx &= - \int_0^T \int_{\Omega} \nabla p^+ \cdot \nabla (1 - \xi_h) \psi dt dx \\ &\geq \int_0^T \int_{\Omega} (\nabla \cdot V - f)(1 - \xi_h) \text{Sign}_0^+(p) \psi dt dx \end{aligned}$$

Then, letting $h \rightarrow 0$ and using the fact that $\xi_h \rightarrow 1$ in $L^\infty(\Omega) - \text{weak}^*$, we deduce that $\liminf_{h \rightarrow 0} \int_0^T \int_{\Omega} \nabla p^+(t) \cdot \nabla \xi_h \psi dt dx \geq 0$. The proof for p^- follows in a similar way by using (2.7). \square

Lemma 2.5. *Under the assumptions of Theorem 2.1, if (u_1, p_1) and (u_2, p_2) are two couples of $L^\infty(Q) \times L^2(0, T; H_D^1(\Omega))$ satisfying (2.21) and (2.22) corresponding to $f_1 \in L^1(Q)$ and $f_2 \in L^1(Q)$ respectively, then there exists $\kappa \in L^\infty(Q)$, such that $\kappa \in \text{Sign}^+(u_1 - u_2)$ a.e. in Q and (2.2) is fulfilled.*

Proof. See that

$$\frac{d}{dt} \int (u_1 - u_2)^+ dx - \int \kappa(f_1 - f_2) dx = \underbrace{\lim_{h \rightarrow 0} \frac{d}{dt} \int (u_1 - u_2)^+ \xi_h dx - \int \kappa(f_1 - f_2) \xi_h dx}_{I(h)}.$$

Taking ξ_h as a test function in (2.23), we have

$$\begin{aligned} I(h) &\leq - \int (\nabla(p_1^+ + p_2^-) - (u_1 - u_2)^+ V) \cdot \nabla \xi_h dx \\ &\leq - \int \nabla(p_1^+ + p_2^-) \cdot \nabla \xi_h dx - \int (u_1 - u_2)^+ V \cdot \nu_h(x) dx. \end{aligned}$$

Using Lemma 2.4, this implies that

$$(2.32) \quad \begin{aligned} \lim_{h \rightarrow 0} I(h) &\leq - \lim_{h \rightarrow 0} \int (u_1 - u_2)^+ V \cdot \nu_h(x) dx \\ &\leq 0 \end{aligned}$$

where we use the outgoing velocity vector field assumption (2.1). Thus (2.2). \square

Proof of Theorem 2.1. It clear that the first part follows by Lemma 2.5. The rest of the theorem is a straightforward consequence of (2.2). \square

Remark 7. *See that Lemma 2.5 is the lonely step of the proof of Theorem 2.1 where we use assumption (2.1) (see (2.32)). Working so enables to avoid all the technicality related to doubling and de-doubling variable by using test functions which does not vanish on the boundary. We do believe that the result of Theorem 2.1 remains to be true under the general assumption (1.6). We think that this works at least for Dirichlet boundary condition following Carrillo's techniques (cf. [20]) but it could be heavy*

and very technical. One sees also that, if the solutions have a trace (like for BV solution), maybe one can weaken this condition by handling (2.32) otherwise.

3. EXISTENCE FOR THE EVOLUTION PROBLEM

Throughout this section, we assume that the assumptions of Section 1.4 and (2.1) are fulfilled. Our main result here concerns existence of a solution.

Theorem 3.2. *For any $f \in L^2(Q)$ and $u_0 \in L^\infty(\Omega)$ be such that $|u_0| \leq 1$ a.e. in Ω , the problem (1.10) has a unique solution $u \in \mathcal{C}([0, T], L^1(\Omega))$ and $u(0) = u_0$ in the sense of Definition 1.1. Moreover, the solution satisfies all the properties of Theorem 2.1.*

Thanks to Theorem 3.2 and Theorem 2.1, we have the following practical particular results.

Corollary 3.2. *Let $f \in L^2(Q)$, $u_0 \in L^\infty(\Omega)$ be such that $|u_0| \leq 1$ a.e. in Ω , and u be the unique weak solution of the problem (1.10).*

(1) *If $u_0 \geq 0$ and $f \geq 0$, then (u, p) is the unique weak solution of the one phase problem*

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = f \\ 0 \leq u \leq 1, p \geq 0, p(u - 1) = 0 \end{array} \right\} \quad \begin{array}{l} \text{in } Q \\ \\ \text{on } \Sigma_D \\ \text{on } \Sigma_N \\ \text{in } \Omega, \end{array}$$

in the sense of Definition 1.1.

(2) *For each $n = 1, 2, \dots$, let $f_n \in L^1(Q)$, $u_{0n} \in L^\infty(\Omega)$ be such that $|u_{0n}| \leq 1$ a.e. in Ω , and u_n be the unique weak solution of the corresponding problem (1.10). If $u_{0n} \rightarrow u_0$ in $L^1(\Omega)$ and $f_n \rightarrow f$ in $L^1(Q)$, then $u_n \rightarrow u$, in $\mathcal{C}([0, T], L^1(\Omega))$, $p_n \rightarrow p$, in $L^2(0, T; H_D^1(\Omega))$ – weak, and (u, p) is the solution corresponding to u_0 and f .*

To study the existence, we process in the framework of nonlinear semigroup theory in $L^1(\Omega)$. In connection with the Euler implicit discretization scheme of the evolution problem (1.10), we consider the following stationary problem :

$$(3.1) \quad \left\{ \begin{array}{l} u - \Delta p + \nabla \cdot (u V) = f \\ u \in \text{Sign}(p) \end{array} \right\} \quad \begin{array}{l} \text{in } \Omega \\ \\ \text{on } \Gamma_D \\ \text{on } \Gamma_N, \end{array}$$

where $f \in L^2(\Omega)$ and $\lambda > 0$ are given. A couple $(u, p) \in L^\infty(\Omega) \times H_D^1(\Omega)$ satisfying $u \in \text{Sign}(p)$ a.e. in Ω , is said to be a weak solution of (3.1) if

$$\int_{\Omega} u \xi + \int_{\Omega} \nabla p \cdot \nabla \xi - \int_{\Omega} u V \cdot \nabla \xi = \int_{\Omega} f \xi, \quad \forall \xi \in H_D^1(\Omega).$$

As for the evolution problem, we'll say simply that u is a solution of (3.1) if there exists p such that the couple (u, p) is a weak solution of (3.1).

As a consequence of Theorem 2.1, we can deduce the following result.

Corollary 3.3. *If u_1 and u_2 are two solutions of (3.1) associated with $f_1, f_2 \in L^1(\Omega)$, respectively, then*

$$\int (u_1 - u_2)^+ \leq \int (f_1 - f_2)^+.$$

In particular, we have

$$\|u_1 - u_2\|_1 \leq \|f_1 - f_2\|_1$$

and, if $f_1 \leq f_2$, a.e. in Ω , then

$$u_1 \leq u_2, \quad \text{a.e. in } \Omega.$$

Proof. This is a simple consequence of the fact that if (u, p) (which is independent of t) is a solution of (3.1), then it can be assimilated to a time-independent solution of the evolution problem (1.10) with f replaced by $f - u$ (which is also independent of t). \square

For the existence, we consider the regularized problem

$$(3.2) \quad \begin{cases} \left. \begin{array}{l} u - \Delta p + \nabla \cdot (u V) = f \\ u = \mathcal{H}_\varepsilon(p) \end{array} \right\} & \text{in } \Omega \\ p = 0 & \text{on } \Gamma_D \\ (\nabla p + u V) \cdot \eta = 0 & \text{on } \Gamma_N, \end{cases}$$

for an arbitrary $\varepsilon > 0$. See that for any $\varepsilon > 0$, $|\mathcal{H}_\varepsilon| \leq 1$, \mathcal{H}_ε is Lipschitz continuous and satisfies

$$(I + \mathcal{H}_\varepsilon)^{-1}(r) \rightarrow (I + \text{Sign})^{-1}(r), \quad \text{as } \varepsilon \rightarrow 0, \quad \text{for any } r \in \mathbb{R}.$$

That is \mathcal{H}_ε converges to Sign in the sense of resolvent, which is equivalent to the convergence in the sense of graph (cf. [19]).

Proposition 3.3. *For any $f \in L^2(\Omega)$ and $\varepsilon > 0$, the problem (3.2) has a weak solution $(u_\varepsilon, p_\varepsilon)$ in the sense that $p_\varepsilon \in H_D^1(\Omega)$, $u_\varepsilon = \mathcal{H}_\varepsilon(p_\varepsilon)$ a.e. in Ω and*

$$(3.3) \quad \int u_\varepsilon \xi \, dx + \int \nabla p_\varepsilon \cdot \nabla \xi \, dx - \int u_\varepsilon V \cdot \nabla \xi \, dx = \int f \xi \, dx, \quad \text{for any } \xi \in H_D^1(\Omega).$$

Moreover, as $\varepsilon \rightarrow 0$, we have

$$(3.4) \quad \mathcal{H}_\varepsilon(p_\varepsilon) \rightarrow u \quad \text{in } L^\infty(\Omega) - \text{weak}^*,$$

$$p_\varepsilon \rightarrow p \quad \text{in } H_D^1(\Omega) - \text{weak}$$

and (u, p) is the weak solution of (3.1).

Proof. The existence of a solution for (3.2) is standard. For completeness we give the arguments. Let us denote by $H_D^1(\Omega)^*$ the topological dual space of $H_D^1(\Omega)$ and $\langle \cdot, \cdot \rangle$ the associate dual bracket. See that the operator $A_\varepsilon : H_D^1(\Omega) \rightarrow H_D^1(\Omega)^*$, given by

$$\langle A_\varepsilon p, \xi \rangle = \int \mathcal{H}_\varepsilon(p) \xi \, dx + \int \nabla p \cdot \nabla \xi \, dx - \int \mathcal{H}_\varepsilon(p) V \cdot \nabla \xi \, dx, \quad \forall \xi \in H_D^1(\Omega),$$

is bounded and weakly continuous. Moreover, A_ε is coercive. Indeed, for any $u \in H_D^1(\Omega)$, we have

$$\begin{aligned} \langle A_\varepsilon p, p \rangle &= \int \mathcal{H}_\varepsilon(p) p \, dx + \int |\nabla p|^2 \, dx - \int \mathcal{H}_\varepsilon(p) V \cdot \nabla p \, dx \\ &\geq \int |\nabla p|^2 \, dx - \int |V \cdot \nabla p| \, dx \\ &\geq \frac{1}{2} \int |\nabla p|^2 \, dx - \frac{1}{2} \int |V|^2 \, dx, \end{aligned}$$

where we use Young inequality. So, for any $f \in H_D^1(\Omega)^*$ the problem $A_\varepsilon p = f$ has a solution $p_\varepsilon \in H_D^1(\Omega)$. To let $\varepsilon \rightarrow 0$, we see that

$$(3.9) \quad \int |\nabla p_\varepsilon|^2 \, dx \leq C(N, \Omega) \left(\int |V|^2 \, dx + \int |f|^2 \, dx \right).$$

Indeed, taking p_ε as a test function we have

$$\int u_\varepsilon p_\varepsilon \, dx + \int |\nabla p_\varepsilon|^2 \, dx = \int u_\varepsilon V \cdot \nabla p_\varepsilon \, dx + \int f p_\varepsilon \, dx.$$

See that using Young inequality we have

$$\int u_\varepsilon V \cdot \nabla p_\varepsilon \, dx \leq \frac{3}{4} \int |V|^2 + \frac{1}{3} \int |\nabla p_\varepsilon|^2 \, dx$$

and, by combining Pincaré inequality with Young inequality we have

$$\int f p_\varepsilon \, dx \leq C(N, \Omega) \int |f|^2 + \frac{1}{3} \int |\nabla p_\varepsilon|^2 \, dx.$$

Using the fact that $u_\varepsilon p_\varepsilon \geq 0$, we deduce (3.9). Now, it is clear that the sequences p_ε and $u_\varepsilon = \mathcal{H}_\varepsilon(p_\varepsilon)$ are bounded respectively in $H_D^1(\Omega)$ and in $L^\infty(\Omega)$. So, there exists a subsequence that we denote again by p_ε such that (3.4) and (3.13) are fulfilled. In particular, using monotonicity argument (see for instance [19]) this implies that $u \in \text{Sign}(p)$, a.e. and letting $\varepsilon \rightarrow 0$ in (3.3), we obtain that (u, p) is a weak solution of (3.1). \square

To prove the existence, of a weak solution to (1.10), we fix $f \in L^2(Q)$, and for an arbitrary $0 < \varepsilon \leq \varepsilon_0$, and $n \in \mathbb{N}^*$ be such that $n\varepsilon = T$, we consider the sequence of (u_i, p_i) given by the the ε -Euler implicit scheme associated with (1.10) :

$$(3.10) \quad \left\{ \begin{array}{ll} \left. \begin{array}{l} u_{i+1} - \varepsilon \Delta p_{i+1} + \varepsilon \nabla \cdot (u_{i+1} V) = u_i + \varepsilon f_i \\ u_{i+1} \in \text{Sign}(p_{i+1}) \end{array} \right\} & \text{in } \Omega \\ p_{i+1} = 0 & \text{on } \Gamma_D \\ (\nabla p_{i+1} + u_{i+1} V) \cdot \eta = 0 & \text{on } \Gamma_N, \end{array} \right. \quad i = 0, 1, \dots, n-1,$$

where, for each $i = 0, \dots, n-1$, f_i is given by

$$f_i = \frac{1}{\varepsilon} \int_{i\varepsilon}^{(i+1)\varepsilon} f(s) \, ds, \quad \text{a.e. in } \Omega.$$

Now, for a given ε -time discretization $0 = t_0 < t_1 < t_2 < \dots < t_n = T$, satisfying $t_{i+1} - t_i = \varepsilon$, we define the ε -approximate solution by

$$u_\varepsilon := \sum_{i=0}^{n-1} u_i \chi_{[t_i, t_{i+1})}, \quad \text{and} \quad p_\varepsilon := \sum_{i=1}^{n-1} p_i \chi_{[t_i, t_{i+1})}.$$

Thanks to Proposition 3.3 and the general theory of evolution problem governed by accretive operator (see for instance [13, 11]), we define the operator \mathcal{A} in $L^1(\Omega)$, by $\mu \in \mathcal{A}(z)$ if and only if $\mu, z \in L^1(\Omega)$ and z is a solution of the problem

$$\begin{cases} \left. \begin{array}{l} -\Delta p + \nabla \cdot (z V) = \mu \\ z \in \text{Sign}(p) \end{array} \right\} & \text{in } \Omega \\ p = 0 & \text{on } \Gamma_D \\ (\nabla p + u V) \cdot \eta = 0 & \text{on } \Gamma_N, \end{cases}$$

in the sense that $z \in L^\infty(\Omega)$, there exists $p \in H_D^1(\Omega)$ satisfying $z \in \text{Sign}(p)$ a.e. in Ω and

$$\int_{\Omega} \nabla p \cdot \nabla \xi - \int_{\Omega} z V \cdot \nabla \xi = \int_{\Omega} \mu \xi, \quad \forall \xi \in H_D^1(\Omega).$$

As a consequence of Corollary 3.3, we know that the operator \mathcal{A} is accretive in $L^1(\Omega)$. Moreover, it is not difficult to see that $\overline{\mathcal{D}(\mathcal{A})} = \{u \in L^\infty(\Omega) : |u| \leq 1 \text{ a.e. in } \Omega\}$. So, thanks to the general theory of nonlinear semigroup governed by accretive operator (see for instance [11]), we know that, as $\varepsilon \rightarrow 0$,

$$(3.11) \quad u_\varepsilon \rightarrow u, \quad \text{in } \mathcal{C}([0, T], L^1(\Omega)),$$

and u is the so called "mild solution" of the evolution problem

$$(3.12) \quad \begin{cases} u_t + \mathcal{A}u \ni f & \text{in } (0, T) \\ u(0) = u_0. \end{cases}$$

To accomplish the proof of existence for the problem (1.10), we prove that the mild solution u is in fact the solution of (1.10). To this aim, we use the limit of the sequence p_ε given by the ε -approximate solution.

Lemma 3.6. *As $\varepsilon \rightarrow 0$,*

$$(3.13) \quad p_\varepsilon \rightarrow p, \quad \text{in } L^2(0, T; H_D^1(\Omega))$$

and (u, p) is a weak solution of (1.10).

Proof. Thanks to Proposition 3, the sequence $(u_i, p_i)_{i=1, \dots, n}$ given by (3.10) is well defined in $L^\infty(\Omega) \times H_D^1(\Omega)$ and satisfies $u_i \in \text{Sign}(p_i)$ and

$$(3.14) \quad \int_{\Omega} u_{i+1} \xi + \varepsilon \int_{\Omega} \nabla p_{i+1} \cdot \nabla \xi - \varepsilon \int_{\Omega} u_{i+1} V \cdot \nabla \xi = \varepsilon \int_{\Omega} f_i \xi, \quad \text{for any } \xi \in H_D^1(\Omega).$$

Taking p_{i+1} as a test function in (3.14), working as for the proof of (3.9) and using the fact that $(u_{i+1} - u_i)p_{i+1} \geq 0$, we get

$$\int |\nabla p_i|^2 dx \leq C(N, \Omega) \left(\int |V|^2 dx + \int |f_i|^2 dx \right).$$

Thus

$$\int |\nabla p_\varepsilon|^2 dx \leq C(N, \Omega) \left(\int |V|^2 dx + \int |f_\varepsilon|^2 dx \right),$$

where $f_\varepsilon = \sum_{i=0}^{n-1} f_i \chi_{[t_i, t_{i+1})}$, in Q . This implies that p_ε is bounded in $L^\infty(0, T; H_D^1(\Omega))$, and that there exists $p \in L^\infty(0, T; H_D^1(\Omega))$ such that, taking a subsequence if necessary,

$$p_\varepsilon \rightarrow p, \quad \text{in } L^2(0, T; H_D^1(\Omega)) - \text{weak.}$$

Combining this with (3.11), we deduce moreover that $u \in \text{Sign}(p)$, a.e. in Q . Now, as usually used with nonlinear semigroup theory for evolution problem, we consider

$$\tilde{u}_\varepsilon = \sum_{i=0}^{n-1} \frac{(t - t_i)u_{i+1} - (t - t_{i+1})u_i}{\varepsilon} \chi_{[t_i, t_{i+1})},$$

which converges to u as well in $\mathcal{C}([0, T]; L^1(\Omega))$. We see that for any test function $\xi \in H_D^1(\Omega)$, we have

$$\frac{d}{dt} \int_\Omega \tilde{u}_\varepsilon \xi + \int_\Omega (\nabla p_\varepsilon - u_\varepsilon V) \cdot \nabla \xi = \int_\Omega f_\varepsilon \xi, \quad \text{in } \mathcal{D}'([0, T]).$$

So, letting $\varepsilon \rightarrow 0$, and using the convergence of $(\tilde{u}_\varepsilon, u_\varepsilon, p_\varepsilon, f_\varepsilon)$ to (u, u, p, f) , we deduce that (u, p) is a weak solution of (1.10). □

Corollary 3.4. *For any $f \in L^2(Q)$ and $u_0 \in L^\infty(\Omega)$ such that $|u_0| \leq 1$ a.e. in Ω , the mild solution of (3.12) is the unique solution of (1.10).*

4. EXTENSION, COMMENTS AND REMARKS

- (1) Thanks to the proof of Proposition 2.1, one sees that we can work with $\xi \in H_D^1(\Omega)$, so that any weak solution (u, p) satisfies

$$\begin{aligned} & \frac{d}{dt} \int \beta(u) \xi dx + \int \nabla p^+ \cdot \nabla \xi dx + \langle V \cdot \nabla \beta(u), \xi \rangle + \int (u \nabla \cdot V(\beta'(u) \chi_{[p=0]} + \text{Sign}_0^+(p)) \xi dx \\ & \leq \int f (\beta'(u) \chi_{[p=0]} + \text{Sign}_0^+(p)) \xi dx \quad \text{in } \mathcal{D}'(0, T), \end{aligned}$$

and

$$\begin{aligned} & \frac{d}{dt} \int \beta(u) \xi dx + \int \nabla p^- \cdot \nabla \xi dx + \langle V \cdot \nabla \beta(u), \xi \rangle + \int (u \nabla \cdot V(\beta'(u) \chi_{[p=0]} - \text{Sign}_0^-(p)) \xi dx \\ & \leq \int f (\beta'(u) \chi_{[p=0]} - \text{Sign}_0^-(p)) \xi dx \quad \text{in } \mathcal{D}'(0, T), \end{aligned}$$

for any $0 \leq \xi \in H_D^1(\Omega)$ and $\beta \in \mathcal{C}^1(\mathbb{R})$ such that $0 \leq \beta' \leq 1$. In particular, for the case of Neumann boundary condition ; i.e. $\Gamma_D = \emptyset$, these formulations imply that entropic inequalities (2.21) and (2.22) hold to be true up to the boundary $\partial\Omega$. That is (2.21) and (2.22) are fulfilled in $\mathcal{D}'((0, T) \times \bar{\Omega})$. However, to de-double the variable by taking into account the boundary becomes unhandy since one needs to handle the trace on the boundary in a strong sense which is not guaranteed in general (one can see for instance [6] where this kind of approach is

developed for elliptic-parabolic equation with homogeneous boundary condition). By the way, let us mention here that this case is fruitful for gradient flow approach in euclidean Wasserstein space. An elegant and promising proof of uniqueness is given in [29] for the one phase problem under a monotonicity assumption on V . The strength of this approach is the absolute continuity of weak solution in the set of probability equipped with Wasserstein distance.

- (2) For the case where $\Omega = \mathbb{R}^N$, one sees that working as in Section 2, it is possible to prove local Kato's inequality of the type (2.23) and achieve L^1 -comparison principle for weak solution like in Theorem (2.1). However, the existence of weak solutions in $L^1(\mathbb{R}^N)$ is not clear yet for us.
- (3) Beyond their significant role for the uniqueness, in our opinion formulations (2.4) and (2.5) remain to be interesting also for qualitative descriptions of the weak solution. As usual for renormalized formulation, they enable to localize the description of the solution with respect to its values. For similar description with respect to the values of p , one can see the following remarks which aims to widen the formulations (2.6) and (2.7).
- (4) The renormalization process we use in this paper concerns essentially u . The formulation (2.6) and (2.7) provide a particular renormalization for p in the region $[p \neq 0]$. Notice here, that it is possible to prove mended renormalized formulations for p . As a matter of fact, under the assumptions of Lemma 2.1, we can prove that for any $H \in C^1(\mathbb{R})$ such that $H' \geq 0$ and $H(0) = 0$, we have

$$(4.15) \quad -\nabla \cdot (H(p)\nabla p) + |H(p)| \nabla \cdot V = -H'(p) |\nabla p|^2 + f H(p) \quad \text{in } \mathcal{D}'(Q).$$

If moreover, $H \geq 0$, then

$$(4.16) \quad -\nabla \cdot (H(p)\nabla p) + |H(p)| \nabla \cdot V \leq -H'(p) |\nabla p|^2 + f H(p) \quad \text{in } \mathcal{D}'((0, T) \times \bar{\Omega}).$$

These formulations improve the description of p with respect to the values of p , and yield renormalized formulations for p . As we see in Section 2, the case where H approximates Sign_0^+ is crucial for the proof of uniqueness. The general case may serve out other qualitative description of the solution. The proof of (4.15) and (4.16) follows more or less similar arguments as for the proof Lemma 2.1.

- (5) Coming back to the approximation (3.2), thanks to Proposition 3.3, we know that weak solutions of the stationary problem of (3.2) depend weak-continuously on the non-linearity \mathcal{H}_ε . For the evolution problem, working as in the proof of Proposition 3.3, one can prove that the problem

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = f \\ u = \mathcal{H}_\varepsilon(p) \end{array} \right\} \quad \text{in } Q$$

$$\left\{ \begin{array}{l} p = 0 \\ (\nabla p + u V) \cdot \nu = 0 \\ u(0) = u_0 \end{array} \right. \quad \begin{array}{l} \text{on } \Sigma_D \\ \text{on } \Sigma_N \\ \text{in } \Omega, \end{array}$$

has a unique weak solution $(u_\varepsilon, p_\varepsilon) \in \mathcal{C}([0, T], L^1(\Omega)) \times L^2(0, T; H_D^1(\Omega))$, with $u_\varepsilon = \mathcal{H}_\varepsilon(p_\varepsilon)$ a.e. in Q , and as $\varepsilon \rightarrow 0$, we have $u_\varepsilon \rightarrow u$, in $L^\infty(Q) - \text{weak}^*$ and $p_\varepsilon \rightarrow p$, in $L^2(0, T; H_D^1(\Omega))$. We do believe that the convergence of u_ε remains to be true strongly in $L^1(\Omega)$, for both problems stationary and evolution one. We did not get into these further questions here. This need

more fine estimates on u (like BV -estimates). We'll touch them in forthcoming works. The case where one replace \mathcal{H}_ε by an arbitrary non-linearity which converges to Sign, in the sense of resolvent, like for instance the porous medium non-linearity $\varphi(r) = |r|^{m-1}r$, for any $r \in \mathbb{R}$, with $m \rightarrow \infty$, is likely concerned.

- (6) It is clear that our approach is based on a renormalization process à la DiPerna-Lions. This corroborates that this concept is still fruitful for the uniqueness of weak solution (solution in the sense of distribution) for nonlinear versions of Fokker-Planck equation of the type (1.1). As we quote in the previous remarks, operating with the absolute continuity of weak solution in the set of probability equipped with Wasserstein distance seems to be promising (cf. [29]) even if it is actually hampered by conceptual technical assumptions like the sign of the solution, conservation of the mass and monotone transport field (connected to some kind of convexity conditions). We do not overlook the L^1 -kinetic approach, which was developed in [23] for Cauchy problems of general degenerate parabolic-hyperbolic equations with non-isotropic non-linearity. The approach could be applied to more general situations and, as far as we know, this approach has not yet been explored for problems of the type (1.1) and (1.2). At last let us quote here the very recent work of [27] (that we learned just when we finish the preparation of this paper through Benoit Perthame) where the authors deal among other questions with the uniqueness of weak solutions for the one phase problem (1.2) in \mathbb{R}^N , by means of Hilbert's duality method. This method turns up to be very restrictive since they need the vector field $V = \nabla\Phi$ to be smooth enough (at least such that $\nabla\Delta\Phi$ is a $L_{loc}^{12/5}$ -Lebesgue function).
- (7) Under the same conditions on V , existence and uniqueness of weak solution remains to be true if the source term is replaced by a reaction terms of the form $g(t, x, u)$ satisfying reasonable assumptions (including continuous Lipschitz dependence with respect to u). This can be extended also to the case where the reaction is in the form $g(\cdot, u)G(p)$. These generalizations are treated separately in the paper [38].

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5. APPENDIX

The aim of this section is to complete the proof of Proposition 2.2 with the proof of de-doubling variable process concerning the vector field part of (2.25), which is

$$\lim_{\lambda \rightarrow 0} \int_{\Omega} \int_{\Omega} u(t, t, x, y)^+ (V(x) - V(y)) \cdot \nabla_y \zeta_\lambda \, dx dy.$$

Remember that

$$\zeta_\lambda(x, y) = \xi \left(\frac{x + y}{2} \right) \delta_\lambda \left(\frac{x - y}{2} \right),$$

for any $x, y \in \Omega$, where $\xi \in \mathcal{D}(\Omega)$ and δ_λ is a sequence of mollifiers in \mathbb{R}^N given by

$$(5.1) \quad \delta_\lambda(x) = \frac{1}{\lambda^N} \delta \left(\frac{|x|^2}{\lambda^2} \right), \quad \text{for any } x \in \mathbb{R}^N,$$

where $0 \leq \delta \in \mathcal{D}(0, 1)$ and $\frac{1}{2^N} \int \delta(|z|^2) dz = 1$. A typical example is given by $\delta(r) = C e^{\frac{-1}{1-r^2}} \chi_{[|r| \leq 1]}$, for any $r \in \mathbb{R}^N$.

Lemma 5.7. *For any $h \in L^1((0, T) \times \Omega \times \Omega)$, $V \in W^{1,1}(\Omega)$ and $\psi \in \mathcal{D}(0, T)$, we have*

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \int_0^T \int_{\Omega} \int_{\Omega} h(t, x, y) (V(x) - V(y)) \cdot \nabla_y \zeta_{\lambda}(x, y) \psi(t) dt dx dy \\ &= \int_0^T \int_{\Omega} h(x, x) \nabla \cdot V(x) \xi(x) \psi(t) dt dx. \end{aligned}$$

Proof. We see that

$$(\nabla_x + \nabla_y) \zeta = \delta_{\lambda} \left(\frac{x-y}{2} \right) \nabla \xi \left(\frac{x+y}{2} \right).$$

Since, for any $\eta \in \mathbb{R}^N$,

$$\nabla \delta_{\lambda}(\eta) = \frac{2}{\lambda^N} \frac{z}{\lambda^2} \delta' \left(\frac{|z|^2}{\lambda^2} \right),$$

we have

$$\nabla \delta_{\lambda} \left(\frac{x-y}{2} \right) = \frac{2}{\lambda^N} \frac{x-y}{2\lambda^2} \delta' \left(\frac{|x-y|^2}{(2\lambda)^2} \right), \quad \text{for any } x, y \in \Omega.$$

So,

$$\begin{aligned} \nabla_y \zeta_{\lambda}(x, y) &= \frac{1}{2} \nabla \xi \left(\frac{x+y}{2} \right) \delta_{\lambda} \left(\frac{x-y}{2} \right) - \frac{1}{2} \xi \left(\frac{x+y}{2} \right) \nabla \delta_{\lambda} \left(\frac{x-y}{2} \right) \\ &= \nabla \xi \left(\frac{x+y}{2} \right) \delta_{\lambda} \left(\frac{x-y}{2} \right) - \frac{1}{\lambda^{N+1}} \xi \left(\frac{x+y}{2} \right) \delta' \left(\frac{|x-y|^2}{(2\lambda)^2} \right) \frac{x-y}{2\lambda} \end{aligned}$$

and

$$\begin{aligned} (V(x) - V(y)) \cdot \nabla_y \zeta &= \frac{1}{2} (V(x) - V(y)) \cdot \nabla \xi \left(\frac{x+y}{2} \right) \delta_{\lambda} \left(\frac{x-y}{2} \right) \\ &\quad - \frac{1}{\lambda^N} \xi \left(\frac{x+y}{2} \right) \delta' \left(\frac{|x-y|^2}{(2\lambda)^2} \right) \frac{V(x) - V(y)}{\lambda} \cdot \frac{x-y}{2\lambda}. \end{aligned}$$

This implies that

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \iint h(x, y) (V(x) - V(y)) \cdot \nabla_y \zeta dx dy \\ &= \lim_{\lambda \rightarrow 0} -\frac{1}{\lambda^N} \iint \xi \left(\frac{x+y}{2} \right) \delta' \left(\frac{|x-y|^2}{(2\lambda)^2} \right) \frac{V(x) - V(y)}{\lambda} \cdot \frac{x-y}{2\lambda} dx dy =: I(\lambda). \end{aligned}$$

Changing the variable by setting

$$z = \frac{x-y}{2\lambda}$$

we get $x = y + 2\lambda z$ and $dx = (2\lambda)^N dz$, so that

$$\begin{aligned} -I(\lambda) &= 2^N \iint h((y + 2\lambda z, y) \xi(y + \lambda z) \delta'(|z|^2) \frac{V(y + 2\lambda z) - V(y)}{\lambda} \cdot z \, dz dy \\ &= 2^N \sum_{i=1}^N \iint h((y + 2\lambda z, y) \xi(y + \lambda z) \delta'(|z|^2) \frac{V_i(y + 2\lambda z) - V_i(y)}{\lambda} z_i \, dz dy. \end{aligned}$$

Letting $\lambda \rightarrow 0$, we get

$$\begin{aligned} \lim_{\lambda \rightarrow 0} I(\lambda) &= -2^{N+1} \sum_{i=1}^N \iint \xi(y) h((y, y) \nabla V_i(y) \cdot z z_i \delta'(|z|^2) \, dz dy \\ &= -2^{N+1} \sum_{i,j=1}^N \iint \xi(y) h((y, y) \frac{\partial V_i(y)}{\partial y_j} z_j z_i \delta'(|z|^2) \, dz dy \\ &= -2^{N+1} \sum_{i=1}^N \iint \xi(y) h((y, y) \frac{\partial V_i(y)}{\partial y_j} z_i^2 \delta'(|z|^2) \, dz dy \\ &= -\frac{2^{N+1}}{N} \int \xi(y) h((y, y) \nabla \cdot V(y) \, dy \int |z|^2 \delta'(|z|^2) \, dz, \end{aligned}$$

where we use the fact that (since $z \in B(0, 1) \rightarrow \delta(|z|^2)$ is symmetric)

$$\int z_j z_i \delta'(|z|^2) \, dz = 0, \quad \text{for any } i \neq j$$

and

$$\sum_{i=1}^N \int z_i^2 \delta'(|z|^2) \, dz = N \int z_1^2 \delta'(|z|^2) \, dz = \int |z|^2 \delta'(|z|^2) \, dz.$$

At last, we use the fact that

$$\frac{2^{N+1}}{N} \int |z|^2 \delta'(|z|^2) \, dz = -1.$$

Indeed, recall that

$$\int |z|^2 \delta'(|z|^2) \, dz = \int_0^\infty \int_{S_{N-1}} |r\theta|^2 \delta'(|r\theta|^2) r^{N-1} dr \, ds_{N-1}(\theta),$$

where s_{N-1} is the probability measure defined on the sphere S_{N-1} by

$$s_{N-1}(A) = |\{rx : r \in [0, 1], x \in A\}|.$$

This implies that

$$\begin{aligned}
 \int |z|^2 \delta'(|z|^2) dz &= |B(0,1)| \int_0^1 r^2 \delta'(r^2) r^{n-1} dr \\
 &= |B(0,1)| \int_0^1 \delta'(r^2) r^{n+1} dr \\
 &= -\frac{N}{2} \int \delta(|z|^2) dz.
 \end{aligned}$$

Thus the result. □

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