L 1 -THEORY FOR REACTION-DIFFUSION HELE-SHAW FLOW WITH LINEAR DRIFT

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L^1-THEORY FOR REACTION-DIFFUSION HELE-SHAW FLOW WITH LINEAR DRIFT

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Abstract. The main goal of this paper is to prove L^1-comparison and contraction principles for weak solutions (in the sense of distributions) of Hele-Shaw flow with a linear Drift. The flow is considered with a general reaction term including the Lipschitz continuous case, and subject to mixed homogeneous boundary conditions : Dirichlet and Neumann. Our approach combines a renormalization proceedings à la DiPerna-Lions with Kruzhkov device of doubling and de-doubling variables. The L^1-contraction principle allows afterwards to handle the problem in a general framework of nonlinear semigroup theory in L^1, taking thus advantage of this strong theory to study existence, uniqueness, comparison of weak solutions, L^1-stability as well as many further questions.

1. Introduction and preliminaries

1.1. **Introduction and main contributions.** Let $\Omega \subset I\!R^N$ be a bounded open set. We are interested in the existence and uniqueness as well as the L^1-comparison principle for the weak solution concerning the PDE of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u\, V) = g(t,x,u) \\ \\ u \in \mathrm{Sign}(p) \end{cases} \qquad \text{in } Q := (0,T) \times \Omega. \tag{1.1}$$

Here, Sign is the maximal monotone graph defined in $I\!R$ by

$$\mathrm{Sign}(r) = \begin{cases} 1 & \text{for any } r > 0 \\ [-1,1] & \text{for } r = 0 \\ -1 & \text{for any } r < 0, \end{cases}$$

V and g are the given velocity field and the reaction term respectively, satisfying besides assumptions we precise next (including at least the common case of Lipschitz continuous reaction term). In the case of nonnegative solution (one phase problem), the problem may be written in the widespread form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u\, V) = g(t,x,u) \\ \\ 0 \le u \le 1,\ p \ge 0,\ p(u-1) = 0 \end{cases} \qquad \text{in } Q. \tag{1.2}$$

The linear version of the problem which ties into the case where Sign's graph is superseded by the identity ; that is to take $p = u$, the problem corresponds to Fokker-Planck equation. Existence, uniqueness and stability of weak solutions for this case is studied in [31] with possibly linear degenerate

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diffusion, additive noise and $BV-$vector fields V in $\mathbb{R}^N$. One can see moreover the work [36] where the case of irregular coefficient and the associated stochastic differential equation are touch on.

In general, the problem is of elliptic-hyperbolic-parabolic type. Despite the broad results on this class of nonlinear PDE, the structure of (1.1) and (1.2) exclude them definitely out the scope of the current literature, at least concerning uniqueness (cf. [2, 7, 8, 20, 34, 42], see also the expository paper [9] for a complete list on refs). In spite of the "hard" non-linearity connected to the sign graph, the Fokker-Planck look of the equation with linear drift seems to be very fruitful for the analysis based on gradient flow in the space of probability measures equipped with Wasserstein distance (the distance arising in the Monge-Kantorovich optimal transport problem) in the one phase case (1.2) with no-flux Neumann boundary condition and no-reaction ; i.e. $g \equiv 0$. Numerous results on the existence, properties and estimates on the weak solution were elaborate in the last decade (one can see for instance [40] and the references therein). Other interesting progress on qualitative properties of a solution were obtained using the notion of viscosity solution (one can see for instance [1] and the references therein). The key utility of the viscosity concept is the skill to describe the pointwise behaviour of the free boundary evolution. Nonetheless, the uniqueness is still an open problem and seems to be harshness in general. In the case of monotone velocity field V, the uniqueness of absolute continuous solution in the set of probability equipped with Wasserstein distance (solutions in $AC([0;T),(P(\Omega),W_2))$ has been obtained craftily in [27] for (1.2), in the framework of gradient flow in euclidean Wasserstein space again in the case of Neumann boundary condition and $g \equiv 0$.

Our main results in this paper concern the L^1-comparison and contraction principles for the reaction-diffusion-transport equation of the type (1.1) subject to mixed homogeneous boundary conditions : Dirichlet and Neumann. This allows to handle the problem in the context of nonlinear semigroup theory in $L^1(\Omega)$ to prove thus existence, uniqueness, comparison of weak solutions, L^1-stability as well as many other questions. We'll not cover all the possibilities with this framework here, but let us mention at least its prospect to tackle the continuous dependence with respect to the Sign graph, including for instance the connection between this problem and the so called incompressible limit of the porous medium equation even in the singular case. This subject would be likely touch on in forthcoming works. We tackle the problem using a renormalisation approach of DiPerna-Lions type (cf. [28]) combined with Kruzhkov device of doubling variables. To avoid much more technicality of the paper, we restrict ourself to the case of ingoing vector field velocity on the boundary which remains practically useful for many applications like in crowd motion. Although, we do believe that our arguments may be generalized and used for general V with reasonable regularity. They can be extended also to the case where $\Omega = \mathbb{R}^N$ as well as to more general maximal monotone graph β instead of Sign.

Among the propose of this paper, let us mention the treatment of the equation with a reaction term g. Besides existence, uniqueness and comparison results for weak solutions, we supply sufficient condition on $g(t,x,1)$ and $g(t,x,-1)$ to avoid a congested regime and cover the solution of (4.1) by the purely transport equation :

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\, V) = g(t,x,u), \quad \text{in } Q. \tag{1.3}$$

The one phase problem (1.2) will be concerned as well with sufficient condition of the type $g(.,0) \geq 0$ a.e. in Q. For the application in crowd motion, the conditions on $g(.,1)$ may be heuristically connected to the reaction of the concerned population at the position $x \in \Omega$ and time $t > 0$ within "congested" circumstance. Performing its value regarding to the divergence of the velocity vector field V, may avoid the congestion in the transportation phenomena for crowed motion.

1.2. **Motivation and related works.** By means of the diffusion term, the nonlinear equation of the type (1.1) is usually called Hele-Shaw Flow. Indeed, in the case where $V \equiv 0$, the equation is a free-boundary problem modeling the evolution of a slow incompressible viscous fluid moving between slightly separated plates (cf. [30, 43, 44, 29, 26, 47] for physical and mathematical formulation). The equation appears also as the viscous incompressible limit of the porous medium equation (see for instance [33, 32, 13] and the references therein).

The study of the Hele-Shaw flow with a reaction term (without drift) goes back to [14] and [15] in the study of the limit of the $m-$porous medium equation as $m \to \infty$. Concrete models with a dissimilar typical reaction term appeared after in the study of the tumor growth (cf. [45]). The emergence of a drift in the Hele-Shaw flow appeared for the first time in the study of congested crowd motion (cf. [37, 38, 39]). In this case, the function u is assumptive to be nonnegative to model the density of population and the constraints on u prevents an evolution beyond a given threshold. Notice also that the occurrence of nonlocal drift in the Hele-Shaw flow appears in some biological applications to describe the so called aggregation phenomena (one can see the paper [19] for a review and references in this direction).

Completed with boundary condition (Dirichlet, Neumann or mixed Dirichlet-Neumann) and initial data, the questions of existence and uniqueness of a weak solution is well understood by now in the case where $V \equiv 0$, under general conditions on g (cf [14, 15], one can see also [45] for a different structure of a reaction term). For general V, though the existence of a solution seems to be more or less well understood, the uniqueness question seems to be very delicate and challenging. The main difficulties comes from the combination between the "hard" non-linearity Sign (so called hard-congestion) and the arbitrary pointing of velocity field for the drift. Although the first order term is linear, one notices the hyperbolic character of the equation (outside the congested region). This motivates the questions of existence and uniqueness of a weak solution, and challenges new as well as standard techniques for first order hyperbolic equations and second order hyperbolic-parabolic structures as well. In particular, one sees that setting

$$b = \left(I + \mathrm{Sign}^{-1}\right)^{-1} \quad \text{and} \quad \beta = (I + \mathrm{Sign})^{-1},$$

$v := u - p$, the problem (1.1) falls into the scope of diffusion-transport equation of the type

$$\frac{\partial b(v)}{\partial t} - \Delta\beta(v) + \nabla \cdot (V(x)\Phi(v)) = f \quad \text{in } Q := (0,\infty) \times \Omega. \tag{1.4}$$

The study of this class of degenerate parabolic problems has under-gone a considerable progress in the last twenty years, thanks to the fundamental paper of J. Carrillo [20] in which the Kruzhkov device of doubling of variables was extended to this class of hyperbolic-parabolic-elliptic problems. In [20], the appropriate notion of solution was established : the so called weak-entropy solution. This definition (or, sometimes, parts of the uniqueness techniques of [20]) led to many developments, we refer to the expository paper [9] for a survey and complete references on this subject. As much as restrictive it is, the approach of [20] serves out primarily and ingeniously the case where V does not depend on space and homogeneous Dirichlet boundary condition. Strengthened with L^1-nonlinear semi-group techniques (cf. [9]), the approach of [20] enables to achieve successful as well the uniqueness of weak solution in the "weakly degenerate" convection-diffusion problems of parabolic-elliptic type ; that is $[\beta = 0] \subseteq [\Phi = 0]$. As far as Dirichlet boundary condition are concerned (cf. [20, 34, 7, 7, 9]), the case of homogeneous Neumann was treated in [6]. Yet, the notion of entropic solution à la Carillo is definitively the suitable notion for general problem of the type (1.4). Nevertheless, following the theory of hyperbolic equation, the case of linear drift is a particular case for which one expects the uniqueness

of weak solution. As far as we are aware of, this case remains to be out of the scope of the current literature on this subject.

In spite of the serving of the second order term, we can not pass over the forcefulness of the first order term, which is here linear and carrying away along a vector field V with an arbitrary pointing in Ω. Heuristically, in the likely case $p \equiv 0$, the PDE turns into a linear initial-boundary value problem for continuity equation. In general, one guess the uniqueness of weak solution under reasonable assumption on V, like bounded total variation coefficients. As far as the regularity of V is involved, the way how to handle the boundary condition is hooked to the proofs of uniqueness and adduce different difficulties to the problem. The treatment of boundary conditions is definitely a crucial step in the proof of uniqueness. This is closely connected in some sens to the regularity of the solution as well as to the pointing of V on the boundary. This affects the manner to treat the weak/strong trace of the flux on the boundary (see the papers [3], [5], [10], [21],[23]and by [5], [8], [6]). The approach based on the concept of renormalized solution introduced in 1989 by Lions and DiPerna [28] for tangential velocity field seems to be unmissable and powerful in general for the proof of uniqueness of weak solution for this kind of problem. This concept was extended to bounded domains in $I\!R^N$ with inflow boundary conditions and velocity field with a kind of Sobolev regularity in [16], [17] and [41]. These results were generalized to velocity fields with BV regularity in [3, 4, 5, 24]. The concept was used also in [31] and [36] to tackle the existence and uniqueness of weak solutions for Fokker-Planck equation and its associated stochastic differential equation in some extreme cases like linear degenerate diffusion with irregular coefficient, additive noise and $BV-$vector fields V.

In this paper, we show how to use this approach in the presence of second order term of Hele-Shaw type to prove the uniqueness of weak solution in the contest of mixed Dirichlet-Neuman boundary condition with an outward pointing (possibly tangential) velocity fields V on the boundary. The case of strict inward pointing velocity fields V on the boundary need more careful, and possible delicate, treatment of the trace of the flux on the boundary. Taught, we do believe that it is possible adapt carefully our techniques to the case of strict inward pointing with the additional assumption $uV \cdot \nu = 0$ on the boundary, which can be heuristically interpreted as (a weak version of) $u = 0$ on the boundary.

1.3. **Plan of the paper.** In the following subsection, we give the main assumptions we'll use throughout the paper and the definition of weak solution we are dealing with. Section 2, is devoted to the L^1-comparison principle for the weak solution. We introduce renormalized like formulations and use them with doubling and de-doubling variable techniques à la Kruzhkov inside Ω, to prove first that weak solutions satisfies some kind of local Kato's inequalities. Then, linking the incoming assumption on the velocity field V on the boundary with the distance-to-boundary function, we prove that we can go with our Kato's inequalities up to the boundary proving thus L^1-comparison principle, and then deduce the uniqueness. In section 3, we prove the existence of a weak solution by using nonlinear semigroup theory governed by L^1-accretive operator. Here the main ingredient is to use the L^1-contraction principle for weak solution of stationary problem associated with the $\epsilon-$Euler implicit time discretization of the evolution problem. Then, we pass to the limit in the so called $\varepsilon-$approximate solution and prove that the limit is the weak solution of the evolution problem. In section 4, we treat the equation with reaction term $g(t, x, u)$. Using again nonlinear semigroup theory in $L^1(\Omega)$, we prove the existence and uniqueness of a weak solution under a general condition on g including the Lipschitz continuous case. Then, using the L^1-comparison principle we show how some natural result concerning the existence and uniqueness of weak solution for some particular cases like one phase problem (1.2) as well the purely transport equation (1.3) (i.e. $p \equiv 0$) may be wrap up by this approach. Section 5 is devoted to some remarks, comments and possible extensions. At last, in Section 6, for completeness we give a

complement for the proof of de-doubling variables process to handle space dependent vector field with arbitrary pointing out.

1.4. **Preliminaries, remarks and and main assumptions.** We assume that $\Omega \subset I\!R^N$ is a bounded open set, with regular boundary splitted into $\partial\Omega = \Gamma_D \cup \Gamma_N$, such that $\Gamma_D \cap \Gamma_N = \emptyset$ and

$$\mathcal{L}^{N-1}(\Gamma_D) > 0.$$

For any $h > 0$, we denote by

$$\xi_h(t,x) = \frac{1}{h}\min\Big\{h, d(x,\partial\Omega)\Big\} \quad \text{and} \quad \nu_h(x) = -\nabla\xi_h, \quad \text{for any } x \in \Omega, \tag{1.1}$$

where $d(.,\partial\Omega)$ names the euclidean distance-to-the-boundary function. We see that $\xi_h \in H^1_0(\Omega)$ is a regular (as well as the boundary is) concave function, $0 \le \xi_h \le 1$ and

$$\nu_h(x) = -\frac{1}{h}\nabla\, d(.,\partial\Omega), \quad \text{for any } x \in \Omega \text{ s.t. } d(x,\partial\Omega) < h \le h_0 \text{ (small anough)}.$$

In particular, for such x, we have $h\nu_h(x) = \nu(\pi(x))$, where $\pi(x)$ design the projection of x on the boundary $\partial\Omega$, and $\nu(y)$ represents the outward unitary normal to the boundary $\partial\Omega$ at y.

We denote by Sign^+ the maximal monotone graph given by

$$\mathrm{Sign}^+(r) = \begin{cases} 1 & \text{for } r > 0 \\ [0,1] & \text{for } r = 0 \\ 0 & \text{for } r < 0. \end{cases}$$

Moreover, we define Sign_0 and $\mathrm{Sign}_0^{\pm}$, the discontinuous applications defined from $I\!R$ to $I\!R$ by

$$\mathrm{Sign}_0(r) = \begin{cases} 1 & \text{for } r > 0 \\ 0 & \text{for } r = 0 \\ -1 & \text{for } r < 0, \end{cases}, \quad \mathrm{Sign}_0^+(r) = \begin{cases} 1 & \text{for } r > 0 \\ 0 & \text{for } r \le 0 \end{cases} \quad \mathrm{Sign}_0^-(r) = \begin{cases} 0 & \text{for } r \ge 0 \\ 1 & \text{for } r < 0 \end{cases}$$

Throughout the paper, we assume that u_0 and the velocity vector filed V satisfy the following assumptions :

- $u_0 \in L^\infty(\Omega)$ satisfies $0 \le |u_0| \le 1$ a.e. in Ω.
- $V \in W^{1,1}_{loc}(\Omega)^N$, $\nabla \cdot V \in L^\infty(\Omega)$ and V satisfies (inward pointing velocity vector field condition on the boundary)

$$V \cdot \nu \le 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Then, we denote by $H^1_D(\Omega)$ the Sobolev space

$$H^1_D(\Omega) = \Big\{u \in H^1(\Omega) \ :\ u = 0,\ \mathcal{L}^{N-1}\text{-a.e. in } \Gamma_D\Big\}.$$

Remark 1. (1) *Since $V \in W^{1,1}_{loc}(\Omega)^N$ and $\nabla \cdot V \in L^\infty(\Omega)$, then $V \cdot \nu$ needs to be understood in a weak sense, like*

$$\int_\Omega V \cdot \nabla\xi \, dx + \int_\Omega \nabla \cdot V \, \xi \, dx \le 0, \quad \textit{for any } 0 \le \xi \in \mathcal{D}(\overline{\Omega}).$$

One can see also the papers [21, 23] *for more details and expressions in this direction. In particular, this condition implies that*

$$\liminf_{h\to 0} \int_{D_h} \xi\, V(x) \cdot \nu_h(x)\, dx \le 0, \quad \textit{for any } 0 \le \xi \in L^\infty(\Omega), \tag{1.3}$$

where, for any $0 < h \leq h_0$, D_h is a neighbour of $\partial\Omega$.

(2) *In some models of congested crowd motion, the vector field velocity may be given by*

$$V = -\nabla d(.,\Gamma_D), \quad \text{in } \Omega. \tag{1.4}$$

This means that the pedestrian choose the euclidean geodesic trajectory to the escape door to get away from the environment Ω. See in this case that the condition (1.4) is completely satisfied. This example constitute a typical practical situation (among many others) for which all the results of this paper may be applied.

To begin with, we consider first the problem

$$\begin{cases} \left.\begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u\,V) = f \\ u \in \mathrm{Sign}(p) \end{array}\right\} & \text{in } Q \\ p = 0 & \text{on } \Sigma_D := (0,T) \times \Gamma_D \\ (\nabla p + u\,V) \cdot \nu = 0 & \text{on } \Sigma_N := (0,T) \times \Gamma_N \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $f \in L^1(Q)$ is given.

Definition 1.1 (Notion of solution)**.** *A couple (u,p) is said to be a weak solution of (1.5) if $(u,p) \in L^\infty(Q) \times L^2\left(0,T; H^1_D(\Omega)\right)$, $u \in Sign(p)$ a.e. in Q and*

$$\frac{d}{dt}\int_\Omega u\,\xi + \int_\Omega (\nabla p - u\,V)\cdot\nabla\xi = \int_\Omega f\,\xi, \quad \text{in } \mathcal{D}'(0,T), \quad \forall\,\xi \in H^1_D(\Omega).$$

We'll say plainly that u is a solution of (1.5) if $u \in \mathcal{C}([0,T), L^1(\Omega))$, $u(0) = u_0$ and there exists $p \in L^2\left(0,T; H^1_D(\Omega)\right)$ such that the couple (u,p) is a weak solution of (1.5).

Remark 2. *The assumptions we suppose on V look alike to be stronger regard to the literature on (linear) continuity equation. We think it is possible to extend our results for weak solution to more general V. But, in the presence of the second order term, the problem seems to be heavy and much more technical. Let us give here at least a short reminder on the uniqueness of weak solution for continuity equation, which could comprise some possible extensions (open questions !) for the Hele-Shaw flow with a linear Drift.*

(1) *The analysis of the continuity equation (without a second order term) in the case when V has low regularity has drawn considerable attention. For an overview of some of the main contributions, we refer to the lecture notes by Ambrosio and Crippa* [4] *(see also* [24] *for more references on this subject). Counter-example for the uniqueness when V is not enough regular (up to the boundary) may be found in* [24]*. Indeed, regardless the orientation of V at the boundary, uniqueness may be violated as soon as V enjoys BV regularity in every open set $\omega \subset\subset \Omega$, but not at the boundary of Ω (cf.* [24] *for more details and discussions on this subject).*

(2) *When the vector filed is outward pointing, heuristically the solution would not be worst at/near the boundary. Indeed, at least in the smooth case the solution may simply "carried out" of the domain along the characteristics and, consequently, the behaviour of the solution inside the domain is not substantially affected by what happens close to the boundary. However, examples*

are given in [24] *discussing that even if* V *is outward pointing at* $\partial\Omega$*, then uniqueness may be violated as soon as the* BV *regularity deteriorates at the boundary.*

2. L^1-Comparison principle

In this section, we focus first on the uniqueness and L^1-Comparison principle of weak solution. Our first main result is the following.

Theorem 2.1. *If* (u_1, p_1) *and* (u_2, p_2) *are two weak solutions of* (1.5) *associated with* $f_1,\ f_2 \in L^1(Q)$ *respectively, then there exists* $\kappa \in L^\infty(Q)$*, such that* $\kappa \in Sign^+(u_1 - u_2)$ *a.e. in* Q *and*

$$\frac{d}{dt}\int_\Omega (u_1 - u_2)^+ \, dx \le \int_\Omega \kappa\,(f_1 - f_2)\, dx, \qquad \text{in } \mathcal{D}'(0,T). \tag{2.1}$$

In particular, we have

$$\frac{d}{dt}\|u_1 - u_2\|_1 \le \|f_1 - f_2\|_1, \qquad \text{in } \mathcal{D}'(0,T).$$

Moreover, if $f_1 \le f_2$*, a.e. in* Q*, and* u_1, u_2 *are two corresponding solutions satisfying* $u_1(0) \le u_2(0)$ *a.e. in* Ω*, then*

$$u_1 \le u_2, \qquad \text{a.e. in } Q.$$

The main tool to prove this result is doubling and de-doubling variables. Since the degeneracy of the problem, we prove first that a weak solution satisfies some kind of (new) renormalized formulation, and then carry out trickily doubling and de-doubling variables device to prove (2.1).

Recall that, the merely transport equation case correspond to the situation where $p \equiv 0$. In this case, the renormalized formulation reads

$$\partial_t \beta(u) + V \cdot \nabla \beta(u) + u\,\nabla \cdot V \beta'(u) = f\,\beta'(u) \quad \text{ in } \mathcal{D}'(Q),$$

for any $\beta \in \mathcal{C}^1(I\!R)$, where from now on, for any $z \in L^1(\Omega)$ the notation $V \cdot \nabla z$ needs to be understood in the sens of distribution as follows

$$V \cdot \nabla z = \nabla \cdot (z\,V) - z\,\nabla \cdot V\,. \tag{2.2}$$

Proposition 2.1. *If* (u, p) *is a weak solution of* (1.5)*, then*

$$\begin{aligned} &\partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + u\,\nabla \cdot V(\beta'(u)\,\chi_{[p=0]} + Sign_0^+(p)) \\ &\qquad \le f\,(\beta'(u)\,\chi_{[p=0]} + Sign_0^+(p)) \quad \text{ in } \mathcal{D}'(Q) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\partial_t \beta(u) - \Delta p^- + V \cdot \nabla \beta(u) + u\,\nabla \cdot V(\beta'(u)\ \chi_{[p=0]} - Sign_0^-(p)) \\ &\qquad \le f\,(\beta'(u)\,\chi_{[p=0]} - Sign_0^-(p)) \quad \text{ in } \mathcal{D}'(Q), \end{aligned} \tag{2.4}$$

for any $\beta \in \mathcal{C}^1(I\!R)$ *such that* $0 \le \beta' \le 1$.

Remark 3. *The formulations* (2.3) *and* (2.4) *describe some kind of renormalized formulation for the solution* u*. For the one phase Hele-Shaw problem this formulation reads simply as follows*

$$\begin{aligned} &\partial_t \beta(u) - \Delta p + V \cdot \nabla \beta(u) + u\,\beta'(u)\,\nabla \cdot V\,\chi_{[p=0]} + \nabla \cdot V \chi_{[p\neq 0]} \\ &\qquad \le f\,(\beta'(u)\,\chi_{[p=0]} + \chi_{[p\neq 0]}) \quad \text{ in } \mathcal{D}'(Q), \end{aligned}$$

for any $\beta \in \mathcal{C}^1(I\!R)$ *such that* $0 \le \beta' \le 1$.

We prove this results in two steps, we begin with the case where $\beta \equiv 0$. Then proceeding skillfully with the positive and negative part of p, and using a smoothing procedure with a commutator à la DiPerna-Lions (cf. [28], see also [3]), we prove that any weak solution u satisfies renormalized formulations (2.3) and (2.4).

Lemma 2.1. *If (u,p) is a weak solution of (1.5), then the renomalized formulations (2.3) and (2.4) are fulfilled with $\beta \equiv 0$ in $\mathcal{D}'((0,T)\times\overline{\Omega})$. That is*

$$-\Delta p^+ + (\nabla\cdot V - f)Sign_0^+(p) \le 0 \quad \text{in } \mathcal{D}'((0,T)\times\overline{\Omega}), \tag{2.5}$$

and

$$-\Delta p^- + (\nabla\cdot V + f)Sign_0^-(p) \le 0 \quad \text{in } \mathcal{D}'((0,T)\times\overline{\Omega}). \tag{2.6}$$

Moreover, we have

$$\partial_t u - \Delta p^+ + \nabla\cdot(u\,V) + \nabla\cdot V\; Sign_0^-(p) \le f\,(1 - Sign_0^-(p)) \quad \text{in } \mathcal{D}'(Q), \tag{2.7}$$

Proof. We extend p onto $\mathbb{R}\times\Omega$ by 0 for any $t \notin (0,T)$. Then, for any $h>0$ and nonnegative $\xi \in H^1(\Omega)$ and $\psi \in \mathcal{D}(\mathbb{R})$, we consider

$$\Phi^h(t) = \xi\,\psi(t)\,\frac{1}{h}\int_t^{t+h} \mathcal{H}_\varepsilon^+(p((s))\,ds,$$

where $\mathcal{H}_\varepsilon^+$ is given by

$$\mathcal{H}_\varepsilon^+(r) = \min\left(\frac{r^+}{\varepsilon}, 1\right), \quad \text{for any } r\in\mathbb{R},$$

for arbitrary $\varepsilon>0$. It is clear that $\Phi_h \in W^{1,2}\Big(0,T;H_D^1(\Omega)\Big)\cap L^\infty(Q)$ is an admissible test function for the weak formulation, so that

$$-\iint_Q u\,\partial_t\Phi^h + \iint_Q (\nabla p - V\,u)\cdot\nabla\Phi^h = \iint_Q f\,\Phi^h. \tag{2.8}$$

See that

$$\iint_Q u\,\partial_t\Phi^h = \iint_Q u\,\partial_t\psi\,\frac{1}{h}\int_t^{t+h}\mathcal{H}_\varepsilon^+(p((s))\,ds + \iint_Q u(t)\,\frac{\mathcal{H}_\varepsilon^+(p(t+h)) - \mathcal{H}_\varepsilon^+(p(t))}{h}\,\psi(t)\,\xi. \tag{2.9}$$

Moreover, using the fact that for a.e. $t\in(0,T)$, $-1\le u(t)\le 1$, $\mathcal{H}_\varepsilon^+\ge 0$ and $\mathcal{H}_\varepsilon^+(0)=0$, we have $u(t,x)\,\mathcal{H}_\varepsilon^+(p(t,x)) = \mathcal{H}_\varepsilon^+(p(t,x))$ and $u(t,x)\,\mathcal{H}_\varepsilon^+(p(t+h,x)) \le \mathcal{H}_\varepsilon^+(p(t+h,x))$ a.e. $(t,x)\in Q$. So, for any $h>0$ (small enough), we have

$$\begin{aligned}\iint_Q u(t)\,\frac{\mathcal{H}_\varepsilon^+(p(t+h)) - \mathcal{H}_\varepsilon^+(p(t))}{h}\,\psi(t)\,\xi &\le \iint_Q \frac{\mathcal{H}_\varepsilon^+(p(t+h)) - \mathcal{H}_\varepsilon^+(p(t))}{h}\,\psi(t)\,\xi \\ &\le \iint_Q \frac{\psi(t-h)-\psi(t)}{h}\,\mathcal{H}_\varepsilon^+(p(t))\,\xi.\end{aligned}$$

This implies that

$$\limsup_{h\to 0}\iint_Q u(t)\,\frac{\mathcal{H}_\varepsilon^+(p(t+h)) - \mathcal{H}_\varepsilon^+(p(t))}{h}\,\psi(t)\,\xi \le -\iint_Q \partial_t\psi\,\mathcal{H}_\varepsilon^+(p(t))\,\xi,$$

so that, by letting $h \to 0$ in (2.9), we get

$$\lim_{h\to 0} \iint_Q u\, \partial_t \Phi^h \le 0.$$

Then, by letting $h \to 0$ in (2.8)

$$\iint_Q (\nabla p - V\, u) \cdot \nabla(\mathcal{H}_\varepsilon^+(p(t))\, \xi) - \iint_Q f\, \mathcal{H}_\varepsilon^+(p(t))\, \xi \le \iint_Q f\, \mathcal{H}_\varepsilon^+(p(t))\, \xi. \tag{2.12}$$

On the other hand, using again the fact that $u\mathcal{H}_\varepsilon^+(p) = \mathcal{H}_\varepsilon^+(p)$, a.e. in Q, we have

$$\begin{aligned}\iint_Q (\nabla p - u\, V) \cdot \nabla\Big(\mathcal{H}_\varepsilon^+(p)\, \xi\Big)\, \psi \;&= \iint_Q \mathcal{H}_\varepsilon^+(p)\, \nabla p \cdot \nabla \xi\, \psi + \iint_Q |\nabla p|^2\, (\mathcal{H}_\varepsilon^+)'(p)\, \xi\, \psi \\ &\quad - \iint_Q V \cdot \nabla(\xi \mathcal{H}_\varepsilon^+(p))\, \psi \\ &\ge \iint_Q \mathcal{H}_\varepsilon^+(p)\, \nabla p \cdot \nabla \xi\, \psi + \iint_Q \nabla \cdot V\, (\xi \mathcal{H}_\varepsilon^+(p))\, \psi,\end{aligned}$$

where we use (1.2) and the fact that $|\nabla p|^2\, (\mathcal{H}_\varepsilon^+)'(p) \ge 0$. Thanks to (2.12), this implies that

$$\iint_Q \nabla p \cdot \nabla \xi\, \mathcal{H}_\varepsilon^+(p)\, \xi\, \psi + \iint_Q \nabla \cdot V\, \xi \mathcal{H}_\varepsilon^+(p)\, \psi \le \iint_Q f\, \mathcal{H}_\varepsilon^+(p(t))\, \xi.$$

Letting now $\varepsilon \to 0$, we get (2.5). As to (2.6), it follows by using the that $(-u, -p)$ is also a solution of (1.5) with f replaced by $-f$, and applying (2.5) to $(-u, -p)$. At last, recall that

$$\partial_t u - \Delta p + \nabla \cdot (u\, V) = f \quad \text{ in } \mathcal{D}'(Q). \tag{2.13}$$

So, summing (2.6) restrained to $\mathcal{D}'(Q)$ and (2.13), we get (2.7). □

Now, in order to prove the proposition by using (2.5) and (2.7), we prove the following technical lemma.

Lemma 2.2. *Let $u \in L^1_{loc}(Q)$, $F \in L^1_{loc}(Q)^N$ and $J_1 \in L^1_{loc}(Q)$ be such that*

$$\partial_t u + V \cdot \nabla u - \nabla \cdot F \le J_1 \quad \text{ in } \mathcal{D}'(Q) \tag{2.14}$$

where $V \cdot \nabla u$ is taken in the sens $V \cdot \nabla u = \nabla \cdot (u\, V) - u\, \nabla \cdot V$, in $\mathcal{D}'(Q)$. If

$$-\nabla \cdot F \le J_2 \quad \text{ in } \mathcal{D}'(Q), \tag{2.15}$$

for some $J_2 \in L^1_{loc}(Q)$, then

$$\partial_t \beta(u) + V \cdot \nabla \beta(u) - \nabla \cdot F \le J_1 \beta'(u) + J_2(1 - \beta'(u)) \quad \text{ in } \mathcal{D}'(\Omega), \tag{2.16}$$

for any $\beta \in \mathcal{C}^1(\mathbb{R})$ such that $0 \le \beta' \le 1$.

Proof. We set $Q_\varepsilon := \{(t,x) \in Q \,:\, d((t,x), \partial Q) > \varepsilon\}$. Moreover, for any $z \in L^1(Q)$, we denote by z_ε the usual regularization of z by convolution given and denoted by

$$z_\varepsilon := z \star \rho_\varepsilon, \quad \text{ in } Q_\varepsilon,$$

where ρ_ε is the usual mollifiers sequence defined here in $\mathbb{R} \times \mathbb{R}^N$. It is not difficult to see that (2.14) and (2.15) implies respectively

$$\partial_t u_\varepsilon + V \cdot \nabla u_\varepsilon - \nabla \cdot F_\varepsilon \le J_{1\varepsilon} + \mathcal{C}_\varepsilon \quad \text{ in } Q_\varepsilon \tag{2.17}$$

and

$$-\nabla \cdot F_\varepsilon \leq J_{2\varepsilon} \quad \text{in } Q_\varepsilon, \tag{2.18}$$

where $\mathcal{C}_\varepsilon$ is the usual commutators given by

$$\mathcal{C}_\varepsilon := V \cdot \nabla u_\varepsilon - (V \cdot \nabla u)_\varepsilon.$$

Using (2.2), here $(V \cdot \nabla u)_\varepsilon$ needs to be understood in the sense

$$(V \cdot \nabla u)_\varepsilon = (u\, V) \star \nabla \rho_\varepsilon - (u\, \nabla \cdot V) \star \rho_\varepsilon, \quad \text{in } Q_\varepsilon.$$

Multiplying (2.17) by $\beta'(u_\varepsilon)$ and (2.18) by $1 - \beta'(u_\varepsilon)$ and adding the resulting equations, we obtain

$$\beta'(u_\varepsilon)\, \partial_t u_\varepsilon + \beta'(u_\varepsilon)\, V \cdot \nabla u_\varepsilon - \nabla \cdot F_\varepsilon \leq \mathcal{C}_\varepsilon\, \beta'(u_\varepsilon) + J_{1\varepsilon}\beta'(u_\varepsilon) + J_{2\varepsilon}(1 - \beta'(u_\varepsilon)) \quad \text{in } Q_\varepsilon$$

and then

$$\partial_t \beta(u_\varepsilon) + V \cdot \nabla \beta(u_\varepsilon) - \nabla \cdot F_\varepsilon \leq \mathcal{C}_\varepsilon\, \beta'(u_\varepsilon) + J_{1\varepsilon}\beta'(u_\varepsilon) + J_{2\varepsilon}(1 - \beta'(u_\varepsilon)) \quad \text{in } Q_\varepsilon. \tag{2.19}$$

Since $V \in W^{1,1}_{loc}(\Omega)$ and $\nabla \cdot V \in L^\infty(\Omega)$, it is well known by now that, by taking a subsequence if necessary, the commutators converges to 0 in $L^1_{loc}(Q)$, as $\varepsilon \to 0$ (see for instance Ambrosio [3]). So, letting $\varepsilon \to 0$ in (2.19), we get (2.16). □

Now, let us prove how to get the renormalized formulations (2.3) and (2.4).

Proof of Proposition 2.1. Thanks to Lemma 2.1, by using the fact that

$$\nabla \cdot V\, \mathrm{Sign}_0^-(p) = -u\, \nabla \cdot V\, \mathrm{Sign}_0^-(p) \text{ and } \nabla \cdot V\, \mathrm{Sign}_0^+(p) = u\, \nabla \cdot V\, \mathrm{Sign}_0^+(p),$$

we see that (2.14) and (2.15) are fulfilled with

$$F := \nabla p^+, \quad J_1 := (f - u\, \nabla \cdot V)(1 - \mathrm{Sign}_0^-(p))$$

and

$$J_2 := (f - u\nabla \cdot V)\, \mathrm{Sign}_0^+(p).$$

Thanks to Lemma 2.1, for any $\beta \in \mathcal{C}^1(I\!R)$ such that $0 \leq \beta' \leq 1$, we deduce that

$$\partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + (u\, \nabla \cdot V - f)(1 - \mathrm{Sign}_0^-(p))\beta'(u)$$

$$+(f - u\nabla \cdot V)\, \mathrm{Sign}_0^+(p)(\beta'(u) - 1) \leq 0 \quad \text{in } \mathcal{D}'(Q),$$

Using again the fact that $\nabla \cdot V\, \mathrm{Sign}_0^+(p) = u\, \nabla \cdot V\, \mathrm{Sign}_0^+(p)$ this implies that

$$\partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + u\, \nabla \cdot V\, (1 - \mathrm{Sign}_0^-(p))\beta'(u) + u\nabla \cdot V\, \mathrm{Sign}_0^+(p)(1 - \beta'(u))$$

$$\leq f(\mathrm{Sign}_0^+(p)(1 - \beta'(u)) + (1 - \mathrm{Sign}_0^-(p))\beta'(u)) \quad \text{in } \mathcal{D}'(Q).$$

That is

$$\partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + u\, \nabla \cdot V(\beta'(u)(1 - \mathrm{Sign}_0^-(p) - \mathrm{Sign}_0^+(p)) + \mathrm{Sign}_0^+(p))$$

$$\leq f(\beta'(u)(1 - \mathrm{Sign}_0^-(p) - \mathrm{Sign}_0^+(p)) + \mathrm{Sign}_0^+(p)) \quad \text{in } \mathcal{D}'(Q),$$

and then

$$\partial_t \beta(u) - \Delta p^+ + V \cdot \nabla \beta(u) + u\, \nabla \cdot V(\beta'(u)\chi_{[p=0]} + \mathrm{Sign}_0^+(p))$$

$$\leq f(\beta'(u)\chi_{[p=0]} + \mathrm{Sign}_0^+(p)) \quad \text{in } \mathcal{D}'(Q).$$

Thus (2.3). At last, using the fact that the couple $(-u,-p)$ is a solution of (1.5) with f replaced by $-f$, and applying (2.3) to $(-u,-p)$, we deduce (2.4).

□

See here that the attendance of a nonlinear second order term creates an obstruction to use standard approaches for the uniqueness of weak solutions based effectively on the linearity of the first order term. In order to prove uniqueness in our case, we proceed by doubling and de-doubling variable techniques. To this aim, we'll use the fact that the renormalized formulation implies some kind of local entropic inequalities. This is the aim of the following Lemma.

Lemma 2.3. *If (u,p) is a weak solution of (1.5), then we have*

$$\begin{aligned}&\partial_t(u-k)^+ - \Delta p^+ + \nabla\cdot((u-k)^+ V) + k\,\nabla\cdot V\; Sign_0^+(u-k)\\ &\leq f\; Sign_0^+(u-k) \quad \text{in } \mathcal{D}'(Q), \quad \text{for any } k<1\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}&\partial_t(u-k)^- - \Delta p^- + \nabla\cdot((k-u)^+ V) - k\,\nabla\cdot V\; Sign_0^+(k-u)\\ &\leq (u-f) Sign_0^+(k-u) \quad \text{in } \mathcal{D}'(Q) \quad \text{for any } k>-1\end{aligned} \tag{2.21}$$

Remark 4. *See here that our entropic inequalities* (2.20) *and* (2.21) *are local and do not proceed up to the boundary like in* [20]. *Indeed, in contrast of Carillo's approach, we are intending here to handle the boundary condition separately. In particular, as we'll see this would reduce the technicality of de-doubling variables process (see Proposition 2.2).*

Proof of Lemma 2.3. Let us consider

$$\beta_\epsilon(r) = \tilde{\mathcal{H}}_\varepsilon(r-k), \quad \text{for any } r\in I\!R,$$

where

$$\tilde{\mathcal{H}}_\varepsilon(r) = \begin{cases} \dfrac{1}{2\varepsilon} r^{+2} & \text{for } r\leq\varepsilon \\ r-\dfrac{\varepsilon}{2} & \text{elsewhere .}\end{cases}$$

In this case, $\beta'_\epsilon(1) = \mathcal{H}^+_\varepsilon(1) = 1$, for any $0<\epsilon\leq\epsilon_0$ (small enough), and we have

$$\beta'_\varepsilon(u) = \mathcal{H}^+_\varepsilon(u-k) \to \mathrm{Sign}_0^+(u-k), \quad \text{as } \varepsilon\to 0.$$

For any $k<1$, since

$$\mathrm{Sign}_0^+(u-k)\,\chi_{[p=0]} + \mathrm{Sign}_0^+(p) = \mathrm{Sign}_0^+(u-k),$$

letting $\varepsilon\to 0$, in (2.3) where we replace β by $\tilde{\mathcal{H}}_\varepsilon$, we get

$$\begin{aligned}&\partial_t(u-k)^+ - \Delta p^+ + V\cdot\nabla(u-k)^+ + u\,\nabla\cdot V\,\mathrm{Sign}_0^+(u-k)\\ &\leq f\,\mathrm{Sign}_0^+(u-k) \quad \text{in } \mathcal{D}'(Q),\end{aligned}$$

which implies (2.20).
For the second part of the Lemma, we use again the fact that see that $(\tilde u := -u, \tilde p := -p)$ is a weak solution of (1.5) with f replaced by $\tilde f := -f$. So, using (2.20) for $(\tilde u,\tilde p)$ with $\tilde f$, for any $k<1$, we have

$$\begin{aligned}&-\Delta\tilde p^+ + \nabla\cdot((\tilde u-k)^+ V) + k\,\nabla\cdot V\,\mathrm{Sign}_0^+(\tilde u-k)\\ &\leq (\tilde f-\tilde u)\mathrm{Sign}_0^+(\tilde u-k) \quad \text{in } \mathcal{D}'(Q)\end{aligned}$$

Which implies that, for any $-1 < s$, we have

$$\partial_t(s-u)^- - \Delta p^- + \nabla\cdot((s-u)^+V) - s\,\nabla\cdot V\,\mathrm{Sign}_0^+(s-u)$$
$$\leq (u-f)\mathrm{Sign}_0^+(s-u) \quad \text{ in } \mathcal{D}'(Q).$$

Thus the result of the lemma. □

Now, we are able to prove some kind of "local" Kato's inequality which mold our L^1-approach for the uniqueness.

Proposition 2.2 (Kato's inequality)**.** *If (u_1,p_1) and (u_2,p_2) are two couples of $L^\infty(Q)\times L^2\left(0,T;H^1_D(\Omega)\right)$ satisfying (2.20) and (2.21) corresponding to $f_1\in L^1(Q)$ and $f_2\in L^1(Q)$ respectively, then there exists $\kappa\in L^\infty(Q)$, such that $\kappa\in Sign^+(u_1-u_2)$ a.e. in Q and*

$$\partial_t(u_1-u_2)^+ - \Delta(p_1^+ + p_2^-) + \nabla\cdot\left((u_1-u_2)^+V\right) \leq \kappa(f_1-f_2) \text{ in } \mathcal{D}'(Q). \tag{2.22}$$

Proof. The proof of this lemma is based on doubling and de-doubling variable techniques. Let us give here briefly the arguments. To double the variables, we fix $\tau>0$, and since $u_1(s,y)-\tau<1$, we use the fact that (u_1,p_1) satisfies (2.20) with $k=u_1(s,y)-\tau$, we have

$$\frac{d}{dt}\int (u_1(t,x)-u_2(s,y)+\tau)^+\,\zeta + \int(\nabla_x p_1^+(t,x) - (u_1(t,x)-u_2(s,y)+\tau)^+V(x)\cdot\nabla_x\zeta$$
$$+\int_\Omega \nabla_x\cdot V\,u_2(s,y)\zeta\mathrm{Sign}_0^+(u_1(t,x)-u_2(s,y)+\tau) \leq \int f_1(t,x)\mathrm{Sign}_0^+(u_1(t,x)-u_2(s,y)+\tau)\,\zeta,$$

for any $0\leq\eta\in\mathcal{D}(\Omega\times\Omega)$. See that $\displaystyle\int\nabla_y p_2^-(s,y)\cdot\nabla_x\zeta\,dx=0$, so that

$$\frac{d}{dt}\int (u_1(t,x)-u_2(s,y)+\tau)^+\,\zeta + \int(\nabla_x p_1^+(t,x)+\nabla_y p_2^-(t,x))\cdot\nabla_x\zeta - \int(u_1(t,x)-u_2(s,y)+\tau)^+V(x)\cdot\nabla_x\zeta$$
$$+\int\nabla_x\cdot V\,u_2(s,y)\,\zeta\,\mathrm{Sign}_0^+(u_1(t,x)-u_2(s,y)+\tau) \leq \int f_1(t,x)\mathrm{Sign}_0^+(u_1(t,x)-u_2(s,y)+\tau)\,\zeta.$$

Denoting by

$$u(t,s,x,y)=u_1(t,x)-u_2(s,y)+\tau, \quad \text{ and } \quad p(t,s,x,y)=p_1^+(t,x)+p_2^-(s,y),$$

and integrating with respect to y, we obtain

$$\frac{d}{dt}\int u(t,s,x,y)^+\,\zeta + \iint(\nabla_x+\nabla_y)p(t,s,x,y)\cdot\nabla_x\zeta - \iint u(t,s,x,y)^+V(x)\cdot\nabla_x\zeta$$
$$+\iint\nabla_x\cdot V\,u_2(s,y)\,\zeta\,\mathrm{Sign}_0^+u(t,s,x,y) \leq \iint f_1(t,x)\mathrm{Sign}_0^+u(t,s,x,y)\,\zeta.$$

On the other hand, since $u_1(t,x)+\tau>-1$, using the fact that (u_2,p_2) satisfies (2.21) with $k=u_1(t,x)+\tau$, we have

$$\frac{d}{ds}\int u(t,s,x,y)^+\,\zeta + \int(\nabla_y p_2^-(s,y) - u(t,s,x,y)^+V(y)\cdot\nabla_y\zeta$$
$$-\int_\Omega\nabla_y\cdot V\,u_1(t,x)\zeta\mathrm{Sign}_0^+(u(t,s,x,y)) \leq -\int f_2(s,y)\mathrm{Sign}_0^+(u(t,s,x,y))\,\zeta.$$

Working in the same way, we get

$$\frac{d}{ds}\int u(t,s,x,y)^+\,\zeta + \iint (\nabla_x+\nabla_y)p(t,s,x,y)\cdot\nabla_y\zeta - \iint u(t,s,x,y)^+\,V(y)\cdot\nabla_y\zeta$$
$$-\iint \nabla_y\cdot V(y)\,u_1(t,x)\,\zeta\,\mathrm{Sign}_0^+(u(t,s,x,y)) \le -\iint f_2(s,y)\mathrm{Sign}_0^+(u(t,s,x,y))\,\zeta.$$

Adding both inequalities, we obtain

$$\begin{aligned}\left(\frac{d}{dt}+\frac{d}{ds}\right)\iint u(t,s,x,y)^+\,\zeta + \iint (\nabla_x+\nabla_y)p(t,s,x,y)\cdot(\nabla_x+\nabla_y)\zeta\\ -\iint u(t,s,x,y)^+\,(V(x)\cdot\nabla_x\zeta + V(y)\cdot\nabla_y\zeta)\\ +\iint (\nabla_x\cdot V(x)\,u_2(s,y) - \nabla_y\cdot V(y)\,u_1(t,x))\;\zeta\,\mathrm{Sign}_0^+(u(t,s,x,y))\\ \le \iint (f_1(t,x)-f_2(s,y))\mathrm{Sign}_0^+(u(t,s,x,y))\,\zeta.\end{aligned}\tag{2.23}$$

Now, we can de-double the variables t and s, as well as x and y, by taking as usual the sequence of test functions

$$\psi_\varepsilon(t,s) = \psi\left(\frac{t+s}{2}\right)\rho_\varepsilon\left(\frac{t-s}{2}\right) \text{ and } \zeta_\lambda(x,y) = \xi\left(\frac{x+y}{2}\right)\delta_\lambda\left(\frac{x-y}{2}\right),$$

for any $t,s\in(0,T)$ and $x,y\in\Omega$. Here $\xi\in\mathcal{D}(\Omega)$, $\psi\in\mathcal{D}(0,T)$, ρ_ε and δ_λ are sequences of usual mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$ respectively. See that

$$\left(\frac{d}{dt}+\frac{d}{ds}\right)\psi_\varepsilon(t,s) = \rho_\varepsilon\left(\frac{t-s}{2}\right)\dot\psi\left(\frac{t+s}{2}\right)$$

and

$$(\nabla_x+\nabla_y)\zeta_\lambda(x,y) = \delta_\lambda\left(\frac{x-y}{2}\right)\nabla\xi\left(\frac{x+y}{2}\right)$$

Moreover, for any $h\in L^1((0,T)^2\times\Omega^2)$ and $\Phi\in L^1((0,T)^2\times\Omega^2)^N$, it is not difficult to prove that

- $\lim\limits_{\lambda\to0}\lim\limits_{\varepsilon\to0}\int_0^T\int_0^T\int_\Omega\int_\Omega h(t,s,x,y)\,\zeta_\lambda(x,y)\,\rho_\varepsilon(t,s) = \int_0^T\int_\Omega h(t,t,x,x)\,\xi(x)\,\psi(t).$
- $\lim\limits_{\lambda\to0}\lim\limits_{\varepsilon\to0}\int_0^T\int_0^T\int_\Omega\int_\Omega h(t,s,x,y)\,\zeta_\lambda(x,y)\,\left(\frac{d}{dt}+\frac{d}{ds}\right)\rho_\varepsilon(t,s) = \int_0^T\int_\Omega h(t,t,x,x)\,\xi(x)\,\dot\psi(t).$
- $\lim\limits_{\lambda\to0}\lim\limits_{\varepsilon\to0}\int_0^T\int_\Omega\int_\Omega \Phi(t,s,x,y)\cdot(\nabla_x+\nabla_y)\zeta_\lambda(x,y)\,\rho_\varepsilon(t,s) = \int_0^T\int_\Omega \Phi(t,t,x,x)\cdot\nabla\xi(x)\,\psi(t)\,dtdx.$

So replacing ζ in (2.23) by ζ_λ, testing with ψ_ε and, letting $\varepsilon\to0$ and $\lambda\to0$, we get

$$\begin{aligned}-\int_0^T\int_\Omega\Big\{(u_1-u_2)^+\,\xi\,\dot\psi + \nabla(p_1^+ + p_2^-)\cdot\nabla\xi\,\psi - (u_1-u_2+\tau)^+\;(V\cdot\nabla\xi\;+\nabla\cdot V\,\xi\,)\,\psi\Big\}\\ \le \int_0^T\int_\Omega \kappa_\tau(x)(f_1-f_2)\,\xi\,\psi + \lim_{\lambda\to0}\int_0^T\int_\Omega\int_\Omega u(t,t,x,y)^+\,(V(x)-V(y))\cdot\nabla_y\zeta\,\psi,\end{aligned}$$

where $\kappa_\tau\in L^\infty(Q)$ is such that $\kappa\in\mathrm{Sign}^+(u_1-u_2+\tau)$ a.e. in Q. Notice here that the emergence of κ_τ follows from the L^1-compactness of the term u and the $L^\infty-$weak* compactness of $\mathrm{Sign}_0^+(u)$, as

well as the monotonicity of the graph Sign^+. Thus

$$\begin{aligned}\frac{d}{dt}\int (u_1-u_2+\tau)^+\,\xi+\int \nabla(p_1^+ + p_2^-)\cdot\nabla\xi-\int (u_1-u_2+\tau)^+\ (V\cdot\nabla\xi+\nabla\cdot V\,\xi)\\ \leq \int \kappa(x)(f_1-f_2)\,\xi+\lim_{\lambda\to 0}\int_\Omega\int_\Omega u(t,t,x,y)^+\,(V(x)-V(y))\cdot\nabla_y\zeta.\end{aligned}\tag{2.24}$$

To pass to the limit in the last term corresponding to the vector filed V, we use moreover the technical result of Lemma 6.8 (cf. Appendix) which is more or less well known. We put back its proof in the Appendix. This implies that

$$\begin{aligned}\frac{d}{dt}\int (u_1(t,x)-u_2(t,x)+\tau)^+\,\xi\,dx+\int \nabla(p_1^+ + p_2^-)\cdot\nabla\xi\,dx\\ -\int (u_1-u_2+\tau)^+\,V\cdot\nabla\xi\,dx\leq\int \kappa_\tau(f_1-f_2)\,\xi\,dx.\end{aligned}$$

Letting then $\tau\to 0$, and using again the fact that $\kappa_\tau\to\kappa$ weakly to $L^\infty(Q)$, with $\kappa\in\mathrm{Sign}^+(u_1-u_2)$, a.e. in Q, the result of the proposition follows. □

The aim now is to process with the sequence of test function ξ_h given by (1.1) in Kato's inequality and let $h\to 0$, to cover (2.1).

Lemma 2.4. *Under the assumption* (1.3)*, if* (u,p) *is a weak solution of* (1.5)*, then for any* $0\leq\psi\in D(0,T)$*, we have*

$$\liminf_{h\to 0}\int_0^T\int_\Omega \nabla p^\pm\cdot\nabla\xi_h\,\psi(t)\,dtdx\geq 0.$$

Proof : Using (2.5), we see that, for any $0\leq\psi\in D(0,T)$ we have

$$\begin{aligned}\int_0^T\int_\Omega \nabla p^+\cdot\nabla\xi_h\,\psi\,dtdx &= -\int_0^T\int_\Omega \nabla p^+\cdot\nabla(1-\xi_h)\,\psi\,dtdx\\ &\geq \int_0^T\int_\Omega (\nabla\cdot V-f)(1-\xi_h)\mathrm{Sign}_0^+(p)\,\psi\,dtdx\end{aligned}$$

Then, letting $h\to 0$ and using the fact that $\xi_h\to 1$ in $L^\infty(\Omega)-\mathrm{weak}^*$, we deduce that $\liminf_{h\to 0}\int_0^T\int_\Omega \nabla p^+(t)\cdot\nabla\xi_h\,\psi\,dtdx\geq 0$. The proof for p^- follows in a similar way by using (2.6).

□

Lemma 2.5. *If* (u_1,p_1) *and* (u_2,p_2) *are two couples of* $L^\infty(Q)\times L^2\left(0,T;H^1_D(\Omega)\right)$ *satisfying* (2.20) *and* (2.21) *corresponding to* $f_1\in L^1(Q)$ *and* $f_2\in L^1(Q)$ *respectively, then there exists* $\kappa\in L^\infty(Q)$*, such that* $\kappa\in Sign^+(u_1-u_2)$ *a.e. in* Q *and* (2.1) *is fulfilled.*

Proof. See that

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u_1-u_2)^+\,dx-\int\kappa(f_1-f_2) &= \lim_{h\to 0}\frac{d}{dt}\int_\Omega (u_1-u_2)^+\,\xi_h\,dx-\int\kappa(f_1-f_2)\xi_h\,dx\\ &=: I(h).\end{aligned}$$

Taking ξ_h as a test function in (2.22), we have

$$I(h) \leq -\int \left(\nabla(p_1^+ + p_2^-) - (u_1 - u_2)^+ V\right) \cdot \nabla \xi_h \, dx.$$

Using Lemma 2.4, this implies that

$$\begin{aligned} \lim_{h\to 0} I(h) &\leq \lim_{h\to 0} \int (u_1 - u_2)^+ V \cdot \nu(x) \, dx \\ &\leq 0 \end{aligned}$$

where we use the ingoing velocity vector field assumption (1.3). Thus (2.1). □

Proof of Theorem 2.1. It clear that the first part follows by Lemma 2.5. The rest of the theorem is a straightforward consequence of (2.1). □

3. Existence for the evolution problem

Our main result here concerns existence of a solution.

Theorem 3.2. *For any $f \in L^2(Q)$ and $u_0 \in L^\infty(\Omega)$ be such that $|u_0| \leq 1$ a.e. in Ω, the problem (1.5) has a unique weak solution (u, p) such that $u \in \mathcal{C}([0,T), L^1(\Omega))$ and $u(0) = u_0$. Moreover, the solution satisfies all the properties of Theorem 2.1.*

Thanks to Theorem 3.2 and Theorem 2.1, we have the following practical particular results.

Corollary 3.1. *Let $f \in L^2(Q)$, $u_0 \in L^\infty(\Omega)$ be such that $|u_0| \leq 1$ a.e. in Ω, and u be the unique weak solution of the problem (1.5).*

(1) *If $u_0 \geq 0$ and $f \geq 0$, then (u, p) is the unique weak solution of the one phase problem*

$$\begin{cases} \left.\begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u\, V) = f \\ 0 \leq u \leq 1,\ p \geq 0,\ p(u-1) = 0 \end{array}\right\} & \text{in } Q \\ p = 0 & \text{on } \Sigma_D \\ (\nabla p + u\, V) \cdot \nu = 0 & \text{on } \Sigma_N \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

in the sense of Definition 1.1.

(2) *For each $n = 1, 2, ...$, let $f_n \in L^2(Q)$, $u_{0n} \in L^\infty(\Omega)$ be such that $|u_{0n}| \leq 1$ a.e. in Ω, and u_n be the unique weak solution of the corresponding problem (1.5). If $u_{0n} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q)$, then $u_n \to u$, in $\mathcal{C}([0,T), L^1(\Omega))$, $p_n \to p$, in $L^2(0,T; H^1_D(\Omega)) - weak$, and (u, p) is the solution corresponding to u_0 and f.*

To study the existence, we process in the framework of nonlinear semigroup theory in $L^1(\Omega)$. In connection with the Euler implicit discretization scheme of the evolution problem (1.5), we consider

the following stationary problem :

$$\left\{\begin{array}{ll} \left.\begin{array}{l} u-\Delta p+\nabla\cdot(u\,V)=f \\ u\in \mathrm{Sign}(p)\end{array}\right\} & \text{in } \Omega \\ p=0 & \text{on } \Gamma_D \\ (\nabla p+u\,V)\cdot\eta=0 & \text{on } \Gamma_N, \end{array}\right. \tag{3.1}$$

where $f\in L^2(\Omega)$ and $\lambda>0$ are given. A couple $(u,p)\in L^\infty(\Omega)\times H^1_D(\Omega)$ satisfying $u\in \mathrm{Sign}(p)$ a.e. in Ω, is said to be a weak solution of (3.1) if

$$\int_\Omega u\,\xi+\int_\Omega \nabla p\cdot\nabla\xi-\int_\Omega u\,V\cdot\nabla\xi=\int_\Omega f\,\xi,\quad \forall\,\xi\in H^1_D(\Omega).$$

As for the evolution problem, we'll say simply that u is a solution of (3.1) if there exists p such that the couple (u,p) is a weak solution of (3.1).

As a consequence of Theorem 2.1, we can deduce the following result.

Corollary 3.2. *If u_1 and u_2 are two solutions of* (3.1) *associated with f_1, $f_2\in L^1(\Omega)$, respectively, then*

$$\int (u_1-u_2)^+\le\int (f_1-f_2)^+.$$

In particular, we have

$$\|u_1-u_2\|_1\le\|f_1-f_2\|_1$$

and, if $f_1\le f_2$, a.e. in Ω, then

$$u_1\le u_2,\qquad \text{a.e. in } \Omega.$$

Proof. This is a simple consequence of the fact that if (u,p) is a solution of (3.1), then it is a solution of (1.5) with f replaced by $f-u$. □

For the existence, we consider the regularized problem

$$\left\{\begin{array}{ll} \left.\begin{array}{l} u-\Delta p+\nabla\cdot(u\,V)=f \\ u=\mathcal{H}_\varepsilon(p)\end{array}\right\} & \text{in } \Omega \\ p=0 & \text{on } \Gamma_D \\ (\nabla p+u\,V)\cdot\eta=0 & \text{on } \Gamma_N, \end{array}\right. \tag{3.2}$$

for an arbitrary for any $\varepsilon>0$. See that for any $\varepsilon>0$, $|\mathcal{H}_\varepsilon|\le 1$, $\mathcal{H}_\varepsilon$ is Lipschitz continuous and satisfies

$$(I+\mathcal{H}_\varepsilon)^{-1}(r)\to(I+\mathrm{Sign})^{-1}(r),\ \text{as } \varepsilon\to 0,\ \text{for any } r\in I\!R.$$

That is $\mathcal{H}_\varepsilon$ converges to Sign in the sense of resolvent, which is equivalent to the convergence in the sens of graph (cf. [18]).

Proposition 3.3. *For any $f \in L^2(\Omega)$ and $\varepsilon > 0$, the problem (3.2) has a weak solution $(u_\varepsilon, p_\varepsilon)$ in the sense that $p_\varepsilon \in H^1_D(\Omega)$, $u_\varepsilon = \mathcal{H}_\varepsilon(p_\varepsilon)$ a.e. in Ω and*

$$\int u_\varepsilon\, \xi\, dx + \int \nabla p_\varepsilon \cdot \nabla \xi\, dx - \int u_\varepsilon\, V \cdot \nabla \xi\, dx = \int f\, \xi\, dx, \quad \text{for any } \xi \in H^1_D(\Omega). \tag{3.3}$$

Moreover, as $\varepsilon \to 0$, we have

$$\mathcal{H}_\varepsilon(p_\varepsilon) \to u \quad \text{in } L^\infty(\Omega) - \text{weak}^*, \tag{3.4}$$

$$p_\varepsilon \to p \quad \text{in } H^1_D(\Omega) - \text{weak}$$

and (u, p) is the weak solution of (3.1).

Proof. The existence of a solution for (3.2) is standard. For completeness we give the arguments. Let us denote by $H^1_D(\Omega)^*$ the topological dual space of $H^1_D(\Omega)$ and $\langle ., . \rangle$ the associate dual bracket. See that the operator $A_\varepsilon : H^1_D(\Omega) \to H^1_D(\Omega)^*$, given by

$$\langle A_\varepsilon p, \xi \rangle = \int \mathcal{H}_\varepsilon(p)\, \xi\, dx + \int \nabla p \cdot \nabla \xi\, dx - \int \mathcal{H}_\varepsilon(p)\, V \cdot \nabla \xi\, dx, \quad \forall\, \xi \in H^1_D(\Omega),$$

is bounded and weakly continuous. Moreover, A_ε is coercive. Indeed, for any $u \in H^1_D(\Omega)$, we have

$$\begin{aligned} \langle A_\varepsilon p, p \rangle &= \int \mathcal{H}_\varepsilon(p)\, p\, dx + \int |\nabla p|^2\, dx - \int \mathcal{H}_\varepsilon(p)\, V \cdot \nabla p\, dx \\ &\geq \int |\nabla p|^2\, dx - \int |V \cdot \nabla p|\, dx \\ &\geq \frac{1}{2} \int |\nabla p|^2\, dx - \frac{1}{2} \int |V|^2\, dx, \end{aligned}$$

where we use Young inequality. So, for any $f \in H^1_D(\Omega)^*$ the problem $A_\varepsilon p = f$ has a solution $p_\varepsilon \in H^1_D(\Omega)$. To let $\varepsilon \to 0$, we see that

$$\int |\nabla p_\varepsilon|^2\, dx \leq C(N, \Omega) \left(\int |V|^2\, dx + \int |f|^2\, dx \right). \tag{3.9}$$

Indeed, taking p_ε as a test function we have

$$\int u_\varepsilon\, p_\varepsilon\, dx + \int |\nabla p_\varepsilon|^2\, dx = \int u_\varepsilon\, V \cdot \nabla p_\varepsilon\, dx + \int f\, p_\varepsilon\, dx.$$

See that using Young inequality we have

$$\int u_\varepsilon\, V \cdot \nabla p_\varepsilon\, dx \leq \frac{3}{4} \int |V|^2 + \frac{1}{3} \int |\nabla p_\varepsilon|^2\, dx$$

and, by combining Pincaré inequality with Young inequality we have

$$\int f\, p_\varepsilon\, dx \leq C(N, \Omega) \int |f|^2 + \frac{1}{3} \int |\nabla p_\varepsilon|^2\, dx.$$

Using the fact that $u_\varepsilon\, p_\varepsilon \geq 0$, we deduce (3.9). Now, it is clear that the sequences p_ε and $u_\varepsilon = \mathcal{H}_\varepsilon(p_\varepsilon)$ are bounded respectively in $H^1_D(\Omega)$ and in $L^\infty(\Omega)$. So, there exists a subsequence that we denote again by p_ε such that (3.4) and (3.12) are fulfilled. In particular, this implies that $u \in \text{Sign}(p)$, a.e. and letting $\varepsilon \to 0$ in (3.3), we obtain that (u, p) is a weak solution of (3.1). □

To prove the existence, of a weak solution to (1.5), we fix $f \in L^2(Q)$, and for an arbitrary $0 < \varepsilon \leq \varepsilon_0$, and $n \in \mathbb{N}^*$ be such that $(n+1)\varepsilon = T$, we consider the sequence of (u_i, p_i) given by the the $\varepsilon-$Euler implicit schema associated with (1.5) :

$$\left\{\begin{array}{lll} \left.\begin{array}{l} u_{i+1} - \varepsilon\,\Delta p_{i+1} + \varepsilon\,\nabla\cdot(u_{i+1}\,V) = u_i + \varepsilon\,f_i \\ u_{i+1} \in \mathrm{Sign}(p_{i+1}) \end{array}\right\} & \text{in } \Omega & \\ & & \\ p_{i+1} = 0 & \text{on } \Gamma_D & i = 0, 1, ...n-1, \\ & & \\ (\nabla p_{i+1} + u_{i+1}\,V)\cdot\eta = 0 & \text{on } \Gamma_N, & \end{array}\right. \tag{3.10}$$

where, for each $i = 0, ...n-1$, f_i is given by

$$f_i = \frac{1}{\varepsilon}\int_{i\varepsilon}^{(i+1)\varepsilon} f(s)\,ds, \quad \text{a.e. in } \Omega.$$

Now, for a given $\varepsilon-$time discretization $0 = t_0 < t_1 < t_1 < ... < t_n < t_{n+1} = T$, satisfying $t_{i+1} - t_i = \varepsilon$, we define the $\varepsilon-$approximate solution by

$$u_\varepsilon := \sum_{i=0}^{n-1} u_i \chi_{[t_i, t_{i+1})}, \quad \text{and} \quad p_\varepsilon := \sum_{i=1}^{n-1} p_i \chi_{[t_i, t_{i+1})}.$$

Thanks to Proposition 3.3 and the general theory of evolution problem governed by accretive operator (see for instance [12, 11]), we define the operator $\mathcal{A}$ in $L^1(\Omega)$, by $\mu \in \mathcal{A}(z)$ if and only if μ, $z \in L^1(\Omega)$ and z is a solution of the problem

$$\left\{\begin{array}{ll} \left.\begin{array}{l} -\Delta p + \nabla\cdot(z\,V) = \mu \\ z \in \mathrm{Sign}(p) \end{array}\right\} & \text{in } \Omega \\ & \\ p = 0 & \text{on } \Gamma_D \\ & \\ (\nabla p + u\,V)\cdot\eta = 0 & \text{on } \Gamma_N, \end{array}\right.$$

in the sens that $z \in L^\infty(\Omega)$, there exists $p \in H^1_D(\Omega)$ satisfying $z \in \mathrm{Sign}(p)$ a.e. in Ω and

$$\int_\Omega \nabla p\cdot\nabla\xi - \int_\Omega z\,V\cdot\nabla\xi = \int_\Omega \mu\,\xi, \quad \forall\,\xi \in H^1_D(\Omega).$$

As a consequence of Corollary 3.2, we know that the operator $\mathcal{A}$ is accretive in $L^1(\Omega)$. So, thanks to the general theory of nonlinear semigroup governed by accretive operator, we know that, as $\varepsilon \to 0$,

$$u_\varepsilon \to u, \quad \text{in } \mathcal{C}([0,T), L^1(\Omega)), \tag{3.11}$$

and u is the so called "mild solution" of the evolution problem

$$\left\{\begin{array}{ll} u_t + \mathcal{A}u \ni f & \text{in } (0,T) \\ & \\ u(0) = u_0. & \end{array}\right.$$

To accomplish the proof of existence for the problem (1.5), we prove that the mild solution u is in fact the solution of (1.5). To this aim, we use the limit of the sequence p_ε given by the $\varepsilon-$approximate solution.

Lemma 3.6. *As* $\varepsilon \to 0$,

$$p_\varepsilon \to p, \qquad \text{in } L^2(0,T;H^1_D(\Omega)) \tag{3.12}$$

and (u,p) *is a weak solution of* (1.5).

Proof. Thanks to Proposition 3, the sequence $(u_i,p_i)_{i=1,\dots n}$ given by (3.10) is well defined in $L^\infty(\Omega)\times H^1_D(\Omega)$ and satisfies $u_i \in \mathrm{Sign}(p_i)$ and

$$\int_\Omega u_{i+1}\,\xi + \varepsilon\int_\Omega \nabla p_{i+1}\cdot\nabla\xi - \varepsilon\int_\Omega u_{i+1}\,V\cdot\nabla\xi = \varepsilon\int_\Omega f_i\,\xi, \qquad \text{for any } \xi\in H^1_D(\Omega). \tag{3.13}$$

Taking p_{i+1} as a test function in (3.13), working as for the proof of (3.9) and using the fact that $(u_{i+1}-u_i)p_{i+1}\ge 0$, we get

$$\int |\nabla p_i|^2\,dx \le C(N,\Omega)\left(\int |V|^2\,dx + \int |f_i|^2\,dx\right).$$

Thus

$$\int |\nabla p_\varepsilon|^2\,dx \le C(N,\Omega)\left(\int |V|^2\,dx + \int |f_\varepsilon|^2\,dx\right),$$

where $f_\varepsilon = \displaystyle\sum_{i=0}^{n-1} f_i\chi_{[t_i,t_{i+1})}$, in Q. This implies that p_ε is bounded in $L^\infty(0,T;H^1_D(\Omega))$, and that there exists $p\in L^\infty(0,T;H^1_D(\Omega))$ such that, taking a subsequence if necessary,

$$p_\varepsilon \to p, \qquad \text{in } L^2(0,T;H^1_D(\Omega)) - \text{weak}.$$

Combining this with (3.11), we deduce moreover that $u\in\mathrm{Sign}(p)$, a.e. in Q. Now, as usually used with nonlinear semigroup theory for evolution problem, we consider

$$\tilde u_\varepsilon = \sum_{i=0}^{n-1}\frac{(t-t_i)u_{i+1}-(t-t_{i+1})u_i}{\varepsilon}\chi_{[t_i,t_{i+1})},$$

which converges to u as well in $\mathcal{C}([0,T);L^1(\Omega))$. We see that for any test function $\xi\in H^1_D(\Omega)$, we have

$$\frac{d}{dt}\int_\Omega \tilde u_\varepsilon\,\xi + \int_\Omega (\nabla p_\varepsilon - u_\varepsilon\,V)\cdot\nabla\xi = \int_\Omega f_\varepsilon\,\xi, \qquad \text{in } \mathcal{D}'([0,T)).$$

So, letting $\varepsilon\to 0$, and using the convergence of $(\tilde u_\varepsilon,u_\varepsilon,p_\varepsilon,f_\varepsilon)$ to (u,u,p,f), we deduce that (u,p) is a weak solution of (1.5).

□

4. Reaction-diffusion case

In this section we consider the general case with a reaction term :

$$\begin{cases} \left.\begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta p + \nabla\cdot(u\,V) = g(.,u) \\ u\in\mathrm{Sign}(p) \end{array}\right\} & \text{in } Q \\ p = 0 & \text{on } \Sigma_D \\ (\nabla p + u\,V)\cdot\nu = 0 & \text{on } \Sigma_N \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $g : Q \times I\!R \to I\!R$ is a Caratheodory function ; i.e. continuous in $r \in I\!R$ and measurable in $(t,x) \in Q$. We assume that g satisfies moreover, the following assumptions :

($\mathcal{G}_1$) $g^+(.,-1) \in L^2(Q)$ and $g^-(.,1) \in L^2(Q)$

($\mathcal{G}_2$) There exists $0 \leq \mathcal{R} \in L^2(0,T)$, s.t. for any $a,\ b\ \in [-1,+1]$, we have

$$\mathrm{Sign}_0^+(b-a)\,(g(t,x,b)-g(t,x,a)) \leq \mathcal{R}(t)\,(b-a)^+, \quad \text{for a.e. } (t,x) \in Q.$$

On sees in particular that ($\mathcal{G}_2$) implies

$$-g^-(.,1) - \mathcal{R}\,(1-r) \leq g(.,r) \leq g^+(.,-1) + \mathcal{R}\,(1+r), \quad \text{for any } r \in [-1,+1], \tag{4.2}$$

so that $g(.,r) \in L^2(Q)$, for any $r \in [-1,+1]$.

Remark 5. *See that the condition (G2) is fulfilled for instance whenever*

$$\frac{\partial g}{\partial r}(t,x,.) \leq \theta \quad \text{ in } \mathcal{D}'([-1,1]]), \text{ for a.a. } (t,x) \in Q \text{ with } 0 \leq \theta \in \mathcal{C}([-1,1]),$$

In this case $\mathcal{R} = \max\limits_{-1\leq r\leq 1} \theta(r)$.

Theorem 4.3. *Under the assumption* ($\mathcal{G}_1$) *and* ($\mathcal{G}_2$), *for any* $u_0 \in L^\infty(\Omega)$ *such that* $|u_0| \leq 1$, *a.e. in* Ω, *the problem* (4.1) *has a unique weak solution* (u,p) *in the sense of Definition 1.1 with* $f = g(.,u)$. *Moreover,*

($\mathcal{P}_1$) *for any* $\omega_1,\ \omega_2 \in W^{1,1}(0,T)$ *such that* $u_0 \leq \omega_2(0)$ *(resp.* $\omega_1(0) \leq u_0$*) and, for any* $t \in (0,T)$,

$$\dot\omega_2(t) + \omega_2(t)\nabla\cdot V \geq g(.,\omega_2(t)) \quad \text{a.e. in } \Omega \tag{4.3}$$

and

$$\text{(rep. } \dot\omega_1(t) + \omega_1(t)\nabla\cdot V \leq g(.,\omega_1(t)) \quad \text{a.e. in } \Omega), \tag{4.4}$$

we have

$$u \leq \omega_2 \quad (\text{resp. } \omega_1 \leq u) \quad \text{a.e. in } Q.$$

($\mathcal{P}_2$) *If* g *satisfies moreover*

$$(\mathcal{G}_3) \quad \nabla\cdot V \geq g(.,1), \text{ a.e. in } Q \quad \text{and} \quad (\mathcal{G}_4) \quad \nabla\cdot V \leq g(.,-1), \text{ a.e. in } Q,$$

then $p \equiv 0$, *and* u *is given by the unique solution of the reaction-transport equation*

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \dfrac{\partial u}{\partial t} + \nabla\cdot(u\,V) = g(t,x,u) \\ |u| \leq 1 \end{array}\right\} & \text{in } Q \\ u\,V\cdot\nu = 0 & \text{on } \Sigma_N \\ u(0) = u_0 & \text{in } \Omega, \end{array}\right. \tag{4.5}$$

in the sense that $u \in \mathcal{C}([0,T), L^1(\Omega))$, $|u| \leq 1$ *a.e. in* Q *and*

$$\frac{d}{dt}\int_\Omega u\,\xi - \int_\Omega u\,V\cdot\nabla\xi = \int_\Omega g(.,u)\,\xi, \quad \text{in } \mathcal{D}'(0,T), \quad \forall\,\xi \in H^1_D(\Omega). \tag{4.6}$$

Corollary 4.3. *Under the assumptions of Theorem 4.3, assume moreover that* $0 \leq u_0 \leq 1$ *a.e. in* Ω *and*

$$(\mathcal{G}_5) \quad 0 \leq g(.,0) \text{ a.e. in } Q.$$

then

$$p \geq 0 \text{ and } 0 \leq u \leq 1, \qquad \text{a.e. in } Q.$$

If, moreover, g satisfies $(\mathcal{G}_3)$*, then the unique solution u is nonnegative and is given by* (4.5).

Remark 6. *See that* $(\mathcal{G}_3)$ *and* $(\mathcal{G}_4)$ *constitute sufficient conditions to lessen the so called congestion phenomena in the transport process. This is an interesting property for the applications in crowed motion modeling. Indeed, heuristically the reaction term* $g(t,x,1)$ *is connected to the reaction of the population at the position* $x \in \Omega$ *and time* t*, in a "congested" circumstance. Performing its value regarding to the divergence of the velocity vector field V, may avoid the congestion in the transportation phenomena for crowed motion.*

To prove Theorem 4.3, we use the following lemma which is a direct consequence of Lemma 1 of [14].

Lemma 4.7. *Let A be an accretive operator in* $L^1(\Omega)$*; such that the closure* $\overline{\mathcal{A}}$ *is m-accretive in* $L^1(\Omega)$ *(i.e.* $R(I+\lambda\overline{\mathcal{A}}) = L^1(\Omega)$*, for any* $\lambda > 0$*),* $u_0 \in \overline{\mathcal{D}}(A)$ *and* $F : (0,T) \times \overline{\mathcal{D}}(A) \to L^1(\Omega)$ *satisfying :*

i) *F is Caratheodory, i.e.* $t \to F(t,z)$ *is measurable for any* $x \in \overline{\mathcal{D}}(A)$*,* $z \to F(t,z)$ *is continuous for a.a.* $t \in (0,T)$

ii) *There exists* $k \in L^1_{loc}(0,T)$ *such that*

$$\int (F(t,z) - F(t,\hat{z}))\, Sign_0(z-\hat{z})\, dx \leq k(t)\, \|z-\hat{z}\|_1, \qquad \text{a.e. } t \in (0,T),$$

for every $z,\ \hat{z} \in \overline{\mathcal{D}}(A)$.

iii) *There exists* $c \in L^1_{loc}(0,T)$ *such that*

$$\|F(t,z)\|_1 \leq c(t), \qquad \text{a.e. } t \in (0,T)$$

for every $z \in \overline{\mathcal{D}}(A)$.

Then, there exists a unique $u \in \mathcal{C}([0,T);X)$ *such that u is the mild solution of*

$$\begin{cases} \dfrac{du}{dt} + Au \ni f & \text{in } (0,T) \\ \\ u(0) = u_0, \end{cases}$$

with $f(t) = F(t,u(t))$ *a.e.* $t \in (0,T)$.

Notice that, by the assumptions of Lemma 4.7, $F(.,u) \in L^1_{loc}(0,T;L^1(\Omega))$ for any $u \in \mathcal{C}([0,T);L^1(\Omega))$. The results of this lemma is well known in the case where $F(t,r) = f(t) + F_0(r)$, with $f(t) \in L^1(\Omega)$, for a.e. $t \in [0,T)$, and F_0 a Lipschitz continuous function in $\mathbb{R}$ (since then $A - F_0 + kI$ is m-accretive in $L^1(\Omega)$). For the general case, one can see Lemma 1 of the paper [14] for the proof.

Proof of Theorem 4.3. Set $F : [0,T) \times L^1(\Omega) \to L^1(\Omega)$ defined by

$$F(t, z(.)) = g(t, ., z(.)) \quad \text{a.e. in } \Omega, \text{ for any } (t,z) \in [0,T) \times L^1(\Omega).$$

See that F satisfies all the assumptions of Lemma 4.7. In particular, $k(t) = \mathcal{R}(t)$. Then, we consider $u \in \mathcal{C}([0,T), L^1(\Omega))$ the mild solution of the evolution problem

$$\begin{cases} u_t + \mathcal{A}u \ni f & \text{in } (0,T) \\ \\ u(0) = u_0, \end{cases}$$

with $f = g(., u)$. Thanks to (4.2), it is clear that $f \in L^2(Q)$. Moreover, using Lemma 3.6, there exists p such that (u,p) is a weak solution of (4.1). The uniqueness follows from the equivalence between weak solution and mild solution as well as the uniqueness result of Theorem 2.1. Now let us prove the properties $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$.

$(\mathcal{P}_1)$ Thanks to the L^1-comparison principal of Theorem 2.1, and using the fact that $(\omega_2, 0)$ is a weak solution of (1.5) with $f = \dot{\omega}_2 + \omega_2 \nabla \cdot V$, we see that

$$\begin{aligned} \frac{d}{dt}\int (u-\omega_2)^+ &\leq \int_{[u \geq \omega_2]} (g(.,u) - \dot{\omega}_2 - \omega_2 \nabla \cdot V) \\ &\leq \int_{[u \geq \omega_2]} (g(.,u) - g(.,\omega_2))^+ \\ &\leq \mathcal{R} \int (u(t) - \omega_2)^+. \end{aligned}$$

Applying Gronwall and using the fact that $u(0) \leq \omega_2(0)$, we obtain $u(t) \leq \omega_2$ a.e. in Q. The proof of $u \geq \omega_1$ in Q follows in the same way by proving that

$$\frac{d}{dt}\int (\omega_1 - u)^+ \leq \mathcal{R} \int (\omega_1 - u)^+.$$

$(\mathcal{P}_2)$ To prove this assertion, it is enough to prove that the problem (4.5) has a solution in the sense of (4.6). Then the conclusion follows by the uniqueness of the solution for (4.1). To this aim, we proceed by using the existence result as well as the comparison principle of the first part of the theorem. For an arbitrary $\alpha > 0$, we consider the reaction term g_α given by

$$g_\alpha(., r) = \alpha\, g(., r/\alpha), \quad \text{a.e. in } Q, \quad \text{for any } r \in I\!R.$$

It is clear that g_α satisfies all the assumptions of Theorem 4.3. So, we can consider (v_α, p_α) be a weak solution of

$$\begin{cases} \left.\begin{array}{l} \dfrac{\partial v_\alpha}{\partial t} - \Delta p_\alpha + \nabla \cdot (v_\alpha\, V) = g_\alpha(., v_\alpha) \\ \tilde{v}_\alpha \in \mathrm{Sign}(p_\alpha) \end{array}\right\} & \text{in } Q \\ p_\alpha = 0 & \text{on } \Sigma_D \\ (\nabla p_\alpha + v_\alpha\, V) \cdot \nu = 0 & \text{on } \Sigma_N \\ v_\alpha(0) = \alpha\, u_0 & \text{in } \Omega. \end{cases}$$

On the other hand, thanks to (4.3), we see that

$$\partial_t \alpha + \alpha \nabla \cdot V \geq g_\alpha(., \alpha), \quad \text{in } Q.$$

Since $v_\alpha(0) \le \alpha$, the assertion $(\mathcal{P}_1)$ of the theorem implies that $v_\alpha \le \alpha$ a.e. in Q. In a similar way, we can prove using (4.4), that $v_\alpha \ge -\alpha$ a.e. in Q. So, under the assumptions (4.3)- (4.4), we have

$$|v_\alpha| \le \alpha, \quad \text{a.e. in } Q.$$

In particular, taking $0 < \alpha < 1$ arbitrary, this implies that $p_\alpha \equiv 0$ and v_α is in fact a solution of

$$\begin{cases} \left.\begin{array}{l} \dfrac{\partial v_\alpha}{\partial t} + \nabla \cdot (v_\alpha\, V) = g_\alpha(t,x,v_\alpha) \\ |v_\alpha| \le \alpha \end{array}\right\} & \text{in } Q \\ v_\alpha\, V \cdot \nu = 0 & \text{on } \Sigma_N \\ v_\alpha(0) = \alpha\, u_0 & \text{in } \Omega. \end{cases}$$

Thus, taking $u = v_\alpha/\alpha$, we generate the solution of (4.5) as claimed.

□

5. Extension, comments and remarks

(1) Notice here that in our case, the entropic inequalities (2.20) and (2.21) are local and do not proceed up to the boundary like in [20]. For the case where $\Gamma_N = \emptyset$, we do believe that one can combined our notion of renormalized solution with the entropic inequality à la Carillo [20] to handle into the entropic formulations the boundary condition $p = 0$ on $\partial\Omega$. Even more, we think also that in this case one can avoid the incoming condition of V on the boundary. Indeed, one sees that (2.3) and (2.4) holds to be true without any condition on $V \cdot \nu$ on the boundary. Using these formulation one can prove that weak solutions satisfy entropic inequality à la Carillo covering the boundary condition $p = 0$. Then, using doubling and de-doubling variables in the manner of [20] one can prove the L^1-comparison principle for two weak solutions. Nevertheless, we assess that the problem would heighten in technicality.

(2) Thanks to the proof of Proposition 2.1, one sees that we can work with $\xi \in H^1_D(\Omega)$, so that any weak solution (u,p) satisfies

$$\frac{d}{dt}\int \beta(u)\,\xi\,dx + \int \nabla p^+ \cdot \nabla\xi\,dx + \langle V\cdot\nabla\beta(u)), \xi\rangle + \int (u\,\nabla\cdot V(\beta'(u)\,\chi_{[p=0]} + \mathrm{Sign}_0^+(p))\,\xi\,dx$$

$$\le \int f\,(\beta'(u)\,\chi_{[p=0]} + \mathrm{Sign}_0^+(p))\,\xi\,dx \quad \text{in } \mathcal{D}'(0,T),$$

and

$$\frac{d}{dt}\int \beta(u)\,\xi\,dx + \int \nabla p^- \cdot \nabla\xi\,dx + \langle V\cdot\nabla\beta(u)), \xi\rangle + \int (u\,\nabla\cdot V(\beta'(u)\ \chi_{[p=0]} - \mathrm{Sign}_0^-(p))\,\xi\,dx$$

$$\le \int f\,(\beta'(u)\,\chi_{[p=0]} - \mathrm{Sign}_0^-(p))\,\xi\,dx \quad \text{in } \mathcal{D}'(0,T),$$

for any $0 \le \xi \in H^1_D(\Omega)$ and $\beta \in \mathcal{C}^1(I\!R)$ such that $0 \le \beta' \le 1$. In particular, for the case of Neuman boundary condition ; i.e. $\Gamma_D = \emptyset$, these formulations implies that entropic inequalities

(2.20) and (2.21) hold to be true up to the boundary $\partial\Omega$. That is (2.20) and (2.21) are fulfilled in $\mathcal{D}'((0,T)\times\overline{\Omega})$). However, to de-double the variable by taking into account the boundary becomes unhandy since one needs to handle the trace on the boundary in a strong sense which is not guaranteed in general (one can see for instance [6] where this kind of approach is developed for elliptic-parabolic equation with homogeneous boundary condition). By the way, let us mention here that this case is fruitful for gradient flow approach in euclidean Wasserstein space. An elegant and promising proof of uniqueness is given in [27] for the one phase problem under a monotonicity assumption on V. The strength of this approach is the absolute continuity of weak solution in the set of probability equipped with Wasserstein distance.

(3) For the case where $\Omega = I\!R^N$, one sees that working as in Section 2, it is possible to prove local Kato's inequality of the type (2.22) and achieve L^1-comparison principle for weak solution like in Theorem (2.1). However, the existence of weak solutions in $L^1(I\!R^N)$ is not clear yet for us.

(4) Beyond their significant role for the uniqueness, in our opinion formulations (2.3) and (2.4) remain to be interesting also for qualitative descriptions of the weak solution. As usual for renormalized formulation, they enable to localize the description of the solution with respect to its values. For similar description with respect to the values of p, one can see the following remarks which aims to widen the formulations (2.5) and (2.6).

(5) The renormalization process we use in this paper concerns essentially u. The formulation (2.5) and (2.6) provide a particular renormalization for p in the region $[p \neq 0]$. Notice here, that it is possible to prove mended renormalized formulations for p. As a matter of fact, under the assumptions of Lemma 2.1, we can prove that for any $H \in \mathcal{C}^1(I\!R)$ such that $H' \geq 0$ and $H(0) = 0$, we have

$$-\nabla\cdot(H(p)\nabla p) + |H(p)|\,\nabla\cdot V = -H'(p)\,|\nabla p|^2 + f\,H(p) \quad \text{in } \mathcal{D}'(Q). \tag{5.10}$$

If moreover, $H \geq 0$, then

$$-\nabla\cdot(H(p)\nabla p) + |H(p)|\,\nabla\cdot V \leq -H'(p)\,|\nabla p|^2 + f\,H(p) \quad \text{in } \mathcal{D}'((0,T)\times\overline{\Omega}). \tag{5.11}$$

These formulations improve the description of p with respect to the values of p, and yield renormalized formulations for p. As we see in Section 2, the case where H approximates Sign_0^+ is crucial for the proof of uniqueness. The general case may serve out other qualitative description of the solution. The proof of (5.10) and (5.11) follows more or less similar arguments as for the proof Lemma 2.1.

(6) Coming back to the approximation (3.2), thanks to Proposition 3.3, we know that weak solutions of the stationary problem of (3.2) depends weak-continuously on the non-linearity $\mathcal{H}_\varepsilon$. For the evolution problem, working as in the proof of Proposition 3.3, one can prove that the problem

$$\begin{cases} \left.\begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta p + \nabla\cdot(u\,V) = f \\ u = \mathcal{H}_\varepsilon(p) \end{array}\right\} & \text{in } Q \\ p = 0 & \text{on } \Sigma_D \\ (\nabla p + u\,V)\cdot\nu = 0 & \text{on } \Sigma_N \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

has a unique weak solution $(u_\varepsilon, p_\varepsilon) \in \mathcal{C}([0,T), L^1(\Omega)) \times L^2(0,T; H^1_D(\Omega))$, with $u_\varepsilon = \mathcal{H}_\varepsilon(p_\varepsilon)$ a.e. in Q, and as $\varepsilon \to 0$, we have $u_\varepsilon \to u$, in $L^\infty(Q) -$ weak* and $p_\varepsilon \to p$, in $L^2(0,T; H^1_D(\Omega))$. We do believe that the convergence of u_ε remains to be true strongly in $L^1(\Omega)$, for both problems stationary and evolution one. We did not get into these further questions here. This need more fine estimates on u (like $BV-$estimates). We'll touch them in forthcoming works. The case where one replace $\mathcal{H}_\varepsilon$ by an arbitrary non-linearity which converges to Sign, in the sense of resolvent, like for instance the porous medium non-linearity $\varphi(r) = |r|^{m-1}r$, for any $r \in I\!R$, with $m \to \infty$, is likely concerned.

(7) It is clear that our approach is based on a renormalization process à la DiPerna-Lions. This corroborates that this concept is still fruitful for the uniqueness of weak solution (solution in the sense of distribution) for nonlinear versions of Fokker-Planck equation of the type (1.1). As we quote in the previous remarks, operating with the absolute continuity of weak solution in the set of probability equipped with Wasserstein distance seems to be promising (cf. [27]) even if it is actually hampered by conceptual technical assumptions like the sign of the solution, conservation of the mass and monotone transport field (connected to some kind of convexity conditions). We do not overlook the L^1-kinetic approach, which was developed in [22] for Cauchy problems of general degenerate parabolic-hyperbolic equations with non-isotropic non-linearity. The approach could be applied to more general situations and, as far as we know, this approach has not yet been explored for problems of the type (1.1) and (1.2). At last let us quote here the very recent work of [25] (that we have just learned through Benoit Perthame) where the authors deal among other questions with the uniqueness of weak solutions for the one phase problem (1.2) in $I\!R^N$, by means of Hilbert's duality method. This method turns up to be very restrictive since they need the vector filed $V = \nabla\Phi$ to be smooth enough (at least such that $\nabla\Delta\Phi$ is a $L^{12/5}_{loc}-$Lebesgue function).

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6. Appendix

The aim of this section is to complete the proof of Proposition 2.2 with the proof of de-doubling variable process concerning the vector field part of (2.24), which is

$$\lim_{\lambda\to 0} \int_\Omega\int_\Omega u(t,t,x,y)^+ \left(V(x) - V(y)\right) \cdot \nabla_y \zeta_\lambda \, dxdy.$$

Remember that

$$\zeta_\lambda(x,y) = \xi\left(\frac{x+y}{2}\right)\delta_\lambda\left(\frac{x-y}{2}\right),$$

for any $x, y \in \Omega$, where $\xi \in \mathcal{D}(\Omega)$, δ_λ is a sequence of usual mollifiers in $I\!R^N$. For instance, take

$$\delta_\lambda(x) = \frac{1}{\lambda^N}\delta\left(\frac{|x|^2}{\lambda^2}\right), \quad \text{for any } x \in I\!R^N$$

where $0 \le \delta \in \mathcal{D}(0,1)$ and $\dfrac{1}{2^N}\displaystyle\int \delta(|z|^2)\, dz = 1$. A typical example is given by $\delta(r) = Ce^{\frac{-1}{1-r^2}}\chi_{[|r|\le 1]}$, for any $r \in I\!R^N$.

Lemma 6.8. *For any $h \in L^1((0,T)\times\Omega\times\Omega)$, $V \in W^{1,1}(\Omega)$ and $\psi \in \mathcal{D}(0,T)$, we have*

$$\lim_{\lambda\to 0}\int_0^T\int_\Omega\int_\Omega h(t,x,y)\,(V(x)-V(y))\cdot\nabla_y\zeta_\lambda(x,y)\,\psi(t)\,dtdxdy$$

$$=\int_0^T\int_\Omega h(x,x)\,\nabla\cdot V(x)\,\xi(x)\,\psi(t)\,dtdx.$$

Proof. We see that

$$(\nabla_x+\nabla_y)\zeta=\delta_\lambda\left(\frac{x-y}{2}\right)\nabla\xi\left(\frac{x+y}{2}\right).$$

Since, for any $\eta\in I\!R^N$,

$$\nabla\delta_\lambda(\eta) \;=\; \frac{2}{\lambda^N}\frac{z}{\lambda^2}\delta'\left(\frac{|z|^2}{\lambda^2}\right),$$

we have

$$\nabla\delta_\lambda\left(\frac{x-y}{2}\right) \;=\; \frac{2}{\lambda^N}\frac{x-y}{2\lambda^2}\delta'\left(\frac{|x-y|^2}{(2\lambda)^2}\right), \quad \text{for any } x,y\in\Omega.$$

So,

$$\begin{aligned}\nabla_y\zeta(x,y) &= \frac{1}{2}\nabla\xi\left(\frac{x+y}{2}\right)\delta_\lambda\left(\frac{x-y}{2}\right)-\frac{1}{2}\xi\left(\frac{x+y}{2}\right)\nabla\delta_\lambda\left(\frac{x-y}{2}\right)\\ &= \nabla\xi\left(\frac{x+y}{2}\right)\delta_\lambda\left(\frac{x-y}{2}\right)-\frac{1}{\lambda^{N+1}}\xi\left(\frac{x+y}{2}\right)\delta'\left(\frac{|x-y|^2}{(2\lambda)^2}\right)\frac{x-y}{2\lambda}\end{aligned}$$

and

$$\begin{aligned}(V(x)-V(y))\cdot\nabla_y\zeta &= \frac{1}{2}(V(x)-V(y))\cdot\nabla\xi\left(\frac{x+y}{2}\right)\delta_\lambda\left(\frac{x-y}{2}\right)\\ &\quad -\frac{1}{\lambda^N}\xi\left(\frac{x+y}{2}\right)\delta'\left(\frac{|x-y|^2}{(2\lambda)^2}\right)\frac{V(x)-V(y)}{\lambda}\cdot\frac{x-y}{2\lambda}.\end{aligned}$$

This implies that

$$\lim_{\lambda\to 0}\iint h(x,y)\,(V(x)-V(y))\cdot\nabla_y\zeta\,dxdy$$

$$=\lim_{\lambda\to 0}-\frac{1}{\lambda^N}\iint\xi\left(\frac{x+y}{2}\right)\delta'\left(\frac{|x-y|^2}{(2\lambda)^2}\right)\frac{V(x)-V(y)}{\lambda}\cdot\frac{x-y}{2\lambda}\,dxdy=:I(\lambda).$$

Changing the variable by setting

$$z=\frac{x-y}{2\lambda}$$

we get $x = y + 2\lambda z$ and $dx = (2\lambda)^N\, dz$, so that

$$\begin{aligned} -I(\lambda) &= 2^N \iint h((y+2\lambda z, y)\, \xi\,(y+\lambda z)\, \delta'\left(|z|^2\right)\, \frac{V(y+2\lambda z) - V(y)}{\lambda} \cdot z\, dzdy \\ &= 2^N \sum_{i=1}^N \iint h((y+2\lambda z, y)\, \xi\,(y+\lambda z)\, \delta'\left(|z|\right) \frac{V_i(y+2\lambda z) - V_i(y)}{\lambda}\, z_i\ dzdy. \end{aligned}$$

Letting $\lambda \to 0$, we get

$$\begin{aligned} \lim_{\lambda\to 0} I(\lambda) &= -2^{N+1} \sum_{i=1}^N \iint \xi(y)\, h((y,y)\, \nabla V_i(y) \cdot z\, z_i \delta'(|z|^2)\, dzdy \\ &= -2^{N+1} \sum_{i,j=1}^N \iint \xi(y)\, h((y,y)\, \frac{\partial V_i(y)}{\partial y_j}\, z_j\, z_i \delta'(|z|^2)\, dzdy \\ &= -2^{N+1} \sum_{i=1}^N \iint \xi(y)\, h((y,y)\, \frac{\partial V_i(y)}{\partial y_j}\, z_i^2 \delta'(|z|^2)\, dzdy \\ &= -\frac{2^{N+1}}{N} \int \xi(y)\, h((y,y)\, \nabla \cdot V(y)\, dy \int |z|^2\, \delta'(|z|^2)\, dz, \end{aligned}$$

where we use the fact that (since $z \in B(0,1) \to \delta(|z|^2)$ is symmetric)

$$\int z_j\, z_i \delta'(|z|^2)\, dz = 0, \qquad \text{for any } i \neq j$$

and

$$\sum_{i=1}^N \int {z_i}^2 \delta'(|z|^2)\, dz = N \int {z_1}^2 \delta'(|z|^2)\, dz = \int |z|^2 \delta'(|z|^2)\, dz.$$

At last, we use the fact that

$$\frac{2^{N+1}}{N} \int |z|^2\, \delta'(|z|^2)\, dz = -1.$$

Indeed, recall that

$$\int |z|^2\, \delta'(|z|^2)\, dz = \int_0^\infty \int_{S^{n-1}} |r\theta|^2\, \delta'(|r\theta|^2) r^{n-1} dr\, ds_{n-1}(\theta),$$

where s_{n-1} is the probability measure defined on the sphere S_{n-1} by

$$s_{n-1}(A) = |\{rx\ :\ r \in [0,1],\ x \in A\}|\,.$$

This implies that

$$\begin{aligned}\int |z|^2\,\delta'(|z|^2)\,dz &= |B(0,1)|\int_0^1 r^2\,\delta'(r^2)r^{n-1}dr\\ &= |B(0,1)|\int_0^1 \delta'(r^2)r^{n+1}dr\\ &= -\frac{N}{2}\int \delta(|z|^2)\,dz.\end{aligned}$$

Thus the result. □

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