Boundary null controllability of some multi-dimensional linear parabolic systems by the moment method

Franck Boyer, Guillaume Olive

Ann. Inst. Fourier, Grenoble
1, 0 (-1) 000-000

BOUNDARY NULL CONTROLLABILITY OF SOME MULTI-DIMENSIONAL LINEAR PARABOLIC SYSTEMS BY THE MOMENT METHOD

by Franck BOYER & Guillaume OLIVE (*)

ABSTRACT. — In this article we study the null controllability of some abstract linear parabolic systems in tensor product spaces. This special structure allows us to reduce our controllability problem to a particular set of equations that looks like a moment problem, but that does not fall into the previous existing results of the literature.

We transform this non standard moment problem into an infinite family of more usual moment problems, yet coupled one from each other. This reformulation is done with enough care to ensure that the resulting set of equations can be solved, with suitable estimates, by using the recent "block moment method". This is based on a careful analysis of the spectral structure of the underlying operator.

We notably apply our abstract result to show how strong the influence of geometry can be: we provide an example of boundary controlled parabolic system on a rectangle domain which is null controllable in arbitrary small time if two perpendicular faces of the boundary are controlled, whereas it is never null controllable if the control acts on only one face.

Keywords: Controllability, Parabolic systems, Geometric control conditions, Block moment method, Tensor products.
2010 *Mathematics Subject Classification:* 93B05, 93C20, 93C25, 30E05, 35K90, 47A80.
(*) The first author was supported by the Agence Nationale de la Recherche, Project TRECOS, under grant ANR-20-CE40-0009. The second author was supported by National Science Centre, Poland UMO-2020/39/D/ST1/01136. For the purpose of Open Access, the authors have applied a CC-BY public copyright licence to any Author Accepted Manuscript (AAM) version arising from this submission. The authors would like to thank the reviewer for his/her very careful reading of the first version of this work.

Résumé. — Dans cet article nous étudions la contrôlabilité à zéro d'une classe de systèmes paraboliques linéaires abstraits dans des produits tensoriels. Cette structure particulière nous permet de réduire la question de la contrôlabilité à un ensemble particulier d'équations qui ressemble à un problème de moments, mais qui ne relève pas des résultats existants de la littérature.

Nous transformons ce problème de moments non standard en une famille infinie de problèmes de moments usuels, mais couplés entre eux. Cette reformulation est choisie avec soin pour que le nouveau système obtenu puisse être résolu, avec des bonnes estimations des solutions, en utilisant la méthode des moments par blocs développée récemment. Tout ce travail est basé sur une analyse spectrale minutieuse de l'opérateur sous-jacent.

Nous appliquons notamment ce résultat abstrait pour montrer que la position du domaine de contrôle pour un problème de contrôle au bord de deux équations de la chaleur couplées peut être déterminante: nous donnons un exemple explicite d'un tel système posé sur un domaine rectangulaire en 2D qui est contrôlable à zéro en tout temps si le contrôle agit sur deux bords perpendiculaires du domaine mais qui n'est jamais contrôlable à zéro si le contrôle n'agit que sur un seul des bords du domaine.

1. Introduction

This article is devoted to the study of the controllability properties of some abstract linear systems of parabolic type. The main difficulty when dealing with the controllability of systems, as opposed to equations, is to try to control a system with less controls than equations. It is sometimes called "indirect controllability". In the last ten years, drastically different controllability behaviors from what happens for a single heat equation have been highlighted: non equivalence between distributed and boundary controllability, non equivalence between null and approximate controllability, existence of a nonzero minimal control time, etc. In the present article the emphasis will be especially laid on the role played by the geometry of the control zone. At the abstract level this will be encoded by a tensor product structure of the state space and of the associated evolution and control operators. Before detailing more precisely the current literature on this subject, we think it is appropriate to discuss a simple prototype of parabolic systems that possesses the aforementioned structure and to which our main result applies.

1.1. Motivating example

A typical example that will be detailed below is provided by the following misleadingly simple-looking 2×2 system:

$$\begin{cases} \dfrac{\partial y_1}{\partial t} - d\Delta y_1 = 0, & \text{in } (0,T) \times \Omega, \\ \dfrac{\partial y_2}{\partial t} - \quad \Delta y_2 = y_1, & \text{in } (0,T) \times \Omega, \\ y_1 = 1_\gamma u, \qquad y_2 = 0, & \text{on } (0,T) \times \partial\Omega, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $d > 0$ is a diffusion coefficient, $\Omega \subset \mathbb{R}^2$ is a rectangle and $\gamma \subset \partial\Omega$ (see Figure 1.1):

$$\Omega = (0, X_1) \times (0, X_2), \qquad \text{for some } X_1, X_2 > 0. \tag{1.2}$$

The main feature of this system, that makes the problem intricate, is that the control u only acts on the first component of the system and is localized on a subpart γ of the boundary.

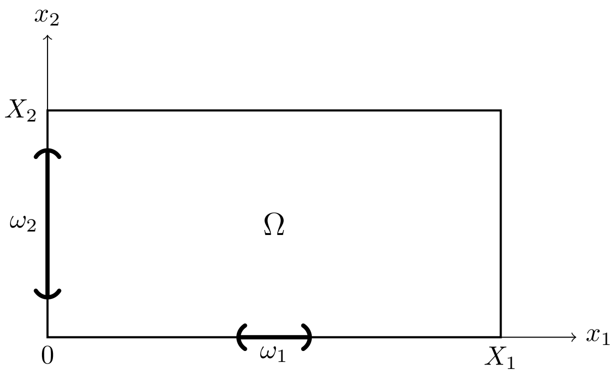

Figure 1.1. Domain and control region

We will establish in particular that such a system is null controllable in any arbitrary small time if the control region γ intersects non trivially two perpendicular faces of the boundary, that is (see also Figure 1.1)

$$\gamma = (\omega_1 \times \{0\}) \cup (\{0\} \times \omega_2)\,, \tag{1.3}$$

for some non empty open subsets $\omega_1 \subset (0, X_1)$ and $\omega_2 \subset (0, X_2)$.

This is radically different from the one-dimensional situation, i.e. when Ω is an interval, in which case it is known since [5] that the minimal null control time, that is denoted by $T_0(d)$, can be any element in $[0, +\infty]$, depending on the value of $d > 0$. We also want to mention that this phenomenon also appears in the two dimension case if the control domain γ is contained in one single face of the boundary of Ω (in this case it is easily seen that the minimal null control time of the 2D problem is at least equal to the one of the corresponding 1D problem).

One of the main achievements of the present paper is thus to show that the particular geometry of the control domain as in Figure 1.1 prevents us from the appearance of a nonzero minimal null control time.

In fact, we will provide our result in a quite abstract form that possibly encompasses many other similar systems. As an example, we will illustrate our analysis on more general parabolic systems than (1.1), in particular concerning the structure of the coupling zero-order terms, for which even in the case $d = 1$ our result is new.

1.2. Influence of the geometry and the moment method in the literature

The influence of the geometry on the boundary controllability properties of parabolic systems already indirectly appeared at a weaker level of strength in the seminal work [22]. It is shown in that article that the controllability of a one-dimensional parabolic system (yet slightly different from (1.1)) is especially depending on the non resonance of the eigenvalues of the associated operator, a condition that involves in particular the eigenvalues of the Laplacian, therefore subject to the length of the interval Ω, and thus to its geometry. A more striking influence of the geometry was then illustrated in [30], which is in part the motivation of the present work. Therein, the author investigated the boundary (approximate) controllability properties of the parabolic system studied in [22] in higher dimension in the particular geometry of a rectangle. It is notably shown there that the controllability properties of such a system strongly depend on the geometry of the control zone γ (and not only on the geometry of the domain Ω as in the aforementioned paper). Let us also add that the influence of the geometry of the control zone is not only restricted to boundary control problems. It was for instance shown in [11] that similar phenomena may occur also in distributed control problems.

The work [22] has then attracted again the attention of numerous researchers on the possible use of the so-called moment method to deal with controllability problems for parabolic systems (see e.g. [4, 5, 6, 16, 31, 13, 34, 9]), a technique initially used in [21] for the boundary null controllability of a one-dimensional heat equation (see also the earlier works [17, 23]). By pursuing the development of this method in view of the controllability of parabolic systems, it was notably shown in [5] that a nonzero minimal time of control may occur, as we have already mentioned before (see also the earlier result [27], with a different approach). Whereas this fact was well-known in the case of the one-dimensional heat equation after the pioneering work [15] on pointwise controllability, the papers [27, 5] showed that a similar complex situation occurs for coupled parabolic systems as well (even for a bounded control operator). The moment method is still developed today since in many situations it seems to be the only robust technique available to tackle controllability issues (in those cases where the other approaches like Carleman estimates or fictitious control, to name a few, fail). The usual range of application of the moment method needs that the spectrum of the underlying operator satisfies a spectral gap estimate (this means that two distinct eigenvalues cannot be arbitrarily close one from each other). In the references [27, 5] the analysis was extended to the case where spectral condensation occurs, which was the reason for the appearance of a minimal null control time. In the even more recent work [9], it was shown that it can be necessary not only to look at the condensation of the eigenvalues but also to the associated eigenfunctions to obtain an accurate description of the controllability properties of such systems: this gave birth to the so-called “block moment method” thanks to which a general formula for the minimal null control time was obtained for a very large class of scalar control problems. The present work crucially relies on this block moment method.

Concerning the study of the controllability of (systems of) equations of parabolic type in particular geometric situation such as rectangular domains, let us mention the pioneering work [20] on the boundary null controllability of the multi-dimensional heat equation in parallelepipedons or cylinders, the work [30], where the above geometric situation described for the system (1.1) has been first considered, and [2, 28] for the introduction of the formalism of tensor product spaces in the controllability of parabolic systems (see also [8]).

Finally, despite not applicable to our geometric situation let us also mention the work [1], where the first multi-dimensional result for the boundary controllability of parabolic systems was derived from the corresponding result on hyperbolic systems by the so-called transmutation method.

1.3. Tensor product formalism

In order to ease the understanding of the problem as well as the associated computations, we will express our evolution operator by making use of a tensor product formulation. This will be a convenient way to handle simultaneously separation of variables and separation of components in our system posed on a cartesian product domain.

We have collected in Appendix A.1 a summary of the main definitions and properties we will need on tensor products of Hilbert spaces and operators, following [33, 32].

Let us introduce this point of view on the example (1.1) in the geometry given by (1.2). First of all, we will see the Laplace operator as follows

$$-\Delta \cong \left(-\frac{\partial^2}{\partial x_1}\right) \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(-\frac{\partial^2}{\partial x_2}\right),$$

where we have made the identification (see Remark A.1)

$$L^2(\Omega) \cong L^2(0,X_1) \mathbin{\widehat{\otimes}} L^2(0,X_2).$$

This let us separate nicely what happens in each of the two space variables of the problem.

In order to take into account the fact that we are dealing with vector-valued unknowns we will proceed to another level of identification by writing

$$\begin{pmatrix} -d\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \cong \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes \left(\left(-\frac{\partial^2}{\partial x_1}\right) \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(-\frac{\partial^2}{\partial x_2}\right)\right),$$

where the state space is now

$$(L^2(\Omega))^2 \cong \mathbb{C}^2 \mathbin{\widehat{\otimes}} L^2(0,X_1) \mathbin{\widehat{\otimes}} L^2(0,X_2).$$

It has to be noted that the pure tensor products in such a space are obtained as a product of a function of x_1, times a function of x_2, times a vector in $\mathbb{C}^2$.

More details of the general framework we consider are given in Section 2.2.

1.4. Outline of the paper

The rest of the paper is organized as follows. In Section 2 we recall some basic facts on abstract control systems (Section 2.1), we describe precisely the functional setting in which our work takes place (Section 2.2) and we state our main result in this quite general abstract framework (Section 2.4). We present applications immediately after in Section 3, where we come back to the actual coupled parabolic systems that motivated this work, in the spirit of the example (1.1) presented above.

Our proofs being based on moment method, it is necessary to accurately describe the spectrum of the operator under study, this is the main purpose of Section 4 in which we particularily highlight a graph structure on this spectrum which will be central in our analysis. With this spectral description at hand we first prove the approximate controllability of our system in Section 5 and we then prove its null controllability at any time horizon in Section 6.

We gather the proofs of few technical results as well as reminders on basic graph theory in appendix.

2. Framework and main result

In this section, we will introduce the standing assumptions on the type of abstract control systems that we consider in this paper and we will state our main result. First of all, let us recall some basic general facts about such systems.

2.1. Background on abstract control systems

All along this section, $-\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H$ is the generator of a C_0-semigroup $(e^{-t\mathcal{A}})_{t\geqslant 0}$ on H and $\mathcal{B} \in \mathcal{L}(U, D(\mathcal{A}^*)')$, where H, U are two complex Hilbert spaces. Here and in what follows, E' denotes the (anti-)dual of the complex space E, that is the complex (Banach) space of all continuous conjugate linear forms (see e.g. [24, Section I.2.2]). We will use the convention that an inner product of a complex Hilbert space is conjugate linear in its second argument.

Let us consider the evolution problem associated with the pair $(-\mathcal{A}, \mathcal{B})$, i.e.

$$\begin{cases} \dfrac{d}{dt}y(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \quad t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{2.1}$$

where $T > 0$, $y(t)$ is the state at time t, y^0 is the initial data and $u(t)$ is the so-called control at time t.

Since $\mathcal{B}^* \in \mathcal{L}(D(\mathcal{A}^*), U)$ we can define a notion of solution in the space $D(\mathcal{A}^*)'$ for the system (2.1).

Definition 2.1 (Solution in $D(\mathcal{A}^*)'$). — *Let $T>0$, $y^0 \in D(\mathcal{A}^*)'$ and $u \in L^2(0,T;U)$. We say that a function $y:[0,T]\to D(\mathcal{A}^*)'$ is a solution to* (2.1) *if $y \in C^0([0,T];D(\mathcal{A}^*)')$ and*

$$\langle y(\tau), z^\tau\rangle_{D(\mathcal{A}^*)',D(\mathcal{A}^*)} - \left\langle y^0, z(0)\right\rangle_{D(\mathcal{A}^*)',D(\mathcal{A}^*)} = \int_0^\tau \langle u(t), \mathcal{B}^* z(t)\rangle_U \, dt, \tag{2.2}$$

for every $\tau \in (0,T]$ and $z^\tau \in D(\mathcal{A}^*)$, where $z \in C^0([0,\tau];D(\mathcal{A}^*))$ is the solution to the so-called adjoint system:

$$\begin{cases} -\dfrac{d}{dt} z(t) + \mathcal{A}^* z(t) = 0, & t \in (0,\tau), \\ z(\tau) = z^\tau, \end{cases} \tag{2.3}$$

i.e. $z(t) = e^{-(\tau-t)\mathcal{A}^*} z^\tau$.

Observe that the maps

$$z^\tau \mapsto \left\langle y^0, z(0)\right\rangle_{D(\mathcal{A}^*)',D(\mathcal{A}^*)}, \qquad z^\tau \mapsto \int_0^\tau \langle u(t), \mathcal{B}^* z(t)\rangle_U \, dt, \tag{2.4}$$

are continuous conjugate linear forms on $D(\mathcal{A}^*)$. Thus, we have a natural definition for the map $\tau \in [0,T] \longmapsto y(\tau) \in D(\mathcal{A}^*)'$ through the formula (2.2). It can be proved that this map is also continuous and that it depends continuously on y^0 and u on compact time intervals (see e.g. [14, Theorem 2.37]). This establishes the so-called well-posedness of the abstract control system $(-\mathcal{A},\mathcal{B})$.

Now that we have a notion of continuous solution for the system (2.1) in the space $D(\mathcal{A}^*)'$, we can speak of its controllability properties in $D(\mathcal{A}^*)'$.

Definition 2.2 (Controllability). — *We say that the system* (2.1) *is:*

- *null controllable in time T if, for every $y^0 \in D(\mathcal{A}^*)'$, there exists a control $u \in L^2(0,T;U)$ such that the corresponding solution $y \in C^0([0,T];D(\mathcal{A}^*)')$ to system* (2.1) *satisfies*

$$y(T) = 0.$$

- *approximately controllable in time T if, for every $\varepsilon>0$ and $y^0, y^T \in D(\mathcal{A}^*)'$, there exists a control $u \in L^2(0,T;U)$ such that the corresponding solution $y \in C^0([0,T];D(\mathcal{A}^*)')$ to system* (2.1) *satisfies*

$$\left\| y(T) - y^T \right\|_{D(\mathcal{A}^*)'} \leqslant \varepsilon.$$

We recall that null controllability implies approximate controllability for analytic semigroups (thanks to the backward uniqueness property of the adjoint system).

When a system is controllable, it is also of interest to measure how much it costs to control it:

Definition 2.3 (Control cost). — *Assume that* $(-\mathcal{A},\mathcal{B})$ *is null controllable in time* T *for some* $T>0$. *We call control cost the quantity* $\mathrm{cost}_T(-\mathcal{A},\mathcal{B})\geqslant 0$ *defined by*

$$\mathrm{cost}_T(-\mathcal{A},\mathcal{B}) = \sup_{\|y^0\|_{D(\mathcal{A}^*)'}=1}\left(\min_{u\in E_T(y^0)}\|u\|_{L^2(0,T;U)}\right),$$

where $E_T(y^0)$ *is the non empty closed convex subset made of the associated null controls, defined by* $E_T(y^0)=\left\{u\in L^2(0,T;U),\quad s.t.\quad y(T)=0\right\}$.

Let us conclude this section with a final remark. As we have seen, since $\mathcal{B}^*\in\mathcal{L}(D(\mathcal{A}^*),U)$, we can always define a notion of solution in the space $D(\mathcal{A}^*)'$ for the system (2.1). However, in practice it often appears that there is a “better” space V' where this system can be considered (see for instance Section 3 below). This motivates the introduction of the following concept.

Definition 2.4 (Admissible subspace). — *For any Banach space* V *(equipped with its own norm* $\|\cdot\|_V$*) such that*

$$D(\mathcal{A}^*)\subset V\subset H,$$

with dense and continuous embeddings, we say that V' *is an admissible subspace for the system* $(-\mathcal{A},\mathcal{B})$ *if we have the following two additional properties:*

(i) *V is invariant through the adjoint semigroup:*

$$e^{-t\mathcal{A}^*}V\subset V,\quad \forall t\geqslant 0.$$

(ii) *The following regularity property holds:*

$$\exists\tau>0,\exists C>0,\quad \int_0^\tau \|\mathcal{B}^*z(t)\|_U^2\,dt\leqslant C\,\|z^\tau\|_V^2,\quad \forall z^\tau\in D(\mathcal{A}^*),$$

where $z\in C^0([0,\tau];D(\mathcal{A}^*))$ *is the solution to adjoint system* (2.3).

The point of view adopted in this definition puts the emphasis on the subspace V and not on the control operator $\mathcal{B}$, contrary to what is done in the current literature. Basic examples of admissible subspaces are $D(\mathcal{A}^*)'$ since $\mathcal{B}^*\in\mathcal{L}(D(\mathcal{A}^*),U)$ and H if $\mathcal{B}\in\mathcal{L}(U,H)$. The previous notions of solution, controllability etc. can then be extended by simply replacing $D(\mathcal{A}^*)$ by V. It is clear that null controllability in the space $D(\mathcal{A}^*)'$ implies null controllability in the space V', whereas approximate controllability in the spaces $D(\mathcal{A}^*)'$ and V' are equivalent for analytic semigroups.

2.2. Standing assumptions on the systems considered in this paper

Let us now describe the kind of abstract control problems that we will deal with in this paper. Of course, this framework will include the motivating example (1.1) given in the introduction as well as its generalisation described in Section 3.

Here and in what follows, $\operatorname{card} E$ denotes the cardinal of a set E. For any subset $S \subset \mathbb{C}$ we define the associated counting function

$$N_S(r) = \operatorname{card}\left\{\lambda \in S, \quad \text{s.t.} \quad |\lambda| \leqslant r\right\}, \quad \forall r \geqslant 0, \tag{2.5}$$

as well as the associated gap

$$\operatorname{Gap}(S) = \inf_{\substack{\lambda,\mu \in S \\ \lambda \neq \mu}} |\lambda - \mu| .$$

We will also use the notation

$$\mathbb{C}_+ = \{z \in \mathbb{C}, \quad \text{s.t.} \quad \mathfrak{Re}\, z > 0\}.$$

2.2.1. The operator A

Let H_1, H_2 be two complex Hilbert spaces.

- For $i = 1, 2$, let $A_i : D(A_i) \subset H_i \longrightarrow H_i$ be an unbounded linear operator satisfying the following properties:
  - (i) A_i is a self-adjoint positive operator with compact resolvent. We denote its spectrum by Λ_i and we observe that we have

$$\Lambda_i \subset \mathbb{R}_+^*. \tag{2.6}$$

    We assume that each eigenvalue is (geometrically) simple.
  - (ii) Λ_i satisfies the following gap condition:

$$\operatorname{Gap}(\Lambda_i) > 0. \tag{2.7}$$

  - (iii) Λ_i satisfies the following asymptotic behavior: there exist $\theta_i \in (0, 1)$ and $\kappa_i > 0$ such that

$$N_{\Lambda_i}(r) \leqslant \kappa_i r^{\theta_i}, \quad \forall r > 0, \tag{2.8}$$

    and

$$|N_{\Lambda_i}(r) - N_{\Lambda_i}(s)| \leqslant \kappa_i(1 + |r - s|^{\theta_i}), \quad \forall r, s > 0. \tag{2.9}$$

- We will in addition always assume that

$$\theta_1 + \theta_2 \leqslant 1. \tag{2.10}$$

  Note that this assumption connects both operators, contrary to the previous ones that only concerned the operators A_1 and A_2 separately.
- Following the ideas briefly discussed in Section 1.3, we finally introduce the state space we will work with all along this work

$$H = \mathbb{C}^2 \mathbin{\widehat{\otimes}} H_1 \mathbin{\widehat{\otimes}} H_2,$$

  where $\widehat{\otimes}$ stands for the tensor product whose main properties are recalled in Appendix A.1.

We can now introduce an operator from which the generator of our control system will be built on. For any $d > 0$, we consider the unbounded operator

$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2), \quad \text{with domain } \mathbb{C}^2 \otimes D(A_1) \otimes D(A_2). \tag{2.11}$$

The definition of the tensor product of linear operators is also recalled in Appendix A.1. This operator has the following important properties:

Proposition 2.5. — *The operator* (2.11) *is closable and its closure, denoted by A_0, is a self-adjoint operator with compact resolvent.*

The proof of these properties is given in Appendix A.2. Concerning material on closable operators we refer for instance to [24, Sections III.5.3 and III.5.5].

The generator of the control system that will be considered in this work is now a bounded perturbation of A_0 defined as follows.

Definition 2.6. — *Let $d > 0$ and $M \in \mathbb{R}^{2\times 2}$. The operator $A : D(A) \subset H \longrightarrow H$ is defined by*

$$A = A_0 - M \otimes \mathrm{Id} \otimes \mathrm{Id}, \quad D(A) = D(A_0).$$

Equivalently,

$$A = \textit{closure of} \quad \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) - M \otimes \mathrm{Id} \otimes \mathrm{Id}.$$

Obviously, $D(A) = D(A_0)$ is dense in H, and a computation shows that the adjoint A^* is simply given by

$$A^* = A_0 - M^* \otimes \mathrm{Id} \otimes \mathrm{Id}, \quad D(A^*) = D(A). \tag{2.12}$$

2.2.2. The control operator B

Let us now introduce the class of control operators that will be considered in this paper.

- For $i = 1, 2$, let $B_i \in \mathcal{L}(\mathbb{C}, D(A_i)')$ be two scalar control operators. We assume that the pair $(-A_i, B_i)$ satisfies the so-called Fattorini-Hautus test, namely

$$\ker(\lambda_i - A_i^*) \cap \ker B_i^* = \{0\}, \quad \forall \lambda_i \in \Lambda_i,$$

and, from now on, we save the notation ϕ_{i,λ_i} to denote the unique eigenfunction of A_i associated with the (simple) eigenvalue $\lambda_i \in \Lambda_i$ that satisfies the condition

$$B_i^* \phi_{i,\lambda_i} = 1. \tag{2.13}$$

This choice of normalization (that has no influence on the results) is maybe not the most conventional one but it simplifies numerous computations below. Since B_i^* is continuous from $D(A_i)$ into $\mathbb{C}$, we deduce the lower bound

$$\|\phi_{i,\lambda_i}\|_{H_i} = \frac{\|\phi_{i,\lambda_i}\|_{D(A_i)}}{\sqrt{1+\lambda_i^2}} \geqslant \frac{1}{\|B_i^*\|_{\mathcal{L}(D(A_i),\mathbb{C})}\sqrt{1+\lambda_i^2}}, \quad \forall \lambda_i \in \Lambda_i. \tag{2.14}$$

We then assume that their norm has the following upper bound: there exist $\nu_i \in [0, 1)$ and $C > 0$ such that

$$\|\phi_{i,\lambda_i}\|_{H_i} \leqslant Ce^{C\lambda_i^{\nu_i}}, \quad \forall \lambda_i \in \Lambda_i. \tag{2.15}$$

We will frequently use the notation

$$\nu_{\max} = \max\{\nu_1, \nu_2\}.$$

- Next, we consider a particular structure of system associated with these scalar operators. Our control space will simply be

$$U = H_1 \times H_2. \tag{2.16}$$

The non scalar control operator $B : U \to D(A^*)'$ that we will finally consider is formally given by

$$B(u_1, u_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (L_1 u_1) \otimes b_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes b_1 \otimes (L_2 u_2),$$

where L_1, L_2 are two bounded operators in H_1 and H_2, respectively and $b_1 \in D(A_1^*)', b_2 \in D(A_2^*)'$ are such that $b_i = B_i 1$. Observe that this operator only acts on the first component of the system.

Its precise definition is given by the following result.

PROPOSITION 2.7. — *For every $L_1 \in \mathcal{L}(H_1)$ and $L_2 \in \mathcal{L}(H_2)$, there exists a unique bounded linear operator $B \in \mathcal{L}(U, D(A^*)')$ such that, on $\mathbb{C}^2 \otimes D(A_1) \otimes D(A_2)$, we have*

$$B^* = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes L_1^* \otimes B_2^* \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B_1^* \otimes L_2^* \end{pmatrix}. \tag{2.17}$$

The proof is postponed to Appendix A.3 to ease the presentation.

2.3. The Kalman condition

With the operators defined above, any solution y to (2.1) can be written

$$y(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes y_1(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes y_2(t),$$

where y_1 and y_2 satisfy, at least at a formal level,

$$\begin{cases} \dfrac{\partial y_1}{\partial t} + d(A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2)y_1 = m_{11}y_1 + m_{12}y_2 \\ \qquad\qquad\qquad\qquad\qquad + (L_1u_1) \otimes b_2 + b_1 \otimes (L_2u_2), \\ \dfrac{\partial y_2}{\partial t} + \quad (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2)y_2 = m_{21}y_1 + m_{22}y_2. \end{cases}$$

We immediately see that a necessary controllability condition for the system is the Kalman condition

$$m_{21} \neq 0. \tag{2.18}$$

Indeed, if $m_{21} = 0$, then we observe that y_2 satisfies the equation

$$\frac{\partial y_2}{\partial t} + (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2)y_2 = m_{22}y_2,$$

that does not depend on the control nor on the first component of the system. In particular any trajectory reaching zero at some time T satisfies $y_2(t) = 0$ for any t, and therefore $y_2(0) = 0$. This proves that not all initial data can be driven to 0.

From now on, we shall always assume this condition (2.18).

2.4. Statement of the main result

The main result of the present paper is the following null controllability result:

Theorem 2.8. — *Let A and B be the operators introduced in Definition 2.6 and Proposition 2.7, respectively. Assume the Kalman condition* (2.18). *Assume also that for $i = 1, 2$ the adjoint of the operator L_i satisfies the following Lebeau-Robbiano type inequality: there exist $\eta_i \in [0, 1)$ and $C > 0$ such that, for every $\mu > 0$ we have*

$$\|z\|_{H_i} \leqslant Ce^{C\mu^{\eta_i}} \|L_i^* z\|_{H_i}, \quad \forall z \in \bigoplus_{\substack{\lambda_i \in \Lambda_i \\ \lambda_i \leqslant \mu}} \ker(\lambda_i - A_i). \tag{2.19}$$

Then, the system $(-A, B)$ is null controllable in time T for every $T > 0$, with control cost satisfying, for some $C > 0$,

$$\mathrm{cost}_T(-A, B) \leqslant C \exp\left(\frac{C}{T^{\frac{p}{1-p}}}\right), \quad \forall T > 0,$$

where $p = \max\{\theta_1, \theta_2, \nu_1, \nu_2, \eta_1, \eta_2\}$.

Remark 2.9. — The same result remains true if we drop the assumption (2.9) on the operators A_1, A_2 but in this case we have to consider p satisfying $p > \max\{\theta_1, \theta_2\}$ and $p \geqslant \max\{\nu_1, \nu_2, \eta_1, \eta_2\}$. This will be explained during the proof below.

Let us mention that we will also prove an independent approximate controllability result in Theorem 5.1 below and that a simple but useful "higher dimensional" version of Theorem 2.8 will be described in Theorem 3.4 below.

3. Application to the boundary null controllability of coupled linear parabolic systems on cartesian geometries

As mentioned in the introduction, our main motivation for the abstract result proved in this paper is its application to actual multi-dimensional boundary null controllability issues for coupled parabolic systems.

3.1. A new 2D result

A typical system to which our general analysis applies is the following 2D two-component system controlled from the boundary by only one control:

$$\left\{\begin{array}{ll}\dfrac{\partial y_1}{\partial t} - d\ \mathrm{div}\,(K(x)\nabla y_1) = m_{11}y_1 + m_{12}y_2 & \text{in } (0,T)\times\Omega,\\ \dfrac{\partial y_2}{\partial t} - \quad \mathrm{div}\,(K(x)\nabla y_2) = m_{21}y_1 + m_{22}y_2 & \text{in } (0,T)\times\Omega,\\ y_1 = 1_\gamma u, \qquad y_2 = 0 & \text{on } (0,T)\times\partial\Omega,\\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0 & \text{in } \Omega,\end{array}\right. \tag{3.1}$$

where the domain $\Omega \subset \mathbb{R}^2$ is the rectangle defined in (1.2) (see Figure 1.1) and the diffusion tensor has the following form

$$K(x) = \begin{pmatrix} k_1(x_1) & 0 \\ 0 & k_2(x_2)\end{pmatrix},$$

with $k_i \in W^{1,\infty}(0,X_i)$, $\inf_{(0,X_i)} k_i > 0$ and $d > 0$ is a parameter accounting for the ratio of diffusion between the two components in the system.

The first equation is controlled from the boundary on a non empty relative open subset γ of $\partial\Omega$. On the other hand, the second equation has no control, but it is coupled to the first equation via a constant internal coupling term, so that it is indirectly controlled, as soon as $m_{21} \neq 0$.

We recall that the case $d = 1$ and $K(x) = \mathrm{Id}$ was studied in the literature:

- The approximate controllability of the system (3.1) was studied in [30] when the underlying operator is the Laplacian.

  Notably, it was established in [30, Theorem 2.15] that this system is approximately controllable in time T for every $T > 0$ if the Kalman condition $m_{21} \neq 0$ holds and if γ satisfies the geometric condition (1.3), which was introduced in this very paper.

  On the other hand, when γ intersects only one face of the boundary $\partial\Omega$, say for instance $\gamma = \omega_1 \times \{0\}$, then it was shown in [30, Theorem 2.14] that the approximate controllability of the system is equivalent to the approximate controllability of the 1D reduced system

$$\left\{\begin{array}{ll}\dfrac{\partial y_1}{\partial t} - \dfrac{\partial^2 y_1}{\partial x^2} = m_{11}y_1 + m_{12}y_2, & \text{in } (0,T)\times(0,X_1),\\ \dfrac{\partial y_2}{\partial t} - \dfrac{\partial^2 y_2}{\partial x^2} = m_{21}y_1 + m_{22}y_2, & \text{in } (0,T)\times(0,X_1),\\ y_1 = 1_{\{0\}}u, \quad y_2 = 0, & \text{on } (0,T)\times\{0,X_1\},\\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \text{in } (0,X_1).\end{array}\right. \tag{3.2}$$

- This second result of [30] was then extended to the null controllability property in [8, Theorem 1.2].
- As already mentioned, the approximate and null controllability of the one-dimensional system (3.2) were studied in the seminal work [22]. More precisely, it was shown in [22, Theorem 1.1] (resp. [22, Theorem 5.2]) that this system is null (resp. approximately) controllable in time T if, and only if, the Kalman condition $m_{21} \neq 0$ holds and we have the following "non resonance" condition:

$$\left(\lambda + \theta = \tilde{\lambda} + \tilde{\theta} \quad \Longrightarrow \quad \lambda = \tilde{\lambda} \text{ and } \theta = \tilde{\theta}\right), \qquad \forall \lambda, \tilde{\lambda} \in \sigma\left(-\frac{\partial^2}{\partial x^2}\right), \forall \theta, \tilde{\theta} \in \sigma(M^*). \tag{3.3}$$

  Note that, if M has only one eigenvalue, then this condition (3.3) is automatically satisfied.

  However, in the case where M has two distinct eigenvalues, the Kalman condition is not sufficient to ensure null controllability and it is needed to assume that (3.3) holds.

On the other hand, Theorem 2.8 leads to the following new null controllability result.

Theorem 3.1. — *Assume that the Kalman condition* $m_{21} \neq 0$ *holds and that* γ *satisfies the geometric condition* (1.3) *(see also Figure 1.1). Then, there exists* $C > 0$ *such that, for any* $T > 0$*, for any* $y^0 \in H^{-1}(\Omega)^2$*, there exists* $u \in L^2((0,T) \times \partial\Omega)$ *with the estimate*

$$\|u\|_{L^2((0,T)\times\partial\Omega)} \leqslant C \exp\left(\frac{C}{T}\right) \|y^0\|_{H^{-1}(\Omega)^2},$$

such that the corresponding solution to the system (3.1) *satisfies* $y_1(T,\cdot) = y_2(T,\cdot) = 0$.

To the best of our knowledge, Theorem 3.1 is the first and only result concerning the controllability properties of the two-dimensional system (3.1) for any value of the ratio of diffusions $d > 0$.

Remark 3.2. — Combined with the results of the literature recalled just before the statement of Theorem 3.1, we see that this result shows a very strong influence of the geometry of the control domain: there are some two-dimensional systems (3.1) which are null controllable in arbitrary small time if two perpendicular faces of the boundary are controlled, whereas they are not even approximately controllable if the control acts on only one face. An explicit example of such systems is

$$\begin{cases} \dfrac{\partial y_1}{\partial t} - \Delta y_1 = 0 & \text{in } (0,T) \times \Omega, \\ \dfrac{\partial y_2}{\partial t} - \Delta y_2 = y_1 + 3y_2 & \text{in } (0,T) \times \Omega, \end{cases}$$

posed on the square domain $\Omega = (0,\pi)^2$, for which we can check that the non resonance condition (3.3) fails.

Proof of Theorem 3.1. —
This result will be a straightforward consequence of Theorem 2.8 once we will have checked that we are under the framework of Section 2.2.

Indeed, the system (3.1) corresponds to the abstract control system (2.1) with the following functional framework

- The state space is
$$H = \mathbb{C}^2 \,\widehat{\otimes}\, L^2(0,X_1) \,\widehat{\otimes}\, L^2(0,X_2).$$
- The operator A is the closure of the operator
$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) - M \otimes \mathrm{Id} \otimes \mathrm{Id},$$
with domain $\mathbb{C}^2 \otimes D(A_1) \otimes D(A_2)$, where, for $i = 1,2$, A_i is the one-dimensional and scalar positive Dirichlet Laplacian on the space $H_i = L^2(0,X_i)$:
$$A_i = -\frac{\partial}{\partial x_i}\left(k_i(x_i)\frac{\partial}{\partial x_i}\,\cdot\,\right),$$
with domain $D(A_i) = H^2(0,X_i) \cap H_0^1(0,X_i)$.
- For γ as in (1.3), the control space can be taken as
$$U = L^2(0,X_1) \times L^2(0,X_2),$$
where for each $(u_1,u_2) \in U$ a control u for (3.1) is simply
$$u(x) = \begin{cases} u_1(x_1), & \text{if } x_1 \in \omega_1 \text{ and } x = (x_1, 0), \\ u_2(x_2), & \text{if } x_2 \in \omega_2 \text{ and } x = (0, x_2), \\ 0, & \text{otherwise.} \end{cases}$$
- For $i = 1,2$, we introduce $B_i^* \in \mathcal{L}(D(A_i),\mathbb{C})$ to be the one dimensional scalar operator
$$B_i^* z = -k_i(0)\frac{\partial z}{\partial x}(0),$$

and $L_i^* \in \mathcal{L}(H_i)$ is simply given by

$$L_i^* z = 1_{\omega_i} z.$$

Then, it is easily checked that the control operator B defined as in (2.17) is such that, for any $u \in U$ and any $\in D(A^*)$, we have

$$\langle Bu, z\rangle_{D(A^*)', D(A^*)} = \\ -\int_{\omega_1} u_1(x_1) k_2(0) \overline{\frac{\partial z_1}{\partial x_2}}(x_1, 0)\, dx_1 - \int_{\omega_2} u_2(x_2) k_1(0) \overline{\frac{\partial z_1}{\partial x_1}}(0, x_2)\, dx_2.$$

Let us now check the assumptions of Section 2.2.

- We recall that A_i is a positive self-adjoint operator with compact resolvent and that its spectrum satisfies
  - the gap condition
  $$\mathrm{Gap}(\Lambda_i) > 0,$$
  - the counting function estimates: there exists $C_i > 0$ such that
  $$N_{\Lambda_i}(r) \leqslant C_i \sqrt{r}, \quad \forall r > 0,$$
  $$|N_{\Lambda_i}(r) - N_{\Lambda_i}(s)| \leqslant C_i (1 + \sqrt{|r-s|}).$$

  Those are very classical results; a self-contained proof is for instance given in [12, Theorem IV.1.3] based on the methodology described in [3, Section 2]).

  Note in particular that, since $\theta_1 = \theta_2 = 1/2$, the condition $\theta_1 + \theta_2 \leqslant 1$ is fulfilled.
- It is clear that $(-A_i, B_i)$ satisfies the Fattorini-Hautus test and that the eigenfunction ϕ_{i,λ_i} of A_i associated with the eigenvalue $\lambda_i \in \Lambda_i$ which is such that $B_i^* \phi_{i,\lambda_i} = 1$, that is such that
  $$-k_i(0)\phi'_{i,\lambda_i}(0) = 1,$$
  satisfies
  $$\|\phi_{i,\lambda_i}\|_{L^2(0,X_i)} \leqslant \frac{C}{\sqrt{\lambda_i}}.$$
  This is also given in [3, Theorem 1.1].

  This estimate implies that the upper bound (2.15) holds with
  $$\nu_i = 0.$$
- The property (2.19) concerning the operators L_i holds with
  $$\eta_i = \frac{1}{2}.$$

This is nothing but the one-dimensional Lebeau-Robbiano spectral inequality [26, Theorem 3] (or Turán's inequality, see [35, Corollary 3.3]).

All the assumptions of Section 2.2 are fulfilled, so that Theorem 2.8 can be applied and shows that the system (3.1) is null controllable in time T for every $T > 0$. Moreover, in the present case, we have $p = \max\{\theta_1, \theta_2, \nu_1, \nu_2, \eta_1, \eta_2\} = 1/2$, which leads to the $e^{C/T}$ estimate stated in Theorem 3.1.

The result is first obtained in the space $D(A^*)' = (H^2(\Omega)^2 \cap H_0^1(\Omega)^2)'$. In addition, note that

$$V' = H^{-1}(\Omega)^2$$

is an admissible subspace for our system (3.1) (see Definition 2.4). This is a direct consequence of the following well-known elliptic regularity estimate satisfied by the solution z to the corresponding adjoint system (2.3) in any time $T > 0$: there exists $C > 0$ such that

$$\begin{aligned}\int_0^T \int_{\partial\Omega} \left\| \frac{\partial z}{\partial n}(t,\sigma) \right\|_{\mathbb{C}^2}^2 d\sigma dt &\leqslant C \int_0^T \|z(t,\cdot)\|_{H^2(\Omega)^2}^2 \, dt \\ &\leqslant C \int_0^T \|Az(t,\cdot)\|_{L^2(\Omega)^2}^2 \, dt \\ &\leqslant C \left\| z^T \right\|_{H_0^1(\Omega)^2}^2, \quad \forall z^T \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2,\end{aligned}$$

where $\partial/\partial n$ denotes the normal derivative.

□

3.2. A 3D result

Let us present here a higher dimensional result. We consider the 3D parallelepiped (see e.g. Figure 3.1)

$$\Omega = (0, X_1) \times (0, X_2) \times (0, X_3), \qquad \text{for some } X_1, X_2, X_3 > 0,$$

and we assume all along this section that the diffusion tensor is of the form

$$K(x) = \begin{pmatrix} k_1(x_1) & 0 & 0 \\ 0 & k_2(x_2) & 0 \\ 0 & 0 & k_3(x_3) \end{pmatrix},$$

for some $k_i \in W^{1,\infty}(0, X_i)$, $\inf_{(0,X_i)} k_i > 0$.

Then, we can control the corresponding system with a boundary control supported on two non parallel faces.

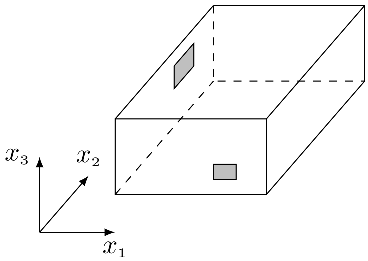

Figure 3.1. The geometry of the boundary control problem in 3D. The control domain is the part of the boundary which is represented in gray.

THEOREM 3.3. — *Assume that the Kalman condition $m_{21} \neq 0$ holds and that γ satisfies the geometric condition (see also Figure 3.1)*

$$\gamma = (\omega_1 \times \{0\} \times \omega_3) \cup (\{0\} \times \omega_2 \times \hat{\omega}_3),$$

for some non empty open subsets $\omega_1 \subset (0, X_1)$, $\omega_2 \subset (0, X_2)$ and $\omega_3, \hat{\omega}_3 \subset (0, X_3)$.

Then, we have the same conclusion as in the statement of Theorem 3.1.

Similarly to the proof of Theorem 3.1, this 3D result is a straightforward application of the following abstract result (applied with $H_3 = L^2(0, X_3)$, $A_3 = -\frac{\partial}{\partial x_3}\left(k_3(x_3)\frac{\partial}{\partial x_3}\,\cdot\,\right)$ and $L_3 = 1_{\omega_3}, \hat{L}_3 = 1_{\hat{\omega}_3}$):

THEOREM 3.4. — *Let A be the operator introduced in Definition 2.6. Assume the Kalman condition (2.18). Let A_3 be a self-adjoint operator with compact resolvent on a complex Hilbert space H_3. Set*

$$\tilde{H} = H \mathbin{\widehat{\otimes}} H_3, \quad \tilde{U} = (H_1 \mathbin{\widehat{\otimes}} H_3) \times (H_2 \mathbin{\widehat{\otimes}} H_3).$$

Let

$$\tilde{A} = \textit{closure of} \quad A \otimes \mathrm{Id} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes A_3.$$

Let $\tilde{B} \in \mathcal{L}(U, D(A^*)')$ be the unique bounded linear operator such that, on $\mathbb{C}^2 \otimes D(A_1) \otimes D(A_2) \otimes D(A_3)$, we have

$$\tilde{B}^* = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes L_1^* \otimes B_2^* \otimes L_3^* \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B_1^* \otimes L_2^* \otimes \hat{L}_3^* \end{pmatrix},$$

where $L_3^*, \hat{L}_3^* \in \mathcal{L}(H_3)$ satisfy the Lebeau-Robbiano type inequality (2.19) (with $i = 3$) for some $\eta_3, \hat{\eta}_3 \in [0, 1)$.

Then, the system $(-\tilde{A}, \tilde{B})$ *is null controllable in time* T *for every* $T > 0$, *with control cost satisfying, for some* $C > 0$,

$$\mathrm{cost}_T(-\tilde{A}, \tilde{B}) \leqslant C \exp\left(\frac{C}{T^{\frac{\tilde{p}}{1-\tilde{p}}}}\right), \quad \forall T > 0,$$

with $\tilde{p} = \max\{p, \eta_3, \hat{\eta}_3\} \in (0,1)$ *(we recall that* p *is defined in Theorem 2.8).*

Theorem 3.4 is easily deduced from Theorem 2.8 after using a Fourier decomposition in the direction of the added dimension. This idea has already been used many times in the literature (see e.g. [28, 8]). For this reason, its proof will be omitted.

Let us mention that the construction of $\tilde{B}$ can be done in the same way as for the operator B (see Proposition 2.7 and Appendix A.3).

4. Spectral analysis

4.1. Description of the spectrum of A^*

First of all let us mention that, by classical perturbation arguments, it is expected that A has a good spectral theory. For instance, it inherits from A_0 the following properties:

- A has a compact resolvent (see e.g. [18, Proposition III.1.12]).
- $-A$ generates an analytic semigroup on H (see e.g. [18, Corollary II.4.7 and Proposition III.1.12]).

Besides, the same statements hold for the adjoint A^* as well since it has the same structure (recall (2.12)).

Let us now describe precisely the structure of the spectrum of A^*.

Definition 4.1. — *For any* $\lambda \in \mathbb{R}$, *we define*

$$\Delta_\lambda = ((1-d)\lambda + m_{11} - m_{22})^2 + 4m_{21}m_{12}, \tag{4.1}$$

and $\sqrt{\Delta_\lambda} \in \mathbb{R}_+ \cup i\mathbb{R}_+$ *will always denote the principal square root of this number.*

Note that

$$d \neq 1 \quad \Longrightarrow \quad \lim_{\lambda \to +\infty} \Delta_\lambda = +\infty. \tag{4.2}$$

On the other hand, when $d = 1$, *we see that* Δ_λ *does not depend on* λ, *and we simply denote this quantity by* Δ. *It is nothing but the discriminant of the characteristic polynomial of* M*:*

$$\Delta = (\mathrm{Tr}\, M)^2 - 4\det M.$$

We shall also introduce the set

$$\widehat{\Lambda} = \{\lambda \in \mathbb{R}, \quad s.t. \quad \Delta_\lambda = 0\},$$

that will play a particular role in the spectral analysis of our problem.

Remark 4.2. — We notice that, when $d \neq 1$, the cardinal of $\widehat{\Lambda}$ is less or equal than 2. However, for $d = 1$, we have either $\widehat{\Lambda} = \emptyset$ or $\widehat{\Lambda} = \mathbb{R}$.

We can now introduce some sets that will be instrumental in our description of the spectrum of A^*.

Definition 4.3. —

(1) *We introduce the set*

$$\Gamma = \{+, -\} \times \Lambda_1 \times \Lambda_2,$$

and for any $\gamma \in \Gamma$, we denote its components by $s(\gamma) \in \{+, -\}$, $\lambda_1(\gamma) \in \Lambda_1$ and $\lambda_2(\gamma) \in \Lambda_2$. We will also use the notation $\lambda(\gamma) = \lambda_1(\gamma) + \lambda_2(\gamma)$.

(2) *We shall use the following particular subsets of Γ*

$$\widehat{\Gamma} = \{\gamma \in \Gamma, \quad s.t. \quad \lambda(\gamma) \in \widehat{\Lambda}\},$$

$$\Gamma^\pm = \{\pm\} \times \Lambda_1 \times \Lambda_2, \quad \text{and} \quad \widehat{\Gamma}^\pm = \widehat{\Gamma} \cap \Gamma^\pm.$$

(3) *For any $\lambda_1 \in \Lambda_1$ and any $\lambda_2 \in \Lambda_2$, we introduce*

$$\Gamma_{1,\lambda_1} = \{+, -\} \times \{\lambda_1\} \times \Lambda_2, \quad \Gamma_{2,\lambda_2} = \{+, -\} \times \Lambda_1 \times \{\lambda_2\},$$

and, for $i = 1, 2$,

$$\Gamma^\pm_{i,\lambda_i} = \Gamma_{i,\lambda_i} \cap \Gamma^\pm, \quad \widehat{\Gamma}_{i,\lambda_i} = \Gamma_{i,\lambda_i} \cap \widehat{\Gamma}.$$

Definition 4.4. — *We introduce $\sigma : \Gamma \to \mathbb{C}$ to be the function defined by*

$$\sigma(\gamma) = \frac{(1+d)\lambda(\gamma) - \mathrm{Tr}(M) + s(\gamma)\sqrt{\Delta_{\lambda(\gamma)}}}{2}, \qquad \forall \gamma \in \Gamma. \tag{4.3}$$

Associated to this function we introduce, for any $\lambda_i \in \Lambda_i$, $i = 1, 2$, the subsets of $\mathbb{C}$ defined by

$$\Sigma_{i,\lambda_i} = \sigma(\Gamma_{i,\lambda_i}), \qquad \Sigma^\pm_{i,\lambda_i} = \sigma(\Gamma^\pm_{i,\lambda_i}), \qquad \widehat{\Sigma}_{i,\lambda_i} = \sigma(\widehat{\Gamma}_{i,\lambda_i}). \tag{4.4}$$

In some computations, we shall need an alternative expression for the function σ given in the following result.

Proposition 4.5. — *There exist* $\sigma^+, \sigma^- \in \mathbb{C}$ *and a function* $\lambda \in \mathbb{R} \mapsto \varepsilon_\lambda \in \mathbb{C}$ *that satisfy*

$$\lim_{\lambda\to+\infty} \varepsilon_\lambda = 0, \tag{4.5}$$

and such that

$$\begin{cases} \sigma(\gamma) = \max(d,1)\lambda(\gamma) + \sigma^+ + \varepsilon_{\lambda(\gamma)}, & \text{if } \gamma \in \Gamma^+, \\ \sigma(\gamma) = \min(d,1)\lambda(\gamma) + \sigma^- - \varepsilon_{\lambda(\gamma)}, & \text{if } \gamma \in \Gamma^-. \end{cases} \tag{4.6}$$

Let us first give a straightforward corollary of this result.

Corollary 4.6. — *There exist* $m, C > 0$ *such that, if one replaces* M *by* $M - m\mathrm{Id}$ *in the definition of the operator* A*, then we have*

$$\mathfrak{Re}\,\sigma(\gamma) \geqslant 1, \quad \forall \gamma \in \Gamma, \tag{4.7}$$

and

$$|\mathfrak{Im}\,\sigma(\gamma)| \leqslant C, \quad \forall \gamma \in \Gamma. \tag{4.8}$$

In particular, we have

$$|\sigma(\gamma)| \leqslant \sqrt{1+C^2}\,\big(\mathfrak{Re}\,\sigma(\gamma)\big), \quad \forall \gamma \in \Gamma. \tag{4.9}$$

Since changing M into $M - m\mathrm{Id}$ does not influence the controllability properties of the system (indeed it amounts to consider the system satisfied by the new unknown $t \mapsto e^{-mt}y(t)$), we will always assume in the sequel that (4.7) and (4.8) hold. Note that this manipulation does not change the values of Δ_λ.

Proof of Proposition 4.5.

- In the case $d = 1$, we recall that $\Delta_\lambda = \Delta$ does not depend on λ, and therefore we simply take $\varepsilon_\lambda = 0$ and $\sigma^\pm = \frac{\pm\sqrt{\Delta} - \mathrm{Tr}(M)}{2}$.
- If $d > 1$ we choose $\sigma^+ = -m_{11}$, $\sigma^- = -m_{22}$ and
$$\varepsilon_\lambda = \frac{(1-d)\lambda + m_{11} - m_{22} + \sqrt{\Delta_\lambda}}{2},$$
for which the required properties can be easily checked (recall (4.2)).
- For $d < 1$, we set $\sigma^+ = -m_{22}$ and $\sigma^- = -m_{11}$ and
$$\varepsilon_\lambda = \frac{(d-1)\lambda - m_{11} + m_{22} + \sqrt{\Delta_\lambda}}{2},$$
and we conclude by a straightforward computation.

□

Remark 4.7. — Observe that:

(i) If $d \neq 1$, then (4.2) implies that $\sigma(\pm, \lambda_1, \lambda_2)$ are real numbers for λ_1, λ_2 large enough.

(ii) There exists $C > 0$ such that

$$\frac{1}{C}\left|\sigma(\gamma)\right| \leqslant \lambda(\gamma) \leqslant C\left|\sigma(\gamma)\right|, \quad \forall \gamma \in \Gamma. \tag{4.10}$$

Remark 4.8. — We will see in the next proposition that the spectrum of A^* is precisely the range $\sigma(\Gamma)$. However, it is crucial to observe that σ is in general not injective, so that an eigenvalue may simultaneously be equal to $\sigma(\gamma)$ and $\sigma(\tilde{\gamma})$ for two different $\gamma \neq \tilde{\gamma}$. In fact, the situation is even more complex than that. Since this will be the source of most technical problems that we will encounter in what follows, let us detail what can happen for instance for system (1.1)-(1.2) with $X_1 = X_2 = \pi$. For any $i = 1, 2$ and $\lambda_i \in \Lambda_i = \left\{k^2\right\}_{k\geqslant 1}$, the following situations may occur:

(i) The map $\gamma \in \Gamma_{i,\lambda_i} \mapsto \sigma(\gamma)$ may not be injective, for infinitely many values of λ_i. As an example, we consider for instance the case $d = 16/25$ and $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. For any $p \in \mathbb{N}^*$ with $p > 1$, we set

$$\begin{aligned} \lambda_{1,p} &= (300(p^2-1))^2 \in \Lambda_1, \\ \lambda_{2,p} &= (360p)^2 \in \Lambda_2, \\ \hat{\lambda}_{2,p} &= (225(p^2+1))^2 \in \Lambda_1. \end{aligned}$$

Then, by a straightforward computation, one can check that

$$\sigma(+, \lambda_{1,p}, \lambda_{2,p}) = \lambda_{1,p} + \lambda_{2,p} = d(\lambda_{1,p} + \hat{\lambda}_{2,p}) = \sigma(-, \lambda_{1,p}, \hat{\lambda}_{2,p}).$$

(ii) Even in the case where the previous map is injective, it will certainly happen some spectral condensation phenomenon, that is the fact that $\sigma(\gamma)$ and $\sigma(\tilde{\gamma})$ may be exponentially close for infinitely many $\tilde{\gamma} \neq \gamma$. More precisely, we can show that there exist $d > 0$ and infinitely many $k \in \mathbb{N}$ such that the inequality

$$0 < \left|\sigma(+, k^2, \ell^2) - \sigma(-, k^2, \tilde{\ell}^2)\right| \leqslant e^{-k^2\left|\sigma(+,k^2,\ell^2)\right|},$$

holds for some $\ell, \tilde{\ell} \in \mathbb{N}$. This can be proved as follows. From [27, Proposition 1], we know that, for every $\varepsilon > 0$, there exists $d > 0$ with $\sqrt{d} \notin \mathbb{Q}$ such that the inequality

$$\left|q\sqrt{d} - p\right| < e^{-q^{2+\varepsilon}},$$

holds for infinitely many $q, p \in \mathbb{N}^*$. Assume for instance that $d \leqslant 1$ (the other case is similar). Then, taking $k = pq$, $\ell = p^2$ and $\tilde{\ell} = q^2$, we have

$$\begin{aligned}\left|\sigma(+,k^2,\ell^2)-\sigma(-,k^2,\tilde{\ell}^2)\right| & \\ &\leqslant \left|k^2-d\tilde{\ell}^2\right|+\left|dk^2-\ell^2\right| \\ &= \left|q\sqrt{d}-p\right|(q\sqrt{d}+p)(p^2+q^2).\end{aligned}$$

Using that $p<(1+\sqrt{d})q$ for q large enough, we obtain the desired estimate taking $\varepsilon>6$. Finally, it is easy to see from the expressions of $k,\ell,\tilde{\ell}$ that the condition $\sigma(+,k^2,\ell^2)\neq\sigma(-,k^2,\tilde{\ell}^2)$ is equivalent to $\sqrt{d}\notin\mathbb{Q}$.

Let us now list all the properties of the operator A^* that result from the previous assumptions and that will be needed in this article. The proof is postponed to Appendix A to ease the presentation.

Proposition 4.9. — *Assume that the Kalman condition* (2.18) *is satisfied. Then, the operator A^* has the following properties:*

(i) The spectrum of A^* is given by

$$\Sigma=\sigma(\Gamma).$$

We recall that it is only made of eigenvalues.

(ii) For any eigenvalue $\hat{\sigma}\in\Sigma$, the eigenspace $\ker(\hat{\sigma}-A^*)$ *is spanned by*

$$\Phi^0_\gamma=\begin{pmatrix}1\\ r_\gamma\end{pmatrix}\otimes\phi_{1,\lambda_1(\gamma)}\otimes\phi_{2,\lambda_2(\gamma)}, \tag{4.11}$$

with

$$r_\gamma=\frac{\sigma(\gamma)-d(\lambda_1(\gamma)+\lambda_2(\gamma))+m_{11}}{-m_{21}},$$

for each $\gamma\in\Gamma$ such that $\sigma(\gamma)=\hat{\sigma}$ (we recall that ϕ_{i,λ_i} is defined in Section 2.2.2). A basis of the eigenspace $\ker(\hat{\sigma}-A^*)$ *is given by considering only the eigenfunctions Φ^0_γ for $\gamma\in\Gamma\setminus\widehat{\Gamma}^+$ (such that $\sigma(\gamma)=\hat{\sigma}$).*

(iii) If $\gamma\notin\widehat{\Gamma}$, the Jordan chain associated with Φ^0_γ is trivial, and we set $k_\gamma=0$.
If $\gamma\in\widehat{\Gamma}$, the function

$$\Phi^1_\gamma=\begin{pmatrix}0\\ -\frac{1}{m_{21}}\end{pmatrix}\otimes\phi_{1,\lambda_1(\gamma)}\otimes\phi_{2,\lambda_2(\gamma)}, \tag{4.12}$$

is a generalized eigenfunction of A^* satisfying

$$(A^*-\sigma(\gamma))\Phi^1_\gamma=\Phi^0_\gamma.$$

In that case, we set $k_\gamma=1$.

(iv) *The counting function N associated with the eigenvalues of the operator A^* defined by*

$$N(r) = \operatorname{card}\left\{\gamma \in \Gamma \setminus \widehat{\Gamma}^+, \quad s.t. \quad |\sigma(\gamma)| \leqslant r\right\}, \quad \forall r > 0, \tag{4.13}$$

satisfies the following asymptotics: there exists $\kappa_0 > 0$ such that

$$N(r) \leqslant \kappa_0 r, \quad \forall r > 0. \tag{4.14}$$

(v) *There exists $C > 0$ such that*

$$\left\|\Phi_\gamma^k\right\|_{D(A^*)} \leqslant C e^{C|\sigma(\gamma)|^{\nu_{\max}}}, \quad \forall \gamma \in \Gamma, \quad \forall 0 \leqslant k \leqslant k_\gamma. \tag{4.15}$$

(vi) *The normalized family $\left\{\Phi_\gamma^k / \left\|\Phi_\gamma^k\right\|_{D(A^*)}\right\}_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+ \\ 0\leqslant k\leqslant k_\gamma}}$ is a Riesz basis of $D(A^*)$ equipped with the graph norm.*

Remark 4.10. — A few remarks are in order.

(1) Let $\gamma = (s, \lambda_1, \lambda_2)$ be an element of $\widehat{\Gamma}$. Then $\tilde{\gamma} = (-s, \lambda_1, \lambda_2)$ also belongs to $\widehat{\Gamma}$ and we have that

$$\sigma(\gamma) = \sigma(\tilde{\gamma}), \quad \Phi_\gamma^0 = \Phi_{\tilde{\gamma}}^0, \quad \Phi_\gamma^1 = \Phi_{\tilde{\gamma}}^1.$$

That is the reason why the Riesz basis introduced in point (vi) is indexed on $\Gamma \setminus \widehat{\Gamma}^+$; this prevents an element from appearing twice in the family (same for the last line of the statement in point (ii)). We could also have chosen to index on $\Gamma \setminus \widehat{\Gamma}^-$.

(2) The counting function N in this theorem takes into account the geometric multiplicities of the eigenvalues and in particular it is not equal to the counting function $N_{\sigma(\Gamma)}$ of the set $\sigma(\Gamma)$ as defined in (2.5).

(3) In fact, in most of this work, we only need that the family $\left\{\Phi_\gamma^k\right\}_{\substack{\gamma\in\Gamma \\ 0\leqslant k\leqslant k_\gamma}}$ is complete in $D(A^*)$. It is only for Theorems 2.8 and 3.4 that we really need that this family forms a basis.

(4) There is of course no uniqueness of the generalized eigenfunction. We have chosen here the ones that satisfy

$$B^*\Phi_\gamma^1 = 0, \quad \forall \gamma \in \widehat{\Gamma}, \tag{4.16}$$

as it can be easily seen from (4.12) and (2.17).

(5) The eigenfunctions and generalized eigenfunctions of A can also be computed in a similar way.

4.2. Graph structures associated to the spectrum of A^*

Let us now define two kind of relationships between elements in Γ and introduce an appropriate structure of graph (we recall the needed elementary graph theory notions in Appendix B). Such relations are motivated by the particular structure (2.17) of our control operator B, this link will be clearer during the proofs below. These relations depend on a small parameter ρ, the choice of which will be very important for our analysis.

Definition 4.11. — *Let $\rho \geqslant 0$ be given.*

- *For $\gamma, \tilde{\gamma} \in \Gamma$ and $i \in \{1,2\}$, we will write*

$$\gamma \underset{\lambda_i}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma} \quad \text{if and only if} \quad \begin{cases} |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho, \\ \lambda_i(\gamma) = \lambda_i(\tilde{\gamma}). \end{cases}$$

- *For $\gamma, \tilde{\gamma} \in \Gamma$, we will write*

$$\gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma} \quad \text{if and only if} \quad \left(\gamma \underset{\lambda_1}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma} \quad \text{or} \quad \gamma \underset{\lambda_2}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma} \right).$$

  If $\gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma}$ and $\tilde{\gamma} \overset{\rho}{\Longleftrightarrow} \tilde{\tilde{\gamma}}$ then we will write

$$\gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma} \overset{\rho}{\Longleftrightarrow} \tilde{\tilde{\gamma}}.$$

  We say that two arrows $\overset{\rho}{\Longleftrightarrow}$ are of different types if one is of type $\underset{\lambda_1}{\overset{\rho}{\longleftrightarrow}}$ and the other one is of type $\underset{\lambda_2}{\overset{\rho}{\longleftrightarrow}}$.

- *To the set $\Gamma \setminus \widehat{\Gamma}^+$, we associate a structure of graph whose edges are defined by*

$$\mathcal{E}_\rho = \left\{ \{\gamma, \tilde{\gamma}\}, \quad s.t. \quad \gamma, \tilde{\gamma} \in \Gamma \setminus \widehat{\Gamma}^+, \quad \gamma \neq \tilde{\gamma}, \quad \gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma} \right\}.$$

- *A cycle of the graph $(\Gamma \setminus \widehat{\Gamma}^+, \mathcal{E}_\rho)$ will be called ρ-cycle to emphasize the dependence on the parameter ρ.*

By commodity, a sequence of edges $(\{\gamma_0, \gamma_1\}, \dots, \{\gamma_{n-1}, \gamma_n\})$ (whether it is a path or a cycle) will be denoted by

$$\gamma_0 \overset{\rho}{\Longleftrightarrow} \gamma_1 \overset{\rho}{\Longleftrightarrow} \cdots \overset{\rho}{\Longleftrightarrow} \gamma_{n-1} \overset{\rho}{\Longleftrightarrow} \gamma_n.$$

5. Approximate controllability

In this section we prove that our system $(-A, B)$ is approximately controllable in time T for any $T > 0$ (provided that the operators L_i defining B satisfy suitable properties). This is a weaker result than the null controllability but its proof motivates the introduction of graph theoretic arguments in a simpler context than the one we will need for the null controllability result in Section 6. We will prove the following:

Theorem 5.1. — *Let A and B be the operators introduced in Definition 2.6 and Proposition 2.7, respectively. Assume the Kalman condition* (2.18). *Assume that for $i = 1, 2$ the adjoint of the operator L_i satisfies*

$$\ker L_i^* \cap \bigoplus_{\lambda_i \in \Lambda_i} \ker(\lambda_i - A_i) = \{0\}\,. \tag{5.1}$$

Then, the system $(-A, B)$ is approximately controllable in time T for every $T > 0$.

Remark 5.2. — We wish to point out that this approximate controllability result requires less assumptions than our null controllability result. To be precise, the conditions (2.7), (2.8), (2.10) and (2.15) are not needed for the proof of Theorem 5.1 (despite being very important in the proof of our main result).

The proof of Theorem 5.1 relies on the so-called Fattorini-Hautus test which allows, under some reasonable assumptions, to completely characterize the approximate controllability of systems with analytic semigroup in terms of the spectral elements of the adjoint operator:

Theorem 5.3 (Fattorini-Hautus test). — *Assume that:*

(i) Each point of the spectrum $\sigma(\mathcal{A})$ is isolated and is a pole of finite order of the resolvent of $\mathcal{A}$.
(ii) The subspace of generalized eigenvectors of $\mathcal{A}$ is dense in H.
(iii) $-\mathcal{A}$ generates an analytic C_0-semigroup.

Then, $(-\mathcal{A}, \mathcal{B})$ is approximately controllable in time T for every $T > 0$, if and only if the Fattorini-Hautus test holds, i.e.

$$\ker(\sigma - \mathcal{A}^*) \cap \ker \mathcal{B}^* = \{0\}\,, \quad \forall \sigma \in \mathbb{C}.$$

This powerful result was established in [19, Corollary 3.3] (see also [7, 30]).

The operator A under study in this paper, introduced in Definition 2.6, satisfies the assumptions (i), (ii) and (iii) of Theorem 5.3. It remains to prove the Fattorini-Hautus test associated with the control operator B introduced in Section 2.2.2.

PROPOSITION 5.4. — *If we assume the Kalman condition* (2.18) *and the unique continuation property* (5.1), *then the pair* $(-A, B)$ *satisfies the Fattorini-Hautus test:*

$$\ker(\sigma - A^*) \cap \ker B^* = \{0\}\,, \quad \forall \sigma \in \mathbb{C}.$$

The proof of this proposition relies on the following crucial properties.

LEMMA 5.5. — *(1) For every* $\gamma_0, \gamma_1 \in \Gamma \setminus \widehat{\Gamma}^+$, $\gamma_0 \neq \gamma_1$, *we cannot have*

$$\gamma_0 \underset{\lambda_1}{\overset{0}{\longleftrightarrow}} \gamma_1 \underset{\lambda_2}{\overset{0}{\longleftrightarrow}} \gamma_0.$$

(2) There is no 0*-cycle* $\gamma_0 \overset{0}{\Longleftrightarrow} \gamma_1 \overset{0}{\Longleftrightarrow} \cdots \overset{0}{\Longleftrightarrow} \gamma_{n-1} \overset{0}{\Longleftrightarrow} \gamma_0$ *made of elements in* $\Gamma \setminus \widehat{\Gamma}^+$.

Let us first show how this implies the proposition above.

Proof of Proposition 5.4. — Let $\hat{\sigma} \in \sigma(\Gamma)$ be fixed. Let $\Phi \in \ker(\hat{\sigma} - A^*) \cap \ker B^*$ and let us show that necessarily $\Phi = 0$. Since in particular $\Phi \in \ker(\hat{\sigma} - A^*)$, we can use the description of the eigenfunctions given in Proposition 4.9 (see also 1 of Remark 4.10) to write

$$\Phi = \sum_{\substack{\gamma \in \Gamma \setminus \widehat{\Gamma}^+ \\ \sigma(\gamma) = \hat{\sigma}}} a_\gamma \Phi^0_\gamma,$$

for some scalars $a_\gamma \in \mathbb{C}$. We introduce the support of Φ defined by

$$\operatorname{Supp} \Phi = \left\{ \gamma \in \Gamma \setminus \widehat{\Gamma}^+, \quad \text{s.t.} \quad \sigma(\gamma) = \hat{\sigma}, \quad a_\gamma \neq 0 \right\}.$$

Our goal is to prove that $\operatorname{Supp} \Phi = \emptyset$. Assume, by contradiction, that $\operatorname{Supp} \Phi \neq \emptyset$. We compute (recall the normalization (2.13))

$$B^* \Phi = \begin{pmatrix} L_1^* \left(\displaystyle\sum_{\gamma \in \operatorname{Supp} \Phi} a_\gamma \phi_{1, \lambda_1(\gamma)} \right) \\ L_2^* \left(\displaystyle\sum_{\gamma \in \operatorname{Supp} \Phi} a_\gamma \phi_{2, \lambda_2(\gamma)} \right) \end{pmatrix}.$$

Therefore, the equation $B^* \Phi = 0$ and the assumption (5.1) yield

$$\sum_{\gamma \in \operatorname{Supp} \Phi} a_\gamma \phi_{1, \lambda_1(\gamma)} = 0, \tag{5.2}$$

and

$$\sum_{\gamma \in \operatorname{Supp} \Phi} a_\gamma \phi_{2,\lambda_2(\gamma)} = 0.$$

Let us now show that this implies the following property, for $i = 1, 2$,

$$\forall \gamma \in \operatorname{Supp} \Phi, \quad \exists \tilde{\gamma} \in \operatorname{Supp} \Phi, \, \tilde{\gamma} \neq \gamma, \quad \gamma \xleftrightarrow[\lambda_i]{0} \tilde{\gamma}. \tag{5.3}$$

Consider for instance $i = 1$. Assume that (5.3) does not hold for some $\gamma \in \operatorname{Supp} \Phi$. This means that the eigenfunction $\phi_{1,\lambda_1(\gamma)}$ only appears once in the sum (5.2). Since the family $(\phi_{1,\lambda_1})_{\lambda_1 \in \Lambda_1}$ is linearly independent, this means that the corresponding coefficient a_γ is equal to 0, which is not possible by definition of the support of Φ. Thus, we have (5.3).

Since $\operatorname{Supp} \Phi \neq \emptyset$ by assumption, there exists an element $\gamma_0 \in \operatorname{Supp} \Phi$, then we apply (5.3) with $i = 1$ to find $\gamma_1 \in \operatorname{Supp} \Phi$ with $\gamma_1 \neq \gamma_0$ such that

$$\gamma_0 \xleftrightarrow[\lambda_1]{0} \gamma_1.$$

Then, we apply (5.3) with $i = 2$ to find a $\gamma_2 \in \operatorname{Supp} \Phi$ with $\gamma_2 \neq \gamma_1$ such that

$$\gamma_1 \xleftrightarrow[\lambda_2]{0} \gamma_2.$$

By the first point of Lemma 5.5, we know that $\gamma_2 \neq \gamma_0$. We can apply again (5.3) with $i = 1$ and so on. Since $\operatorname{Supp} \Phi$ is a finite set, we can repeat the process until we select en element that was already selected in the process. Therefore we end up with the following situation

$$\gamma_0 \overset{0}{\Longleftrightarrow} \gamma_1 \overset{0}{\Longleftrightarrow} \cdots \overset{0}{\Longleftrightarrow} \gamma_{i-2} \overset{0}{\Longleftrightarrow} \gamma_{i-1} \overset{0}{\Longleftrightarrow} \gamma_i,$$

where $\gamma_i \in \{\gamma_0, \ldots, \gamma_{i-1}\}$. By construction we have $\gamma_{i-1} \neq \gamma_i$, and moreover, since the kind of the arrows alternate in our construction, we also know that $\gamma_{i-2} \neq \gamma_i$ thanks to the first point of Lemma 5.5. We have finally found a 0-cycle

$$\gamma_k \overset{0}{\Longleftrightarrow} \ldots \overset{0}{\Longleftrightarrow} \gamma_{i-2} \overset{0}{\Longleftrightarrow} \gamma_{i-1} \overset{0}{\Longleftrightarrow} \gamma_i, \text{ with } \gamma_k = \gamma_i \text{ and } 0 \leqslant k < i-2,$$

made of elements in $\operatorname{Supp} \Phi \subset \Gamma \setminus \widehat{\Gamma}^+$. This is a contradiction with the second point of Lemma 5.5. $\square$

Let us now turn out to the proof of this crucial lemma.

Proof of Lemma 5.5. —

(1) It follows from the assumption that

$$\lambda_1(\gamma_0) = \lambda_1(\gamma_1), \text{ and } \lambda_2(\gamma_0) = \lambda_2(\gamma_1).$$

Since $\gamma_0 \neq \gamma_1$, we necessarily have $s(\gamma_0) = -s(\gamma_1)$. Moreover, we also have $\sigma(\gamma_0) = \sigma(\gamma_1)$, which implies by using (4.3), that necessarily

$$\Delta_{\lambda(\gamma_0)} = \Delta_{\lambda(\gamma_1)} = 0.$$

This proves that one of the two elements γ_0 or γ_1 belongs to $\widehat{\Gamma}^+$ which is excluded.

(2) Assume that such a 0-cycle exists

$$\gamma_0 \overset{0}{\Longleftrightarrow} \gamma_1 \overset{0}{\Longleftrightarrow} \cdots \overset{0}{\Longleftrightarrow} \gamma_{n-1} \overset{0}{\Longleftrightarrow} \gamma_0,$$

where, by definition, $\gamma_i \neq \gamma_j$ for $i \neq j$. For convenience, we shall set $\gamma_n = \gamma_0$. We will denote by $\hat{\sigma}$ the common value of all the $\sigma(\gamma_i)$.

- Observe first that we have

$$\lambda(\gamma_i) \neq \lambda(\gamma_{i+1}), \quad \forall i \in \{0, \dots, n-1\}\,. \tag{5.4}$$

Indeed, if it were not the case, recalling that $\lambda(\gamma) = \lambda_1(\gamma) + \lambda_2(\gamma)$, we would have for some $i \in \{0, \dots, n-1\}$

$$\lambda_1(\gamma_i) = \lambda_1(\gamma_{i+1}), \quad \lambda_2(\gamma_i) = \lambda_2(\gamma_{i+1}),$$

that can be written

$$\gamma_i \underset{\lambda_1}{\overset{0}{\longleftrightarrow}} \gamma_{i+1} \underset{\lambda_2}{\overset{0}{\longleftrightarrow}} \gamma_i.$$

This is excluded by the first point of the lemma.

- By (4.3) and (4.1) we have, for any $i \in \{0, \dots, n-1\}$,

$$\begin{aligned}(2\hat{\sigma} - (1+d)\lambda(\gamma_i) + \mathrm{Tr}(M))^2 &= \Delta_{\lambda(\gamma_i)} \\ &= ((1-d)\lambda(\gamma_i) + m_{11} - m_{22})^2 + 4m_{21}m_{12}.\end{aligned}$$

This is a second order polynomial equation for $\lambda(\gamma_i)$ and therefore there exists at most two possible values for this quantity. By (5.4), we deduce that n is necessarily even (we write $n = 2\ell$ with $\ell \geqslant 1$) and that there exist $\lambda' \neq \lambda''$ such that

$$\lambda(\gamma_{2j}) = \lambda', \quad \lambda(\gamma_{2j+1}) = \lambda'', \quad \forall j \in \{0, \dots, \ell-1\}\,. \tag{5.5}$$

- Let us prove now that two consecutive arrows in the cycle cannot be of the same kind. Assume, by contradiction, that we have

$$\gamma_i \underset{\lambda_j}{\overset{0}{\longleftrightarrow}} \gamma_{i+1} \underset{\lambda_j}{\overset{0}{\longleftrightarrow}} \gamma_{i+2}, \tag{5.6}$$

for some $i \in \{0, \dots, n-2\}$ and some $j \in \{1,2\}$. By (5.5) we know that $\lambda(\gamma_i) = \lambda(\gamma_{i+2})$ and using (5.6) it follows that

$$\lambda_1(\gamma_i) = \lambda_1(\gamma_{i+2}), \quad \lambda_2(\gamma_i) = \lambda_2(\gamma_{i+2}),$$

so that we can write

$$\gamma_i \xleftrightarrow[\lambda_1]{0} \gamma_{i+2} \xleftrightarrow[\lambda_2]{0} \gamma_i,$$

which is excluded by the first point of the lemma.

- We assume now that $\gamma_0 \xleftrightarrow[\lambda_2]{0} \gamma_1$, the other case being similar. By the discussion above we know that

$$\gamma_{2j} \xleftrightarrow[\lambda_2]{0} \gamma_{2j+1} \xleftrightarrow[\lambda_1]{0} \gamma_{2j+2}, \quad \forall j \in \{0, \dots, \ell-1\},$$

that is to say

$$\lambda_2(\gamma_{2j}) = \lambda_2(\gamma_{2j+1}), \quad \lambda_1(\gamma_{2j+1}) = \lambda_1(\gamma_{2j+2}), \quad \forall j \in \{0, \dots, \ell-1\}. \tag{5.7}$$

Using (5.5), we can now compute

$$\begin{aligned}
\ell\lambda' &= \sum_{j=0}^{\ell-1} \lambda(\gamma_{2j}) \\
&= \sum_{j=0}^{\ell-1} \lambda_1(\gamma_{2j}) + \sum_{j=0}^{\ell-1} \lambda_2(\gamma_{2j}) \\
&= \sum_{j=0}^{\ell-2} \lambda_1(\gamma_{2j+2}) + \lambda_1(\gamma_0) + \sum_{j=0}^{\ell-1} \lambda_2(\gamma_{2j}) \\
&= \sum_{j=0}^{\ell-1} \lambda_1(\gamma_{2j+1}) + \sum_{j=0}^{\ell-1} \lambda_2(\gamma_{2j+1})
\end{aligned}$$

(since $\gamma_0 = \gamma_{2\ell}$ and using (5.7))

$$\begin{aligned}
&= \sum_{j=0}^{\ell-1} \lambda(\gamma_{2j+1}) \\
&= \ell\lambda''.
\end{aligned}$$

This is a contradiction with $\lambda' \neq \lambda''$ and the proof is complete.

$\square$

6. Null controllability

The first five parts of this section are devoted to the proof of Theorem 2.8 in the case where the operators L_i are simply

$$L_i = \mathrm{Id}, \quad i = 1, 2.$$

The corresponding control operator B will be denoted by B_{ref}. The operator B_{ref} will play the role of a reference operator and the hardest task of the work is actually to establish controllability properties for our system with this B_{ref}.

More precisely, we will first establish the following result:

THEOREM 6.1. — *Let A be the operator introduced in Definition 2.6 and let B_{ref} be the operator B introduced in Proposition 2.7 with $L_i = \mathrm{Id}$ for $i = 1, 2$. Assume the Kalman condition* (2.18)*.*

Let $p_0 = \max\{\theta_1, \theta_2, \nu_1, \nu_2\}$. Then, the system $(-A, B_{\mathrm{ref}})$ is null controllable in time T for every $T > 0$, with control cost satisfying, for some $C > 0$,

$$\mathrm{cost}_T(-A, B_{\mathrm{ref}}) \leqslant C \exp\left(\frac{C}{T^{\frac{p_0}{1-p_0}}}\right), \quad \forall T > 0.$$

We will then show in Section 6.5 how to deduce the extension Theorem 2.8 by using the so-called Lebeau-Robbiano method ([25, 29]).

Remark 6.2. — The same result remains true if we drop the assumption (2.9) on the operators A_1, A_2 but in this case we have to consider p_0 satisfying $p_0 > \max\{\theta_1, \theta_2\}$ and $p_0 \geqslant \max\{\nu_1, \nu_2\}$. This will be explained in Remark 6.13 below.

All along this section we assume that we are under the assumptions of Theorem 6.1.

6.1. A non standard moment problem

We start by reformulating the null controllability problem into a moment problem. Let $T > 0$ be fixed.

1) From the very definition of the notion of solution (2.2), we see that $(-A, B_{\mathrm{ref}})$ is null controllable in time T if, and only if, for every $y^0 \in D(A^*)'$, there exists $u \in L^2(0, T; U)$ such that

$$-\langle y^0, z(0)\rangle_{D(A^*)', D(A^*)} = \int_0^T \langle u(t), B_{\mathrm{ref}}^* z(t)\rangle_U \, dt, \quad \forall z^T \in D(A^*), \tag{6.1}$$

with z solution to the adjoint problem (2.3) with $\tau = T$.

Let $y^0 \in D(A^*)'$ be fixed from now on. Since the conjugate linear forms involved in the previous identity (i.e. (2.4)) are continuous on $D(A^*)$, it is sufficient to check this identity on a dense subset of $D(A^*)$. We know that span $\left\{\Phi_\gamma^k\right\}_{\substack{\gamma\in\Gamma\\ 0\leqslant k\leqslant k_\gamma}}$ is dense in $D(A^*)$ by point (vi) of Proposition 4.9 (see also point (3) in Remark 4.10).

Therefore, by linearity, it is enough to test (6.1) with $z^T = \Phi_\gamma^0$ for every $\gamma \in \Gamma$, and $z^T = \Phi_\gamma^1$ for every $\gamma \in \widehat{\Gamma}$.

2) For an eigenfunction $z^T = \Phi_\gamma^0$, with $\gamma \in \Gamma$, the corresponding solution to the adjoint system (2.3) is given by

$$z(t) = e^{-(T-t)\sigma(\gamma)}\Phi_\gamma^0,$$

while for a generalized eigenfunction $z^T = \Phi_\gamma^1$, with $\gamma \in \widehat{\Gamma}$, the solution to the adjoint system is

$$z(t) = e^{-(T-t)\sigma(\gamma)}\Phi_\gamma^1 - (T-t)e^{-(T-t)\sigma(\gamma)}\Phi_\gamma^0.$$

Using (4.16), it follows that the property (6.1) is equivalent to

$$\begin{cases} -e^{-T\overline{\sigma(\gamma)}}\left\langle y^0, \Phi_\gamma^0\right\rangle_{D(A^*)',D(A^*)} \\ \qquad\qquad = \displaystyle\int_0^T e^{-(T-t)\overline{\sigma(\gamma)}}\left\langle u(t), B_{\mathrm{ref}}^*\Phi_\gamma^0\right\rangle_U dt, \quad \forall\gamma\in\Gamma, \\ -e^{-T\overline{\sigma(\gamma)}}\left\langle y^0, \Phi_\gamma^1 - T\Phi_\gamma^0\right\rangle_{D(A^*)',D(A^*)} \\ \qquad\qquad = \displaystyle\int_0^T (T-t)e^{-(T-t)\overline{\sigma(\gamma)}}\left\langle u(t), B_{\mathrm{ref}}^*\Phi_\gamma^0\right\rangle_U dt, \quad \forall\gamma\in\widehat{\Gamma}. \end{cases}$$

Finding a function u such that the above system is satisfied for every γ is a so-called “moment problem”. By assumption (2.16) on the structure of the control space, we can write

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_i \in L^2(0,T;H_i), \quad i = 1,2.$$

Using now the structure (2.17) of B_{ref}^* and the structure (4.11) of the eigenfunctions of A^* with (2.13), we see that (6.1) is equivalent to

$$\begin{cases} -e^{-T\overline{\sigma(\gamma)}}\left\langle y^0, \Phi^0_\gamma \right\rangle_{D(A^*)',D(A^*)} \\ \qquad = \displaystyle\int_0^T e^{-(T-t)\overline{\sigma(\gamma)}}\left\langle u_1(t), \phi_{1,\lambda_1(\gamma)}\right\rangle_{H_1} dt \\ \qquad\quad + \displaystyle\int_0^T e^{-(T-t)\overline{\sigma(\gamma)}}\left\langle u_2(t), \phi_{2,\lambda_2(\gamma)}\right\rangle_{H_2} dt, \quad \forall \gamma \in \Gamma, \\ -e^{-T\overline{\sigma(\gamma)}}\left\langle y^0, \Phi^1_\gamma - T\Phi^0_\gamma \right\rangle_{D(A^*)',D(A^*)} \\ \qquad = \displaystyle\int_0^T (T-t)e^{-(T-t)\overline{\sigma(\gamma)}}\left\langle u_1(t), \phi_{1,\lambda_1(\gamma)}\right\rangle_{H_1} dt \\ \qquad\quad + \displaystyle\int_0^T (T-t)e^{-(T-t)\overline{\sigma(\gamma)}}\left\langle u_2(t), \phi_{2,\lambda_2(\gamma)}\right\rangle_{H_2} dt, \quad \forall \gamma \in \widehat{\Gamma}. \end{cases}$$

Since for $i = 1, 2$, $\{\phi_{i,\lambda_i}\}_{\lambda_i\in\Lambda_i}$ is an orthogonal basis of H_i, it is equivalent to look for u_i in the form of the series

$$u_i(t) = \sum_{\lambda_i\in\Lambda_i} \tilde{u}_{i,\lambda_i}(T-t)\frac{\phi_{i,\lambda_i}}{\|\phi_{i,\lambda_i}\|^2_{H_i}},$$

with $\tilde{u}_{i,\lambda_i} \in L^2(0,T)$ such that

$$\sum_{\lambda_i\in\Lambda_i} \frac{\|\tilde{u}_{i,\lambda_i}\|^2_{L^2(0,T)}}{\|\phi_{i,\lambda_i}\|^2_{H_i}} < +\infty, \quad i = 1, 2. \tag{6.2}$$

Thus, the goal is to find $\{\tilde{u}_{1,\lambda_1}\}_{\lambda_1\in\Lambda_1} \subset L^2(0,T)$ and $\{\tilde{u}_{2,\lambda_2}\}_{\lambda_2\in\Lambda_2} \subset L^2(0,T)$ with (6.2) and such that

$$\begin{cases} -e^{-T\overline{\sigma(\gamma)}}\left\langle y^0, \Phi^0_\gamma \right\rangle_{D(A^*)',D(A^*)} \\ \qquad = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}}\tilde{u}_{1,\lambda_1(\gamma)}(t)\, dt \\ \qquad\quad + \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}}\tilde{u}_{2,\lambda_2(\gamma)}(t)\, dt, \quad \forall \gamma \in \Gamma, \\ -e^{-T\overline{\sigma(\gamma)}}\left\langle y^0, \Phi^1_\gamma - T\Phi^0_\gamma \right\rangle_{D(A^*)',D(A^*)} \\ \qquad = \displaystyle\int_0^T te^{-t\overline{\sigma(\gamma)}}\tilde{u}_{1,\lambda_1(\gamma)}(t)\, dt \\ \qquad\quad + \displaystyle\int_0^T te^{-t\overline{\sigma(\gamma)}}\tilde{u}_{2,\lambda_2(\gamma)}(t)\, dt, \quad \forall \gamma \in \widehat{\Gamma}. \end{cases} \tag{6.3}$$

In summary, we have shown the following:

Proposition 6.3. — *Let $T > 0$ and $y^0 \in D(A^*)'$ be fixed. Then, the following are equivalent:*

(i) *There exists $u \in L^2(0,T;U)$ such that the corresponding solution y to the system $(-A, B_{\mathrm{ref}})$ satisfies $y(T) = 0$.*

(ii) *There exist $\{\tilde{u}_{1,\lambda_1}\}_{\lambda_1 \in \Lambda_1} \subset L^2(0,T)$ and $\{\tilde{u}_{2,\lambda_2}\}_{\lambda_2 \in \Lambda_2} \subset L^2(0,T)$ with* (6.2) *and satisfying* (6.3).

Moreover, if (ii) holds, then u in (i) can be chosen so that

$$\|u\|^2_{L^2(0,T;U)} = \sum_{i \in \{1,2\}} \sum_{\lambda_i \in \Lambda_i} \frac{\|\tilde{u}_{i,\lambda_i}\|^2_{L^2(0,T)}}{\|\phi_{i,\lambda_i}\|^2_{H_i}}.$$

The system of equations (6.3) looks like a family of coupled moment problems. The main difficulty in solving this system comes from the following facts (below, $i = 1,2$ and $\lambda_i \in \Lambda_i$ are fixed):

- The unknown function $\tilde{u}_{i,\lambda_i}$ appears in an infinite subset of those equations, namely the ones corresponding to the parameters γ belonging to the set Γ_{i,λ_i} (defined in point (3) of Definition 4.3).
- The map $\gamma \in \Gamma_{i,\lambda_i} \mapsto \sigma(\gamma)$ may not be injective (see Remark 4.8), so that the same integral term may appear in many of those equations.
- Even in the case where the previous map is injective, it will certainly happen some spectral condensation phenomenon (see again Remark 4.8). In general, this condensation may be an obstacle to the small time null controllability of the system as mentioned in the introduction. The block moment method was precisely introduced in [9] to carefully analyze this phenomenon in a quite general setting.

  In the sequel of the paper, we will show how to apply this block moment approach in order to prove the small time null controllability of our system.

The strategy to solve (6.3) is to build separate sets of equations for the families $(\tilde{u}_{1,\lambda_1})_{\lambda_1 \in \Lambda_1}$ and $(\tilde{u}_{2,\lambda_2})_{\lambda_2 \in \Lambda_2}$. In that perspective, we will first state the following key result that will induce the existence of a suitable splitting of the source terms into two parts. The proof of this result is postponed to Section 6.4 below.

Theorem 6.4. — *There exists $\hat{\rho} > 0$ small enough and there exist two families $(\Phi^k_{\gamma,1})_{\substack{\gamma \in \Gamma \\ 0 \leqslant k \leqslant k_\gamma}} \subset D(A^*)$, $(\Phi^k_{\gamma,2})_{\substack{\gamma \in \Gamma \\ 0 \leqslant k \leqslant k_\gamma}} \subset D(A^*)$, such that:*

(i) *We have*

$$\Phi^k_\gamma = \Phi^k_{\gamma,1} + \Phi^k_{\gamma,2}, \quad \forall \gamma \in \Gamma, \quad \forall 0 \leqslant k \leqslant k_\gamma.$$

(ii) For every $\gamma, \tilde{\gamma} \in \Gamma$, we have

$$\left.\begin{aligned} |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \hat{\rho} \\ \lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) \end{aligned}\right\} \implies \begin{cases} \Phi^0_{\gamma,1} = \Phi^0_{\tilde{\gamma},1}, \\ \Phi^1_{\gamma,1} = \Phi^1_{\tilde{\gamma},1}, & \text{if } \gamma, \tilde{\gamma} \in \widehat{\Gamma}, \end{cases}$$

and

$$\left.\begin{aligned} |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \hat{\rho} \\ \lambda_2(\gamma) = \lambda_2(\tilde{\gamma}) \end{aligned}\right\} \implies \begin{cases} \Phi^0_{\gamma,2} = \Phi^0_{\tilde{\gamma},2}, \\ \Phi^1_{\gamma,2} = \Phi^1_{\tilde{\gamma},2}, & \text{if } \gamma, \tilde{\gamma} \in \widehat{\Gamma}. \end{cases}$$

(iii) There exists $C > 0$ such that

$$\left\|\Phi^k_{\gamma,1}\right\|_{D(A^*)} + \left\|\Phi^k_{\gamma,2}\right\|_{D(A^*)} \leqslant C\,|\sigma(\gamma)| \exp\left(C\,|\sigma(\gamma)|^{\nu_{\max}}\right), \qquad \forall \gamma \in \Gamma, \quad \forall 0 \leqslant k \leqslant k_\gamma.$$

Let us explain how to use this result to solve the system (6.3). We introduce the complex numbers

$$\begin{cases} \omega^k_{\gamma,1} = (-1)^{k+1} e^{-T\overline{\sigma(\gamma)}} \left\langle y^0, \Phi^k_{\gamma,1}\right\rangle_{D(A^*)',D(A^*)}, \\ \omega^k_{\gamma,2} = (-1)^{k+1} e^{-T\overline{\sigma(\gamma)}} \left\langle y^0, \Phi^k_{\gamma,2}\right\rangle_{D(A^*)',D(A^*)}. \end{cases} \tag{6.4}$$

The idea is to split the problem into two independent sets of equations as follows

$$\begin{cases} \omega^0_{\gamma,1} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}} \tilde{u}_{1,\lambda_1(\gamma)}(t)\,dt, & \forall \gamma \in \Gamma, \\ \omega^1_{\gamma,1} + T\omega^0_{\gamma,1} = \displaystyle\int_0^T t e^{-t\overline{\sigma(\gamma)}} \tilde{u}_{1,\lambda_1(\gamma)}(t)\,dt, & \forall \gamma \in \widehat{\Gamma}, \end{cases}$$

$$\begin{cases} \omega^0_{\gamma,2} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}} \tilde{u}_{2,\lambda_2(\gamma)}(t)\,dt, & \forall \gamma \in \Gamma, \\ \omega^1_{\gamma,2} + T\omega^0_{\gamma,2} = \displaystyle\int_0^T t e^{-t\overline{\sigma(\gamma)}} \tilde{u}_{2,\lambda_2(\gamma)}(t)\,dt, & \forall \gamma \in \widehat{\Gamma}. \end{cases}$$

If we manage to solve those problems, by summing the equations and using point (i) of Theorem 6.4 we immediately get that (6.3) is solved.

Solving those problems amounts to ask that, each function $\tilde{u}_{1,\lambda_1}$, for $\lambda_1 \in \Lambda_1$, satisfies

$$\begin{cases} \omega^0_{\gamma,1} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}} \tilde{u}_{1,\lambda_1}(t)\,dt, & \forall \gamma \in \Gamma_{1,\lambda_1}, \\ \omega^1_{\gamma,1} + T\omega^0_{\gamma,1} = \displaystyle\int_0^T t e^{-t\overline{\sigma(\gamma)}} \tilde{u}_{1,\lambda_1}(t)\,dt, & \forall \gamma \in \widehat{\Gamma}_{1,\lambda_1}, \end{cases} \tag{6.5}$$

and each function $\tilde{u}_{2,\lambda_2}$, for $\lambda_2 \in \Lambda_2$, satisfies

$$(6.6) \qquad \begin{cases} \omega^0_{\gamma,2} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}} \tilde{u}_{2,\lambda_2}(t)\,dt, & \forall \gamma \in \Gamma_{2,\lambda_2}, \\ \omega^1_{\gamma,2} + T\omega^0_{\gamma,2} = \displaystyle\int_0^T te^{-t\overline{\sigma(\gamma)}} \tilde{u}_{2,\lambda_2}(t)\,dt, & \forall \gamma \in \widehat{\Gamma}_{2,\lambda_2}. \end{cases}$$

We have now to solve an infinite set of uncoupled moment problems (one for each $\tilde{u}_{1,\lambda_1}$ and one for each $\tilde{u}_{2,\lambda_2}$), each of them being associated with a different family of (generalized) exponential functions corresponding to the eigenvalues in Σ_{1,λ_1} and Σ_{2,λ_2}, respectively. Of course, those moment problems are in fact coupled through the construction of the families $(\Phi^k_{\gamma,1})_{\substack{\gamma\in\Gamma \\ 0\leqslant k\leqslant k_\gamma}}$, $(\Phi^k_{\gamma,2})_{\substack{\gamma\in\Gamma \\ 0\leqslant k\leqslant k_\gamma}}$, given by Theorem 6.4.

We are now led to prove that all those moment problems can be solved with appropriate estimates on the solutions to ensure the convergence of the series and thus the existence of the control for our initial problem.

Consequently, the proof of Theorem 6.1 will be complete if we manage to prove the following result.

Proposition 6.5. — *Let $T > 0$ and $y^0 \in D(A^*)'$ be fixed and $p_0 \in (0,1)$ be such that $p_0 > \max\{\theta_1, \theta_2, \nu_1, \nu_2\}$.*

There exist $\{\tilde{u}_{1,\lambda_1}\}_{\lambda_1\in\Lambda_1} \subset L^2(0,T)$ and $\{\tilde{u}_{2,\lambda_2}\}_{\lambda_2\in\Lambda_2} \subset L^2(0,T)$ that satisfy (6.5) and (6.6) respectively, and such that, for some $C > 0$ not depending on T and y^0,

$$(6.7) \qquad \sum_{i\in\{1,2\}} \sum_{\lambda_i\in\Lambda_i} \frac{\|\tilde{u}_{i,\lambda_i}\|^2_{L^2(0,T)}}{\|\phi_{i,\lambda_i}\|^2_{H_i}} \leqslant C \exp\left(\frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|^2_{D(A^*)'}.$$

The rest of the section is organized as follows. We first summarize in Section 6.2 some useful definitions and results coming from the so-called block moment method. Then we proceed in Section 6.3 to the proof of the Proposition 6.5 and finally we conclude with the proof of the key Theorem 6.4, in Section 6.4.

6.2. Background on the block moment method

Let us introduce some elements taken from [12] that will be useful in our analysis (see also [9, 10] where slightly different definitions were used).

Definition 6.6. — *Let $n \in \mathbb{N}^*$, $\rho > 0$, $\alpha > 0$, $\theta \in (0,1)$ and $\kappa > 0$ be fixed. We denote by $\mathcal{L}_w(\alpha, \kappa, \theta, \rho, n)$ the class of subsets $S \subset \mathbb{C}_+$ satisfying the following three conditions:*

- *Sector condition:*

$$|\mathfrak{Im}\, z| \leqslant (\sinh \alpha)(\mathfrak{Re}\, z), \quad \forall z \in S.$$

- *Counting function asymptotics:*

$$N_S(r) \leqslant \kappa r^\theta, \quad \forall r > 0, \tag{6.8}$$

$$|N_S(r) - N_S(s)| \leqslant \kappa \left(1 + |r-s|^\theta\right), \quad \forall r, s > 0. \tag{6.9}$$

- *Weak gap condition:*

$$\operatorname{card}\,(S \cap D(\mu, \rho/2)) \leqslant n, \quad \forall \mu \in \mathbb{C}, \tag{6.10}$$

  where $D(\mu, \rho)$ is the open disk of center μ and radius ρ in the complex plane.

We shall use the following results taken from [12, Propositions V.5.26 and A.5.32] (for the first one, see also a slightly different version in [9, Proposition 7.1]).

Proposition 6.7. — *Assume that S satisfies the weak gap condition* (6.10) *for some $\alpha > 0$, $n \in \mathbb{N}^*$ and $\rho > 0$. Then, there exists a countable family $\mathcal{G}$ made of non empty disjoint subsets of S satisfying the following three properties:*

(i) Covering:

$$S = \bigcup_{G \in \mathcal{G}} G.$$

(ii) Uniform bound on the cardinality and the diameter

$$\operatorname{card} G \leqslant n \quad \textit{and} \quad \operatorname{diam} G \leqslant \rho, \quad \forall G \in \mathcal{G}.$$

(iii) Gap condition:

$$\operatorname{dist}\,(\operatorname{conv}(G), S \setminus G) \geqslant \frac{\rho}{2.4^{n-1}}, \quad \forall G \in \mathcal{G},$$

where $\operatorname{conv}(G)$ is the convex hull of G.

Proposition 6.8. — *Let $S \subset \mathbb{C}_+$ be a family satisfying the counting function asymptotic* (6.8). *Then, there exist $C > 0$ depending only on θ, such that*

$$\sum_{z \in S} e^{-\tau |z|} \leqslant C \frac{\kappa}{\tau^\theta} e^{-\tau \inf |S|/2}, \quad \forall \tau > 0.$$

We recall, also with adapted notation, the following result from [12, Theorem V.4.25] (which is an improved version of [9, Theorem 2.5]), that we specialized in the particular case where $n = 2$, which is sufficient in the present work.

Theorem 6.9. — *Let $S \in \mathcal{L}_w(\alpha,\kappa,\theta,\rho,2)$ and $(G)_{G\in\mathcal{G}}$ be a grouping as introduced in Proposition 6.7. There exists $C>0$ depending only on α,κ,θ, and ρ such that, for any $T>0$ and $G\in\mathcal{G}$, the following assertions hold.*

- *If $\operatorname{card} G = 1$, say $G=\{\sigma_G\}$, then for any $\omega^0_G,\omega^1_G\in\mathbb{C}$, there exists $q_G\in L^2(0,T;\mathbb{C})$ such that*

$$\begin{cases} \begin{pmatrix}\omega^0_G\\ \omega^1_G\end{pmatrix} = \displaystyle\int_0^T \begin{pmatrix}1\\ t\end{pmatrix} e^{-t\overline{\sigma_G}} q_G(t)\,dt, \\ \begin{pmatrix}0\\ 0\end{pmatrix} = \displaystyle\int_0^T \begin{pmatrix}1\\ t\end{pmatrix} e^{-t\overline{\sigma}} q_G(t)\,dt, \quad \forall \sigma\in S\setminus G, \end{cases}$$

and

$$\|q_G\|_{L^2(0,T)} \leqslant C\exp\left(\frac{(\mathfrak{Re}\,\sigma_G)T}{2} + C|\sigma_G|^\theta + \frac{C}{T^{\frac{\theta}{1-\theta}}}\right)\left(\left|\omega^0_G\right| + \left|\omega^1_G\right|\right).$$

- *If $\operatorname{card} G = 2$, say $G=\{\sigma_G,\hat\sigma_G\}$, with $\mathfrak{Re}\,\sigma_G\leqslant\mathfrak{Re}\,\hat\sigma_G$, then for any $\omega^0_G,\hat\omega^0_G\in\mathbb{C}$, there exists $q_G\in L^2(0,T)$ such that*

$$\begin{cases} \omega^0_G = \displaystyle\int_0^T e^{-t\overline{\sigma_G}} q_G(t)\,dt, \\ \hat\omega^0_G = \displaystyle\int_0^T e^{-t\overline{\hat\sigma_G}} q_G(t)\,dt, \\ \begin{pmatrix}0\\ 0\end{pmatrix} = \displaystyle\int_0^T \begin{pmatrix}1\\ t\end{pmatrix} e^{-t\overline{\sigma}} q_G(t)\,dt, \quad \forall \sigma\in S\setminus G, \end{cases}$$

and

$$\|q_G\|_{L^2(0,T)} \leqslant C\exp\left(\frac{(\mathfrak{Re}\,\sigma_G)T}{2} + C|\sigma_G|^\theta + \frac{C}{T^{\frac{\theta}{1-\theta}}}\right)\left(\left|\omega^0_G\right| + \left|\frac{\omega^0_G-\hat\omega^0_G}{\sigma_G-\hat\sigma_G}\right|\right).$$

Remark 6.10. — As shown in [12], the same result remains true if we drop the assumption (6.9) on the family S but in this case θ in the previous estimates has to be replaced by an arbitrary $\tilde\theta\in(\theta,1)$.

6.3. Proof of Proposition 6.5

To begin with, we establish that the families of eigenvalues introduced in (4.4) belong in a uniform way to suitable classes as defined in Definition 6.6. The precise statement is as follows.

Proposition 6.11. — *There exist $\alpha, \rho_1, \rho_2 > 0$ and $\tilde{\kappa}_1, \tilde{\kappa}_2 > 0$ such that, we have*

$$\Sigma_{1,\lambda_1} \in \mathcal{L}_w(\alpha, \tilde{\kappa}_2, \theta_2, \rho_2, 2), \quad \textit{for any } \lambda_1 \in \Lambda_1,$$

$$\Sigma_{2,\lambda_2} \in \mathcal{L}_w(\alpha, \tilde{\kappa}_1, \theta_1, \rho_1, 2), \quad \textit{for any } \lambda_2 \in \Lambda_2.$$

We recall that $\theta_i \in (0,1)$ is such that (2.8) *and* (2.9) *hold.*

Proof. — We focus on Σ_{1,λ_1}, the other case being similar. To this end, it is enough to show that there exist $\rho_2 > 0$ and $\overline{\kappa}_2 > 0$ such that, for any $\lambda_1 \in \Lambda_1$,

$$\Sigma^+_{1,\lambda_1}, \Sigma^-_{1,\lambda_1} \in \mathcal{L}_w(\alpha, \overline{\kappa}_2, \theta_2, \rho_2, 1). \tag{6.11}$$

Indeed, from [12, Lemma V.4.20], we know that the union $\Sigma_{1,\lambda_1} = \Sigma^+_{1,\lambda_1} \cup \Sigma^-_{1,\lambda_1}$ of two families satisfying (6.11) belongs to the class $\mathcal{L}_w(\alpha, \tilde{\kappa}_2, \theta_2, \rho_2, 2)$ for some $\tilde{\kappa}_2 > 0$.

Let us prove (6.11) for Σ^+_{1,λ_1}, the other case being similar.

- The sector condition is clearly satisfied thanks to (4.7) and (4.8).
- By (2.7) and (4.5), we can find $\hat{\lambda} > 0$ large enough such that

$$|\varepsilon_\lambda| \leqslant \frac{1}{4} \max(d,1) \operatorname{Gap}(\Lambda_2), \quad \forall \lambda > \hat{\lambda}. \tag{6.12}$$

- Let $\lambda_1 \in \Lambda_1$ and $\lambda_2, \tilde{\lambda}_2 \in \Lambda_2$ such that $\lambda_2 \neq \tilde{\lambda}_2$, that satisfy

$$\lambda_1 + \lambda_2 > \hat{\lambda}, \quad \lambda_1 + \tilde{\lambda}_2 > \hat{\lambda}. \tag{6.13}$$

  From (4.6), we have

$$\sigma(+,\lambda_1,\lambda_2) - \sigma(+,\lambda_1,\tilde{\lambda}_2) = \max(d,1)(\lambda_2 - \tilde{\lambda}_2) + \varepsilon_{\lambda_1+\lambda_2} - \varepsilon_{\lambda_1+\tilde{\lambda}_2},$$

  and therefore

$$\begin{aligned} &|\sigma(+,\lambda_1,\lambda_2) - \sigma(+,\lambda_1,\tilde{\lambda}_2)| \\ &\qquad\qquad \geqslant \max(d,1) \operatorname{Gap}(\Lambda_2) - |\varepsilon_{\lambda_1+\lambda_2}| - |\varepsilon_{\lambda_1+\tilde{\lambda}_2}| \\ &\qquad\qquad \geqslant \frac{1}{2} \max(d,1) \operatorname{Gap}(\Lambda_2), \end{aligned}$$

  by using (6.12).
- For any $\lambda_1 \in \Lambda_1$ such that $\lambda_1 > \hat{\lambda}$ the conditions (6.13) are automatically satisfied since we have (2.6). Therefore, we have

$$\operatorname{Gap}\left(\Sigma^+_{1,\lambda_1}\right) \geqslant \frac{1}{2} \max(d,1) \operatorname{Gap}(\Lambda_2), \quad \forall \lambda_1 \in \Lambda_1,\ \lambda_1 > \hat{\lambda}.$$

- For any $\lambda_1 \in \Lambda_1$ (in particular, such that $\lambda_1 \leqslant \hat{\lambda}$), the conditions (6.13) are satisfied for λ_2 and $\tilde{\lambda}_2$ large enough, so that we have

$$\mathrm{Gap}\left(\Sigma^+_{1,\lambda_1}\right) > 0.$$

Since $\Lambda_1 \cap (-\infty, \hat{\lambda}]$ is a finite set, we have finally proved the existence of a $\rho_2 > 0$ such that

$$\mathrm{Gap}\left(\Sigma^+_{1,\lambda_1}\right) > \rho_2, \quad \forall \lambda_1 \in \Lambda_1.$$

- For any $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, we have (see (4.6))

$$\begin{aligned}\lambda_2 &\leqslant |\max(d,1)(\lambda_1+\lambda_2)| = |\sigma(+,\lambda_1,\lambda_2) - \sigma^+ - \varepsilon_{\lambda_1+\lambda_2}| \\ &\leqslant |\sigma(+,\lambda_1,\lambda_2)| + C,\end{aligned}$$

with

$$C = |\sigma^+| + \sup_{\lambda \in [0,+\infty[} |\varepsilon_\lambda|.$$

Therefore, for any $r > 0$ the condition $|\sigma(+,\lambda_1,\lambda_2)| \leqslant r$ implies that $\lambda_2 \leqslant r + C$. It follows that the counting function associated with Σ^+_{1,λ_1} satisfies (recall (2.8))

$$N_{\Sigma^+_{1,\lambda_1}}(r) \leqslant N_{\Lambda_2}(r+C) \leqslant \kappa_2 (r+C)^{\theta_2}.$$

By (4.7), we know that $N_{\Sigma^+_{1,\lambda_1}}(r) = 0$ for $r < 1$, in such a way that the estimate above leads to

$$N_{\Sigma^+_{1,\lambda_1}}(r) \leqslant \kappa_2 \left(1+C\right)^{\theta_2} r^{\theta_2}, \quad \forall r > 0,$$

and the claim for Σ^+_{1,λ_1} is proved.

- Let $0 < s < r$. The same reasoning as before shows that if $s < |\sigma(+,\lambda_1,\lambda_2)| \leqslant r$ then we have

$$\frac{1}{\max(d,1)}(s-C) < \lambda_1 + \lambda_2 \leqslant \frac{1}{\max(d,1)}(r+C),$$

so that

$$\frac{1}{\max(d,1)}(s-C) - \lambda_1 < \lambda_2 \leqslant \frac{1}{\max(d,1)}(r+C) - \lambda_1.$$

Using (2.9), it follows that

$$\begin{aligned}|N_{\Sigma^+_{1,\lambda_1}}(r) - N_{\Sigma^+_{1,\lambda_1}}(s)| \leqslant & N_{\Lambda_2}\left(\frac{1}{\max(d,1)}(r+C) - \lambda_1\right) \\ & - N_{\Lambda_2}\left(\frac{1}{\max(d,1)}(s-C) - \lambda_1\right) \\ \leqslant & \kappa_2\left(1 + \left|\frac{1}{\max(d,1)}(r-s+2C)\right|^{\theta_2}\right) \\ \leqslant & \hat{\kappa}_2\left(1+|r-s|^{\theta_2}\right),\end{aligned}$$

with $\hat{\kappa}_2 = \kappa_2\left(1+\left(\frac{2C}{\max\{d,1\}}\right)^{\theta_2}\right)$. This leads to (6.11) for some $\overline{\kappa}_2 > 0$.

□

We will also need the following estimates.

Proposition 6.12. — *There exists $C > 0$ depending only on the operator A and there exists $C_1 > 0$ (resp. $C_2 > 0$) depending only on κ_1, θ_1 (resp. κ_2, θ_2) such that, for any $\tau > 0$, we have*

$$\sum_{z\in\Sigma_{1,\lambda_1}} e^{-\tau|z|} \leqslant \frac{C_2}{\tau^{\theta_2}} e^{-C\tau\lambda_1}, \quad \forall \lambda_1 \in \Lambda_1,$$

$$\sum_{z\in\Sigma_{2,\lambda_2}} e^{-\tau|z|} \leqslant \frac{C_1}{\tau^{\theta_1}} e^{-C\tau\lambda_2}, \quad \forall \lambda_2 \in \Lambda_2.$$

This result immediately follows from Proposition 6.8 and Remark 4.7 (which yields $\inf|\Sigma_{i,\lambda_i}| \geqslant \lambda_i/C$ for some $C > 0$ not depending on λ_i).

We can now move to the proof of the desired proposition.

Proof of Proposition 6.5. — The main ingredients for the proof are Theorem 6.9 and the estimates of Theorem 6.4. Without loss of generality, we shall assume that

$$T \leqslant 1. \tag{6.14}$$

- Once the values $\omega^k_{\gamma,1}$, $\omega^k_{\gamma,2}$ have been defined by (6.4) thanks to Theorem 6.4, it is clear that it is enough to consider only one family of problems, for instance (6.5), the other one being treated in a similar way.

  For each $\lambda_1 \in \Lambda_1$ fixed, (6.5) looks like a classical moment problem in $L^2(0,T)$ associated with the family of functions

$$\left\{t \mapsto e^{-t\overline{\sigma}}, \quad \text{s.t.} \quad \sigma \in \Sigma_{1,\lambda_1}\right\} \cup \left\{t \mapsto te^{-t\overline{\sigma}}, \quad \text{s.t.} \quad \sigma \in \widehat{\Sigma}_{1,\lambda_1}\right\}.$$

However, there are two reasons why this is not really a standard moment problem. The first one comes from the fact that Σ_{1,λ_1} may not satisfy a uniform gap property and may present some condensation phenomenon. The second one comes from the fact that the map σ is not injective and therefore we need to ensure that if $\gamma \neq \tilde{\gamma}$ satisfy $\sigma(\gamma) = \sigma(\tilde{\gamma})$, then the left-hand side in the corresponding two equations in (6.5) are exactly the same. We refer again to Remark 4.8 for concrete examples where these problems occur.

In order to solve those two issues simultaneously, we will use the block moment approach, as described in Theorem 6.9.

- To this end, we first use Proposition 6.11, that shows that there exist $\alpha, \rho_2 > 0$ and $\tilde{\kappa}_2 > 0$ such that for every $\lambda_1 \in \Lambda_1$, we have

$$\Sigma_{1,\lambda_1} \in \mathcal{L}_w(\alpha, \tilde{\kappa}_2, \theta_2, \rho_2, 2).$$

It is very important to notice that the parameters of this class do not depend on λ_1.

From the definition of the class $\mathcal{L}_w$ it is then clear that we also have

$$\Sigma_{1,\lambda_1} \in \mathcal{L}_w(\alpha, \tilde{\kappa}_2, \theta_2, \hat{\rho}_2, 2), \quad \hat{\rho}_2 = \min(\rho_2/2, \hat{\rho}),$$

where $\hat{\rho}$ is provided by Theorem 6.4.

- Let now $\mathcal{G}_{\lambda_1}$ be a grouping associated with the family Σ_{1,λ_1} as given by Proposition 6.7.

In particular, each $G \in \mathcal{G}_{\lambda_1}$ has at most two elements and its diameter is at most $\hat{\rho}_2$. Consequently, we are in one of the following two configurations:

  – Case 1: $\operatorname{card} G = 1$, say $G = \{\sigma_G\}$.
  We observe that, if $\gamma, \tilde{\gamma} \in \Gamma_{1,\lambda_1}$ satisfy $\sigma_G = \sigma(\gamma) = \sigma(\tilde{\gamma})$, then by using item (ii) of Theorem 6.4 and (6.4), we have

$$\omega^0_{\tilde{\gamma},1} = \omega^0_{\gamma,1},$$

  since, by definition of Γ_{1,λ_1} we have $\lambda_1(\gamma) = \lambda_1(\tilde{\gamma})$. Therefore, we can simply define

$$\omega^0_G = \omega^0_{\gamma,1},$$

  for any $\gamma \in \Gamma_{1,\lambda_1}$ such that $\sigma(\gamma) = \sigma_G$, since this value does not depend on the choice of γ.
  Similarly, we can define

$$\omega^1_G = \begin{cases} \omega^1_{\gamma,1} + T\omega^0_{\gamma,1}, & \text{if } \exists \gamma \in \widehat{\Gamma}_{1,\lambda_1} \text{ s.t. } \sigma(\gamma) = \sigma_G, \\ 0 & \text{otherwise.} \end{cases}$$

Applying now Theorem 6.9, we know that there exists a function $q_G \in L^2(0,T)$ such that, for every $\sigma \in \Sigma_{1,\lambda_1}$ we have

$$\begin{cases} \begin{pmatrix} \omega_G^0 \\ \omega_G^1 \end{pmatrix} = \displaystyle\int_0^T \begin{pmatrix} 1 \\ t \end{pmatrix} e^{-t\overline{\sigma_G}} q_G(t)\,dt, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \displaystyle\int_0^T \begin{pmatrix} 1 \\ t \end{pmatrix} e^{-t\overline{\sigma}} q_G(t)\,dt \qquad \text{if } \sigma \in \Sigma_{1,\lambda_1} \backslash G, \end{cases}$$

as well as the estimate

$$\|q_G\|_{L^2(0,T)} \leqslant C \exp\left(\frac{(\mathfrak{Re}\,\sigma_G)T}{2} + C|\sigma_G|^{\theta_2} + \frac{C}{T^{\frac{\theta_2}{1-\theta_2}}} \right) \left(|\omega_G^0| + |\omega_G^1| \right).$$

By construction of the values of ω_G^0 and ω_G^1 above, this system of equations implies that, for any $\gamma \in \Gamma_{1,\lambda_1}$ we have

$$\begin{cases} \omega_{\gamma,1}^0 = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}} q_G(t)\,dt & \text{if } \sigma(\gamma) \in G, \\ \omega_{\gamma,1}^1 + T\omega_{\gamma,1}^0 = \displaystyle\int_0^T t e^{-t\overline{\sigma(\gamma)}} q_G(t)\,dt & \text{if } \sigma(\gamma) \in G \text{ and } \gamma \in \widehat{\Gamma}_{1,\lambda_1}, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \displaystyle\int_0^T \begin{pmatrix} 1 \\ t \end{pmatrix} e^{-t\overline{\sigma(\gamma)}} q_G(t)\,dt & \text{if } \sigma(\gamma) \notin G. \end{cases}$$

– Case 2: $\operatorname{card} G = 2$, say $G = \{\sigma_G, \hat{\sigma}_G\}$ with $\mathfrak{Re}\,\sigma_G \leqslant \mathfrak{Re}\,\hat{\sigma}_G$.
By the same reasoning as in the previous case, we can define

$$\omega_G^0 = \omega_{\gamma,1}^0,$$

where γ is any element in Γ_{1,λ_1}, such that $\sigma(\gamma) = \sigma_G$, and

$$\hat{\omega}_G^0 = \omega_{\tilde{\gamma},1}^0,$$

where $\tilde{\gamma}$ is any element in Γ_{1,λ_1} such that $\sigma(\tilde{\gamma}) = \hat{\sigma}_G$. Those values do not depend on the choices of γ and $\tilde{\gamma}$ respectively.
Observe now that such elements γ and $\tilde{\gamma}$ satisfy necessarily $s(\gamma) = -s(\tilde{\gamma})$. Indeed, if for instance we have $s(\gamma) = s(\tilde{\gamma}) = +$ for a given choice of γ and $\tilde{\gamma}$, then we deduce that σ_G and $\hat{\sigma}_G$ both belong to the family Σ_{1,λ_1}^+. Since $\sigma_G \neq \hat{\sigma}_G$, we deduce by (6.11), that $|\sigma_G - \hat{\sigma}_G| \geqslant \rho_2$ which is a contradiction with the fact that the diameter of G is less than or equal to $\rho_2/2$ by construction.
This remark implies that, for any choice of $\gamma = (s, \lambda_1, \lambda_2)$ satisfying $\sigma(\gamma) = \sigma_G$, we have $\gamma \notin \widehat{\Gamma}_{1,\lambda_1}$. Indeed, if it were not the case, we know that $\hat{\gamma} = (-s, \lambda_1, \lambda_2)$ would also satisfy $\sigma(\hat{\gamma}) = \sigma_G$ but with $s(\hat{\gamma}) = s(\tilde{\gamma})$, which is a contradiction.

All in all, this means that the functions $t \mapsto te^{-\sigma_G t}$ and $t \mapsto te^{-\hat{\sigma}_G t}$ will not appear in our moment problem.

Applying again Theorem 6.9 we obtain a function $q_G \in L^2(0,T)$ such that, for every $\sigma \in \Sigma_{1,\lambda_1}$ we have

$$\begin{cases} \omega_G^0 = \displaystyle\int_0^T e^{-t\overline{\sigma_G}} q_G(t)\,dt, \\ \hat{\omega}_G^0 = \displaystyle\int_0^T e^{-t\overline{\hat{\sigma}_G}} q_G(t)\,dt, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \displaystyle\int_0^T \begin{pmatrix} 1 \\ t \end{pmatrix} e^{-t\overline{\sigma}} q_G(t)\,dt \quad \text{if } \sigma \in \Sigma_{1,\lambda_1} \backslash G, \end{cases}$$

as well as the estimate

$$\|q_G\|_{L^2(0,T)} \leqslant C \exp\left(\frac{(\mathfrak{Re}\,\sigma_G)T}{2} + C|\sigma_G|^{\theta_2} + \frac{C}{T^{\frac{\theta_2}{1-\theta_2}}} \right) \times \left(|\omega_G^0| + \left| \frac{\omega_G^0 - \hat{\omega}_G^0}{\sigma_G - \hat{\sigma}_G} \right| \right).$$

By definition of ω_G^0 and $\hat{\omega}_G^0$, we see that this system of equations implies that, for any $\gamma \in \Gamma_{1,\lambda_1}$ we have

$$\begin{cases} \omega_{\gamma,1}^0 = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}} q_G(t)\,dt & \text{if } \sigma(\gamma) \in G, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \displaystyle\int_0^T \begin{pmatrix} 1 \\ t \end{pmatrix} e^{-t\overline{\sigma(\gamma)}} q_G(t)\,dt & \text{if } \sigma(\gamma) \notin G. \end{cases}$$

- In the first case, from (6.4) and item (iii) of Theorem 6.4, we have for $k = 0, 1$,

  $$\left|\omega_G^k\right| \leqslant C\,|\sigma_G| \exp\left(-(\mathfrak{Re}\,\sigma_G)T + C\,|\sigma_G|^{\nu_{\max}}\right) \left\|y^0\right\|_{D(A^*)'}.$$

  In the second case, from (6.4), and using items (ii) and (iii) of Theorem 6.4, which is allowed since we have

  $$|\sigma(\gamma) - \sigma(\tilde{\gamma})| = |\sigma_G - \hat{\sigma}_G| \leqslant \hat{\rho}_2 \leqslant \hat{\rho},$$

  and $\lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) = \lambda_1$ by definition of Γ_{1,λ_1}, we have

$$\begin{aligned} \left| \frac{\omega_G^0 - \hat{\omega}_G^0}{\sigma_G - \hat{\sigma}_G} \right| &= \left| \frac{e^{-T\sigma_G} - e^{-T\hat{\sigma}_G}}{\sigma_G - \hat{\sigma}_G} \right| \left| \left\langle y^0, \Phi_{\gamma,1}^0 \right\rangle_{D(A^*)',D(A^*)} \right| \\ &\leqslant CT\,|\sigma_G| \exp\left(-(\mathfrak{Re}\,\sigma_G)T + C\,|\sigma_G|^{\nu_{\max}}\right) \left\|y^0\right\|_{D(A^*)'}. \end{aligned}$$

With these estimates we see that, in any of the two cases above, for $p_0 = \max\{\theta_2, \nu_{\max}\}$, the following estimate holds

$$\|q_G\|_{L^2(0,T)} \leqslant C \exp\left(-\frac{(\mathfrak{Re}\,\sigma_G)T}{2} + C|\sigma_G|^{p_0} + \frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|_{D(A^*)'}.$$

Note that we used (6.14) and (4.7).

By using (4.9) and Young's inequality, we end up with

$$\|q_G\|_{L^2(0,T)} \leqslant C \exp\left(-\frac{\tilde{C}T}{2}|\sigma_G| + \frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|_{D(A^*)'}. \tag{6.15}$$

- Let us now form the series

$$\tilde{u}_{1,\lambda_1}(t) = \sum_{G\in\mathcal{G}_{\lambda_1}} q_G(t).$$

Thanks to the estimate (6.15) and the estimate provided by Proposition 6.12, this series converges normally in $L^2(0,T)$ with

$$\begin{aligned}
\|\tilde{u}_{1,\lambda_1}\|_{L^2(0,T)} &\leqslant C \left(\sum_{G\in\mathcal{G}_{\lambda_1}} \exp\left(-\frac{\tilde{C}T}{2}|\sigma_G|\right)\right) \exp\left(\frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|_{D(A^*)'} \\
&\leqslant C \left(\sum_{z\in\Sigma_{1,\lambda_1}} \exp\left(-\frac{\tilde{C}T}{2}|z|\right)\right) \exp\left(\frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|_{D(A^*)'} \\
&\leqslant \frac{C}{T^{\theta_2}} \exp\left(-\hat{C}\lambda_1 T + \frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|_{D(A^*)'},
\end{aligned}$$

where C and $\hat{C}$ do not depend on λ_1 and T. This inequality and the lower bound (2.14) clearly lead to the claimed estimate (6.7).

Finally, $\tilde{u}_{1,\lambda_1}$ solves (6.5) for any $\lambda_1 \in \Lambda_1$ by construction.

A similar argument gives the existence of suitable functions $\tilde{u}_{2,\lambda_2}$ for any $\lambda_2 \in \Lambda_2$, using this time the values of $\omega^k_{\gamma,2}$. The proof is complete. □

Remark 6.13. — If we drop the assumption (2.9) on the operators A_1, A_2 then, thanks to Remark 6.10, we see that the same proof remains valid but in this case we have to consider p_0 satisfying $p_0 > \max\{\theta_1, \theta_2\}$ and $p_0 \geqslant \nu_{\max}$. This explains what was announced in Remark 6.2.

6.4. Proof of Theorem 6.4

We first observe that, by point (1) of Remark 4.10, it is enough to determine $\Phi^0_{\gamma,1}, \Phi^0_{\gamma,2}$ for $\gamma \in \Gamma \setminus \widehat{\Gamma}^+$ and $\Phi^1_{\gamma,1}, \Phi^1_{\gamma,2}$ for $\gamma \in \widehat{\Gamma}^-$. Indeed, the missing values can simply be defined, for $\gamma = (+,\lambda_1,\lambda_2) \in \widehat{\Gamma}^+$, by

$$\Phi^0_{(+,\lambda_1,\lambda_2),1} = \Phi^0_{(-,\lambda_1,\lambda_2),1}, \quad \Phi^0_{(+,\lambda_1,\lambda_2),2} = \Phi^0_{(-,\lambda_1,\lambda_2),2},$$

$$\Phi^1_{(+,\lambda_1,\lambda_2),1} = \Phi^1_{(-,\lambda_1,\lambda_2),1}, \quad \Phi^1_{(+,\lambda_1,\lambda_2),2} = \Phi^1_{(-,\lambda_1,\lambda_2),2}.$$

It is straightforward to see that the required properties will be satisfied.

Since $\widehat{\Gamma}^- \subset \Gamma \setminus \widehat{\Gamma}^+$, we are led to study carefully the structure of $\Gamma \setminus \widehat{\Gamma}^+$. More precisely, the idea of the proof is to show that $\Gamma \setminus \widehat{\Gamma}^+$ can be written as a disjoint infinite union of finite subsets such that we can easily solve by induction, in each of those sets, the required equations of point (i) of Theorem 6.4 with the desired conditions given in point (ii). Moreover, we need to ensure that elements belonging to two different such subsets are never concerned by the condition. This analysis will make use of elementary graph theory notions. We recall that we have associated to our set $\Gamma \setminus \widehat{\Gamma}^+$ a structure of graph in Definition 4.11.

The goal of this section is to establish the following result.

Theorem 6.14. — *There exists $\hat{\rho} > 0$ small enough such that the graph $(\Gamma \setminus \widehat{\Gamma}^+, \mathcal{E}_{\hat{\rho}})$ is a forest and such that, for any path $\gamma_0 \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_1 \overset{\hat{\rho}}{\Longleftrightarrow} \cdots \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_{n-1} \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_n$, we have*

$$n \leqslant 2\kappa_0 \min_{0 \leqslant i \leqslant n} |\sigma(\gamma_i)|, \tag{6.16}$$

($\kappa_0 > 0$ is introduced in (4.14)) and

$$\max_{0 \leqslant i \leqslant n} |\sigma(\gamma_i)| \leqslant 2 \min_{0 \leqslant i \leqslant n} |\sigma(\gamma_i)|. \tag{6.17}$$

Theorem 6.4 will then be a consequence of this result.

Proof of Theorem 6.4. — We construct $\Phi^0_{\gamma,1}$ and $\Phi^0_{\gamma,2}$ in each tree as follows.

- Pick any node γ of a tree to serve as a root (represented in gray in Figure 6.1) and define arbitrarily the corresponding values, for instance as follows
$$\Phi^0_{\gamma,1} = 0, \quad \Phi^0_{\gamma,2} = \Phi^0_{\gamma}.$$
- If the tree is reduced to one node, we are done. Otherwise, consider any node $\tilde{\gamma} \neq \gamma$ such that $\tilde{\gamma} \overset{\hat{\rho}}{\Longleftrightarrow} \gamma$ and solve accordingly the corresponding equations:
$$\begin{cases} \Phi^0_{\tilde{\gamma},1} = \Phi^0_{\gamma,1}, \quad \Phi^0_{\tilde{\gamma},2} = \Phi^0_{\tilde{\gamma}} - \Phi^0_{\tilde{\gamma},1}, \quad \text{if } \gamma \underset{\lambda_1}{\overset{\hat{\rho}}{\longleftrightarrow}} \tilde{\gamma}, \\ \Phi^0_{\tilde{\gamma},2} = \Phi^0_{\gamma,2}, \quad \Phi^0_{\tilde{\gamma},1} = \Phi^0_{\tilde{\gamma}} - \Phi^0_{\tilde{\gamma},2}, \quad \text{if } \gamma \underset{\lambda_2}{\overset{\hat{\rho}}{\longleftrightarrow}} \tilde{\gamma}. \end{cases}$$

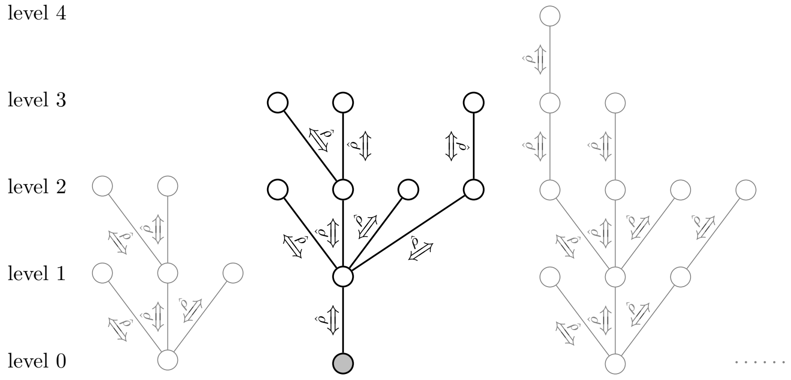

Figure 6.1. The forest structure of $(\Gamma \setminus \widehat{\Gamma}^+, \mathcal{E}_{\hat\rho})$*. In each tree, a root node is fixed, and the other nodes are organised by levels corresponding to their distance to the root. One of the trees is emphasized, as well as the chosen root.*

This way, we can determine all the values associated to the nodes at distance 1 of the root (first level of nodes in Figure 6.1).

Repeating this process for each level, we construct $\Phi^0_{\gamma,1}$ and $\Phi^0_{\gamma,2}$ in such a way that the properties (i) and (ii) of Theorem 6.4 are satisfied. Note that this construction is not ambiguous precisely because of the fact that our graph is a forest, so that there is a unique path in the graph that connects any two nodes.

Besides, if all the trees are either single nodes or reduced to a path of length one with an arrow of the same type $\xleftrightarrow[\lambda_1]{\hat\rho}$ (resp. $\xleftrightarrow[\lambda_2]{\hat\rho}$), then we can take $\Phi^0_{\gamma,1} = 0$ (resp. $\Phi^0_{\gamma,2} = 0$) for every γ.

- It remains to check the corresponding estimate stated in (iii). Below, we denote by C a positive number that may change from line to line but that does not depend on $\gamma, \tilde\gamma$. First of all, for the first picked node γ we have the common estimate (recall (4.9))

$$\max\left\{\left\|\Phi^0_{\gamma,1}\right\|_{D(A^*)}, \left\|\Phi^0_{\gamma,2}\right\|_{D(A^*)}\right\} \leqslant \left\|\Phi^0_{\gamma}\right\|_{D(A^*)}.$$

Then, for any node $\tilde{\gamma} \neq \gamma$ such that $\tilde{\gamma} \overset{\hat{\rho}}{\Longleftrightarrow} \gamma$, we have

$$\begin{aligned}\max\left\{ \left\|\Phi^0_{\tilde{\gamma},1}\right\|_{D(A^*)}, \left\|\Phi^0_{\tilde{\gamma},2}\right\|_{D(A^*)} \right\} \\ &\leqslant \left\|\Phi^0_{\tilde{\gamma}}\right\|_{D(A^*)} + \max\left\{\left\|\Phi^0_{\gamma,1}\right\|_{D(A^*)}, \left\|\Phi^0_{\gamma,2}\right\|_{D(A^*)}\right\} \\ &\leqslant \left\|\Phi^0_{\tilde{\gamma}}\right\|_{D(A^*)} + \left\|\Phi^0_{\gamma}\right\|_{D(A^*)}.\end{aligned}$$

Repeating this process, we see that, for any node $\tilde{\gamma}$ in the same tree as γ, we have (recall the estimate (4.15))

$$\begin{aligned}\max\left\{\left\|\Phi^0_{\tilde{\gamma},1}\right\|_{D(A^*)}, \left\|\Phi^0_{\tilde{\gamma},2}\right\|_{D(A^*)}\right\} &\leqslant \sum_{i=0}^{n} \left\|\Phi^0_{\gamma_i}\right\|_{D(A^*)} \\ &\leqslant C\sum_{i=0}^{n} e^{C|\sigma(\gamma_i)|^{\nu}_{\max}},\end{aligned}$$

where

$$\gamma_0 \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_1 \overset{\hat{\rho}}{\Longleftrightarrow} \cdots \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_{n-1} \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_n,$$

is the unique path from $\gamma_0 = \gamma$ to $\gamma_n = \tilde{\gamma}$. In particular, by using (6.17) and (6.16), we deduce

$$\max\left\{\left\|\Phi^0_{\tilde{\gamma},1}\right\|_{D(A^*)}, \left\|\Phi^0_{\tilde{\gamma},2}\right\|_{D(A^*)}\right\} \leqslant C\left|\sigma(\tilde{\gamma})\right| e^{C|\sigma(\tilde{\gamma})|^{\nu}_{\max}}.$$

The proof is similar for $\Phi^1_{\gamma,1}$ and $\Phi^1_{\gamma,2}$ for $\gamma \in \widehat{\Gamma}$.

□

To prove Theorem 6.14, we need the following basic lemma concerning the edges of our graph.

Lemma 6.15. — *There exist $\rho^* > 0$ small enough and $\sigma^* \geqslant 0$ large enough such that, for any $\gamma, \tilde{\gamma} \in \Gamma \setminus \widehat{\Gamma}^+$ with $\gamma \overset{\rho^*}{\Longleftrightarrow} \tilde{\gamma}$, we have:*

(i) if $|\sigma(\gamma)| \geqslant \sigma^*$ and $|\sigma(\tilde{\gamma})| \geqslant \sigma^*$, then

$$s(\gamma) = s(\tilde{\gamma}) \quad \Longrightarrow \quad \gamma = \tilde{\gamma}.$$

(ii) if $s(\gamma) = +$ and $s(\tilde{\gamma}) = -$, then

$$\begin{cases}\lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) \quad \Longrightarrow \quad \lambda_2(\gamma) < \lambda_2(\tilde{\gamma}), \\ \lambda_2(\gamma) = \lambda_2(\tilde{\gamma}) \quad \Longrightarrow \quad \lambda_1(\gamma) < \lambda_1(\tilde{\gamma}).\end{cases}$$

Proof of Lemma 6.15. —

(i) We consider the case $\gamma = (+, \lambda_1, \lambda_2)$ and $\tilde{\gamma} = (+, \lambda_1, \tilde{\lambda}_2)$, since the other cases are similar. From (4.6), we have

$$\sigma(\gamma) - \sigma(\tilde{\gamma}) = \max(d, 1)(\lambda_2 - \tilde{\lambda}_2) + \varepsilon_{\lambda_1+\lambda_2} - \varepsilon_{\lambda_1+\tilde{\lambda}_2},$$

and since by assumption we have $|\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho^*$, we end up with

$$\max(d, 1)|\lambda_2 - \tilde{\lambda}_2| \leqslant \rho^* + |\varepsilon_{\lambda_1+\lambda_2}| + \left|\varepsilon_{\lambda_1+\tilde{\lambda}_2}\right|.$$

Using (4.10) and (4.5), we see that we can choose σ^* large enough, depending on ρ^*, to ensure that

$$|\varepsilon_{\lambda_1+\lambda_2}| + \left|\varepsilon_{\lambda_1+\tilde{\lambda}_2}\right| \leqslant \rho^*.$$

Choosing first $2\rho^* < \max(d, 1) \operatorname{Gap}(\Lambda_2)$ and then σ^* as above, we obtain

$$|\lambda_2 - \tilde{\lambda}_2| < \operatorname{Gap}(\Lambda_2),$$

which, by definition of $\operatorname{Gap}(\Lambda_2)$, implies that $\lambda_2 = \tilde{\lambda}_2$.

(ii) We consider the case $\gamma = (+, \lambda_1, \lambda_2)$ and $\tilde{\gamma} = (-, \lambda_1, \tilde{\lambda}_2)$, since the other case is similar.

We write $\sqrt{\Delta_\lambda} = a_\lambda + ib_\lambda$ with $a_\lambda, b_\lambda \geqslant 0$. From the expression (4.3), we see that the condition $|\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho^*$ implies that

$$\left|(1 + d)(\lambda_2 - \tilde{\lambda}_2) + a_{\lambda_1+\lambda_2} + a_{\lambda_1+\tilde{\lambda}_2}\right| \leqslant 2\rho^*, \tag{6.18}$$

and

$$\left|b_{\lambda_1+\lambda_2} + b_{\lambda_1+\tilde{\lambda}_2}\right| \leqslant 2\rho^*. \tag{6.19}$$

First of all, observe that b_λ does not depend on λ for λ large enough (recall (4.2)). Taking then $\rho^* > 0$ small enough, we see that the condition (6.19) implies that $b_{\lambda_1+\lambda_2} = b_{\lambda_1+\tilde{\lambda}_2} = 0$. Since $s(\gamma) = +$ and since no node in the graph belongs to $\widehat{\Gamma}^+$, we have $\gamma \notin \widehat{\Gamma}$ and thus $\Delta_{\lambda_1+\lambda_2} \neq 0$. As $b_{\lambda_1+\lambda_2} = 0$ this means that we necessarily have

$$a_{\lambda_1+\lambda_2} \neq 0.$$

Introducing

$$\delta = \inf_{\substack{\lambda \in \Lambda_1+\Lambda_2 \\ a_\lambda \neq 0}} a_\lambda,$$

we have $\delta > 0$ (recall (4.2)), and the condition (6.18) yields

$$\lambda_2 - \tilde{\lambda}_2 \leqslant \frac{1}{1+d}\left(2\rho^* - \delta\right).$$

It follows that we can find a positive $\rho^* < \delta/2$ to obtain the claimed inequality $\lambda_2 < \tilde{\lambda}_2$.

□

Let us now finally prove the desired result.

Proof of Theorem 6.14. —

1) Let us first prove that the graph is a forest. By definition, we have to show that it has no $\hat{\rho}$-cycle. Assume by contradiction that there exists a $\hat{\rho}$-cycle:

$$\gamma_0 \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_1 \overset{\hat{\rho}}{\Longleftrightarrow} \cdots \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_{n-1} \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_0,$$

where, by definition, $n \geqslant 3$ and $\gamma_i \neq \gamma_j$ for $i \neq j$. Let $\gamma_{\min} \in \{\gamma_0, \dots, \gamma_{n-1}\}$ be such that

$$|\sigma(\gamma_{\min})| = \min_{i \in \{0,\dots,n-1\}} |\sigma(\gamma_i)|.$$

We distinguish two cases.

- Case 1 : $|\sigma(\gamma_{\min})| < \sigma^*$.

  We claim that in this case, for $\hat{\rho}$ well chosen, we have a 0-cycle, that is

$$\sigma(\gamma_i) = \sigma(\gamma_{\min}), \quad \forall i \in \{0, \dots, n-1\}. \tag{6.20}$$

  This will establish a contradiction with point 2 of Lemma 5.5. If (6.20) is not true, we can take the smallest index $i_0 \in \{0, \dots, n-2\}$ such that

$$\begin{cases} \sigma(\gamma_{i_0}) = \sigma(\gamma_{\min}), \\ \sigma(\gamma_{i_0+1}) \neq \sigma(\gamma_{\min}). \end{cases}$$

  In particular we have

$$|\sigma(\gamma_{i_0})| < \sigma^*.$$

  Moreover, by definition of $\overset{\hat{\rho}}{\Longleftrightarrow}$, we have

$$|\sigma(\gamma_{i_0+1}) - \sigma(\gamma_{i_0})| \leqslant \hat{\rho}, \tag{6.21}$$

  and, taking $\hat{\rho} \leqslant \sigma^*$, we deduce that

$$|\sigma(\gamma_{i_0+1})| < 2\sigma^*.$$

  Therefore, $|\sigma(\gamma_{i_0}) - \sigma(\gamma_{i_0+1})| \geqslant \delta^*$, where

$$\delta^* = \inf \Big\{ |\sigma(\gamma) - \sigma(\tilde{\gamma})|, \quad \text{s.t.} \quad \gamma, \tilde{\gamma} \in \Gamma \setminus \widehat{\Gamma}^+, \\ |\sigma(\gamma)| < \sigma^*, \quad |\sigma(\tilde{\gamma})| < 2\sigma^*, \quad \sigma(\gamma) \neq \sigma(\tilde{\gamma}) \Big\}.$$

Since this quantity is the minimum of a finite number of positive values, it satisfies $\delta^* > 0$. Therefore we can impose on the parameter $\hat{\rho}$ the additional condition $\hat{\rho} < \delta^*$ to obtain a contradiction with (6.21). This establishes (6.20).

- Case 2 : $|\sigma(\gamma_{\min})| \geqslant \sigma^*$.
  By item (i) of Lemma 6.15, we have

$$s(\gamma_{i+1}) = -s(\gamma_i), \quad \forall i \in \{0, \ldots, n-1\}, \tag{6.22}$$

  where we introduced $\gamma_n = \gamma_0$ for convenience. Note in particular that n is even.

  – Let us now show that the kinds of the arrows in the $\hat{\rho}$-cycle necessarily alternate, that is

$$\begin{cases} \gamma_i \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_{i+1} & \Longrightarrow \quad \gamma_{i+1} \xleftrightarrow[\lambda_1]{\hat{\rho}} \gamma_{i+2}, \\ \gamma_i \xleftrightarrow[\lambda_1]{\hat{\rho}} \gamma_{i+1} & \Longrightarrow \quad \gamma_{i+1} \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_{i+2}. \end{cases} \tag{6.23}$$

  We will show the first of these properties, the proof of the other being similar.
  Assume, by contradiction, that we have

$$\gamma_i \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_{i+1} \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_{i+2}.$$

  Since $\gamma_i \neq \gamma_{i+2}$, we deduce that $\gamma_i \xleftrightarrow[\lambda_2]{2\hat{\rho}} \gamma_{i+2}$ and, choosing $2\hat{\rho} \leqslant \rho^*$, we can use again item (i) of Lemma 6.15 to deduce that $s(\gamma_i) \neq s(\gamma_{i+2})$, which is a contradiction with (6.22).

  – Let us assume for instance that $s(\gamma_0) = +$ and $\gamma_0 \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_1$, the other cases being similar. Recalling (6.22), and since $\hat{\rho} \leqslant \rho^*$, by item (ii) of Lemma 6.15, we thus have

$$\lambda_1(\gamma_0) < \lambda_1(\gamma_1).$$

  By using (6.23) we see that the second arrow in the cycle is $\gamma_1 \xleftrightarrow[\lambda_1]{\hat{\rho}} \gamma_2$, which gives $\lambda_1(\gamma_1) = \lambda_1(\gamma_2)$. By induction, we eventually obtain

$$\lambda_1(\gamma_0) < \lambda_1(\gamma_1) = \lambda_1(\gamma_2) < \ldots < \lambda_1(\gamma_{n-1}) = \lambda_1(\gamma_n).$$

  This is impossible since, by definition of the cycle we have $\gamma_n = \gamma_0$.
  This shows that there is no $\hat{\rho}$-cycle in the case $|\sigma(\gamma_{\min})| \geqslant \sigma^*$ either.

2) Let us now prove the estimates (6.16) and (6.17). We consider a path:

$$\gamma_0 \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_1 \overset{\hat{\rho}}{\Longleftrightarrow} \cdots \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_n,$$

where $n \geqslant 1$ and $\gamma_i \neq \gamma_j$ for $i \neq j$ and we define

$$m = \min_{0 \leqslant i \leqslant n} |\sigma(\gamma_i)|.$$

Let $i_0 \in \{0, \dots, n\}$ such that $|\sigma(i_0)| = m$. By definition of $\overset{\hat{\rho}}{\Longleftrightarrow}$ we see that, for any $i \in \{0, \dots, n\}$, we have

$$|\sigma(\gamma_i) - \sigma(\gamma_{i_0})| \leqslant \hat{\rho}|i - i_0| \leqslant \hat{\rho}n,$$

and therefore

$$|\sigma(\gamma_i)| \leqslant m + \hat{\rho}n. \tag{6.24}$$

Since by definition all the elements in the path are distinct, we have

$$N(m + \hat{\rho}n) \geqslant n + 1,$$

where N is the counting function defined in (4.13). Using (4.14), we get

$$n \leqslant \kappa_0(m + \hat{\rho}n),$$

and, choosing $\hat{\rho} \leqslant 1/(2\kappa_0)$, we obtain the first estimate (6.16):

$$n \leqslant 2\kappa_0 m.$$

Plugging this inequality into (6.24), we obtain the second estimate (6.17).

□

6.5. Proof of the main result

In this section, we prove our main result (Theorem 2.8), which is an extension of Theorem 6.1 to more general control operators $B \in \mathcal{L}(U, D(A^*)')$ whose adjoint is of the following form on $\mathbb{C}^2 \otimes D(A_1) \otimes D(A_2)$:

$$B^* = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes L_1^* \otimes B_2^* \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B_1^* \otimes L_2^* \end{pmatrix},$$

for some operators $L_i^* \in \mathcal{L}(H_i)$ $(i = 1, 2)$ subject to the Lebeau-Robbiano type estimates (2.19).

This case is much harder to handle than the previous case of control operator B_{ref} (corresponding to $L_1 = L_2 = \mathrm{Id}$) because we lose some important orthogonal properties. For this operator B, the proof given in

Section 6.1 cannot be simply adapted because the expansions of the controls u_i into Fourier series no longer seem usable.

However, thanks to the estimate of the control cost previously obtained for the system $(-A, B_{\mathrm{ref}})$, we can use the so-called Lebeau-Robbiano method to deal with this more general case.

We recall that the purpose of the Lebeau-Robbiano method is precisely to allow the change of the control operator for null controllable systems under some conditions. More precisely,

THEOREM 6.16 (Lebeau-Robbiano method). — *Let $\mathcal{B}_{\mathrm{ref}} \in \mathcal{L}(U, D(\mathcal{A}^*)')$ and assume that the system $(-\mathcal{A}, \mathcal{B}_{\mathrm{ref}})$ is null controllable at any time $T > 0$, with control cost satisfying, for some $p_0 \in (0,1)$ and $C_1 > 0$,*

$$\mathrm{cost}_T(-\mathcal{A}, \mathcal{B}_{\mathrm{ref}}) \leqslant C_1 \exp\left(\frac{C_1}{T^{\frac{p_0}{1-p_0}}}\right), \quad \forall T > 0.$$

Assume in addition that there exists a family of operators $\{P_\mu\}_{\mu>0} \subset \mathcal{L}(D(\mathcal{A}^*))$ such that

$$\begin{cases} e^{-t\mathcal{A}^*}(\mathrm{Ran}\, P_\mu) \subset \mathrm{Ran}\, P_\mu, & \forall t \geqslant 0, \forall \mu > 0, \\ \sup\limits_{\mu>0} \|P_\mu\|_{\mathcal{L}(D(\mathcal{A}^*))} < +\infty, \end{cases} \tag{6.25}$$

and that satisfies the following two key properties for some $C_2, C_3 > 0$:

(i) ***A relative observability property*** *(of the operator $\mathcal{B}^*$ with respect to the reference operator $\mathcal{B}^*_{\mathrm{ref}}$): there exists $\eta \in [0,1)$ such that*

$$\|\mathcal{B}^*_{\mathrm{ref}} z\|_U \leqslant C_2 e^{C_2 \mu^\eta} \|\mathcal{B}^* z\|_U, \quad \forall \mu > 0, \quad \forall z \in \mathrm{Ran}\, P_\mu, \tag{6.26}$$

(ii) ***A dissipation property:***

$$\left\| e^{-t\mathcal{A}^*} z \right\|_{D(\mathcal{A}^*)} \leqslant C_2 e^{-C_3 \mu t} \|z\|_{D(\mathcal{A}^*)}, \qquad \forall t \geqslant 0, \quad \forall \mu > 0, \quad \forall z \in \mathrm{Ran}\,(\mathrm{Id} - P_\mu). \tag{6.27}$$

Then, the system $(-\mathcal{A}, \mathcal{B})$ is null controllable in time T for every $T > 0$, with control cost satisfying, for some $C > 0$,

$$\mathrm{cost}_T(-\mathcal{A}, \mathcal{B}) \leqslant C \exp\left(\frac{C}{T^{\frac{p}{1-p}}}\right), \quad \forall T > 0,$$

with $p = \max(p_0, \eta)$.

Theorem 6.16 is a simple adaptation of the abstract Lebeau-Robbiano method established in [29, Theorem 2.2] (see pp. 1469-1470). The main difference is that we consider here operators P_μ which are not necessarily orthogonal projections, which is an important feature when considering systems of PDEs since they are not self-adjoint in general. However, the last property in (6.25) is enough to make the same proof work. For this reason, we omit the details.

Let us now turn to the proof of the desired result:

Proof of Theorem 2.8. — We simply show that we are in the configuration of Theorem 6.16 with $\mathcal{A} = A$, $\mathcal{B}_{\mathrm{ref}} = B_{\mathrm{ref}}$, $\mathcal{B} = B$ and

$$P_\mu z = \sum_{\substack{\gamma\in\Gamma\backslash\widehat{\Gamma}^+\\ |\sigma(\gamma)|\leqslant\mu\\ 0\leqslant k\leqslant k_\mu}} \left\langle z, \Phi_\gamma^{k,*}\right\rangle_{D(A^*)} \tilde{\Phi}_\gamma^k,$$

where $\left\{\Phi_\gamma^{k,*}\right\}_{\substack{\gamma\in\Gamma\backslash\widehat{\Gamma}^+\\ 0\leqslant k\leqslant k_\gamma}}$ is the biorthogonal family, in $D(A^*)$, of the Riesz basis $\left\{\tilde{\Phi}_\gamma^k\right\}_{\substack{\gamma\in\Gamma\backslash\widehat{\Gamma}^+\\ 0\leqslant k\leqslant k_\gamma}}$, where $\tilde{\Phi}_\gamma^k = \Phi_\gamma^k / \left\|\Phi_\gamma^k\right\|_{D(A^*)}$ (see Proposition 4.9).

- The controllability of the reference system was established in Theorem 6.1 with $p_0 = \max\{\theta_1, \theta_2, \nu_1, \nu_2\}$ (and any p_0 satisfying $p_0 > \max\{\theta_1, \theta_2\}$ and $p_0 \geqslant \max\{\nu_1, \nu_2\}$ if we drop the assumption (2.9) on the operators A_1, A_2, see Remark 6.2).
- Note that P_μ is not an orthogonal projection since we only have a Riesz basis and not necessarily a Hilbert basis ($P_\mu^* \neq P_\mu$, unless $M^* = M$). However, this family of projections clearly satisfies the conditions in (6.25).
- The relative observability property (6.26) holds with $\eta = \max\{\eta_1, \eta_2\}$ thanks to our assumption (2.19) on the operators L_i. Note that this has to be checked only on linear combinations of eigenfunctions of A^* since the observation operators B^* and B^*_{ref} do not see the generalized eigenfunctions: $B^*\Phi_\gamma^1 = B^*_{\mathrm{ref}}\Phi_\gamma^1 = 0$ for $\gamma \in \widehat{\Gamma}^+$ (see (4.12)).
- Finally, the dissipation property (6.27) is easy to check because, from the Riesz basis property, the semigroup of $-A^*$ is explicitely given for every $t \geqslant 0$ and $z \in \mathrm{Ran}\,(\mathrm{Id} - P_\mu)$ by

$$\begin{aligned} e^{-tA^*} z = &\sum_{\substack{\gamma\in\Gamma\backslash\widehat{\Gamma}^+\\ |\sigma(\gamma)|>\mu}} e^{-\sigma(\gamma)t}\left\langle z, \Phi_\gamma^{0,*}\right\rangle_{D(A^*)} \tilde{\Phi}_\gamma^0 \\ &+ \sum_{\substack{\gamma\in\widehat{\Gamma}^+\\ |\sigma(\gamma)|>\mu}} e^{-\sigma(\gamma)t}\left\langle z, \Phi_\gamma^{1,*}\right\rangle_{D(A^*)} \left(\tilde{\Phi}_\gamma^1 - t(A^* - \sigma(\gamma))\tilde{\Phi}_\gamma^1\right). \end{aligned}$$

□

Appendix A. Properties of A and B

A.1. Tensor products

Let us briefly recall some basic facts about tensor products. We adopt the abstract point of view given for instance in [33, Chapters II.4 and VIII.10] and [32, Chapter XIII.9]. Concrete examples of interest are recalled in Remark A.1.

For $\varphi_1 \in H_1$ and $\varphi_2 \in H_2$, we denote by $\varphi_1 \otimes \varphi_2 : H_1 \times H_2 \longrightarrow \mathbb{C}$ the pure tensor product of φ_1 with φ_2, that is the continuous bilinear form defined, for every $(h_1, h_2) \in H_1 \times H_2$, by

$$(\varphi_1 \otimes \varphi_2)(h_1, h_2) = \langle \varphi_1, h_1 \rangle_{H_1} \langle \varphi_2, h_2 \rangle_{H_2}.$$

Then, the so-called algebraic tensor space is

$$H_1 \otimes H_2 = \operatorname{span}\left\{\varphi_1 \otimes \varphi_2, \quad \text{s.t.} \quad \varphi_1 \in H_1, \quad \varphi_2 \in H_2\right\}.$$

We will denote by $E_1 \otimes E_2 = \operatorname{span}\left\{\varphi_1 \otimes \varphi_2, \quad \varphi_1 \in E_1, \quad \varphi_2 \in E_2\right\}$ for any subspaces $E_1 \subset H_1$ and $E_2 \subset H_2$. On the vector space $H_1 \otimes H_2$ we introduce the following inner product, first defined on pure tensor products by

$$\langle \varphi_1 \otimes \varphi_2, \widetilde{\varphi}_1 \otimes \widetilde{\varphi}_2 \rangle_{H_1 \otimes H_2} = \langle \varphi_1, \widetilde{\varphi}_1 \rangle_{H_1} \langle \varphi_2, \widetilde{\varphi}_2 \rangle_{H_2},$$

and then extended by linearity to all of $H_1 \otimes H_2$. It can be checked that this inner product is well defined (i.e. that two different writing of an element in $H_1 \otimes H_2$ yields the same value in the computation of the inner product). This makes $H_1 \otimes H_2$ a pre-Hilbert space but this space is in general not complete (because H_1 and H_2 are infinite dimensional). This motivates the introduction of its completion with respect to this inner product, which will be denoted by

$$H_1 \widehat{\otimes} H_2.$$

Remark A.1. — In the main applications we have in mind (see the introduction or Section 3) we will mainly use two kinds of tensor products that can be easily explicited (up to an isomorphism). Details are given in [33, Theorem II.10].

- The first one concerns separation of variables in a cartesian product domain $\Omega = \Omega_1 \times \Omega_2$. Considering $H_1 = L^2(\Omega_1)$ and $H_2 = L^2(\Omega_2)$, we can simply identify the pure tensor product $\varphi_1 \otimes \varphi_2$ in $H_1 \otimes H_2$ with the function in $L^2(\Omega)$ defined by

$$(x_1, x_2) \in \Omega_1 \times \Omega_2 \mapsto \varphi_1(x_1)\varphi_2(x_2).$$

It is easily seen that this map can be extended by bilinearity to a bijective isometry that maps $L^2(\Omega_1) \widehat{\otimes} L^2(\Omega_2)$ onto $L^2(\Omega)$.

- The second example concerns the separation of components in a vector-valued function. In that case we set $H_1 = \mathbb{C}^n$ and $H_2 = L^2(\Omega, \mathbb{C})$ for a given integer $n \geqslant 1$ and for any $v \in \mathbb{C}^n$ and $\varphi \in L^2(\Omega, \mathbb{C})$, we identify the pure tensor product $v \otimes \varphi$ with the function in $L^2(\Omega, \mathbb{C}^n)$ defined by

$$x \in \Omega \mapsto \varphi(x) v \in \mathbb{C}^n.$$

Here also this construction leads to a natural bijective isometry that maps $\mathbb{C}^n \widehat{\otimes} L^2(\Omega)$ with $L^2(\Omega, \mathbb{C}^n)$.

Let now $\tilde{H}_1$, $\tilde{H}_2$ be two complex Hilbert spaces and $A_1 : H_1 \to \tilde{H}_1$, $A_2 : H_2 \to \tilde{H}_2$ be two bounded linear operators. There exists a unique bounded linear operator from $H_1 \widehat{\otimes} H_2$ into $\tilde{H}_1 \widehat{\otimes} \tilde{H}_2$, denoted by $A_1 \otimes A_2$, that satisfies

$$(A_1 \otimes A_2)(\phi_1 \otimes \phi_2) = (A_1 \phi_1) \otimes (A_2 \phi_2), \quad \forall \phi_1 \in H_1, \phi_2 \in H_2.$$

Moreover, we have

$$\|A_1 \otimes A_2\|_{\mathcal{L}(H_1 \widehat{\otimes} H_2, \tilde{H}_1 \widehat{\otimes} \tilde{H}_2)} = \|A_1\|_{\mathcal{L}(H_1, \tilde{H}_1)} \|A_2\|_{\mathcal{L}(H_2, \tilde{H}_2)}.$$

We refer for instance to [33, Proposition VIII.10, p.299].

A.2. Proofs of the properties of A

Let us start by showing that our operator A is a bounded perturbation of a self-adjoint operator with compact resolvent.

Proof of Proposition 2.5. — Let us denote by A_{00} the operator (2.11). First, it is clear that $D(A_{00})$ is dense in H and that the operator A_{00} is symmetric. In particular, it is closable. Moreover, its closure is self-adjoint if, and only if, both $\operatorname{Ran}(A_{00} + i)$ and $\operatorname{Ran}(A_{00} - i)$ are dense in H (see e.g. [33, Corollary VIII.2, p.257]). This clearly holds here since, in fact, any $f \in H$ can be written as

$$f = \sum_{\substack{\lambda_1 \in \Lambda_1 \\ \lambda_2 \in \Lambda_2}} \left(\frac{A_{00} \pm i}{d(\lambda_1 + \lambda_2) \pm i} \left((P^1_{\lambda_1, \lambda_2} f) \otimes \phi_{1, \lambda_1} \otimes \phi_{2, \lambda_2} \right) \right.$$
$$\left. + \frac{A_{00} \pm i}{\lambda_1 + \lambda_2 \pm i} \left((P^2_{\lambda_1, \lambda_2} f) \otimes \phi_{1, \lambda_1} \otimes \phi_{2, \lambda_2} \right) \right),$$

for some finite dimensional operators $P^1_{\lambda_1,\lambda_2}, P^2_{\lambda_1,\lambda_2} \in \mathcal{L}(H,\mathbb{C}^2)$ (recall that $\Lambda_1, \Lambda_2 \subset \mathbb{R}$). This also shows that the closure of $A_{00} \pm i$, and thus A_0, has a compact resolvent. □

Let us now prove the claimed spectral properties of A. We recall that a family in a Hilbert space is a Riesz basis if it is the image of an orthonormal basis through an invertible bounded linear operator. We refer for instance to [37, Section 1.8] for material on Riesz basis.

Proof of Proposition 4.9. —

1) Let $\hat{\sigma} \in \mathbb{C}$ be fixed and let $\Phi \in D(A^*)$. Writing

$$\Phi = \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} v_{\lambda_1,\lambda_2} \otimes \phi_{1,\lambda_1} \otimes \phi_{2,\lambda_2}, \tag{A.1}$$

for some vectors $v_{\lambda_1,\lambda_2} \in \mathbb{C}^2$, a computation shows that $\Phi \in \ker(\hat{\sigma} - A^*)$ if, and only if,

$$v_{\lambda_1,\lambda_2} \in \ker G_\lambda, \quad G_\lambda = \begin{pmatrix} d\lambda - m_{11} - \hat{\sigma} & -m_{21} \\ -m_{12} & \lambda - m_{22} - \hat{\sigma} \end{pmatrix},$$

for every $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, where we use our standard notation $\lambda = \lambda_1 + \lambda_2$. Therefore, $\hat{\sigma}$ is an eigenvalue of A^* if, and only if, $\ker G_\lambda \neq \{0\}$ for some $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. Since $m_{21} \neq 0$ by assumption, this is equivalent to

$$\hat{\sigma} = d\lambda - m_{11} - m_{21} r, \tag{A.2}$$

where r is a root of

$$m_{21} r^2 + ((1-d)\lambda + m_{11} - m_{22})\, r - m_{12} = 0.$$

In addition, the discriminant of this equation is exactly Δ_λ defined in (4.1) and its two complex roots are

$$\left\{ \frac{-((1-d)\lambda + m_{11} - m_{22}) - \sqrt{\Delta_\lambda}}{2m_{21}}, \frac{-((1-d)\lambda + m_{11} - m_{22}) + \sqrt{\Delta_\lambda}}{2m_{21}} \right\}. \tag{A.3}$$

It is then clear that, given an eigenvalue $\hat{\sigma}$, there is only a finite number of $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ that satisfy (A.2), so that the series in (A.1) is in fact over a finite set. Moreover, the first component of each $v_{\lambda_1,\lambda_2} \in \ker G_\lambda \setminus \{0\}$ is necessarily nonzero since $m_{21} \neq 0$. This shows that Φ can be written as a linear combination of the Φ^0_γ defined in (4.11) (a simple computation shows that the expression of the eigenvalues coincides with the one in (4.3), depending on the sign $s(\gamma)$).

The proof of the statement concerning the generalized eigenspaces simply relies on computations and will be omitted.

2) The proof of the estimates in (v) is a straightforward computation

$$\left\|\Phi_\gamma^0\right\|_{D(A^*)} = (1+|\sigma(\gamma)|)\left\|\Phi_\gamma^0\right\|_H$$

(since Φ_γ^0 is an eigenfunction)

$$= (1+|\sigma(\gamma)|)\sqrt{1+|r_\gamma|^2}\left\|\phi_{1,\lambda_1(\gamma)}\right\|_{H_1}\left\|\phi_{2,\lambda_2(\gamma)}\right\|_{H_2}$$

(recall (4.11))

$$\leqslant C_1(1+|\sigma(\gamma)|)\sqrt{1+|r_\gamma|^2}e^{2C_1\lambda(\gamma)^{\nu_{\max}}}$$

(by (2.15))

$$\leqslant C_2\,|\sigma(\gamma)|^2\,e^{C_2|\sigma(\gamma)|^{\nu_{\max}}}$$

(by (4.7) and (4.10)),

where $C_1, C_2 > 0$ do not depend on γ. The reasoning is similar for the estimate of the norm of Φ_γ^1.

3) Let us now show that the family $\left\{\Phi_\gamma^k\right\}_{\substack{\gamma\in\Gamma\\ 0\leqslant k\leqslant k_\gamma}}$ is complete in H. Let then $z \in H$ be such that

$$\left\langle z, \Phi_\gamma^k\right\rangle_H = 0, \quad \forall\gamma\in\Gamma, \quad \forall 0\leqslant k\leqslant k_\gamma, \tag{A.4}$$

and let us show that necessarily $z = 0$. Since H is by definition the completion of $\mathbb{C}^2 \otimes H_1 \otimes H_2$, it is equivalent to prove that (A.4) implies that

$$\langle z, h\rangle_H = 0, \quad \forall h \in \mathbb{C}^2 \otimes H_1 \otimes H_2,$$

and since $\{\phi_{i,\lambda_i}\}_{\lambda_i\in\Lambda_i}$ is complete in H_i $(i = 1, 2)$, this is equivalent by linearity to

$$\left\langle z, \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \otimes \phi_{1,\lambda_1} \otimes \phi_{2,\lambda_2}\right\rangle_H = 0, \qquad \forall \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2,\ \forall\lambda_1\in\Lambda_1,\ \forall\lambda_2\in\Lambda_2. \tag{A.5}$$

Let $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ be given. We define $\gamma^+ = (+, \lambda_1, \lambda_2)$ and $\gamma^- = (-, \lambda_1, \lambda_2)$.

- If $\gamma^+ \notin \widehat{\Gamma}$ (we automatically have $\gamma^- \notin \widehat{\Gamma}$), then the two vectors of $\mathbb{C}^2$
$$\begin{pmatrix} 1 \\ r_{\gamma^+} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ r_{\gamma^-} \end{pmatrix},$$
are linearly independent. Therefore the conditions $\left\langle z, \Phi^0_{\gamma^+} \right\rangle_H = \left\langle z, \Phi^0_{\gamma^-} \right\rangle_H = 0$ from (A.4) imply (A.5).
- If $\gamma^+ \in \widehat{\Gamma}$, then the two vectors of $\mathbb{C}^2$
$$\begin{pmatrix} 1 \\ r_{\gamma^+} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -\frac{1}{m_{21}} \end{pmatrix},$$
are linearly independent. Therefore the conditions $\left\langle z, \Phi^0_{\gamma^+} \right\rangle_H = \left\langle z, \Phi^1_{\gamma^+} \right\rangle_H = 0$ from (A.4) imply (A.5).

4) Let $\tilde{\Phi}^k_\gamma = \Phi^k_\gamma / \left\|\Phi^k_\gamma\right\|_H$. Proving that the family F is a Riesz basis of H is equivalent to show that (see e.g. [37, Theorem 1.9]) there exist $m, M > 0$ such that, for every scalars $(\alpha^k_\gamma)_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+ \\ 0\leqslant k\leqslant k_\gamma}} \subset \mathbb{C}$, cofinitely many of them being equal to 0, we have

$$m \sum_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+ \\ 0\leqslant k\leqslant k_\gamma}} \left|\alpha^k_\gamma\right|^2 \leqslant \left\| \sum_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+ \\ 0\leqslant k\leqslant k_\gamma}} \alpha^k_\gamma \tilde{\Phi}^k_\gamma \right\|^2_H \leqslant M \sum_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+ \\ 0\leqslant k\leqslant k_\gamma}} \left|\alpha^k_\gamma\right|^2. \tag{A.6}$$

For every $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ we set $\gamma^+ = (+, \lambda_1, \lambda_2)$ and $\gamma^- = (-, \lambda_1, \lambda_2)$, and
$$T_{\lambda_1,\lambda_2} = \begin{cases} \alpha^0_{\gamma^-} \tilde{\Phi}^0_{\gamma^-} + \alpha^0_{\gamma^+} \tilde{\Phi}^0_{\gamma^+}, & \text{if } \gamma^+ \notin \widehat{\Gamma}^+, \\ \alpha^0_{\gamma^-} \tilde{\Phi}^0_{\gamma^-} + \alpha^1_{\gamma^-} \tilde{\Phi}^1_{\gamma^-}, & \text{if } \gamma^+ \in \widehat{\Gamma}^+. \end{cases}$$
The sum we have to estimate simply reads $\displaystyle\sum_{\lambda_1\in\Lambda_1} \sum_{\lambda_2\in\Lambda_2} T_{\lambda_1,\lambda_2}$. Since it is easily seen from the definitions of the (generalized) eigenfunctions that all those terms are pairwise orthogonal, we are finally led to find two numbers $m, M > 0$, independent of λ_1, λ_2 and of the coefficients $\alpha^{\bullet}_{\bullet}$ such that (A.6) hold for each term T_{λ_1,λ_2}.

Let us consider such a pair $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$.

- Case 1 : $\lambda \in \widehat{\Lambda}$.
We have $T_{\lambda_1,\lambda_2} = \alpha^0_{\gamma^-} \tilde{\Phi}^0_{\gamma^-} + \alpha^1_{\gamma^-} \tilde{\Phi}^1_{\gamma^-}$. The normalization condition immediately gives
$$\|T_{\lambda_1,\lambda_2}\|^2_H \leqslant 2(|\alpha^0_{\gamma^-}|^2 + |\alpha^1_{\gamma^-}|^2).$$

For the lower bound, we use the estimate (proved at the end of this section)

$$\left|\alpha\frac{1}{\sqrt{1+|a|^2}}\begin{pmatrix}1\\a\end{pmatrix}+\beta\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \geqslant \frac{1}{\sqrt{2}}(|\alpha|^2+|\beta|^2)\frac{1}{1+|a|^2},\quad \forall a,\alpha,\beta\in\mathbb{C}, \tag{A.7}$$

to deduce

$$\|T_{\lambda_1,\lambda_2}\|_H^2 \geqslant \frac{1}{\sqrt{2}}(|\alpha^0_{\gamma^-}|^2+|\alpha^1_{\gamma^-}|^2)\frac{1}{1+|r_{\gamma^+}|^2}.$$

If $\widehat{\Lambda}$ is finite, we immediately deduce a lower bound for the terms of this kind. Otherwise, by Remark 4.2, we necessarily are in the case $d=1$, $\Delta=0$, in which case it is easily seen from (A.3) that

$$r_{\gamma^+}=\frac{m_{22}-m_{11}}{2m_{21}},$$

which is obviously a bounded quantity. The claim is proved.

- Case 2 : $\lambda\notin\widehat{\Lambda}$.
  We know that $\gamma^+=(+,\lambda_1,\lambda_2)\notin\widehat{\Gamma}^+$ and therefore we have $T_{\lambda_1,\lambda_2}=\alpha^0_{\gamma^-}\tilde{\Phi}^0_{\gamma^-}+\alpha^0_{\gamma^+}\tilde{\Phi}^0_{\gamma^+}$.
  Here also the normalization condition gives the upper bound

$$\|T_{\lambda_1,\lambda_2}\|_H^2\leqslant 2(|\alpha^0_{\gamma^-}|^2+|\alpha^0_{\gamma^+}|^2).$$

For the lower bound, we use the estimate (proved at the end of this section)

$$\left|\alpha\frac{1}{\sqrt{1+|a|^2}}\begin{pmatrix}1\\a\end{pmatrix}+\beta\frac{1}{\sqrt{1+|b|^2}}\begin{pmatrix}1\\b\end{pmatrix}\right|^2 \geqslant \frac{1}{\sqrt{2}}(|\alpha|^2+|\beta|^2)\frac{|a-b|^2}{(1+|a|^2)(1+|b|^2)},\quad \forall a,b,\alpha,\beta\in\mathbb{C}, \tag{A.8}$$

to get

$$\|T_{\lambda_1,\lambda_2}\|_H^2 \geqslant \frac{1}{\sqrt{2}}(|\alpha^0_{\gamma^-}|^2+|\alpha^0_{\gamma^+}|^2)\frac{|r_{\gamma^+}-r_{\gamma^-}|^2}{(1+|r_{\gamma^+}|^2)(1+|r_{\gamma^-}|^2)}.$$

By the definition of r_γ in (4.11), and the expressions (4.3), (4.6), we have

$$|r_{\gamma^+}-r_{\gamma^-}|^2=\frac{|\Delta_\lambda|}{|m_{21}|^2}\neq 0.$$

– If $d = 1$, then $\Delta_\lambda = \Delta$ and we obtain from (A.3) that

$$r_\gamma = \frac{2s(\gamma)\sqrt{\Delta} + m_{11} - m_{22}}{-2m_{21}}, \quad \forall \gamma \in \Gamma,$$

and in particular the values of r_{γ^+} and r_{γ^-} only depend on M, which proves the claim.

– In the case $d \neq 1$, we write

$$|r_{\gamma^+} r_{\gamma^-}| = \frac{|(\max(d,1) - 1)\lambda + \sigma^+ + m_{11} + \varepsilon_\lambda|}{|m_{21}|} \times \frac{|(\min(d,1) - 1)\lambda + \sigma^- + m_{11} - \varepsilon_\lambda|}{|m_{21}|}.$$

It follows from (4.1) that

$$|r_{\gamma^+} - r_{\gamma^-}|^2 \underset{\lambda \to +\infty}{\sim} C_1(d-1)^2\lambda^2,$$

$$|r_{\gamma^+} r_{\gamma^-}|^2 \underset{\lambda \to +\infty}{\sim} C_2(d-1)^2\lambda^2,$$

for some $C_1, C_2 > 0$ that do not depend on γ^+, γ^-, and thus

$$|r_{\gamma^+}|^2 + |r_{\gamma^-}|^2 \underset{\lambda \to +\infty}{\sim} C_1(d-1)^2\lambda^2.$$

Therefore, the quantity

$$\frac{|r_{\gamma^+} - r_{\gamma^-}|^2}{(1 + |r_{\gamma^+}|^2)(1 + |r_{\gamma^-}|^2)},$$

has a positive limit when $\lambda \to +\infty$, which concludes the proof.

5) Since $A^*\Phi_\gamma^0 = \sigma(\gamma)\Phi_\gamma^0$ and $A^*\Phi_\gamma^1 = \sigma(\gamma)\Phi_\gamma^1 + \Phi_\gamma^0$ if $\gamma \in \widehat{\Gamma}$, it is not difficult to deduce from what precedes that the same family normalized in $D(A^*)$ is also a Riesz basis of $D(A^*)$.

6) We have seen in step 1) that $\sigma(\Gamma) \subset \sigma(A^*)$. Let us now prove the reverse inclusion.

Let then $\mu \in \mathbb{C} \backslash \sigma(\Gamma)$ and $f \in H$. Let $\left\{\Phi_\gamma^{k,*}\right\}_{\substack{\gamma \in \Gamma \backslash \widehat{\Gamma}^+ \\ 0 \leqslant k \leqslant k_\gamma}}$ be the biorthogonal family in H to the Riesz basis $\left\{\tilde{\Phi}_\gamma^k\right\}_{\substack{\gamma \in \Gamma \backslash \widehat{\Gamma}^+ \\ 0 \leqslant k \leqslant k_\gamma}}$ (see e.g. [37, Theorem 1.9] for its existence). We set

$$z = \sum_{\gamma \in \Gamma \backslash \widehat{\Gamma}^+} \frac{\left\langle f, \Phi_\gamma^{0,*} \right\rangle_H}{\mu - \sigma(\gamma)} \tilde{\Phi}_\gamma^0 + \sum_{\gamma \in \widehat{\Gamma}^-} \frac{\left\langle f, \Phi_\gamma^{1,*} \right\rangle_H}{\mu - \sigma(\gamma)} \left(\tilde{\Phi}_\gamma^1 - \frac{\left\|\Phi_\gamma^0\right\|_H}{\left\|\Phi_\gamma^1\right\|_H} \tilde{\Phi}_\gamma^0 \right). \tag{A.9}$$

Note that, $\operatorname{dist}(\mu, \sigma(\Gamma)) > 0$ as $\sigma(\Gamma)$ is obviously closed and $\mu \notin \sigma(\Gamma)$, and additionally for any $\gamma \in \widehat{\Gamma}^-$, we have

$$\frac{\left\|\Phi_\gamma^0\right\|_H}{\left\|\Phi_\gamma^1\right\|_H} = |m_{21}|\sqrt{1+|r_\gamma|^2} \leqslant C(1+|\sigma(\gamma)|).$$

Therefore, both sums in (A.9) are absolutely convergent in H, and using the closedness of A^*, we can check that $z \in D(A^*)$ with $(\mu - A^*)z = f$, which proves the claim.

7) It remains to prove the asymptotic property (4.14) of the counting function N. Note that it is enough to consider $r \geqslant 1$ since $N(r) = 0$ otherwise, still thanks to (4.7). First of all, we obviously have

$$N(r) \leqslant N_+(r) + N_-(r),$$

where

$$N_\pm(r) = \operatorname{card}\left\{\gamma \in \Gamma^\pm, \quad \text{s.t.} \quad |\sigma(\gamma)| \leqslant r\right\}.$$

Let us for instance estimate N_+. Let $\gamma \in \Gamma^+$. From the formula (4.6), we get

$$\begin{aligned} |\lambda(\gamma)| &\leqslant \frac{1}{\max(d,1)}\left(|\sigma(\gamma)| + \left|\sigma^+\right| + \left|\varepsilon_{\lambda(\gamma)}\right|\right) \\ &\leqslant \frac{|\sigma(\gamma)|}{\max(d,1)}\left(1 + \left|\sigma^+\right| + \left|\varepsilon_{\lambda(\gamma)}\right|\right), \end{aligned}$$

by using (4.7). It follows that the condition $|\sigma(\gamma)| \leqslant r$ implies

$$|\lambda(\gamma)| \leqslant Cr, \quad C = \frac{1}{\max(d,1)}\left(1 + \left|\sigma^+\right| + \sup_{\lambda \geqslant 0}|\varepsilon_\lambda|\right).$$

It follows that

$$N_+(r) \leqslant \operatorname{card}\left\{(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2, \quad \text{s.t.} \quad |\lambda_1 + \lambda_2| \leqslant Cr\right\},$$

and since $\lambda_1, \lambda_2 \geqslant 0$ this gives

$$N_+(r) \leqslant N_{\Lambda_1}(Cr) N_{\Lambda_2}(Cr),$$

with the same estimate for N_-. It then follows from the asymptotics (2.8) of N_{Λ_1} and N_{Λ_2} that

$$N(r) \leqslant \kappa_0 r^\theta, \quad \forall r \geqslant 1,$$

where

$$\kappa_0 = 2\kappa_1\kappa_2 C^\theta, \qquad \theta = \theta_1 + \theta_2.$$

Since $\theta \leqslant 1$ by assumption (2.10) and $r \geqslant 1$, this yields the desired asymptotic (4.14).

$\square$

It remains to prove the technical estimate we used during the proof.

Proof of the estimates (A.7) *and* (A.8). — The inequality (A.7) immediately follows from (A.8) by taking the limit $b \to +\infty$. Hence, we focus now on the proof of (A.8).

Let

$$x = \frac{1}{\sqrt{1+|a|^2}}\begin{pmatrix}1\\a\end{pmatrix}, \quad y = \frac{1}{\sqrt{1+|b|^2}}\begin{pmatrix}1\\b\end{pmatrix}.$$

We have

$$|\alpha x + \beta y|^2 = \left\langle G\begin{pmatrix}\alpha\\\beta\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\end{pmatrix}\right\rangle,$$

where G is the Gram matrix of x and y

$$G = \begin{pmatrix} 1 & \frac{1+a\bar{b}}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}} \\ \frac{1+\bar{a}b}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}} & 1 \end{pmatrix}.$$

Since all the entries in G have a modulus less than 1, we have

$$\left\langle G\begin{pmatrix}\alpha\\\beta\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\end{pmatrix}\right\rangle \geqslant \frac{|\det(G)|}{\sqrt{2}}\left\|\begin{pmatrix}\alpha\\\beta\end{pmatrix}\right\|_{\mathbb{C}^2}^2.$$

A straightforward computation shows that

$$\det(G) = \frac{|a-b|^2}{(1+|a|^2)(1+|b|^2)},$$

which concludes the proof. □

A.3. Construction of B

We conclude this appendix with the proof of the proposition defining our control operator B.

Proof of Proposition 2.7. — We will denote by B' the operator in the right-hand side of (2.17) with domain

$$D_0 = \mathbb{C}^2 \otimes D(A_1) \otimes D(A_2).$$

Note that D_0 is dense in $D(A_0)$ by the very definition of the domain of the closure (recall that A_0 is the closure of (2.11)) and $D(A^*) = D(A) = D(A_0)$, therefore D_0 is dense in $D(A^*)$.

Consequently, the claim is equivalent to show that there exists $C > 0$ such that

$$\|B'z\|_U \leqslant C\left(\|A^*z\|_H + \|z\|_H\right), \quad \forall z \in D_0.$$

Below, we denote by C a positive number that may change from line to line but that does not depend on z. Still by a density argument, it is equivalent to prove such an estimate for any $z \in D_0$ of the form

$$z = \sum_{\lambda_1\in\Lambda_1}\sum_{\lambda_2\in\Lambda_2} \begin{pmatrix}\alpha_{\lambda_1,\lambda_2}\\ \beta_{\lambda_1,\lambda_2}\end{pmatrix} \otimes \phi_{1,\lambda_1} \otimes \phi_{2,\lambda_2},$$

with $(\alpha_{\lambda_1,\lambda_2})_{\substack{\lambda_1\in\Lambda_1\\ \lambda_2\in\Lambda_2}}, (\beta_{\lambda_1,\lambda_2})_{\substack{\lambda_1\in\Lambda_1\\ \lambda_2\in\Lambda_2}} \subset \mathbb{C}$, cofinitely many of them being equal to 0. By definition of B', we have

$$B'z = \begin{pmatrix} L_1^*\left(\displaystyle\sum_{\lambda_1\in\Lambda_1}\left(\sum_{\lambda_2\in\Lambda_2}\alpha_{\lambda_1,\lambda_2}B_2^*\phi_{2,\lambda_2}\right)\phi_{1,\lambda_1}\right)\\ L_2^*\left(\displaystyle\sum_{\lambda_2\in\Lambda_2}\left(\sum_{\lambda_1\in\Lambda_1}\alpha_{\lambda_1,\lambda_2}B_1^*\phi_{1,\lambda_1}\right)\phi_{2,\lambda_2}\right)\end{pmatrix}.$$

Since $L_i^* \in \mathcal{L}(H_i)$, we have

$$\|B'z\|_U^2 \leqslant C\left(\left\|\sum_{\lambda_1\in\Lambda_1}\left(\sum_{\lambda_2\in\Lambda_2}\alpha_{\lambda_1,\lambda_2}B_2^*\phi_{2,\lambda_2}\right)\phi_{1,\lambda_1}\right\|_{H_1}^2 + \left\|\sum_{\lambda_2\in\Lambda_2}\left(\sum_{\lambda_1\in\Lambda_1}\alpha_{\lambda_1,\lambda_2}B_1^*\phi_{1,\lambda_1}\right)\phi_{2,\lambda_2}\right\|_{H_2}^2\right).$$

Since the family $(\phi_{i,\lambda_i})_{\lambda_i\in\Lambda_i}$ is orthogonal in H_i we have

$$\|B'z\|_U^2 = \sum_{\lambda_1\in\Lambda_1}\left|B_2^*\sum_{\lambda_2\in\Lambda_2}\alpha_{\lambda_1,\lambda_2}\phi_{2,\lambda_2}\right|^2\|\phi_{1,\lambda_1}\|_{H_1}^2 + \sum_{\lambda_2\in\Lambda_2}\left|B_1^*\sum_{\lambda_1\in\Lambda_1}\alpha_{\lambda_1,\lambda_2}\phi_{1,\lambda_1}\right|^2\|\phi_{2,\lambda_2}\|_{H_2}^2.$$

Using that $B_i^* \in L(D(A_i),\mathbb{C})$, we deduce that

$$\|B'z\|_U^2 \leqslant C\sum_{\lambda_1\in\Lambda_1}\left\|A_2\sum_{\lambda_2\in\Lambda_2}\alpha_{\lambda_1,\lambda_2}\phi_{2,\lambda_2}\right\|_{H_2}^2\|\phi_{1,\lambda_1}\|_{H_1}^2 + C\sum_{\lambda_2\in\Lambda_2}\left\|A_1\sum_{\lambda_1\in\Lambda_1}\alpha_{\lambda_1,\lambda_2}\phi_{1,\lambda_1}\right\|_{H_1}^2\|\phi_{2,\lambda_2}\|_{H_2}^2,$$

and still by orthogonality of the family of eigenfunctions we get

$$\begin{aligned}\|B'z\|_U^2 &\leqslant C \sum_{\lambda_1\in\Lambda_1}\sum_{\lambda_2\in\Lambda_2} |\alpha_{\lambda_1,\lambda_2}|^2 (\lambda_1^2+\lambda_2^2) \|\phi_{1,\lambda_1}\|_{H_1}^2 \|\phi_{2,\lambda_2}\|_{H_2}^2 \\ &\leqslant C \sum_{\lambda_1\in\Lambda_1}\sum_{\lambda_2\in\Lambda_2} |\alpha_{\lambda_1,\lambda_2}|^2 (\lambda_1+\lambda_2)^2 \|\phi_{1,\lambda_1}\|_{H_1}^2 \|\phi_{2,\lambda_2}\|_{H_2}^2 \\ &= C \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (A_1\otimes \mathrm{Id} + \mathrm{Id}\otimes A_2) z \right\|_H^2 .\end{aligned}$$

It follows that

$$\begin{aligned}\|B'z\|_U^2 \leqslant C &\left\| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1\otimes \mathrm{Id} + \mathrm{Id}\otimes A_2) z \right\|_H^2 \\ &= C \|A^*z + (M^*\otimes \mathrm{Id}\otimes \mathrm{Id})z\|_H^2 \\ &\leqslant C(\|A^*z\|_H + \|z\|_H)^2.\end{aligned}$$

The proof is complete.

□

Appendix B. Basic elements from graph theory

We recall here some very basic definitions and one result coming from graph theory. We refer for instance to [36] for the details.

Definition B.1. —

- *A (simple, undirected) graph is a pair of two sets $(\mathcal{N},\mathcal{E})$ with*

$$\mathcal{E} \subset \{\{\gamma,\tilde\gamma\}, \quad s.t. \quad \gamma,\tilde\gamma\in\mathcal{N}, \quad \gamma\neq\tilde\gamma\}.$$

  The elements of $\mathcal{N}$ are called the nodes and the elements of $\mathcal{E}$ are called the egdes.
- *Let $\gamma,\tilde\gamma\in\mathcal{N}$. A path from γ to $\tilde\gamma$ is a finite sequence of distinct edges of the form*

$$(\{\gamma_0,\gamma_1\},\dots,\{\gamma_{n-1},\gamma_n\}),$$

  where

$$\begin{gathered} n\geqslant 1, \\ \{\gamma_i,\gamma_{i+1}\}\in\mathcal{E}, \quad \forall i\in\{0,\dots,n-1\}, \\ \gamma_0=\gamma, \quad \gamma_n=\tilde\gamma, \\ \gamma_i\neq\gamma_j, \quad \forall i,j\in\{0,\dots,n\}, \quad i\neq j. \end{gathered}$$

  The integer n is called the length of the path.

A graph $(\mathcal{N},\mathcal{E})$ is connected if for every $\gamma,\tilde{\gamma}\in\mathcal{N}$ there exists a path from γ to $\tilde{\gamma}$.

- *A cycle is a finite sequence of edges of the form*

$$(\{\gamma_0,\gamma_1\},\{\gamma_1,\gamma_2\},\dots,\{\gamma_{n-2},\gamma_{n-1}\},\{\gamma_{n-1},\gamma_0\}),$$

where

$$\begin{gathered} n\geqslant 3,\\ \{\gamma_i,\gamma_{i+1}\}\in\mathcal{E},\quad \forall i\in\{0,\dots,n-2\},\\ \gamma_i\neq\gamma_j,\quad \forall i,j\in\{0,\dots,n-1\},\quad i\neq j. \end{gathered}$$

A graph with no cycles is called a forest. A connected graph with no cycles is called a tree.

The only result that we need is the following classical and simple one (see e.g. [36, Remark 1.2.7]):

Proposition B.2. — *The relation "$\gamma=\tilde{\gamma}$ or there exists a path from γ to $\tilde{\gamma}$" is an equivalence relation over $\mathcal{N}\times\mathcal{N}$.*

It follows that any graph can be partitioned into connected subgraphs. Indeed, for $\gamma\in\mathcal{N}$, let us denote by $\mathcal{N}_\gamma$ its equivalence class. Thus, we have the natural partition of $\mathcal{N}$:

$$\mathcal{N}=\bigcup_{\gamma\in\mathcal{N}}\mathcal{N}_\gamma,$$

and each subgraph $(\mathcal{N}_\gamma,\mathcal{E}_\gamma)$ is connected, where

$$\mathcal{E}_\gamma=\left\{\{\tilde{\gamma},\tilde{\tilde{\gamma}}\}\in\mathcal{E},\quad \text{s.t.}\quad \tilde{\gamma},\tilde{\tilde{\gamma}}\in\mathcal{N}_\gamma\right\}.$$

In particular, a forest is partitioned into trees (the union of two graphs $(\mathcal{N}_1,\mathcal{E}_1)$ and $(\mathcal{N}_2,\mathcal{E}_2)$ is by definition the graph $(\mathcal{N}_1\cup\mathcal{N}_2,\mathcal{E}_2\cup\mathcal{E}_2)$; the intersection of graphs is defined similarly).

BIBLIOGRAPHY

[1] F. Alabau-Boussouira & M. Léautaud, "Indirect controllability of locally coupled wave-type systems and applications", *J. Math. Pures Appl. (9)* **99** (2013), no. 5, p. 544-576.

[2] D. Allonsius & F. Boyer, "Boundary null-controllability of semi-discrete coupled parabolic systems in some multi-dimensional geometries", *Math. Control Relat. Fields* **10** (2020), no. 2, p. 217-256.

[3] D. Allonsius, F. Boyer & M. Morancey, "Spectral analysis of discrete elliptic operators and applications in control theory", *Numerische Mathematik* **140** (2018), no. 4, p. 857–911.

[4] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos & L. de Teresa, "The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials", *J. Math. Pures Appl. (9)* **96** (2011), no. 6, p. 555-590.

[5] ———, "Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences", *J. Funct. Anal.* **267** (2014), no. 7, p. 2077-2151.

[6] ———, "New phenomena for the null controllability of parabolic systems: minimal time and geometrical dependence", *J. Math. Anal. Appl.* **444** (2016), no. 2, p. 1071-1113.

[7] M. Badra & T. Takahashi, "On the Fattorini criterion for approximate controllability and stabilizability of parabolic systems", *ESAIM Control Optim. Calc. Var.* **20** (2014), no. 3, p. 924-956.

[8] A. Benabdallah, F. Boyer, M. González-Burgos & G. Olive, "Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null controllability in cylindrical domains", *SIAM J. Control Optim.* **52** (2014), no. 5, p. 2970-3001.

[9] A. Benabdallah, F. Boyer & M. Morancey, "A block moment method to handle spectral condensation phenomenon in parabolic control problems", *Annales Henri Lebesgue* **3** (2020) (en), p. 717-793.

[10] F. Boyer & M. Morancey, "Analysis of non scalar control problems for parabolic systems by the block moment method", *preprint* (2021), `https://hal.archives-ouvertes.fr/hal-02397706`.

[11] F. Boyer & G. Olive, "Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients", *Math. Control Relat. Fields* **4** (2014), no. 3, p. 263-287.

[12] F. Boyer, "Controllability of linear parabolic equations and systems", 2022, Lecture Notes, `https://hal.archives-ouvertes.fr/hal-02470625v3`.

[13] N. Carreño, E. Cerpa & A. Mercado, "Boundary controllability of a cascade system coupling fourth- and second-order parabolic equations", *Systems & Control Letters* **133** (2019), p. 104542.

[14] J.-M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs, vol. 136, American Mathematical Society, Providence, RI, 2007, xiv+426 pages.

[15] S. Dolecki, "Observability for the one-dimensional heat equation", *Studia Math.* **48** (1973), p. 291-305.

[16] M. Duprez, "Controllability of a 2×2 parabolic system by one force with space-dependent coupling term of order one", *ESAIM Control Optim. Calc. Var.* **23** (2017), no. 4, p. 1473-1498.

[17] J. V. Egorov, "Some problems in the theory of optimal control", *Ž. Vyčisl. Mat i Mat. Fiz.* **3** (1963), p. 887-904.

[18] K. Engel & R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000, With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt, xxii+586 pages.

[19] H. O. Fattorini, "Some remarks on complete controllability", *SIAM J. Control* **4** (1966), p. 686-694.

[20] ———, "Boundary control of temperature distributions in a parallelepipedon", *SIAM J. Control* **13** (1975), p. 1-13.

[21] H. O. Fattorini & D. L. Russell, "Exact controllability theorems for linear parabolic equations in one space dimension", *Arch. Rational Mech. Anal.* **43** (1971), p. 272-292.

[22] E. Fernández-Cara, M. González-Burgos & L. de Teresa, "Boundary controllability of parabolic coupled equations", *J. Funct. Anal.* **259** (2010), no. 7, p. 1720-1758.

[23] L. I. Gal'chuk, "Optimal control of systems described by parabolic equations", *SIAM J. Control* **7** (1969), p. 546-558.

[24] T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag New York, Inc., New York, 1966, xix+592 pages.

[25] G. Lebeau & L. Robbiano, "Contrôle exact de l'équation de la chaleur", *Comm. Partial Differential Equations* **20** (1995), no. 1-2, p. 335-356.

[26] G. Lebeau & E. Zuazua, "Null-controllability of a system of linear thermoelasticity", *Arch. Rational Mech. Anal.* **141** (1998), no. 4, p. 297-329.

[27] F. Luca & L. de Teresa, "Control of coupled parabolic systems and Diophantine approximations", *SeMA J.* **61** (2013), p. 1-17.

[28] L. Miller, "On the null-controllability of the heat equation in unbounded domains", *Bull. Sci. Math.* **129** (2005), no. 2, p. 175-185.

[29] ———, "A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups", *Discrete Contin. Dyn. Syst. Ser. B* **14** (2010), no. 4, p. 1465-1485.

[30] G. Olive, "Boundary approximate controllability of some linear parabolic systems", *Evol. Equ. Control Theory* **3** (2014), no. 1, p. 167-189.

[31] L. Ouaili, "Minimal time of null controllability of two parabolic equations", *Math. Control Relat. Fields* **10** (2020), no. 1, p. 89-112.

[32] M. Reed & B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, xv+396 pages.

[33] ———, *Methods of modern mathematical physics. I*, second ed., Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, Functional analysis, xv+400 pages.

[34] E. H. Samb, "Boundary null-controllability of two coupled parabolic equations: simultaneous condensation of eigenvalues and eigenfunctions", *ESAIM Control Optim. Calc. Var.* **27** (2021), no. suppl., p. Paper No. S29, 43.

[35] G. Tenenbaum & M. Tucsnak, "On the null-controllability of diffusion equations", *ESAIM Control Optim. Calc. Var.* **17** (2011), no. 4, p. 1088-1100.

[36] D. B. West, *Introduction to graph theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 2001, xx+588 pages.

[37] R. M. Young, *An introduction to nonharmonic Fourier series*, Pure and Applied Mathematics, vol. 93, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980, x+246 pages.

Franck BOYER
Institut de Mathématiques de Toulouse & Institut Universitaire de France, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France

E-mail address (F. Boyer):
franck.boyer@math.univ-toulouse.fr

Guillaume OLIVE
Faculty of Mathematics and Computer Science,
Jagiellonian University, ul. Łojasiewicza 6, 30-348
Kraków, Poland
E-mail address (G. Olive):
math.golive@gmail.com