Boundary null-controllability of some multi-dimensional linear parabolic systems by the moment method

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BOUNDARY NULL CONTROLLABILITY OF SOME MULTI-DIMENSIONAL LINEAR PARABOLIC SYSTEMS BY THE MOMENT METHOD

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Abstract. In this article we study the null controllability of some abstract linear parabolic systems in tensor product spaces. This special structure allows us to reduce our controllability problem to a particular set of equations that looks like a moment problem, but that does not fall into the previous existing results of the literature. We transform this non standard moment problem into an infinite family of more usual moment problems, yet coupled one from each other. This reformulation is done with enough care to ensure that the resulting set of equations can be solved, with suitable estimates, by using the recent "block moment method". This is based on a careful analysis of the spectral structure of the underlying operator.

We notably apply our abstract result to show how strong the influence of geometry can be: we provide an example of boundary controlled parabolic system on a rectangle domain which is null controllable in arbitrary small time if two perpendicular faces of the boundary are controlled, whereas it is never null controllable if the control acts on only one face.

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1. Introduction

This article is devoted to the study of the controllability properties of some abstract linear systems of parabolic type. The main difficulty when dealing with the controllability of systems, as opposed to equations, is to try to control a system with less controls than equations. It is sometimes called "indirect controllability". In the last ten years, drastically different controllability behaviors from what happens for a single heat equation have been highlighted: non equivalence between distributed and boundary controllability, non equivalence between null and approximate controllability, existence of a nonzero minimal control time, etc. In the present article the emphasis will be especially laid on the role played by the geometry of the control zone. At the abstract level this will be encoded by a tensor product structure of the state space and of the associated evolution and control operators. Before detailing more precisely the current literature on this subject, we think it is appropriate to discuss a simple prototype of parabolic systems that possesses the aforementioned structure and to which our main result applies.

Keywords and phrases: Controllability, Parabolic systems, Geometric control conditions, Block moment method, Tensor products.

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1.1. **Motivating example**

A typical example that will be detailed below is provided by the following misleadingly simple-looking 2×2 system:

$$\begin{cases} \dfrac{\partial y_1}{\partial t} - d\Delta y_1 = 0, & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial y_2}{\partial t} - \ \Delta y_2 = y_1, & \text{in } (0,T)\times\Omega, \\ y_1 = 1_\gamma u, \quad y_2 = 0, & \text{on } (0,T)\times\partial\Omega, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $d > 0$ is a diffusion coefficient, $\Omega \subset \mathbb{R}^2$ is a rectangle and $\gamma \subset \partial\Omega$ (see Figure 1):

$$\Omega = (0, X_1) \times (0, X_2), \qquad \text{for some } X_1, X_2 > 0. \tag{1.2}$$

The main feature of this system, that makes the problem intricate, is that the control u only acts on the first component of the system and is localized on a subpart γ of the boundary.

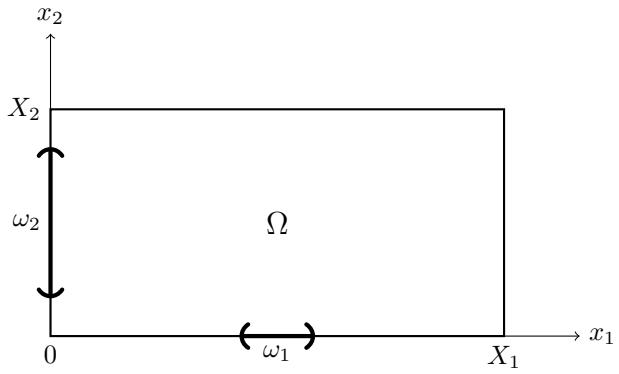

FIGURE 1. Domain Ω and control region $\gamma = \big(\omega_1 \times \{0\}\big) \bigcup \big(\{0\} \times \omega_2\big)$

We will establish in particular that such a system is null controllable in any arbitrary small time if the control region γ intersects non trivially two perpendicular faces of the boundary. This is radically different from the one-dimensional situation, i.e. when Ω is an interval, in which case it is known since [6] that the minimal null control time, that is denoted by $T_0(d)$, can be any element in $[0, +\infty]$, depending on the value of $d > 0$. We also want to mention that this phenomenon also appears in the two dimension case if the control domain γ is contained in one single face of the boundary of Ω (in this case it is easily seen that the minimal null control time of the 2D problem is at least equal to the one of the corresponding 1D problem).

One of the main achievements of the present paper is thus to show that the particular geometry of the control domain as in Figure 1 prevents us from the appearance of a nonzero minimal null control time.

In fact, we will provide our results in a quite abstract form that possibly encompasses many other similar systems. As an example, we will illustrate our analysis on more general parabolic systems than (1.1), in particular concerning the structure of the coupling zero-order terms, for which even in the case $d = 1$ our results are new.

1.2. **Influence of the geometry and the moment method in the literature**

The influence of the geometry on the boundary controllability properties of parabolic systems already indirectly appeared at a weaker level of strength in the seminal work [23]. It is shown in that article that the controllability of a one-dimensional parabolic system (yet slightly different from (1.1)) is especially depending on the non resonance of the eigenvalues of the associated operator, a condition that involves in particular the eigenvalues of the Laplacian, therefore subject to the length of the interval Ω, and thus to its geometry. A more striking influence of the geometry was then illustrated in [31], which is in part the motivation of the present work. Therein, the author investigated the boundary (approximate) controllability properties of the parabolic system studied in [23] in higher dimension in the particular geometry of a rectangle. It is notably shown there that the controllability properties of such a system strongly depend on the geometry of the control zone γ (and not only on the geometry of the domain Ω as in the aforementioned paper). Let us also add that the influence of the geometry of the control zone is not only restricted to boundary control problems. It was for instance shown in [13] that similar phenomena may occur also in distributed control problems.

The work [23] has then attracted again the attention of numerous researchers on the possible use of the so-called moment method to deal with controllability problems for parabolic systems (see e.g. [5–7,10,14,17,32,35]), a technique initially used in [22] for the boundary null controllability of a one-dimensional heat equation (see also the earlier works [18,24]). By pursuing the development of this method in view of the controllability of parabolic systems, it was notably shown in [6] that a nonzero minimal time of control may occur, as we have already mentioned before (see also the earlier result [28], with a different approach). Whereas this fact was well-known in the case of the one-dimensional heat equation after the pioneering work [16] on pointwise controllability, the papers [6, 28] showed that a similar complex situation occurs for coupled parabolic systems as well (even for a bounded control operator). The moment method is still developed today since in many situations it seems to be the only robust technique available to tackle controllability issues (in those cases where the other approaches like Carleman estimates or fictitious control, to name a few, fail). The usual range of application of the moment method needs that the spectrum of the underlying operator satisfies a spectral gap estimate (this means that two distinct eigenvalues cannot be arbitrarily close one from each other). In the references [6, 28] the analysis was extended to the case where spectral condensation occurs, which was the reason for the appearance of a minimal null control time. In the even more recent work [10], it was shown that it can be necessary not only to look at the condensation of the eigenvalues but also to the associated eigenfunctions to obtain an accurate description of the controllability properties of such systems: this gave birth to the so-called "block moment method" thanks to which a general formula for the minimal null control time was obtained for a very large class of scalar control problems. The present work crucially relies on this block moment method.

Concerning the study of the controllability of (systems of) equations of parabolic type in particular geometric situation such as rectangular domains, let us mention the pioneering work [21] on the boundary null controllability of the multi-dimensional heat equation in parallelepipedons or cylinders, the work [31], where the above geometric situation described for the system (1.1) has been first considered, and [2] for the introduction of the formalism of tensor product spaces in the controllability of parabolic systems (see also [9, 29]).

Finally, despite not applicable to our geometric situation let us also mention the work [1], where the first multi-dimensional result for the boundary controllability of parabolic systems was derived from the corresponding result on hyperbolic systems by the so-called transmutation method.

1.3. **Tensor product formalism**

In order to ease the understanding of the problem as well as the associated computations, we will express our evolution operator by making use of a tensor product formulation. We refer to Appendix A.1 for a summary of the main definitions and properties we will need on tensor products of Hilbert spaces and operators.

Let us introduce this on the example (1.1) in the geometry given by (1.2). First of all, we will see the Laplace operator as follows

$$-\Delta \cong \left(-\frac{\partial^2}{\partial x_1}\right) \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(-\frac{\partial^2}{\partial x_2}\right),$$

where we have made the identification

$$L^2(\Omega) \cong L^2(0, X_1) \,\widehat{\otimes}\, L^2(0, X_2).$$

This let us separate nicely what happens in each of the two space variables of the problem.

In order to take into account the fact that we are dealing with vector-valued unknowns we will proceed to another level of identification by writing

$$\begin{pmatrix} -d\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \cong \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes \left(\left(-\frac{\partial^2}{\partial x_1}\right) \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(-\frac{\partial^2}{\partial x_2}\right)\right),$$

where the state space is now

$$(L^2(\Omega))^2 \cong \mathbb{C}^2 \,\widehat{\otimes}\, L^2(0, X_1) \,\widehat{\otimes}\, L^2(0, X_2).$$

It has to be noted that the pure tensor products in such a space are obtained as a product of a function of x_1, times a function of x_2, times a vector in $\mathbb{C}^2$.

More details of the general framework we consider are given in 2.2.

1.4. **Outline of the paper**

The rest of the paper is organized as follows. In Section 2 we recall some basic facts on abstract control systems (Section 2.1) and we describe precisely the functional setting in which our work takes place (Section 2.2). We state our main results in this quite general abstract framework in Section 3.

Our proofs being based on moment method, it is necessary to accurately describe the spectrum of the operator under study, this is the main purpose of Section 4 in which we particularily highlight a graph structure on this spectrum which will be central in our analysis. With this spectral description at hand we manage to prove the approximate controllability of our system in Section 5 and its null controllability at any time horizon in Section 6. Finally, in Section 7, we come back to the actual coupled parabolic systems that motivated this work, in the spirit of the example (1.1) presented above.

We gather the proofs of few technical results as well as reminders on basic graph theory in appendix.

2. Framework

In this section, we will introduce the standing assumptions on the type of abstract control systems that we consider in this paper. First of all, let us recall some basic general facts about such systems.

2.1. **Background on abstract control systems**

All along this section, $-\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H$ is the generator of a C_0-semigroup $(e^{-t\mathcal{A}})_{t\geq 0}$ on H and $\mathcal{B} \in \mathcal{L}(U, D(\mathcal{A}^*)')$, where H, U are two complex Hilbert spaces. Here and in what follows, E' denotes the (anti-)dual of the complex space E, that is the complex (Banach) space of all continuous conjugate linear forms (see e.g. [26, Section I.2.2]). We will use the convention that an inner product of a complex Hilbert space is conjugate linear in its second argument.

Let us consider the evolution problem associated with the pair $(-\mathcal{A}, \mathcal{B})$, i.e.

$$\begin{cases} \dfrac{d}{dt}y(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \quad t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{2.1}$$

where $T > 0$, $y(t)$ is the state at time t, y^0 is the initial data and $u(t)$ is the so-called control at time t.

Since $\mathcal{B}^* \in \mathcal{L}(D(\mathcal{A}^*), U)$ we can define a notion of solution in the space $D(\mathcal{A}^*)'$ for the system (2.1).

Definition 2.1 (Solution in $D(\mathcal{A}^*)'$)**.** Let $T > 0$, $y^0 \in D(\mathcal{A}^*)'$ and $u \in L^2(0,T;U)$. We say that a function $y : [0,T] \to D(\mathcal{A}^*)'$ is a solution to (2.1) if $y \in C^0([0,T]; D(\mathcal{A}^*)')$ and

$$\langle y(\tau), z^\tau \rangle_{D(\mathcal{A}^*)', D(\mathcal{A}^*)} - \big\langle y^0, z(0) \big\rangle_{D(\mathcal{A}^*)', D(\mathcal{A}^*)} = \int_0^\tau \langle u(t), \mathcal{B}^* z(t) \rangle_U \, dt, \tag{2.2}$$

for every $\tau \in (0,T]$ and $z^\tau \in D(\mathcal{A}^*)$, where $z \in C^0([0,\tau]; D(\mathcal{A}^*))$ is the solution to the so-called adjoint system:

$$\begin{cases} -\dfrac{d}{dt}z(t) + \mathcal{A}^* z(t) = 0, \quad t \in (0,\tau), \\ z(\tau) = z^\tau, \end{cases} \tag{2.3}$$

i.e. $z(t) = e^{-(\tau - t)\mathcal{A}^*} z^\tau$.

Observe that the maps

$$z^\tau \mapsto \big\langle y^0, z(0) \big\rangle_{D(\mathcal{A}^*)', D(\mathcal{A}^*)}, \qquad z^\tau \mapsto \int_0^\tau \langle u(t), \mathcal{B}^* z(t) \rangle_U \, dt, \tag{2.4}$$

are continuous conjugate linear forms on $D(\mathcal{A}^*)$. Thus, we have a natural definition for the map $\tau \in [0,T] \longmapsto y(\tau) \in D(\mathcal{A}^*)'$ through the formula (2.2). It can be proved that this map is also continuous and that it depends continuously on y^0 and u on compact time intervals (see e.g. [15, Theorem 2.37]). This establishes the so-called well-posedness of the abstract control system $(-\mathcal{A}, \mathcal{B})$.

Now that we have a notion of continuous solution for the system (2.1) in the space $D(\mathcal{A}^*)'$, we can speak of its controllability properties in $D(\mathcal{A}^*)'$.

Definition 2.2 (Controllability)**.** We say that the system (2.1) is:

- null controllable in time T if, for every $y^0 \in D(\mathcal{A}^*)'$, there exists $u \in L^2(0,T;U)$ such that the corresponding solution $y \in C^0([0,T]; D(\mathcal{A}^*)')$ to system (2.1) satisfies
$$y(T) = 0.$$
- approximately controllable in time T if, for every $\varepsilon > 0$ and $y^0, y^T \in D(\mathcal{A}^*)'$, there exists $u \in L^2(0,T;U)$ such that the corresponding solution $y \in C^0([0,T]; D(\mathcal{A}^*)')$ to system (2.1) satisfies
$$\big\| y(T) - y^T \big\|_{D(\mathcal{A}^*)'} \leq \varepsilon.$$

We recall that null controllability implies approximate controllability for analytic semigroups.

When a system is controllable, it is also of interest to measure how much it costs to control it:

Definition 2.3 (Control cost)**.** Assume that $(-\mathcal{A},\mathcal{B})$ is null controllable in time T for some $T>0$. We call control cost the quantity $\mathrm{cost}_T(-\mathcal{A},\mathcal{B})\geq 0$ defined by

$$\mathrm{cost}_T(-\mathcal{A},\mathcal{B})=\sup_{\|y^0\|_{D(\mathcal{A}^*)'}=1}\left(\min_{u\in E_T(y^0)}\|u\|_{L^2(0,T;U)}\right),$$

where $E_T(y^0)$ is the non empty closed convex subset made of the associated null controls, defined by $E_T(y^0)=\left\{u\in L^2(0,T;U),\quad \text{s.t.}\quad y(T)=0\right\}$.

Let us conclude this section with a final remark. As we have seen, since $\mathcal{B}^*\in\mathcal{L}(D(\mathcal{A}^*),U)$, we can always define a notion of solution in the space $D(\mathcal{A}^*)'$ for the system (2.1). However, in practice it often appears that there is a "better" space V' where this system can be considered (see for instance Section 7 below). This motivates the introduction of the following concept.

Definition 2.4 (Admissible subspace)**.** For a Banach space V (equipped with its own norm $\|\cdot\|_V$) such that

$$D(\mathcal{A}^*)\subset V\subset H,$$

with dense and continuous embeddings, we say that V' is an admissible subspace for the system $(-\mathcal{A},\mathcal{B})$ if we have the following two additional properties:

(i) V is invariant through the adjoint semigroup:

$$e^{-t\mathcal{A}^*}V\subset V,\quad \forall t\geq 0.$$

(ii) The following regularity property holds:

$$\exists\tau>0,\exists C>0,\quad \int_0^\tau\|\mathcal{B}^*z(t)\|_U^2\,dt\leq C\,\|z^\tau\|_V^2\,,\quad \forall z^\tau\in D(\mathcal{A}^*),$$

where $z\in C^0([0,\tau];D(\mathcal{A}^*))$ is the solution to adjoint system (2.3).

The point of view adapted in this definition puts the emphasis on the subspace V and not on the control operator $\mathcal{B}$, contrary to what is done in the current literature. Basic examples of admissible subspaces are $D(\mathcal{A}^*)'$ since $\mathcal{B}^*\in\mathcal{L}(D(\mathcal{A}^*),U)$ and H if $\mathcal{B}\in\mathcal{L}(U,H)$. The previous notions of solution, controllability etc. can then be extended by simply replacing $D(\mathcal{A}^*)$ by V. It is clear that null controllability in the space $D(\mathcal{A}^*)'$ implies null controllability in the space V', whereas approximate controllability in the spaces $D(\mathcal{A}^*)'$ and V' are equivalent for analytic semigroups.

2.2. **Standing assumptions on the systems considered in this paper**

Let us now describe the kind of abstract control problems that we will deal with in this paper. Of course, this framework will include the motivating example (1.1) given in the introduction as well as its generalisation described in Section 7.

Here and in what follows, $\mathrm{card}\,E$ denotes the cardinal of a set E. For any subset $S\subset\mathbb{C}$ we define the associated counting function

$$N_S(r)=\mathrm{card}\,\{\lambda\in S,\quad \text{s.t.}\quad |\lambda|<r\}\,,\quad \forall r\geq 0,\tag{2.5}$$

as well as the associated gap

$$\mathrm{Gap}(S)=\inf_{\substack{\lambda,\mu\in S\\ \lambda\neq\mu}}|\lambda-\mu|\,.$$

We will also use the notation

$$\mathbb{C}_+ = \{z \in \mathbb{C}, \quad \text{s.t.} \quad \mathfrak{Re}\, z \geq 0\}.$$

2.2.1. *The operator A*

Let H_1, H_2 be two complex Hilbert spaces.

- For $i = 1, 2$, let $A_i : D(A_i) \subset H_i \longrightarrow H_i$ be an unbounded linear operator satisfying the following properties:
  (i) A_i is a self-adjoint positive operator with compact resolvent. We denote its spectrum by Λ_i and we observe that we have
  $$\Lambda_i \subset \mathbb{R}_+^*. \tag{2.6}$$
  We assume that each eigenvalue is (geometrically) simple.
  (ii) Λ_i satisfies the following gap condition:
  $$\mathrm{Gap}(\Lambda_i) > 0. \tag{2.7}$$
  (iii) Λ_i satisfies the following asymptotic behavior: there exist $\theta_i \in (0, 1)$ and $\kappa_i > 0$ such that
  $$N_{\Lambda_i}(r) \leq \kappa_i r^{\theta_i}, \quad \forall r > 0. \tag{2.8}$$
- We will in addition always assume that $\theta = \theta_1 + \theta_2$ satisfies
  $$\theta \leq 1. \tag{2.9}$$
  Note that this assumption connects both operators, contrary to the previous ones that only concerned the operators A_1 and A_2 separately.
- Following the ideas briefly discussed in Section 1.3, we finally introduce the state space we will work with all along this work
  $$H = \mathbb{C}^2 \,\widehat{\otimes}\, H_1 \,\widehat{\otimes}\, H_2,$$
  where $\widehat{\otimes}$ stands for the tensor product whose main properties are recalled in Appendix A.1.

We can now introduce an operator from which the generator of our control system will be built on. For any $d > 0$, we consider the unbounded operator

$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2), \quad \text{with domain } \mathbb{C}^2 \otimes D(A_1) \otimes D(A_2). \tag{2.10}$$

The definition of the tensor product of linear operators is also recalled in Appendix A.1. This operator has the following important properties:

Proposition 2.5. *The operator* (2.10) *is closable and its closure, denoted by A_0, is a self-adjoint operator with compact resolvent.*

The proof of these properties is given in Appendix A.2. Concerning material on closable operators we refer for instance to [26, Sections III.5.3 and III.5.5].

The generator of the control system that will be considered in this work is now a bounded perturbation of A_0 defined as follows.

Definition 2.6. Let $d > 0$ and $M \in \mathbb{R}^{2\times 2}$. The operator $A : D(A) \subset H \longrightarrow H$ is defined by

$$A = A_0 - M \otimes \mathrm{Id} \otimes \mathrm{Id}, \quad D(A) = D(A_0).$$

Equivalently,

$$A = \text{closure of} \quad \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) - M \otimes \mathrm{Id} \otimes \mathrm{Id}.$$

Obviously, $D(A) = D(A_0)$ is dense in H, and a computation shows that the adjoint A^* is simply given by

$$A^* = A_0 - M^* \otimes \mathrm{Id} \otimes \mathrm{Id}, \quad D(A^*) = D(A). \tag{2.11}$$

2.2.2. *The control operators B and B_{ref}*

Let us now introduce the class of control operators that will be considered in this paper.

- For $i = 1, 2$, let $B_i \in \mathcal{L}(\mathbb{C}, D(A_i)')$ be two scalar control operators. We assume that the pair $(-A_i, B_i)$ satisfies the so-called Fattorini-Hautus test, namely

$$\ker(\lambda_i - A_i^*) \cap \ker B_i^* = \{0\}, \quad \forall \lambda_i \in \Lambda_i, \tag{2.12}$$

  and, from now on, we save the notation ϕ_{i,λ_i} to denote the eigenfunction of A_i associated with the (simple) eigenvalue $\lambda_i \in \Lambda_i$ which is normalized such that

$$B_i^* \phi_{i,\lambda_i} = 1. \tag{2.13}$$

  This choice of normalization (that has no influence on the results) is maybe not the most conventional one but it simplifies numerous computations below. Since B_i^* is continuous from $D(A_i)$ into $\mathbb{C}$, we deduce the lower bound

$$\|\phi_{i,\lambda_i}\|_{H_i} = \frac{\|\phi_{i,\lambda_i}\|_{D(A_i)}}{\sqrt{1+\lambda_i^2}} \geq \frac{1}{\|B_i^*\|_{\mathcal{L}(D(A_i),\mathbb{C})}\sqrt{1+\lambda_i^2}}, \quad \forall \lambda_i \in \Lambda_i. \tag{2.14}$$

  We then assume that their norm has the following upper bound: there exist $\nu_i \in [0, 1)$ and $C > 0$ such that

$$\|\phi_{i,\lambda_i}\|_{H_i} \leq C e^{C\lambda_i^{\nu_i}}, \quad \forall \lambda_i \in \Lambda_i. \tag{2.15}$$

  We will frequently use the notation

$$\nu = \max\{\nu_1, \nu_2\}.$$

- Next, we consider a particular structure of system associated with these scalar operators. Our control space will simply be

$$U = H_1 \times H_2. \tag{2.16}$$

The non scalar control operator $B : U \to D(A^*)'$ that we will finally consider is formally given by

$$B(u_1, u_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (L_1 u_1) \otimes b_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes b_1 \otimes (L_2 u_2),$$

where L_1, L_2 are two bounded operators in H_1 and H_2, respectively and $b_1 \in D(A_1^*)', b_2 \in D(A_2^*)'$ are such that $b_i = B_i 1$. Observe that this operator only acts on the first component of the system.

Its precise definition is given by the following result.

Proposition 2.7. *For every $L_1 \in \mathcal{L}(H_1)$ and $L_2 \in \mathcal{L}(H_2)$, there exists a unique bounded linear operator $B \in \mathcal{L}(U, D(A^*)')$ such that, on $\mathbb{C}^2 \otimes D(A_1) \otimes D(A_2)$, we have*

$$B^* = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes L_1^* \otimes B_2^* \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B_1^* \otimes L_2^* \end{pmatrix}. \tag{2.17}$$

We denote by B_{ref} the operator obtained when $L_i = \mathrm{Id}$ for $i = 1, 2$.

The proof is postponed to Appendix A.3 to ease the presentation.

The operator B_{ref} will play the role of a reference operator and the hardest task of the work will be to establish controllability properties for our system with this B_{ref}.

2.3. **The Kalman condition**

With the operators defined above, any solution y to (2.1) can be written

$$y(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes y_1(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes y_2(t),$$

where y_1 and y_2 satisfy, at least at a formal level,

$$\begin{cases} \dfrac{\partial y_1}{\partial t} + d(A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) y_1 = m_{11} y_1 + m_{12} y_2 + (L_1 u_1) \otimes b_2 + b_1 \otimes (L_2 u_2) \\ \dfrac{\partial y_2}{\partial t} + \quad (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) y_2 = m_{21} y_1 + m_{22} y_2, \end{cases}$$

We immediately see that a necessary controllability condition for the system is the Kalman condition

$$m_{21} \neq 0. \tag{2.18}$$

Indeed, if $m_{21} = 0$, then we observe that y_2 satisfies the equation

$$\frac{\partial y_2}{\partial t} + (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) y_2 = m_{22} y_2,$$

that does not depend on the control nor on the first component of the system. In particular any trajectory reaching zero at some time T satisfies $y_2(t) = 0$ for any t, and therefore $y_2(0) = 0$. This proves that not all initial data can be driven to 0.

From now on, we shall always assume this condition (2.18).

3. Statements of the main results

Our first result concerns the approximate controllability, for which we point out that the conditions (2.7), (2.8), (2.9), (2.12) and (2.15) are not needed.

Theorem 3.1. *Let A and B be the operators introduced in Definition 2.6 and Proposition 2.7, respectively. Assume the Kalman condition* (2.18)*. Assume that for $i = 1, 2$ the adjoint of the operator L_i satisfies*

$$\ker L_i^* \cap \bigoplus_{\lambda_i \in \Lambda_i} \ker(\lambda_i - A_i) = \{0\}. \tag{3.1}$$

Then, the system $(-A, B)$ is approximately controllable in time T for every $T > 0$.

The other results of the present paper are concerned with the null controllability. The second main result is the following.

Theorem 3.2. *Let A and B_{ref} be the operators introduced in Definition 2.6 and Proposition 2.7, respectively. Assume the Kalman condition* (2.18)*. Let $p_0 \in (0,1)$ be such that $p_0 > \max\{\theta_1, \theta_2, \nu\}$.*

Then, the system $(-A, B_{\mathrm{ref}})$ is null controllable in time T for every $T > 0$, with control cost satisfying, for some $C > 0$,

$$\mathrm{cost}_T(-A, B_{\mathrm{ref}}) \leq C \exp\left(\frac{C}{T^{\frac{p_0}{1-p_0}}}\right), \quad \forall T > 0. \tag{3.2}$$

A combination of this result and the so-called Lebeau-Robbiano method will give the following important consequence:

Theorem 3.3. *Let A and B be the operators introduced in Definition 2.6 and Proposition 2.7, respectively. Assume that for $i = 1, 2$ the adjoint of the operator L_i satisfies the following Lebeau-Robbiano type inequality: there exists $\eta_i \in [0,1)$ and $C > 0$ such that, for every $\mu > 0$ we have*

$$\|z\|_{H_i} \leq C e^{C\mu^{\eta_i}} \|L_i^* z\|_{H_i}, \quad \forall z \in \bigoplus_{\substack{\lambda_i \in \Lambda_i \\ \lambda_i \leq \mu}} \ker(\lambda_i - A_i). \tag{3.3}$$

Then, the system $(-A, B)$ is null controllable in time T for every $T > 0$, with control cost satisfying, for some $C > 0$,

$$\mathrm{cost}_T(-A, B) \leq C \exp\left(\frac{C}{T^{\frac{p}{1-p}}}\right), \quad \forall T > 0,$$

with $p = \max\{p_0, \eta_1, \eta_2\} \in (0,1)$.

Finally, some "higher dimensional" results can also be obtained similarly after using a Fourier decomposition in the direction of the added dimension (for this reason, its proof will be omitted):

Theorem 3.4. *Let A be the operator introduced in Definition 2.6. Assume the Kalman condition* (2.18)*. Let A_3 be a self-adjoint operator with compact resolvent on a complex Hilbert space H_3. Set*

$$\tilde{H} = H \mathbin{\widehat{\otimes}} H_3, \quad \tilde{U} = U \times H_3.$$

Let

$$\tilde{A} = \textit{closure of} \quad A \otimes \mathrm{Id} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes A_3.$$

Let $\tilde{B} \in \mathcal{L}(U, D(A^*)')$ be the unique bounded linear operator such that, on $\mathbb{C}^2 \otimes D(A_1) \otimes D(A_2) \otimes D(A_3)$, we have

$$\tilde{B}^* = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes L_1^* \otimes B_2^* \otimes L_3^* \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B_1^* \otimes L_2^* \otimes \tilde{L}_3^* \end{pmatrix},$$

where $L_3^*, \tilde{L}_3^* \in \mathcal{L}(H_3)$ satisfy the Lebeau-Robbiano type inequality (3.3) *(with $i = 3$) for some $\eta_3, \tilde{\eta}_3 \in [0,1)$.*

Then, the system $(-\tilde{A}, \tilde{B})$ is null controllable in time T for every $T > 0$, with control cost satisfying, for some $C > 0$,

$$\mathrm{cost}_T(-\tilde{A}, \tilde{B}) \leq C \exp\left(\frac{C}{T^{\frac{\tilde{p}}{1-\tilde{p}}}}\right), \quad \forall T > 0,$$

with $\tilde{p} = \max\{p, \eta_3, \tilde{\eta}_3\} \in (0,1)$.

The existence of $\hat{B}$ can be established in the same way as the existence of B (Proposition 2.7).

4. Spectral analysis

4.1. Description of the spectrum of A^*

First of all let us mention that, by classical perturbation arguments, it is expected that A has a good spectral theory. For instance, it inherits from A_0 the following properties:

- A has a compact resolvent (see e.g. [19, Proposition III.1.12]).
- $-A$ generates an analytic semigroup on H (see e.g. [19, Corollary II.4.7 and Proposition III.1.12]).

Besides, the same statements hold for the adjoint A^* as well since it has the same structure (recall (2.11)).

Let us now describe precisely the structure of the spectrum of A^*.

Definition 4.1. For any $\lambda \in \mathbb{R}$, we define

$$\Delta_\lambda = ((1-d)\lambda + m_{11} - m_{22})^2 + 4m_{21}m_{12}, \tag{4.1}$$

and $\sqrt{\Delta_\lambda} \in \mathbb{R}_+ \cup i\mathbb{R}_+$ will always denote the principal square root of this number.

Note that

$$d \neq 1 \quad \Longrightarrow \quad \lim_{\lambda \to +\infty} \Delta_\lambda = +\infty. \tag{4.2}$$

On the other hand, when $d = 1$, we see that Δ_λ does not depend on λ, and we simply denote this quantity by Δ. It is nothing but the discriminant of the characteristic polynomial of M:

$$\Delta = (\operatorname{Tr} M)^2 - 4 \det M. \tag{4.3}$$

We shall also introduce the set

$$\widehat{\Lambda} = \{\lambda \in \mathbb{R}, \quad \text{s.t.} \quad \Delta_\lambda = 0\}, \tag{4.4}$$

that will play a particular role in the spectral analysis of our problem.

Remark 4.2. We notice that, when $d \neq 1$, the cardinal of $\widehat{\Lambda}$ is less or equal than 2. However, for $d = 1$, we have either $\widehat{\Lambda} = \emptyset$ or $\widehat{\Lambda} = \mathbb{R}$.

We can now introduce some sets that will be instrumental in our description of the spectrum of A^*.

Definition 4.3.

(1) We introduce the set

$$\Gamma = \{+,-\} \times \Lambda_1 \times \Lambda_2,$$

and for any $\gamma \in \Gamma$, we denote its components by $s(\gamma) \in \{+,-\}$, $\lambda_1(\gamma) \in \Lambda_1$ and $\lambda_2(\gamma) \in \Lambda_2$. We will also use the notation $\lambda(\gamma) = \lambda_1(\gamma) + \lambda_2(\gamma)$.

(2) We shall use the following particular subsets of Γ

$$\Gamma^\pm = \{\pm\} \times \Lambda_1 \times \Lambda_2, \quad \widehat{\Gamma} = \{\gamma \in \Gamma, \quad \text{s.t.} \quad \lambda(\gamma) \in \widehat{\Lambda}\}, \quad \widehat{\Gamma}^\pm = \widehat{\Gamma} \cap \Gamma^\pm.$$

(3) For any $\lambda_1 \in \Lambda_1$ and any $\lambda_2 \in \Lambda_2$, we introduce

$$\Gamma_{1,\lambda_1} = \{+,-\} \times \{\lambda_1\} \times \Lambda_2, \quad \Gamma_{2,\lambda_2} = \{+,-\} \times \Lambda_1 \times \{\lambda_2\},$$

and, for $i = 1, 2$,

$$\Gamma^\pm_{i,\lambda_i} = \Gamma_{i,\lambda_i} \cap \Gamma^\pm, \quad \widehat{\Gamma}_{i,\lambda_i} = \Gamma_{i,\lambda_i} \cap \widehat{\Gamma}.$$

Definition 4.4. We introduce $\sigma : \Gamma \to \mathbb{C}$ to be the function defined by

$$\sigma(\gamma) = \frac{(1+d)\lambda(\gamma) - \mathrm{Tr}(M) + s(\gamma)\sqrt{\Delta_{\lambda(\gamma)}}}{2}, \qquad \forall \gamma \in \Gamma. \tag{4.5}$$

Associated to this function we introduce, for any $\lambda_i \in \Lambda_i$, $i = 1, 2$, the subsets of $\mathbb{C}$ defined by

$$\Sigma_{i,\lambda_i} = \sigma(\Gamma_{i,\lambda_i}), \qquad \Sigma^{\pm}_{i,\lambda_i} = \sigma(\Gamma^{\pm}_{i,\lambda_i}), \qquad \widehat{\Sigma}_{i,\lambda_i} = \sigma(\widehat{\Gamma}_{i,\lambda_i}). \tag{4.6}$$

In some computations, we shall need an alternative expression for the function σ given in the following result.

Proposition 4.5. *There exist $\sigma^+, \sigma^- \in \mathbb{C}$ and a function $\lambda \in \mathbb{R} \mapsto \varepsilon_\lambda \in \mathbb{C}$ that satisfy*

$$\lim_{\lambda \to +\infty} \varepsilon_\lambda = 0, \tag{4.7}$$

and such that

$$\begin{cases} \sigma(\gamma) = \max(d,1)\lambda(\gamma) + \sigma^+ + \varepsilon_{\lambda(\gamma)}, & \text{if } \gamma \in \Gamma^+, \\ \sigma(\gamma) = \min(d,1)\lambda(\gamma) + \sigma^- - \varepsilon_{\lambda(\gamma)}, & \text{if } \gamma \in \Gamma^-. \end{cases} \tag{4.8}$$

Let us first give a straightforward corollary of this result.

Corollary 4.6. *There exist $m, C > 0$ such that, if one replaces M by $M - m\mathrm{Id}$ in the definition of the operator A, then we have*

$$\mathfrak{Re}\, \sigma(\gamma) \geq 1, \quad \forall \gamma \in \Gamma, \tag{4.9}$$

and

$$|\mathfrak{Im}\, \sigma(\gamma)| \leq C, \quad \forall \gamma \in \Gamma. \tag{4.10}$$

In particular, we have

$$|\sigma(\gamma)| \leq \sqrt{1+C^2}\, \big(\mathfrak{Re}\, \sigma(\gamma)\big), \quad \forall \gamma \in \Gamma. \tag{4.11}$$

Since changing M into $M - m\mathrm{Id}$ does not influence the controllability properties of the system (indeed it amounts to consider the system satisfied by the new unknown $t \mapsto e^{-mt}y(t)$), we will always assume in the sequel that (4.9) and (4.10) hold. Note that this manipulation does not change the values of Δ_λ.

Proof of Proposition 4.5.

- In the case $d = 1$, we recall that $\Delta_\lambda = \Delta$ does not depend on λ, and therefore we simply take $\varepsilon_\lambda = 0$ and $\sigma^\pm = \frac{\pm\sqrt{\Delta} - \mathrm{Tr}(M)}{2}$.
- If $d > 1$ we choose $\sigma^+ = -m_{11}$, $\sigma^- = -m_{22}$ and

  $$\varepsilon_\lambda = \frac{(1-d)\lambda + m_{11} - m_{22} + \sqrt{\Delta_\lambda}}{2},$$

  for which the required properties can be easily checked (recall (4.2)).
- For $d < 1$, we set $\sigma^+ = -m_{22}$ and $\sigma^- = -m_{11}$ and

  $$\varepsilon_\lambda = \frac{(d-1)\lambda - m_{11} + m_{22} + \sqrt{\Delta_\lambda}}{2},$$

  and we conclude by a straightforward computation.

□

Remark 4.7. Observe that:

(i) If $d \neq 1$, then (4.2) implies that $\sigma(\pm, \lambda_1, \lambda_2)$ are real numbers for λ_1, λ_2 large enough.

(ii) There exists $C > 0$ such that

$$\frac{1}{C} |\sigma(\gamma)| \leq \lambda(\gamma) \leq C |\sigma(\gamma)|, \quad \forall \gamma \in \Gamma. \tag{4.12}$$

Remark 4.8. We will in the next proposition see that the spectrum of A^* is precisely the range $\sigma(\Gamma)$. However, it is crucial to observe that σ is in general not injective, so that an eigenvalue may simultaneously be equal to $\sigma(\gamma)$ and $\sigma(\tilde{\gamma})$ for two different $\gamma \neq \tilde{\gamma}$. This is the source of most technical problems that we will encounter in what follows.

Let us now list all the properties of the operator A^* that result from the previous assumptions and that will be needed in this article. The proof is postponed to Appendix A to ease the presentation.

Proposition 4.9. *Assume that the Kalman condition* (2.18) *is satisfied. Then, the operator A^* has the following properties:*

(i) The spectrum of A^* is given by

$$\Sigma = \sigma(\Gamma).$$

We recall that it is only made of eigenvalues.

(ii) For any eigenvalue $\hat{\sigma} \in \Sigma$, the eigenspace $\ker(\hat{\sigma} - A^*)$ is spanned by

$$\Phi^0_\gamma = \begin{pmatrix} 1 \\ r_\gamma \end{pmatrix} \otimes \phi_{1,\lambda_1(\gamma)} \otimes \phi_{2,\lambda_2(\gamma)}, \quad r_\gamma = \frac{\sigma(\gamma) - d(\lambda_1(\gamma) + \lambda_2(\gamma)) + m_{11}}{-m_{21}}, \tag{4.13}$$

for each $\gamma \in \Gamma$ such that $\sigma(\gamma) = \hat{\sigma}$ (we recall that ϕ_{i,λ_i} is defined in Section 2.2.2). A basis of the eigenspace $\ker(\hat{\sigma} - A^*)$ is given by considering only the eigenfunctions Φ^0_γ for $\gamma \in \Gamma \setminus \widehat{\Gamma}^+$ (such that $\sigma(\gamma) = \hat{\sigma}$).

(iii) If $\gamma \notin \widehat{\Gamma}$, the Jordan chain associated with Φ^0_γ is trivial, and we set $k_\gamma = 0$.
If $\gamma \in \widehat{\Gamma}$, the function

$$\Phi^1_\gamma = \begin{pmatrix} 0 \\ -\frac{1}{m_{21}} \end{pmatrix} \otimes \phi_{1,\lambda_1(\gamma)} \otimes \phi_{2,\lambda_2(\gamma)}, \tag{4.14}$$

is a generalized eigenfunction of A^* satisfying

$$(A^* - \sigma(\gamma))\Phi^1_\gamma = \Phi^0_\gamma.$$

In that case, we set $k_\gamma = 1$.

(iv) The counting function N associated with the eigenvalues of the operator A^* defined by

$$N(r) = \text{ card } \left\{ \gamma \in \Gamma \setminus \widehat{\Gamma}^+, \quad s.t. \quad |\sigma(\gamma)| < r \right\}, \quad \forall r > 0, \tag{4.15}$$

satisfies the following asymptotics: there exists $\kappa_0 > 0$ such that

$$N(r) \leq \kappa_0 r, \quad \forall r > 0. \tag{4.16}$$

(v) There exists $C > 0$ such that

$$\left\| \Phi^k_\gamma \right\|_{D(A^*)} \leq C e^{C|\sigma(\gamma)|^\nu}, \quad \forall \gamma \in \Gamma, \quad \forall 0 \leq k \leq k_\gamma. \tag{4.17}$$

(vi) The normalized family $\left\{\Phi_\gamma^k / \left\|\Phi_\gamma^k\right\|_{D(A^*)}\right\}_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+\\ 0\le k\le k_\gamma}}$ *is a Riesz basis of* $D(A^*)$ *equipped with the graph norm.*

Remark 4.10. A few remarks are in order.

(1) Let $\gamma = (s, \lambda_1, \lambda_2)$ be an element of $\widehat{\Gamma}$. Then $\tilde{\gamma} = (-s, \lambda_1, \lambda_2)$ also belongs to $\widehat{\Gamma}$ and we have that

$$\sigma(\gamma) = \sigma(\tilde{\gamma}), \quad \Phi_\gamma^0 = \Phi_{\tilde{\gamma}}^0, \quad \Phi_\gamma^1 = \Phi_{\tilde{\gamma}}^1.$$

That is the reason why the Riesz basis introduced in point (vi) is indexed on $\Gamma \setminus \widehat{\Gamma}^+$; this prevents an element from appearing twice in the family (same for the last line of the statement in point (ii)). We could also have chosen to index on $\Gamma \setminus \widehat{\Gamma}^-$.

(2) The counting function N in this theorem takes into account the geometric multiplicities of the eigenvalues and in particular it is not equal to the counting function $N_{\sigma(\Gamma)}$ of the set $\sigma(\Gamma)$ as defined in (2.5).

(3) In fact, in most of this work, we only need that the family $\left\{\Phi_\gamma^k\right\}_{\substack{\gamma\in\Gamma\\ 0\le k\le k_\gamma}}$ is complete in $D(A^*)$. It is only for Theorems 3.3 and 3.4 that we really need that this family forms a basis.

(4) There is of course no uniqueness of the generalized eigenfunction. We have chosen here the ones that satisfy

$$B^*\Phi_\gamma^1 = 0, \quad \forall \gamma \in \widehat{\Gamma}, \tag{4.18}$$

as it can be easily seen from (4.14) and (2.17).

(5) The eigenfunctions and generalized eigenfunctions of A can also be computed in a similar way.

4.2. **Graph structures associated to the spectrum of** A^*

Let us now define two kind of relationships between elements in Γ and introduce an appropriate structure of graph (we recall the needed elementary graph theory notions in Appendix B). Such relations are motivated by the particular structure (2.17) of our control operator B, this link will be clearer during the proofs below. These relations depend on a small parameter ρ the choice of which will be very important for our analysis.

Definition 4.11. Let $\rho \ge 0$ be given.

- For $\gamma, \tilde{\gamma} \in \Gamma$ and $i \in \{1, 2\}$, we will write

$$\gamma \underset{\lambda_i}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma} \quad \text{if and only if} \quad \begin{cases} |\sigma(\gamma) - \sigma(\tilde{\gamma})| \le \rho, \\ \lambda_i(\gamma) = \lambda_i(\tilde{\gamma}). \end{cases}$$

- For $\gamma, \tilde{\gamma} \in \Gamma$, we will write

$$\gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma} \quad \text{if and only if} \quad \left(\gamma \underset{\lambda_1}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma} \quad \text{or} \quad \gamma \underset{\lambda_2}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma}\right).$$

If $\gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma}$ and $\tilde{\gamma} \overset{\rho}{\Longleftrightarrow} \tilde{\tilde{\gamma}}$ then we will write

$$\gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma} \overset{\rho}{\Longleftrightarrow} \tilde{\tilde{\gamma}}.$$

We say that two arrows $\overset{\rho}{\Longleftrightarrow}$ are of different types if one is of type $\underset{\lambda_1}{\overset{\rho}{\longleftrightarrow}}$ and the other one is of type $\underset{\lambda_2}{\overset{\rho}{\longleftrightarrow}}$.

- To the set $\Gamma \setminus \widehat{\Gamma}^+$, we associate a structure of graph whose edges are defined by

$$\mathcal{E}_\rho = \left\{\{\gamma, \tilde{\gamma}\}, \quad \text{s.t.} \quad \gamma, \tilde{\gamma} \in \Gamma \setminus \widehat{\Gamma}^+, \quad \gamma \ne \tilde{\gamma}, \quad \gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma}\right\}.$$

- By commodity, a sequence of edges $(\{\gamma_0,\gamma_1\},\dots,\{\gamma_{n-1},\gamma_n\})$ (whether it is a path or a cycle) will be denoted by
$$\gamma_0 \overset{\rho}{\Longleftrightarrow} \gamma_1 \overset{\rho}{\Longleftrightarrow} \cdots \overset{\rho}{\Longleftrightarrow} \gamma_{n-1} \overset{\rho}{\Longleftrightarrow} \gamma_n.$$
- A cycle of the graph $(\Gamma\setminus\widehat{\Gamma}^+,\mathcal{E}_\rho)$ will be called ρ-cycle to emphasize the dependence on the parameter ρ.

5. Approximate controllability

In this section we prove that our system $(-A,B)$ is approximately controllable in time T for any $T>0$ (Theorem 3.1), provided that the adjoint of the operator L_i satisfies (3.1). This is a weaker result than the null controllability (because the semigroup generated by A is analytic). However, their proofs rely on different general methods, namely the Fattorini-Hautus test for the approximate controllability and the moment method for the null controllability, and thus both deserve to be presented. Note as well that the condition (3.1) is weaker than (3.3). Last but not least, the following proof also motivates the introduction of graph theoretic arguments in a simpler context than the one we will need in Section 6.

We emphasize again that, all along this section, the conditions (2.7), (2.8), (2.9), (2.12) and (2.15) are not needed.

The Fattorini-Hautus test allows, under some reasonable assumptions, to completely characterize the approximate controllability of systems with analytic semigroup in terms of the spectral elements of the adjoint operator:

Theorem 5.1 (Fattorini-Hautus test)**.** *Assume that:*

(i) Each point of the spectrum $\sigma(\mathcal{A})$ is isolated and is a pole of finite order of the resolvent of $\mathcal{A}$.
(ii) The subspace of generalized eigenvectors of $\mathcal{A}$ is dense in H.
(iii) $-\mathcal{A}$ generates an analytic C_0-semigroup.

Then, $(-\mathcal{A},\mathcal{B})$ is approximately controllable in time T for every $T>0$, if and only if the Fattorini-Hautus test holds, i.e.
$$\ker(\sigma-\mathcal{A}^*)\cap\ker\mathcal{B}^*=\{0\},\quad\forall\sigma\in\mathbb{C}.$$

This powerful result was established in [20, Corollary 3.3] (see also [8, 31]).

The operator A under study in this paper, introduced in Definition 2.6, satisfies the assumptions (i), (ii) and (iii) of Theorem 5.1. It remains to prove the Fattorini-Hautus test associated with the control operator B introduced in Section 2.2.2.

Proposition 5.2. *If we assume the Kalman condition* (2.18) *and the unique continuation property* (3.1)*, then the pair $(-A,B)$ satisfies the Fattorini-Hautus test:*
$$\ker(\sigma-A^*)\cap\ker B^*=\{0\},\quad\forall\sigma\in\mathbb{C}.$$

The proof of this proposition relies on the following crucial properties.

Lemma 5.3. *(1) For every $\gamma_0,\gamma_1\in\Gamma\setminus\widehat{\Gamma}^+$, $\gamma_0\neq\gamma_1$, we cannot have*
$$\gamma_0 \overset{0}{\underset{\lambda_1}{\longleftrightarrow}} \gamma_1 \overset{0}{\underset{\lambda_2}{\longleftrightarrow}} \gamma_0.$$

(2) There is no 0-cycle $\gamma_0 \overset{0}{\Longleftrightarrow} \gamma_1 \overset{0}{\Longleftrightarrow} \cdots \overset{0}{\Longleftrightarrow} \gamma_{n-1} \overset{0}{\Longleftrightarrow} \gamma_0$ made of elements in $\Gamma\setminus\widehat{\Gamma}^+$.

Let us first show how this implies the proposition above.

Proof of Proposition 5.2. Let $\hat{\sigma} \in \sigma(\Gamma)$ be fixed. Let $\Phi \in \ker(\hat{\sigma} - A^*) \cap \ker B^*$ and let us show that necessarily $\Phi = 0$. Since in particular $\Phi \in \ker(\hat{\sigma} - A^*)$, we can use the description of the eigenfunctions given in Proposition 4.9 (see also 1 of Remark 4.10) to write

$$\Phi = \sum_{\substack{\gamma \in \Gamma \setminus \widehat{\Gamma}^+ \\ \sigma(\gamma) = \hat{\sigma}}} a_\gamma \Phi^0_\gamma,$$

for some scalars $a_\gamma \in \mathbb{C}$. We introduce the support of Φ defined by

$$\operatorname{Supp} \Phi = \left\{ \gamma \in \Gamma \setminus \widehat{\Gamma}^+, \quad \text{s.t.} \quad \sigma(\gamma) = \hat{\sigma}, \quad a_\gamma \neq 0 \right\}.$$

Our goal is to prove that $\operatorname{Supp} \Phi = \emptyset$. Assume, by contradiction, that $\operatorname{Supp} \Phi \neq \emptyset$. We compute (recall the normalization (2.13))

$$B^* \Phi = \begin{pmatrix} L_1^* \left(\displaystyle\sum_{\gamma \in \operatorname{Supp} \Phi} a_\gamma \phi_{1,\lambda_1(\gamma)} \right) \\ L_2^* \left(\displaystyle\sum_{\gamma \in \operatorname{Supp} \Phi} a_\gamma \phi_{2,\lambda_2(\gamma)} \right) \end{pmatrix}.$$

Therefore, the equation $B^* \Phi = 0$ and the assumption (3.1) yield

$$\sum_{\gamma \in \operatorname{Supp} \Phi} a_\gamma \phi_{1,\lambda_1(\gamma)} = 0, \tag{5.1}$$

and

$$\sum_{\gamma \in \operatorname{Supp} \Phi} a_\gamma \phi_{2,\lambda_2(\gamma)} = 0.$$

Let us now show that this implies the following property, for $i = 1, 2$,

$$\forall \gamma \in \operatorname{Supp} \Phi, \quad \exists \tilde{\gamma} \in \operatorname{Supp} \Phi, \, \tilde{\gamma} \neq \gamma, \quad \gamma \overset{0}{\underset{\lambda_i}{\longleftrightarrow}} \tilde{\gamma}. \tag{5.2}$$

Consider for instance $i = 1$. Assume that (5.2) does not hold for some $\gamma \in \operatorname{Supp} \Phi$. This means that the eigenfunction $\phi_{1,\lambda_1(\gamma)}$ only appears once in the sum (5.1). Since the family $(\phi_{1,\lambda_1})_{\lambda_1 \in \Lambda_1}$ is linearly independent, this means that the corresponding coefficient a_γ is equal to 0, which is not possible by definition of the support of Φ. Thus, we have (5.2).

Since $\operatorname{Supp} \Phi \neq \emptyset$ by assumption, there exists an element $\gamma_0 \in \operatorname{Supp} \Phi$, then we apply (5.2) with $i = 1$ to find $\gamma_1 \in \operatorname{Supp} \Phi$ with $\gamma_1 \neq \gamma_0$ such that

$$\gamma_0 \overset{0}{\underset{\lambda_1}{\longleftrightarrow}} \gamma_1.$$

Then, we apply (5.2) with $i = 2$ to find a $\gamma_2 \in \operatorname{Supp} \Phi$ with $\gamma_2 \neq \gamma_1$ such that

$$\gamma_1 \overset{0}{\underset{\lambda_2}{\longleftrightarrow}} \gamma_2.$$

By the first point of Lemma 5.3, we know that $\gamma_2 \neq \gamma_0$. We can apply again (5.2) with $i = 1$ and so on. Since $\operatorname{Supp} \Phi$ is a finite set, we can repeat the process until we select en element that was already selected in the process. Therefore we end up with the following situation

$$\gamma_0 \overset{0}{\Longleftrightarrow} \gamma_1 \overset{0}{\Longleftrightarrow} \cdots \overset{0}{\Longleftrightarrow} \gamma_{i-2} \overset{0}{\Longleftrightarrow} \gamma_{i-1} \overset{0}{\Longleftrightarrow} \gamma_i,$$

where $\gamma_i \in \{\gamma_0, \dots, \gamma_{i-1}\}$. By construction we have $\gamma_{i-1} \neq \gamma_i$, and moreover, since the kind of the arrows alternate in our construction, we also know that $\gamma_{i-2} \neq \gamma_i$ thanks to the first point of Lemma 5.3. We have finally found a 0-cycle

$$\gamma_k \overset{0}{\Longleftrightarrow} \dots \overset{0}{\Longleftrightarrow} \gamma_{i-2} \overset{0}{\Longleftrightarrow} \gamma_{i-1} \overset{0}{\Longleftrightarrow} \gamma_i, \text{ with } \gamma_k = \gamma_i \text{ and } 0 \leq k < i-2,$$

made of elements in $\operatorname{Supp}\Phi \subset \Gamma \setminus \widehat{\Gamma}^+$. This is a contradiction with the second point of Lemma 5.3. □

Let us now turn out to the proof of this crucial lemma.

Proof of Lemma 5.3. (1) It follows from the assumption that

$$\lambda_1(\gamma_0) = \lambda_1(\gamma_1), \text{ and } \lambda_2(\gamma_0) = \lambda_2(\gamma_1).$$

Since $\gamma_0 \neq \gamma_1$, we necessarily have $s(\gamma_0) = -s(\gamma_1)$. Moreover, we also have $\sigma(\gamma_0) = \sigma(\gamma_1)$, which implies by using (4.5), that necessarily

$$\Delta_{\lambda(\gamma_0)} = \Delta_{\lambda(\gamma_1)} = 0.$$

This proves that one of the two elements γ_0 or γ_1 belongs to $\widehat{\Gamma}^+$ which is excluded.

(2) Assume that such a 0-cycle exists

$$\gamma_0 \overset{0}{\Longleftrightarrow} \gamma_1 \overset{0}{\Longleftrightarrow} \cdots \overset{0}{\Longleftrightarrow} \gamma_{n-1} \overset{0}{\Longleftrightarrow} \gamma_0,$$

where, by definition, $\gamma_i \neq \gamma_j$ for $i \neq j$. For convenience, we shall set $\gamma_n = \gamma_0$. We will denote by $\hat{\sigma}$ the common value of all the $\sigma(\gamma_i)$.

- Observe first that we have

$$\lambda(\gamma_i) \neq \lambda(\gamma_{i+1}), \quad \forall i \in \{0, \dots, n-1\}. \tag{5.3}$$

Indeed, if it were not the case, recalling that $\lambda(\gamma) = \lambda_1(\gamma) + \lambda_2(\gamma)$, we would have for some $i \in \{0, \dots, n-1\}$

$$\lambda_1(\gamma_i) = \lambda_1(\gamma_{i+1}), \quad \lambda_2(\gamma_i) = \lambda_2(\gamma_{i+1}),$$

that can be written

$$\gamma_i \overset{0}{\underset{\lambda_1}{\longleftrightarrow}} \gamma_{i+1} \overset{0}{\underset{\lambda_2}{\longleftrightarrow}} \gamma_i.$$

This is excluded by the first point of the lemma.

- By (4.5) and (4.1) we have, for any $i \in \{0, \dots, n-1\}$,

$$(2\hat{\sigma} - (1+d)\lambda(\gamma_i) + \operatorname{Tr}(M))^2 = \Delta_{\lambda(\gamma_i)} = ((1-d)\lambda(\gamma_i) + m_{11} - m_{22})^2 + 4m_{21}m_{12}.$$

This is a second order polynomial equation for $\lambda(\gamma_i)$ and therefore there exists at most two possible values for this quantity. By (5.3), we deduce that n is necessarily even (we write $n = 2\ell$ with $\ell \geq 1$) and that there exist $\lambda' \neq \lambda''$ such that

$$\lambda(\gamma_{2j}) = \lambda', \quad \lambda(\gamma_{2j+1}) = \lambda'', \quad \forall j \in \{0, \dots, \ell-1\}. \tag{5.4}$$

- Let us prove now that two consecutive arrows in the cycle cannot be of the same kind. Assume, by contradiction, that we have

$$\gamma_i \overset{0}{\underset{\lambda_j}{\longleftrightarrow}} \gamma_{i+1} \overset{0}{\underset{\lambda_j}{\longleftrightarrow}} \gamma_{i+2}, \tag{5.5}$$

for some $i \in \{0, \dots, n-2\}$ and some $j \in \{1, 2\}$. By (5.4) we know that $\lambda(\gamma_i) = \lambda(\gamma_{i+2})$ and using (5.5) it follows that

$$\lambda_1(\gamma_i) = \lambda_1(\gamma_{i+2}), \quad \lambda_2(\gamma_i) = \lambda_2(\gamma_{i+2}),$$

so that we can write

$$\gamma_i \xleftrightarrow[\lambda_1]{0} \gamma_{i+2} \xleftrightarrow[\lambda_2]{0} \gamma_i,$$

which is excluded by the first point of the lemma.

- We assume now that $\gamma_0 \xleftrightarrow[\lambda_2]{0} \gamma_1$, the other case being similar. By the discussion above we know that

$$\gamma_{2j} \xleftrightarrow[\lambda_2]{0} \gamma_{2j+1} \xleftrightarrow[\lambda_1]{0} \gamma_{2j+2}, \quad \forall j \in \{0, \dots, \ell-1\},$$

that is to say

$$\lambda_2(\gamma_{2j}) = \lambda_2(\gamma_{2j+1}), \quad \lambda_1(\gamma_{2j+1}) = \lambda_1(\gamma_{2j+2}), \quad \forall j \in \{0, \dots, \ell-1\}. \tag{5.6}$$

Using (5.4), we can now compute

$$\begin{aligned}
\ell\lambda' &= \sum_{j=0}^{\ell-1} \lambda(\gamma_{2j}) \\
&= \sum_{j=0}^{\ell-1} \lambda_1(\gamma_{2j}) + \sum_{j=0}^{\ell-1} \lambda_2(\gamma_{2j}) \\
&= \sum_{j=0}^{\ell-2} \lambda_1(\gamma_{2j+2}) + \lambda_1(\gamma_0) + \sum_{j=0}^{\ell-1} \lambda_2(\gamma_{2j}) \\
&= \sum_{j=0}^{\ell-1} \lambda_1(\gamma_{2j+1}) + \sum_{j=0}^{\ell-1} \lambda_2(\gamma_{2j+1}) \quad (\text{since } \gamma_0 = \gamma_{2\ell} \text{ and using (5.6)}) \\
&= \sum_{j=0}^{\ell-1} \lambda(\gamma_{2j+1}) \\
&= \ell\lambda''.
\end{aligned}$$

This is a contradiction with $\lambda' \neq \lambda''$ and the proof is complete.

□

6. Null controllability

The first five parts of this section are devoted to the proof of Theorem 3.2. We will then show in Section 6.6 how to deduce the extension Theorem 3.3.

All along this section we assume that we are under the assumptions of Theorem 3.2.

6.1. **A non standard moment problem**

We start by reformulating the null controllability problem into a moment problem. Let $T > 0$ be fixed.

1) From the very definition of the notion of solution (2.2), we see that $(-A, B_{\text{ref}})$ is null controllable in time T if, and only if, for every $y^0 \in D(A^*)'$, there exists $u \in L^2(0,T;U)$ such that

$$-\left\langle y^0, z(0)\right\rangle_{D(A^*)',D(A^*)} = \int_0^T \langle u(t), B_{\text{ref}}^* z(t)\rangle_U \, dt, \quad \forall z^T \in D(A^*), \tag{6.1}$$

with z solution to the adjoint problem (2.3) with $\tau = T$.

Let $y^0 \in D(A^*)'$ be fixed from now on. Since the conjugate linear forms involved in the previous identity (i.e. (2.4)) are continuous on $D(A^*)$, it is sufficient to check this identity on a dense subset of $D(A^*)$. We know that span $\left\{\Phi_\gamma^k\right\}_{\substack{\gamma\in\Gamma \\ 0\le k\le k_\gamma}}$ is dense in $D(A^*)$ by point (vi) of Proposition 4.9 (see also point (3) in Remark 4.10).

Therefore, by linearity, it is enough to test (6.1) with $z^T = \Phi_\gamma^0$ for every $\gamma \in \Gamma$, and $z^T = \Phi_\gamma^1$ for every $\gamma \in \widehat{\Gamma}$.

2) For an eigenfunction $z^T = \Phi_\gamma^0$, with $\gamma \in \Gamma$, the corresponding solution to the adjoint system (2.3) is given by

$$z(t) = e^{-(T-t)\sigma(\gamma)}\Phi_\gamma^0,$$

while for a generalized eigenfunction $z^T = \Phi_\gamma^1$, with $\gamma \in \widehat{\Gamma}$, the solution to the adjoint system is

$$z(t) = e^{-(T-t)\sigma(\gamma)}\Phi_\gamma^1 - (T-t)e^{-(T-t)\sigma(\gamma)}\Phi_\gamma^0.$$

Using (4.18), it follows that the property (6.1) is equivalent to

$$\begin{cases} -e^{-T\overline{\sigma(\gamma)}}\left\langle y^0, \Phi_\gamma^0\right\rangle_{D(A^*)',D(A^*)} = \displaystyle\int_0^T e^{-(T-t)\overline{\sigma(\gamma)}}\left\langle u(t), B_{\text{ref}}^*\Phi_\gamma^0\right\rangle_U dt, \quad \forall \gamma \in \Gamma, \\ -e^{-T\overline{\sigma(\gamma)}}\left\langle y^0, \Phi_\gamma^1 - T\Phi_\gamma^0\right\rangle_{D(A^*)',D(A^*)} = \displaystyle\int_0^T (T-t)e^{-(T-t)\overline{\sigma(\gamma)}}\left\langle u(t), B_{\text{ref}}^*\Phi_\gamma^0\right\rangle_U dt, \quad \forall \gamma \in \widehat{\Gamma}. \end{cases}$$

Finding a function u such that the above system is satisfied for every γ is a so-called "moment problem". By assumption (2.16) on the structure of the control space, we can write

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_i \in L^2(0,T;H_i), \quad i = 1,2.$$

Using now the structure (2.17) of B^*_{ref} and the structure (4.13) of the eigenfunctions of A^* with (2.13), we see that (6.1) is equivalent to

$$\begin{cases} -e^{-T\overline{\sigma(\gamma)}}\langle y^0, \Phi^0_\gamma\rangle_{D(A^*)',D(A^*)} = \displaystyle\int_0^T e^{-(T-t)\overline{\sigma(\gamma)}}\big\langle u_1(t), \phi_{1,\lambda_1(\gamma)}\big\rangle_{H_1}\,dt \\ \qquad\qquad\qquad\qquad + \displaystyle\int_0^T e^{-(T-t)\overline{\sigma(\gamma)}}\big\langle u_2(t), \phi_{2,\lambda_2(\gamma)}\big\rangle_{H_2}\,dt, \quad \forall \gamma \in \Gamma, \\ -e^{-T\overline{\sigma(\gamma)}}\langle y^0, \Phi^1_\gamma - T\Phi^0_\gamma\rangle_{D(A^*)',D(A^*)} = \displaystyle\int_0^T (T-t)e^{-(T-t)\overline{\sigma(\gamma)}}\big\langle u_1(t), \phi_{1,\lambda_1(\gamma)}\big\rangle_{H_1}\,dt \\ \qquad\qquad\qquad\qquad + \displaystyle\int_0^T (T-t)e^{-(T-t)\overline{\sigma(\gamma)}}\big\langle u_2(t), \phi_{2,\lambda_2(\gamma)}\big\rangle_{H_2}\,dt, \quad \forall \gamma \in \widehat{\Gamma}. \end{cases}$$

Since for $i = 1, 2$, $\{\phi_{i,\lambda_i}\}_{\lambda_i\in\Lambda_i}$ is an orthogonal basis of H_i satisfying the lower bound (2.14), we can look for u_i in the form of the series

$$u_i(t) = \sum_{\lambda_i\in\Lambda_i} \tilde{u}_{i,\lambda_i}(T-t)\frac{\phi_{i,\lambda_i}}{\|\phi_{i,\lambda_i}\|^2_{H_i}}, \tag{6.2}$$

with $\tilde{u}_{i,\lambda_i} \in L^2(0,T)$ such that

$$\sum_{\lambda_i\in\Lambda_i} (1+\lambda_i^2)\,\|\tilde{u}_{i,\lambda_i}\|^2_{L^2(0,T)} < +\infty, \quad i = 1, 2. \tag{6.3}$$

Thus, the goal is to find $\{\tilde{u}_{1,\lambda_1}\}_{\lambda_1\in\Lambda_1} \subset L^2(0,T)$ and $\{\tilde{u}_{2,\lambda_2}\}_{\lambda_2\in\Lambda_2} \subset L^2(0,T)$ with (6.3) and such that

$$\begin{cases} -e^{-T\overline{\sigma(\gamma)}}\langle y^0, \Phi^0_\gamma\rangle_{D(A^*)',D(A^*)} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}}\tilde{u}_{1,\lambda_1(\gamma)}(t)\,dt + \int_0^T e^{-t\overline{\sigma(\gamma)}}\tilde{u}_{2,\lambda_2(\gamma)}(t)\,dt, \quad \forall \gamma \in \Gamma, \\ -e^{-T\overline{\sigma(\gamma)}}\langle y^0, \Phi^1_\gamma - T\Phi^0_\gamma\rangle_{D(A^*)',D(A^*)} = \displaystyle\int_0^T te^{-t\overline{\sigma(\gamma)}}\tilde{u}_{1,\lambda_1(\gamma)}(t)\,dt + \int_0^T te^{-t\overline{\sigma(\gamma)}}\tilde{u}_{2,\lambda_2(\gamma)}(t)\,dt, \quad \forall \gamma \in \widehat{\Gamma}. \end{cases} \tag{6.4}$$

In summary, we have shown the following:

Proposition 6.1. *Let $T > 0$ and $y^0 \in D(A^*)'$ be fixed. The system $(-A, B_{\text{ref}})$ starting from y^0 is null controllable in time T if, and only if, there exist $\{\tilde{u}_{1,\lambda_1}\}_{\lambda_1\in\Lambda_1} \subset L^2(0,T)$ and $\{\tilde{u}_{2,\lambda_2}\}_{\lambda_2\in\Lambda_2} \subset L^2(0,T)$ with* (6.3) *and satisfying* (6.4).

In that case, there is a null control u that satisfies, for some $C > 0$ not depending on T and y^0, the estimate

$$\|u\|^2_{L^2(0,T)} \le C \sum_{i\in\{1,2\}} \sum_{\lambda_i\in\Lambda_i} (1+\lambda_i^2)\,\|\tilde{u}_{i,\lambda_i}\|^2_{L^2(0,T)}\,.$$

The system of equations (6.4) looks like a family of coupled moment problems. The main difficulty in solving this system comes from the following facts:

- For any $i = 1, 2$ and any $\lambda_i \in \Lambda_i$, the unknown function $\tilde{u}_{i,\lambda_i}$ appears in an infinite subset of those equations, namely the ones corresponding to the parameters γ belonging to the set Γ_{i,λ_i}, as introduced in point (3) of Definition 4.3.

- The map $\gamma \mapsto \sigma(\gamma)$ may not be injective, so that the same exponential function may appear in many of those equations. Moreover, even in the case where this map is injective, it will certainly happen some spectral condensation phenomenon, that is the fact that $\sigma(\gamma)$ and $\sigma(\tilde{\gamma})$ may be exponentially close for infinitely many $\tilde{\gamma} \neq \gamma$. In general, this condensation may be an obstacle to the small time null controllability of the system as mentioned in the introduction. The block moment method was precisely introduced in [10] to carefully analyze this phenomenon in a quite general setting.

  In the sequel of the paper, we will show how to apply this block moment approach in order to prove the small time null controllability of our system.

The strategy to solve (6.4) is to build separate sets of equations for the families $(\tilde{u}_{1,\lambda_1})_{\lambda_1\in\Lambda_1}$ and $(\tilde{u}_{2,\lambda_2})_{\lambda_2\in\Lambda_2}$. In that perspective, we will first state the following key result that will induce the existence of a suitable splitting of the source terms into two parts. The proof of this result is postponed to Section 6.5 below.

Theorem 6.2. *There exists $\hat{\rho} > 0$ small enough and there exist two families $(\Phi^k_{\gamma,1})_{\substack{\gamma\in\Gamma\\ 0\leq k\leq k_\gamma}} \subset D(A^*)$, $(\Phi^k_{\gamma,2})_{\substack{\gamma\in\Gamma\\ 0\leq k\leq k_\gamma}} \subset D(A^*)$, such that:*

(i) We have

$$\Phi^k_\gamma = \Phi^k_{\gamma,1} + \Phi^k_{\gamma,2}, \quad \forall \gamma \in \Gamma, \quad \forall 0 \leq k \leq k_\gamma.$$

(ii) For every $\gamma, \tilde{\gamma} \in \Gamma$, we have

$$\left.\begin{aligned} |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leq \hat{\rho} \\ \lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) \end{aligned}\right\} \quad \Longrightarrow \quad \begin{cases} \Phi^0_{\gamma,1} = \Phi^0_{\tilde{\gamma},1}, \\ \Phi^1_{\gamma,1} = \Phi^1_{\tilde{\gamma},1}, \quad \text{if } \gamma, \tilde{\gamma} \in \widehat{\Gamma}, \end{cases}$$

and

$$\left.\begin{aligned} |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leq \hat{\rho} \\ \lambda_2(\gamma) = \lambda_2(\tilde{\gamma}) \end{aligned}\right\} \quad \Longrightarrow \quad \begin{cases} \Phi^0_{\gamma,2} = \Phi^0_{\tilde{\gamma},2}, \\ \Phi^1_{\gamma,2} = \Phi^1_{\tilde{\gamma},2}, \quad \text{if } \gamma, \tilde{\gamma} \in \widehat{\Gamma}. \end{cases}$$

(iii) There exists $C > 0$ such that

$$\left\|\Phi^k_{\gamma,1}\right\|_{D(A^*)} + \left\|\Phi^k_{\gamma,2}\right\|_{D(A^*)} \leq C\,|\sigma(\gamma)| \exp\left(C\,|\sigma(\gamma)|^\nu\right), \quad \forall \gamma \in \Gamma, \quad \forall 0 \leq k \leq k_\gamma.$$

Besides, if $d = 1$ and if we have the following non resonance condition:

$$\inf_{\substack{\lambda_i,\tilde{\lambda}_i\in\Lambda_i\\ \lambda_i\neq\tilde{\lambda}_i\\ \theta,\tilde{\theta}\in\sigma(M^*)}} \left|\lambda_i + \theta - (\tilde{\lambda}_i + \tilde{\theta})\right| > 0, \tag{6.5}$$

for some $i \in \{1,2\}$, then we can take $\Phi^k_{\gamma,i} = 0$ for every $\gamma \in \Gamma$ and $k \in \{0,\dots,k_\gamma\}$.

Let us explain how to use this result to solve the system (6.4). We introduce the complex numbers

$$\omega^k_{\gamma,1} = (-1)^{k+1} e^{-T\overline{\sigma(\gamma)}} \left\langle y^0, \Phi^k_{\gamma,1}\right\rangle_{D(A^*)',D(A^*)}, \quad \omega^k_{\gamma,2} = (-1)^{k+1} e^{-T\overline{\sigma(\gamma)}} \left\langle y^0, \Phi^k_{\gamma,2}\right\rangle_{D(A^*)',D(A^*)}. \tag{6.6}$$

The idea is to split the problem into two independent sets of equations as follows

$$\begin{cases} \omega^0_{\gamma,1} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}} \tilde{u}_{1,\lambda_1(\gamma)}(t)\,dt, \quad \forall \gamma \in \Gamma, \\ \omega^1_{\gamma,1} + T\omega^0_{\gamma,1} = \displaystyle\int_0^T te^{-t\overline{\sigma(\gamma)}} \tilde{u}_{1,\lambda_1(\gamma)}(t)\,dt, \quad \forall \gamma \in \widehat{\Gamma}, \end{cases}$$

$$\begin{cases} \omega^0_{\gamma,2} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}}\tilde{u}_{2,\lambda_2(\gamma)}(t)\,dt, \quad \forall \gamma \in \Gamma, \\ \omega^1_{\gamma,2} + T\omega^0_{\gamma,2} = \displaystyle\int_0^T te^{-t\overline{\sigma(\gamma)}}\tilde{u}_{2,\lambda_2(\gamma)}(t)\,dt, \quad \forall \gamma \in \widehat{\Gamma}. \end{cases}$$

If we manage to solve those problems, by summing the equations and using point (i) of Theorem 6.2 we immediately get that (6.4) is solved.

Solving those problems amounts to ask that, each function $\tilde{u}_{1,\lambda_1}$, for $\lambda_1 \in \Lambda_1$, satisfies

$$\begin{cases} \omega^0_{\gamma,1} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}}\tilde{u}_{1,\lambda_1}(t)\,dt, \quad \forall \gamma \in \Gamma_{1,\lambda_1}, \\ \omega^1_{\gamma,1} + T\omega^0_{\gamma,1} = \displaystyle\int_0^T te^{-t\overline{\sigma(\gamma)}}\tilde{u}_{1,\lambda_1}(t)\,dt, \quad \forall \gamma \in \widehat{\Gamma}_{1,\lambda_1}, \end{cases} \tag{6.7}$$

and each function $\tilde{u}_{2,\lambda_2}$, for $\lambda_2 \in \Lambda_2$, satisfies

$$\begin{cases} \omega^0_{\gamma,2} = \displaystyle\int_0^T e^{-t\overline{\sigma(\gamma)}}\tilde{u}_{2,\lambda_2}(t)\,dt, \quad \forall \gamma \in \Gamma_{2,\lambda_2}, \\ \omega^1_{\gamma,2} + T\omega^0_{\gamma,2} = \displaystyle\int_0^T te^{-t\overline{\sigma(\gamma)}}\tilde{u}_{2,\lambda_2}(t)\,dt, \quad \forall \gamma \in \widehat{\Gamma}_{2,\lambda_2}. \end{cases} \tag{6.8}$$

We have now to solve an infinite set of uncoupled moment problems (one for each $\tilde{u}_{1,\lambda_1}$ and one for each $\tilde{u}_{2,\lambda_2}$), each of them being associated with a different family of (generalized) exponential functions corresponding to the eigenvalues in Σ_{1,λ_1} and Σ_{2,λ_2}, respectively. Of course, those moment problems are in fact coupled through the construction of the families $(\Phi^k_{\gamma,1})_{\substack{\gamma\in\Gamma \\ 0\le k\le k_\gamma}}$, $(\Phi^k_{\gamma,2})_{\substack{\gamma\in\Gamma \\ 0\le k\le k_\gamma}}$, given by Theorem 6.2.

We are now led to prove that all those moment problems can be solved with appropriate estimates on the solutions to ensure the convergence of the series and thus the existence of the control for our initial problem.

Consequently, the proof of Theorem 3.2 will be complete if we manage to prove the following result.

Proposition 6.3. *Let $p_0 \in (0,1)$ be such that $p_0 > \max\{\theta_1, \theta_2, \nu\}$. Let $T > 0$ and $y^0 \in D(A^*)'$ be fixed. There exist $\{\tilde{u}_{1,\lambda_1}\}_{\lambda_1\in\Lambda_1} \subset L^2(0,T)$ and $\{\tilde{u}_{2,\lambda_2}\}_{\lambda_2\in\Lambda_2} \subset L^2(0,T)$ that satisfy (6.7) and (6.8) respectively, and such that, for some $C > 0$ not depending on T and y^0,*

$$\sum_{i\in\{1,2\}}\sum_{\lambda_i\in\Lambda_i}(1+\lambda_i^2)\,\|\tilde{u}_{i,\lambda_i}\|^2_{L^2(0,T)} \le C\exp\left(\frac{C}{T^{\frac{p_0}{1-p_0}}}\right)\|y^0\|^2_{D(A^*)'}.$$

The rest of the section is organized as follows. We first summarize in Section 6.2 some useful definitions and results coming from the so-called block moment method. Optimal estimates of the cost of the control being only available in the case of real eigenvalues, we describe in Section 6.3 a process that will help us to deal with the presence of some complex eigenvalues in our problem. Then we proceed in Section 6.4 to the proof of the Proposition 6.3 and finally we conclude with the proof of the key Theorem 6.2, in Section 6.5.

6.2. **Background on the block moment method**

Let us introduce some elements taken from [10, 12] that will be useful in our analysis.

Definition 6.4. Let $n \in \mathbb{N}^*$, $\rho > 0$, $\theta \in (0,1)$ and $\kappa > 0$ be fixed. We denote by $\mathcal{L}_w(n,\rho,\theta,\kappa)$ the class of subsets $S \subset \mathbb{C}_+$ satisfying the following two conditions:

(i) Counting function condition:

$$N_S(r) \leq \kappa r^\theta, \quad \forall r > 0.$$

(ii) Weak gap condition:

$$\text{card } (S \cap ([\mu, \mu+\rho] + i\mathbb{R})) \leq n, \quad \forall \mu > 0.$$

We shall use the following result taken from [10, Proposition 7.1], yet with slightly different notation.

Proposition 6.5. *Assume that S satisfies the weak gap condition (ii) of Definition 6.4 for some $n \in \mathbb{N}^*$ and $\rho > 0$. Then, there exists a grouping $(G_k)_{k\geq 1} \subset S$ associated with S satisfying the following five properties:*

(i) Covering:

$$S = \bigcup_{k\geq 1} G_k.$$

(ii) Uniform bound on the cardinality:

$$\text{card}\, G_k \leq n, \quad \forall k \geq 1.$$

(iii) Increasing order:

$$\sup(\mathfrak{Re}\, G_k) < \inf(\mathfrak{Re}\, G_{k+1}), \quad \forall k \geq 1.$$

(iv) Gap condition:

$$\text{dist}\,(G_k, G_{k+1}) \geq \frac{\rho}{n}, \quad \forall k \geq 1.$$

(v) Uniform bound on the diameter:

$$\text{diam}\, G_k < \rho, \quad \forall k \geq 1.$$

We recall, also with adapted notation, the following result from [10, Theorem 2.5] and [12], that we specialized in the particular case where $n = 2$, which is sufficient in the present work.

Theorem 6.6. *Let $S \subset [1, +\infty)$ such that $S \in \mathcal{L}_w(2,\rho,\theta,\kappa)$ and $(G_k)_{k\geq 1}$ be a grouping as introduced in Proposition 6.5. There exists $C > 0$ such that, for any $T > 0$ and $k \geq 1$, the following assertions hold.*

- *If $\text{card}\, G_k = 1$, say $G_k = \{\sigma_k\}$, then for any $\omega_k \in \mathbb{R}$, there exists $q_k \in L^2(0,T)$ such that*

$$\begin{cases} \omega_k = \displaystyle\int_0^T e^{-\sigma_k t} q_k(t)\, dt, \\ 0 = \displaystyle\int_0^T e^{-\sigma t} q_k(t)\, dt, \quad \forall \sigma \in S \setminus G_k, \end{cases}$$

and

$$\|q_k\|_{L^2(0,T)} \leq C \exp\left(C\sigma_k^\theta (1 + \log \sigma_k) + \frac{C}{T^{\frac{\theta}{1-\theta}}} \right) |\omega_k|\,.$$

- *If* $\operatorname{card} G_k = 2$, *say* $G_k = \{\sigma_k, \tilde{\sigma}_k\}$, *then for any* $\omega_k, \tilde{\omega}_k \in \mathbb{R}$, *there exists* $q_k \in L^2(0,T)$ *such that*

$$\begin{cases} \omega_k = \displaystyle\int_0^T e^{-\sigma_k t} q_k(t)\, dt, \\ \tilde{\omega}_k = \displaystyle\int_0^T e^{-\tilde{\sigma}_k t} q_k(t)\, dt, \\ 0 = \displaystyle\int_0^T e^{-\sigma t} q_k(t)\, dt, \quad \forall \sigma \in S \setminus G_k, \end{cases}$$

and

$$\|q_k\|_{L^2(0,T)} \leq C \exp\left(C\sigma_k^\theta (1 + \log \sigma_k) + \frac{C}{T^{\frac{\theta}{1-\theta}}} \right) \left(|\omega_k| + \left| \frac{\omega_k - \tilde{\omega}_k}{\sigma_k - \tilde{\sigma}_k} \right| \right). \tag{6.9}$$

6.3. **Splitting between high and low frequencies**

To the best of our knowledge, Theorem 6.6 above is unfortunately only available for real families of eigenvalues (a version with complex eigenvalues exists but without the sharp estimate with respect to the time T as given in (6.9) that we will need in the present work to obtain our more general result, namely Theorem 3.3). However, it appears that the operator A^* under study in this paper only has a finite number of complex eigenvalues (at least for $d \neq 1$). This was for instance mentioned in Remark 4.7.

The following result will let us deal with this situation, while preserving the short time estimate we need.

Proposition 6.7. *Let* $S \subset \mathbb{C}_+$ *be a family of complex numbers such that* $S \in \mathcal{L}_w(2, \rho, \theta, \kappa)$, *and* $S_0 \subset S$ *a finite subset of* S. *Let* $(\omega_\sigma^0)_{\sigma \in S}$, $(\omega_\sigma^1)_{\sigma \in S_0}$ *two families of complex numbers, and* $T > 0$ *be fixed. Assume that there exists* $\hat{v} \in L^2(0, T/2)$ *such that*

$$\omega_\sigma^0 = \int_0^{T/2} e^{-t\overline{\sigma}} \hat{v}(t)\, dt, \quad \forall \sigma \in S \backslash S_0.$$

Then, there exists $v \in L^2(0,T)$ *such that*

$$\begin{cases} \omega_\sigma^0 = \displaystyle\int_0^T e^{-t\overline{\sigma}} v(t)\, dt, \quad \forall \sigma \in S, \\ \omega_\sigma^1 = \displaystyle\int_0^T t e^{-t\overline{\sigma}} v(t)\, dt, \quad \forall \sigma \in S_0, \end{cases}$$

with, for some $C > 0$ *not depending on* T, $\hat{v}$ *and* $(\omega_\sigma^0)_{\sigma \in S}$, $(\omega_\sigma^1)_{\sigma \in S_0}$,

$$\|v\|_{L^2(0,T)} \leq C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}} \right) \left(\|\hat{v}\|_{L^2(0,T/2)} + \sum_{\sigma \in S_0} \left(|\omega_\sigma^0| + |\omega_\sigma^1| \right) \right).$$

The proof relies on the following result:

Lemma 6.8. *Let* $T > 0$. *Let* $S \in \mathcal{L}_w(2, \rho, \theta, \kappa)$. *Then, there exists a family* $(q_\nu^i)_{\substack{\nu \in S \\ i \in \{0,1\}}} \subset L^2(T/2, T)$ *such that*

$$\int_{T/2}^T q_\nu^i(t) t^j e^{-\sigma t}\, dt = \delta_{i,j} \delta_{\sigma,\nu}, \quad \forall \sigma, \nu \in S, \quad \forall i, j \in \{0,1\},$$

($\delta_{i,j}$ denotes the Kronecker delta) and that satisfies, for some $C > 0$ not depending on T,

$$\left\|q_\nu^i\right\|_{L^2(T/2,T)} \leq C \exp\left(C\left(\frac{1}{T^{\frac{\theta}{1-\theta}}} + \nu^\theta\right)\right) \frac{1+\delta(\nu,S)}{\delta(\nu,S)}, \quad \forall \nu \in S, \quad \forall i \in \{0,1\},$$

where $\delta(\nu,S) = \operatorname{dist}(\nu, S \setminus \{\nu\})$.

As far as we know, the proof of this lemma is not available in the literature in this precise form. However, we will not give the details since it can be obtained in a very similar way as in [25] (where a more general case is considered, but only for $\theta = 1/2$), see also [9]. Note that, if $S \subset \mathbb{R}$, the result follows from the computations in [4], [11].

Proof of Proposition 6.7. It is sufficient to take $v = \hat{v}$ in $(0, T/2)$ and then to build v on $(T/2, T)$ such that

$$\begin{cases} 0 = \displaystyle\int_{T/2}^{T} e^{-t\overline{\sigma}} v(t)\, dt, \quad \forall \sigma \in S \backslash S_0, \\ c_\sigma^0 = \displaystyle\int_{T/2}^{T} e^{-t\sigma} v(t)\, dt, \quad \forall \sigma \in S_0, \\ c_\sigma^1 = \displaystyle\int_{T/2}^{T} t e^{-t\sigma} v(t)\, dt, \quad \forall \sigma \in S_0, \end{cases} \tag{6.10}$$

where we set

$$\begin{cases} c_\sigma^0 = \omega_\sigma^0 - \displaystyle\int_0^{T/2} e^{-t\sigma} \hat{v}(t)\, dt, \\ c_\sigma^1 = \omega_\sigma^1 - \displaystyle\int_0^{T/2} t e^{-t\sigma} \hat{v}(t)\, dt. \end{cases}$$

To solve this new moment problem (6.10), we look for $v_{|(T/2,T)}$ in the form of the following finite sum

$$v = \sum_{\nu \in S_0} \left(c_\nu^0 q_\nu^0 + c_\nu^1 q_\nu^1\right), \quad \text{in } (T/2, T),$$

where $(q_\nu^i)_{\substack{\nu \in S \\ i \in \{0,1\}}}$ is given by Lemma 6.8. Below, we denote by C a positive number that may change from line to line but that does not depend on T, $\hat{v}$ and $(\omega_\sigma^0)_{\sigma \in S}$, $(\omega_\sigma^1)_{\sigma \in S_0}$. Since S_0 is finite, we have

$$\frac{1+\delta(\nu,S)}{\delta(\nu,S)} \leq C, \quad \forall \nu \in S_0,$$

so that we get

$$\begin{aligned} \|v\|_{L^2(T/2,T)} &\leq C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right) \sum_{\nu \in S_0} \left(|c_\nu^0| + |c_\nu^1|\right) \exp\left(C\nu^\theta\right) \\ &\leq C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right) \sum_{\nu \in S_0} \left(|c_\nu^0| + |c_\nu^1|\right) \\ &\leq C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right) \left(\|\hat{v}\|_{L^2(0,T/2)} + \sum_{\nu \in S_0} \left(|\omega_\nu^0| + |\omega_\nu^1|\right)\right). \end{aligned}$$

The proof is complete. □

6.4. **Proof of Proposition 6.3**

To begin with, we establish that the families of eigenvalues introduced in (4.6) belong to suitable classes as defined in Definition 6.4 in a uniform way. The precise statement is as follows.

Proposition 6.9. *There exist $\rho_1, \rho_2 > 0$ and $\tilde{\kappa}_1, \tilde{\kappa}_2 > 0$ such that, we have*

$$\Sigma_{1,\lambda_1} \in \mathcal{L}_w(2, \rho_2, \theta_2, \tilde{\kappa}_2), \quad \textit{for any } \lambda_1 \in \Lambda_1,$$

$$\Sigma_{2,\lambda_2} \in \mathcal{L}_w(2, \rho_1, \theta_1, \tilde{\kappa}_1), \quad \textit{for any } \lambda_2 \in \Lambda_2.$$

We recall that $\theta_i \in (0,1)$ is such that (2.8) *holds.*

Proof. We focus on Σ_{1,λ_1}, the other case being similar. To this end, we first show that there exist $\rho_2 > 0$ and $\tilde{\kappa}_2 > 0$ such that, for any $\lambda_1 \in \Lambda_1$,

$$\Sigma^+_{1,\lambda_1} \in \mathcal{L}_w(1, \rho_2, \theta_2, \tilde{\kappa}_2/2), \quad \Sigma^-_{1,\lambda_1} \in \mathcal{L}_w(1, \rho_2, \theta_2, \tilde{\kappa}_2/2).$$

Let us prove the claim for Σ^+_{1,λ_1}, the other case being similar.

- By (2.7) and (4.7), we can find $\hat{\lambda} > 0$ large enough such that

$$|\varepsilon_\lambda| \leq \frac{1}{4} \max(d,1) \operatorname{Gap}(\Lambda_2), \quad \forall \lambda > \hat{\lambda}. \tag{6.11}$$

- Let $\lambda_1 \in \Lambda_1$ and $\lambda_2, \tilde{\lambda}_2 \in \Lambda_2$ such that $\lambda_2 \neq \tilde{\lambda}_2$, that satisfy

$$\lambda_1 + \lambda_2 > \hat{\lambda}, \quad \lambda_1 + \tilde{\lambda}_2 > \hat{\lambda}. \tag{6.12}$$

  From (4.8), we have

$$\sigma(+, \lambda_1, \lambda_2) - \sigma(+, \lambda_1, \tilde{\lambda}_2) = \max(d,1)(\lambda_2 - \tilde{\lambda}_2) + \varepsilon_{\lambda_1+\lambda_2} - \varepsilon_{\lambda_1+\tilde{\lambda}_2},$$

  and therefore

$$|\sigma(+, \lambda_1, \lambda_2) - \sigma(+, \lambda_1, \tilde{\lambda}_2)| \geq \max(d,1) \operatorname{Gap}(\Lambda_2) - |\varepsilon_{\lambda_1+\lambda_2}| - |\varepsilon_{\lambda_1+\tilde{\lambda}_2}| \geq \frac{1}{2} \max(d,1) \operatorname{Gap}(\Lambda_2),$$

  by using (6.11).
- For any $\lambda_1 \in \Lambda_1$ such that $\lambda_1 > \hat{\lambda}$ the conditions (6.12) are automatically satisfied since we have (2.6)). Therefore, we have

$$\operatorname{Gap}\left(\Sigma^+_{1,\lambda_1}\right) \geq \frac{1}{2} \max(d,1) \operatorname{Gap}(\Lambda_2), \quad \forall \lambda_1 \in \Lambda_1,\ \lambda_1 > \hat{\lambda}.$$

- For any $\lambda_1 \in \Lambda_1$ (in particular, such that $\lambda_1 \leq \hat{\lambda}$), the conditions (6.12) are satisfied for λ_2 and $\tilde{\lambda}_2$ large enough, so that we have

$$\operatorname{Gap}\left(\Sigma^+_{1,\lambda_1}\right) > 0.$$

  Since $\Lambda_1 \cap (-\infty, \hat{\lambda}]$ is a finite set, we have finally proved the existence of a $\rho_2 > 0$ such that

$$\operatorname{Gap}\left(\Sigma^+_{1,\lambda_1}\right) > \rho_2, \quad \forall \lambda_1 \in \Lambda_1.$$

- For any $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, we have (see (4.8))

$$\lambda_2 \leq |\max(d,1)(\lambda_1+\lambda_2)| = |\sigma(+,\lambda_1,\lambda_2) - \sigma^+ - \varepsilon_{\lambda_1+\lambda_2}| \leq |\sigma(+,\lambda_1,\lambda_2)| + C,$$

with

$$C = |\sigma^+| + \sup_{\lambda\in[0,+\infty[} |\varepsilon_\lambda|.$$

Therefore, for any $r>0$ the condition $|\sigma(+,\lambda_1,\lambda_2)|<r$ implies that $\lambda_2 < r+C$. It follows that the counting function associated with Σ^+_{1,λ_1} satisfies (recall (2.8))

$$N_{\Sigma^+_{1,\lambda_1}}(r) \leq N_{\Lambda_2}(r+C) \leq \kappa_2 (r+C)^{\theta_2}.$$

By (4.9), we know that $N_{\Sigma^+_{1,\lambda_1}}(r) = 0$ for $r<1$, in such a way that the estimate above leads to

$$N_{\Sigma^+_{1,\lambda_1}}(r) \leq \kappa_2 \left(1+C\right)^{\theta_2} r^{\theta_2}, \quad \forall r>0,$$

and the claim for Σ^+_{1,λ_1} is proved.
- Since $\Sigma_{1,\lambda_1} = \Sigma^+_{1,\lambda_1} \cup \Sigma^-_{1,\lambda_1}$ we obtain that Σ_{1,λ_1} satisfies the weak gap condition with $n=2$ and $\rho = \rho_2$. Moreover we have, for any $r>0$,

$$N_{\Sigma_{1,\lambda_1}}(r) \leq N_{\Sigma^+_{1,\lambda_1}}(r) + N_{\Sigma^-_{1,\lambda_1}}(r) \leq \tilde{\kappa}_2 r^{\theta_2},$$

for some $\tilde{\kappa}_2$ independent of r. □

We can know move to the proof of the proposition. To this end, we will discuss two cases:

$$d \neq 1 \quad \text{or} \quad (d=1 \quad \text{and} \quad \Delta > 0), \tag{C1}$$

and

$$d = 1 \quad \text{and} \quad \Delta \leq 0, \tag{C2}$$

where we recall that Δ is the discriminant of the characteristic polynomial of M (see (4.3)).

The reason for this separation is because in the case (C1) we have (see Remark 4.7 and Remark 4.2)

$$\text{the number of complex eigenvalues is finite and } \widehat{\Lambda} \text{ is finite.} \tag{6.13}$$

We start with the proof in the most intricate case, namely (C1). The case (C2), which is somehow easier, will be considered just after.

Proof of Proposition 6.3 in the case (C1). The main ingredients for the proof are Theorem 6.6 and the estimates of Theorem 6.2. Without loss of generality, we shall assume that

$$T \leq 1. \tag{6.14}$$

(1) Once the values $\omega^k_{\gamma,1}, \omega^k_{\gamma,2}$ have been defined by (6.6) thanks to Theorem 6.2, it is clear that it is enough to consider only one family of problems, for instance (6.7), the other one being treated in a similar way.

For each $\lambda_1 \in \Lambda_1$ fixed, (6.7) is a classical moment problem in $L^2(0,T)$ associated with the family of exponentials

$$\left\{t \mapsto e^{-\overline{\sigma} t}, \quad \text{s.t.} \quad \sigma \in \Sigma_{1,\lambda_1}\right\} \cup \left\{t \mapsto t e^{-\overline{\sigma} t}, \quad \text{s.t.} \quad \sigma \in \widehat{\Sigma}_{1,\lambda_1}\right\}.$$

We would like to solve each of these independent problems. The main difficulty is that Σ_{1,λ_1} may not satisfy a uniform gap property and may present some condensation phenomenon. The other difficulty comes from the fact that, for a finite number of values of λ_1, Σ_{1,λ_1} may contain complex numbers (but at most a finite number). We will first focus on the problem in $L^2(0,T/2)$ and restricted to the following real exponentials

$$\left\{t \mapsto e^{-\sigma t}, \quad \text{s.t.} \quad \sigma \in \Sigma^{\mathbb{R}}_{1,\lambda_1}\right\},$$

with

$$\Sigma^{\mathbb{R}}_{1,\lambda_1} = \mathbb{R} \cap \Sigma_{1,\lambda_1}.$$

We will also note $\Gamma^{\mathbb{R}}_{1,\lambda_1} = \sigma^{-1}(\Sigma^{\mathbb{R}}_{1,\lambda_1})$.

In other words, for any $\lambda_1 \in \Lambda_1$, we will first look for a function $\hat{u}_{1,\lambda_1} \in L^2(0,T/2)$ such that

$$\omega^0_{\gamma,1} = \int_0^{T/2} e^{-t\sigma(\gamma)} \hat{u}_{1,\lambda_1}(t)\, dt, \quad \forall \gamma \in \Gamma^{\mathbb{R}}_{1,\lambda_1}. \tag{6.15}$$

This will be done by applying Theorem 6.6.

(2) To this end, we first use Proposition 6.9, that shows that there exist $\rho_2 > 0$ and $\tilde{\kappa}_2 > 0$ such that for every $\lambda_1 \in \Lambda_1$, we have

$$\Sigma^{\mathbb{R}}_{1,\lambda_1} \in \mathcal{L}_w(2, \rho_2, \theta_2, \tilde{\kappa}_2). \tag{6.16}$$

It is very important to notice that the parameters of this class do not depend on λ_1.

From the definition of the class $\mathcal{L}_w$ it is then clear that we also have

$$\Sigma^{\mathbb{R}}_{1,\lambda_1} \in \mathcal{L}_w(2, \hat{\rho}_2, \theta_2, \tilde{\kappa}_2),$$

where $\hat{\rho}_2 = \min(\rho_2, \hat{\rho})$, where $\hat{\rho}$ is provided by Theorem 6.2.

(3) Let now $(G_{\lambda_1,k})_{k\geq 1}$ be a grouping associated with the family $\Sigma^{\mathbb{R}}_{1,\lambda_1}$ as given by Proposition 6.5.

In particular, each $G_{\lambda_1,k}$ has at most two elements and its diameter is at most $\hat{\rho}_2$. Consequently, we are in one of the following two configurations:

(a) $\operatorname{card} G_{\lambda_1,k} = 1$, in which case it is of the form $G_{\lambda_1,k} = \{\sigma(\gamma)\}$ for some $\gamma \in \Gamma^{\mathbb{R}}_{1,\lambda_1}$. Note that it may exist other elements $\tilde{\gamma} \in \Gamma^{\mathbb{R}}_{1,\lambda_1}$ such that $G_{\lambda_1,k} = \{\sigma(\tilde{\gamma})\}$.

(b) $\operatorname{card} G_{\lambda_1,k} = 2$, in which case it is necessarily of the form $G_{\lambda_1,k} = \{\sigma(\gamma), \sigma(\tilde{\gamma})\}$ for some $\gamma, \tilde{\gamma} \in \Gamma^{\mathbb{R}}_{1,\lambda_1}$ such that $0 < |\sigma(\gamma) - \sigma(\tilde{\gamma})| = \operatorname{diam} G_{\lambda_1,k} < \hat{\rho}_2$.

(4) Below, we denote by $C, \tilde{C}$ positive numbers that may change from line to line but that do not depend on T, y^0 and λ_1. For any $\hat{\sigma} \in G_{\lambda_1,k}$, there exists at least one $\gamma \in \Gamma^{\mathbb{R}}_{1,\lambda_1}$ such that $\hat{\sigma} = \sigma(\gamma)$. We set $\omega_{\hat{\sigma}} = \omega^0_{\gamma,1}$. It is crucial to note that, if there exists another $\tilde{\gamma} \in \Gamma^{\mathbb{R}}_{1,\lambda_1}$ such that $\hat{\sigma} = \sigma(\tilde{\gamma})$ then we necessarily have $\omega^0_{\tilde{\gamma},1} = \omega^0_{\gamma,1}$ by item (ii) of Theorem 6.2, so that $\omega_{\hat{\sigma}}$ is well defined.

Applying now Theorem 6.6, we know that there exists a sequence $\{q_{\lambda_1,k}\}_{k\geq 1} \subset L^2(0,T/2)$ such that, for every $\gamma \in \Gamma^{\mathbb{R}}_{1,\lambda_1}$ we have

$$\forall k \geq 1, \quad \begin{cases} \omega^0_{\gamma,1} &= \displaystyle\int_0^{T/2} e^{-t\sigma(\gamma)} q_{\lambda_1,k}(t)\, dt \quad \text{if } \sigma(\gamma) \in G_{\lambda_1,k}, \\ 0 &= \displaystyle\int_0^{T/2} e^{-t\sigma(\gamma)} q_{\lambda_1,k}(t)\, dt \quad \text{if } \sigma(\gamma) \in \Sigma^{\mathbb{R}}_{1,\lambda_1} \backslash G_{\lambda_1,k}, \end{cases} \tag{6.17}$$

and with the following properties:

(a) If $\operatorname{card} G_{\lambda_1,k} = 1$, with $G_{\lambda_1,k} = \{\sigma(\gamma)\}$ for some $\gamma \in \Gamma^{\mathbb{R}}_{1,\lambda_1}$, then

$$\|q_{\lambda_1,k}\|_{L^2(0,T/2)} \leq C \exp\left(C\sigma(\gamma)^{\theta_2}(1+\log\sigma(\gamma)) + \frac{C}{T^{\frac{\theta_2}{1-\theta_2}}} \right) \left|\omega^0_{\gamma,1}\right|.$$

(b) If $\operatorname{card} G_{\lambda_1,k} = 2$, with $G_{\lambda_1,k} = \{\sigma(\gamma), \sigma(\tilde{\gamma})\}$ for some $\gamma, \tilde{\gamma} \in \Gamma^{\mathbb{R}}_{1,\lambda_1}$ then

$$\|q_{\lambda_1,k}\|_{L^2(0,T/2)} \leq C \exp\left(C\sigma(\gamma)^{\theta_2}(1+\log\sigma(\gamma)) + \frac{C}{T^{\frac{\theta_2}{1-\theta_2}}} \right) \left(\left|\omega^0_{\gamma,1}\right| + \left| \frac{\omega^0_{\gamma,1} - \omega^0_{\tilde{\gamma},1}}{\sigma(\gamma) - \sigma(\tilde{\gamma})} \right| \right).$$

(5) From (6.6), (4.11) and item (iii) of Theorem 6.2, we have

$$\left|\omega^0_{\gamma,1}\right| \leq C\left|\sigma(\gamma)\right| \exp\left(-\tilde{C}T\left|\sigma(\gamma)\right| + C\left|\sigma(\gamma)\right|^{\nu} \right) \left\|y^0\right\|_{D(A^*)'}.$$

On the other hand, from (6.6), and using items (ii) and (iii) of Theorem 6.2, which is allowed since we have $|\sigma(\gamma) - \sigma(\tilde{\gamma})| < \hat{\rho}_2 \leq \hat{\rho}$ and $\lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) = \lambda_1$ by definition of Γ_{1,λ_1}, we have (recall as well that $\sigma(\gamma)$ are positive real numbers in this part)

$$\begin{aligned} \left| \frac{\omega^0_{\gamma,1} - \omega^0_{\tilde{\gamma},1}}{\sigma(\gamma) - \sigma(\tilde{\gamma})} \right| &= \left| \frac{e^{-T\sigma(\gamma)} - e^{-T\sigma(\tilde{\gamma})}}{\sigma(\gamma) - \sigma(\tilde{\gamma})} \right| \left| \left\langle y^0, \Phi^0_{\gamma,1} \right\rangle_{D(A^*)',D(A^*)} \right| \\ &\leq CT\left|\sigma(\gamma)\right| \exp\left(C\left|\sigma(\gamma)\right|^{\nu} \right) \left\|y^0\right\|_{D(A^*)'}. \end{aligned}$$

With these estimates we see that, in any of the two cases above, for any $p_0 \in (0,1)$ such that $p_0 > \max\{\nu, \theta_2\}$, the following estimate holds

$$\|q_{\lambda_1,k}\|_{L^2(0,T/2)} \leq C \exp\left(-\tilde{C}T\sigma(\gamma) + C\sigma(\gamma)^{p_0} + \frac{C}{T^{\frac{p_0}{1-p_0}}} \right) \|y^0\|_{D(A^*)'},$$

where we used (6.14) and (4.9).

By using Young's inequality, we obtain

$$\|q_{\lambda_1,k}\|_{L^2(0,T/2)} \leq C \exp\left(-\frac{\tilde{C}T}{2}\sigma(\gamma) + \frac{C}{T^{\frac{p_0}{1-p_0}}} \right) \|y^0\|_{D(A^*)'}.$$

Since $\sigma(\gamma)$ belongs to the k-th group, using point (iv) of Proposition 6.5 and (4.12), we deduce that

$$\sigma(\gamma) \geq (k-1)\frac{\hat{\rho}_2}{2} + C\lambda_1.$$

The estimate above becomes

$$\|q_{\lambda_1,k}\|_{L^2(0,T/2)} \leq C \exp\left(-\frac{\tilde{C}\lambda_1 T}{2} - (k-1)\frac{\hat{\rho}_2\tilde{C}T}{4} + \frac{C}{T^{\frac{p_0}{1-p_0}}} \right) \|y^0\|_{D(A^*)'}. \tag{6.18}$$

(6) Let us form the series

$$\hat{u}_{1,\lambda_1}(t) = \sum_{k=1}^{+\infty} q_{\lambda_1,k}(t).$$

Thanks to the estimate (6.18), this series converges normally in $L^2(0,T/2)$ with

$$\|\hat{u}_{1,\lambda_1}\|_{L^2(0,T/2)} \leq C \exp\left(-\frac{\tilde{C}\lambda_1 T}{2} + \frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \left(\sum_{k=1}^{+\infty} e^{-(k-1)\hat{\rho}_2 \tilde{C}T/4}\right) \|y^0\|_{D(A^*)'}.$$

By construction, we conclude that $\hat{u}_{1,\lambda_1}$ solves (6.15) for any $\lambda_1 \in \Lambda_1$, with the estimate

$$\|\hat{u}_{1,\lambda_1}\|_{L^2(0,T/2)} \leq C \exp\left(-\frac{\tilde{C}\lambda_1 T}{2} + \frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|_{D(A^*)'}. \tag{6.19}$$

(7) For any $\lambda_1 \in \Lambda_1$ we now have two situations:

- Case 1 : $\Sigma_{1,\lambda_1} = \Sigma^{\mathbb{R}}_{1,\lambda_1}$ and $\widehat{\Sigma}_{1,\lambda_1} = \emptyset$.
  We can define $\tilde{u}_{1,\lambda_1} \in L^2(0,T)$ as the extension by 0 of $\hat{u}_{1,\lambda_1}$ on $(0,T)$. It is clear that this function $\tilde{u}_{1,\lambda_1}$ solves the original moment problem (6.7) with the same estimate as in (6.19).
- Case 2 : $\Sigma_{1,\lambda_1} \neq \Sigma^{\mathbb{R}}_{1,\lambda_1}$ or $\widehat{\Sigma}_{1,\lambda_1} \neq \emptyset$. We introduce $S_0 = \left(\Sigma_{1,\lambda_1} \setminus \Sigma^{\mathbb{R}}_{1,\lambda_1}\right) \cup \widehat{\Sigma}_{1,\lambda_1}$ and we remark that this set is finite in the case (C1) we are considering here (see (6.13)).
  In that case, we apply Proposition 6.7 with $S = \Sigma_{1,\lambda_1}$ and S_0 defined above to build from $\hat{u}_{1,\lambda_1}$ a function $\tilde{u}_{1,\lambda_1} \in L^2(0,T)$ that solves (6.7) and with the estimate

$$\|\tilde{u}_{1,\lambda_1}\|_{L^2(0,T)} \leq \frac{f(\lambda_1)}{T} \exp\left(-\frac{\tilde{C}\lambda_1 T}{2} + \frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|_{D(A^*)'},$$

  where we have used that $p_0 > \theta_2$ and (6.14).
  The number $f(\lambda_1)$ in this estimate now depends on λ_1 but this Case 2 only happens for a finite number of values of λ_1 (indeed, for λ_1 large enough, we necessarily are in Case 1, see Remark 4.7).

All in all, we have solved (6.7) for any $\lambda_1 \in \Lambda_1$ with a uniform estimate

$$\|\tilde{u}_{1,\lambda_1}\|_{L^2(0,T)} \leq \frac{C}{T} \exp\left(-\frac{\tilde{C}\lambda_1 T}{2} + \frac{C}{T^{\frac{p_0}{1-p_0}}}\right) \|y^0\|_{D(A^*)'}.$$

This implies in particular the convergence of the series $\sum_{\lambda_1 \in \Lambda_1} (1+\lambda_1)^2 \|\tilde{u}_{1,\lambda_1}\|^2_{L^2(0,T)}$, with the expected estimate.

A similar proof gives the existence of suitable functions $\tilde{u}_{2,\lambda_2}$ for any $\lambda_2 \in \Lambda_2$, using this time the values of $\omega^k_{\gamma,2}$.

□

The proof of the remaining case is easier.

Proof of Proposition 6.3 in the case (C2). In that case we have $\sqrt{\Delta} \in i\mathbb{R}$ and it follows that the non resonance condition (6.5) is satisfied for $i = 1, 2$:

$$\left|\lambda_i + \theta - (\tilde{\lambda}_i + \tilde{\theta})\right|^2 = \left|\lambda_i - \tilde{\lambda}_i\right|^2 + \left|\sqrt{\Delta}\right|^2 \geq \left|\mathrm{Gap}(\Lambda_i)\right|^2.$$

Therefore, by Theorem 6.2 we can for instance impose that $\Phi^k_{\gamma,2}=0$ for every γ,k, so that (see (6.6))

$$\omega^k_{\gamma,1}=(-1)^{k+1}e^{-T\overline{\sigma(\gamma)}}\langle y^0,\Phi^k_{\gamma,1}\rangle_{D(A^*)',D(A^*)},\quad \omega^k_{\gamma,2}=0,\quad \forall\gamma\in\Gamma,\forall 0\le k\le k_\gamma. \tag{6.20}$$

We also could have imposed that $\Phi^k_{\gamma,1}=0$ for every γ,k, so that

$$\omega^k_{\gamma,1}=0,\quad \omega^k_{\gamma,2}=(-1)^{k+1}e^{-T\overline{\sigma(\gamma)}}\langle y^0,\Phi^k_{\gamma,2}\rangle_{D(A^*)',D(A^*)},\quad \forall\gamma\in\Gamma,\forall 0\le k\le k_\gamma, \tag{6.21}$$

The overall strategy developed in the proof of the previous case can then be used again, yet in a simple way since we can check that all the eigenvalue families appearing in the moment problems (6.7) and (6.8) satisfy usual gap conditions that read

$$\Sigma_{1,\lambda_1},\Sigma_{2,\lambda_2}\in\mathcal{L}_w(1,\hat\rho,\hat\theta,\hat\kappa),\quad \forall\lambda_1\in\Lambda_1,\lambda_2\in\Lambda_2.$$

Compare this with Proposition 6.9 in the general case.

We can therefore avoid using the block moment approach and simply employ the standard argument using estimates of biorthogonal families of complex exponentials as given in [23], [9] (yet slightly generalized in the same way as we discussed in Section 6.3).

Moreover, we see that the condition (6.20) (resp. (6.21)) leads to a control (u_1,u_2) satisfying $u_1=0$ (resp. $u_2=0$). This concludes the proof in that case as well. □

6.5. **Proof of Theorem 6.2**

We first observe that, by point (1) of Remark 4.10, it is enough to determine $\Phi^0_{\gamma,1},\Phi^0_{\gamma,2}$ for $\gamma\in\Gamma\setminus\widehat\Gamma^+$ and $\Phi^1_{\gamma,1},\Phi^1_{\gamma,2}$ for $\gamma\in\widehat\Gamma^-$. Indeed, the missing values can simply be defined, for $\gamma=(+,\lambda_1,\lambda_2)\in\widehat\Gamma^+$, by

$$\Phi^0_{(+,\lambda_1,\lambda_2),1}=\Phi^0_{(-,\lambda_1,\lambda_2),1},\quad \Phi^0_{(+,\lambda_1,\lambda_2),2}=\Phi^0_{(-,\lambda_1,\lambda_2),2},$$

$$\Phi^1_{(+,\lambda_1,\lambda_2),1}=\Phi^1_{(-,\lambda_1,\lambda_2),1},\quad \Phi^1_{(+,\lambda_1,\lambda_2),2}=\Phi^1_{(-,\lambda_1,\lambda_2),2}.$$

It is straightforward to see that the required properties will be satisfied.

Since $\widehat\Gamma^-\subset\Gamma\setminus\widehat\Gamma^+$, we are led to study carefully the structure of $\Gamma\setminus\widehat\Gamma^+$. More precisely, the idea of the proof is to show that $\Gamma\setminus\widehat\Gamma^+$ can be written as a disjoint infinite union of finite subsets such that we can easily solve by induction, in each of those sets, the required equations of point (i) of Theorem 6.2 with the desired conditions given in point (ii). Moreover, we need to ensure that elements belonging to two different such subsets are never concerned by the condition. This analysis will make use of elementary graph theory notions. We recall that we have associated to our set $\Gamma\setminus\widehat\Gamma^+$ a structure of graph in Definition 4.11.

The goal of this section is to establish the following result.

Theorem 6.10. *There exists $\hat\rho>0$ small enough such that the graph $(\Gamma\setminus\widehat\Gamma^+,\mathcal{E}_{\hat\rho})$ is a forest and such that, for any path $\gamma_0\overset{\hat\rho}{\iff}\gamma_1\overset{\hat\rho}{\iff}\cdots\overset{\hat\rho}{\iff}\gamma_{n-1}\overset{\hat\rho}{\iff}\gamma_n$, we have*

$$n\le 2\kappa_0\min_{0\le i\le n}|\sigma(\gamma_i)|, \tag{6.22}$$

($\kappa_0>0$ is introduced in (4.16)) and

$$\max_{0\le i\le n}|\sigma(\gamma_i)|\le 2\min_{0\le i\le n}|\sigma(\gamma_i)|. \tag{6.23}$$

Besides, when $d = 1$ and the non resonance condition (6.5) is satisfied for some $i \in \{1, 2\}$, then $\hat{\rho}$ can be chosen in such a way that all the trees are either single nodes or reduced to a path of length one with an arrow of type $\xleftrightarrow[\lambda_i]{\hat{\rho}}$.

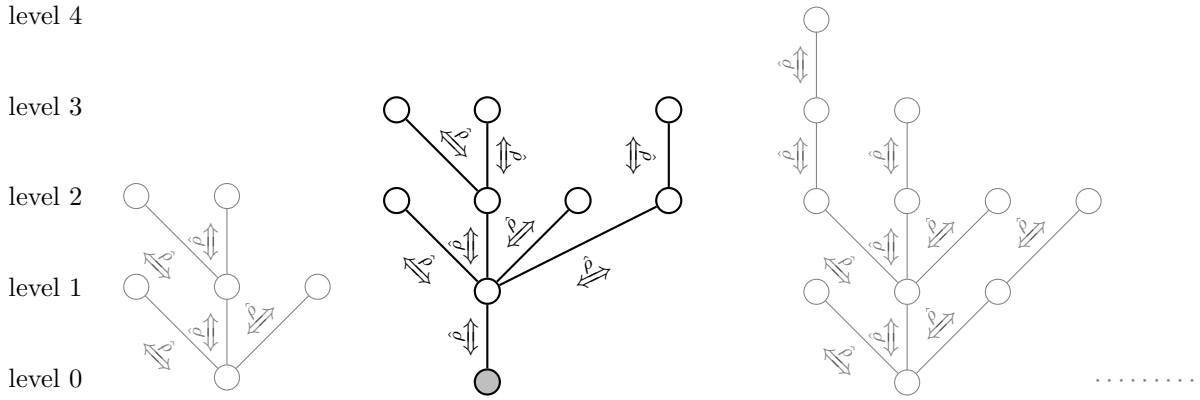

FIGURE 2. The forest structure of $(\Gamma \setminus \widehat{\Gamma}^+, \mathcal{E}_{\hat{\rho}})$. In each tree, a root node is fixed, and the other nodes are organised by levels corresponding to their distance to the root. One of the trees is emphasized, as well as the chosen root.

Theorem 6.2 will then be a consequence of this result.

Proof of Theorem 6.2. We construct $\Phi^0_{\gamma,1}$ and $\Phi^0_{\gamma,2}$ in each tree as follows.

- Pick any node γ of a tree to serve as a root (represented in gray in Figure 2) and define arbitrarily the corresponding values, for instance as follows

$$\Phi^0_{\gamma,1} = 0, \quad \Phi^0_{\gamma,2} = \Phi^0_\gamma.$$

- If the tree is reduced to one node, we are done. Otherwise, consider any node $\tilde{\gamma} \neq \gamma$ such that $\tilde{\gamma} \xleftrightarrow{\hat{\rho}} \gamma$ and solve accordingly the corresponding equations:

$$\begin{cases} \Phi^0_{\tilde{\gamma},1} = \Phi^0_{\gamma,1}, \quad \Phi^0_{\tilde{\gamma},2} = \Phi^0_{\tilde{\gamma}} - \Phi^0_{\tilde{\gamma},1}, & \text{if } \gamma \xleftrightarrow[\lambda_1]{\hat{\rho}} \tilde{\gamma}, \\ \Phi^0_{\tilde{\gamma},2} = \Phi^0_{\gamma,2}, \quad \Phi^0_{\tilde{\gamma},1} = \Phi^0_{\tilde{\gamma}} - \Phi^0_{\tilde{\gamma},2}, & \text{if } \gamma \xleftrightarrow[\lambda_2]{\hat{\rho}} \tilde{\gamma}. \end{cases}$$

  This way, we can determine all the values associated to the nodes at distance 1 of the root (first level of nodes in Figure 2).

  Repeating this process for each level, we construct $\Phi^0_{\gamma,1}$ and $\Phi^0_{\gamma,2}$ in such a way that the properties (i) and (ii) of Theorem 6.2 are satisfied. Note that this construction is not ambiguous precisely because of the fact that our graph is a forest, so that there is a unique path in the graph that connects any two nodes.

  Besides, if all the trees are either single nodes or reduced to a path of length one with an arrow of the same type $\xleftrightarrow[\lambda_1]{\hat{\rho}}$ (resp. $\xleftrightarrow[\lambda_2]{\hat{\rho}}$), then we can take $\Phi^0_{\gamma,1} = 0$ (resp. $\Phi^0_{\gamma,2} = 0$) for every γ.

- It remains to check the corresponding estimate stated in (iii). Below, we denote by C a positive number that may change from line to line but that does not depend on $\gamma, \tilde{\gamma}$. First of all, for the first picked node γ we have the common estimate (recall (4.11))

$$\max\left\{\left\|\Phi^0_{\gamma,1}\right\|_{D(A^*)}, \left\|\Phi^0_{\gamma,2}\right\|_{D(A^*)}\right\} \leq \left\|\Phi^0_{\gamma}\right\|_{D(A^*)}.$$

Then, for any node $\tilde{\gamma} \neq \gamma$ such that $\tilde{\gamma} \overset{\hat{\rho}}{\Longleftrightarrow} \gamma$, we have

$$\begin{aligned}\max\left\{\left\|\Phi^0_{\tilde{\gamma},1}\right\|_{D(A^*)}, \left\|\Phi^0_{\tilde{\gamma},2}\right\|_{D(A^*)}\right\} &\leq \left\|\Phi^0_{\tilde{\gamma}}\right\|_{D(A^*)} + \max\left\{\left\|\Phi^0_{\gamma,1}\right\|_{D(A^*)}, \left\|\Phi^0_{\gamma,2}\right\|_{D(A^*)}\right\} \\ &\leq \left\|\Phi^0_{\tilde{\gamma}}\right\|_{D(A^*)} + \left\|\Phi^0_{\gamma}\right\|_{D(A^*)}.\end{aligned}$$

Repeating this process, we see that, for any node $\tilde{\gamma}$ in the same tree as γ, we have (recall the estimate (4.17))

$$\max\left\{\left\|\Phi^0_{\tilde{\gamma},1}\right\|_{D(A^*)}, \left\|\Phi^0_{\tilde{\gamma},2}\right\|_{D(A^*)}\right\} \leq \sum_{i=0}^{n} \left\|\Phi^0_{\gamma_i}\right\|_{D(A^*)} \leq C \sum_{i=0}^{n} e^{C|\sigma(\gamma_i)|^\nu},$$

where

$$\gamma_0 \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_1 \overset{\hat{\rho}}{\Longleftrightarrow} \cdots \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_{n-1} \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_n,$$

is the unique path from $\gamma_0 = \gamma$ to $\gamma_n = \tilde{\gamma}$. In particular, by using (6.23) and (6.22), we deduce

$$\max\left\{\left\|\Phi^0_{\tilde{\gamma},1}\right\|_{D(A^*)}, \left\|\Phi^0_{\tilde{\gamma},2}\right\|_{D(A^*)}\right\} \leq C\,|\sigma(\tilde{\gamma})|\, e^{C|\sigma(\tilde{\gamma})|^\nu}.$$

The proof is similar for $\Phi^1_{\gamma,1}$ and $\Phi^1_{\gamma,2}$ for $\gamma \in \widehat{\Gamma}$.

□

To prove Theorem 6.10, we need the following basic lemma concerning the edges of our graph.

Lemma 6.11. *There exist $\rho^* > 0$ small enough and $\sigma^* \geq 0$ large enough such that, for any $\gamma, \tilde{\gamma} \in \Gamma \setminus \widehat{\Gamma}^+$ with $\gamma \overset{\rho^*}{\Longleftrightarrow} \tilde{\gamma}$, we have:*

(i) if $|\sigma(\gamma)| \geq \sigma^*$ and $|\sigma(\tilde{\gamma})| \geq \sigma^*$, then

$$s(\gamma) = s(\tilde{\gamma}) \quad \Longrightarrow \quad \gamma = \tilde{\gamma}.$$

Besides, if $d = 1$ we can take $\sigma^* = 0$.

(ii) if $s(\gamma) = +$ and $s(\tilde{\gamma}) = -$, then

$$\begin{cases}\lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) & \Longrightarrow \quad \lambda_2(\gamma) < \lambda_2(\tilde{\gamma}), \\ \lambda_2(\gamma) = \lambda_2(\tilde{\gamma}) & \Longrightarrow \quad \lambda_1(\gamma) < \lambda_1(\tilde{\gamma}).\end{cases}$$

Besides, if $d = 1$ and if the non resonance condition (6.5) is satisfied for some $i \in \{1, 2\}$, then in fact the situation $\lambda_{3-i}(\gamma) = \lambda_{3-i}(\tilde{\gamma})$ never occurs.

Note already that the additional property mentioned in the statement of Theorem 6.10 when $d = 1$ immediately follows from this lemma.

Proof of Lemma 6.11. (i) We consider the case $\gamma = (+, \lambda_1, \lambda_2)$ and $\tilde{\gamma} = (+, \lambda_1, \tilde{\lambda}_2)$, since the other cases are similar. From (4.8), we have

$$\sigma(\gamma) - \sigma(\tilde{\gamma}) = \max(d, 1)(\lambda_2 - \tilde{\lambda}_2) + \varepsilon_{\lambda_1+\lambda_2} - \varepsilon_{\lambda_1+\tilde{\lambda}_2},$$

and since by assumption we have $|\sigma(\gamma) - \sigma(\tilde{\gamma})| \leq \rho^*$, we end up with

$$\max(d, 1)|\lambda_2 - \tilde{\lambda}_2| \leq \rho^* + |\varepsilon_{\lambda_1+\lambda_2}| + \left|\varepsilon_{\lambda_1+\tilde{\lambda}_2}\right|.$$

Using (4.12) and (4.7), we see that we can choose σ^* large enough, depending on ρ^*, to ensure that

$$|\varepsilon_{\lambda_1+\lambda_2}| + \left|\varepsilon_{\lambda_1+\tilde{\lambda}_2}\right| \leq \rho^*. \tag{6.24}$$

Choosing first $2\rho^* < \max(d, 1)\,\mathrm{Gap}(\Lambda_2)$ and then σ^* as above, we obtain

$$|\lambda_2 - \tilde{\lambda}_2| < \mathrm{Gap}(\Lambda_2),$$

which, by definition of $\mathrm{Gap}(\Lambda_2)$, implies that $\lambda_2 = \tilde{\lambda}_2$.

Additionally, when $d = 1$ we have $\varepsilon_\lambda = 0$ for any λ by construction (see the proof of Proposition 4.5). Thus, (6.24) holds for any $\rho^* \geq 0$ and we can simply take $\sigma^* = 0$.

(ii) We consider the case $\gamma = (+, \lambda_1, \lambda_2)$ and $\tilde{\gamma} = (-, \lambda_1, \tilde{\lambda}_2)$, since the other case is similar.

We write $\sqrt{\Delta_\lambda} = a_\lambda + ib_\lambda$ with $a_\lambda, b_\lambda \geq 0$. From the expression (4.5), we see that the condition $|\sigma(\gamma) - \sigma(\tilde{\gamma})| \leq \rho^*$ implies that

$$\left|(1 + d)(\lambda_2 - \tilde{\lambda}_2) + a_{\lambda_1+\lambda_2} + a_{\lambda_1+\tilde{\lambda}_2}\right| \leq 2\rho^*, \tag{6.25}$$

and

$$\left|b_{\lambda_1+\lambda_2} + b_{\lambda_1+\tilde{\lambda}_2}\right| \leq 2\rho^*. \tag{6.26}$$

First of all, observe that b_λ does not depend on λ for λ large enough (recall (4.2)). Taking then $\rho^* > 0$ small enough, we see that the condition (6.26) implies that $b_{\lambda_1+\lambda_2} = b_{\lambda_1+\tilde{\lambda}_2} = 0$. Since $s(\gamma) = +$ and since no node in the graph belongs to $\widehat{\Gamma}^+$, we have $\gamma \notin \widehat{\Gamma}$ and thus $\Delta_{\lambda_1+\lambda_2} \neq 0$. As $b_{\lambda_1+\lambda_2} = 0$ this means that we necessarily have

$$a_{\lambda_1+\lambda_2} \neq 0.$$

Introducing

$$\delta = \inf_{\substack{\lambda \in \Lambda_1+\Lambda_2 \\ a_\lambda \neq 0}} a_\lambda,$$

we have $\delta > 0$ (recall (4.2)), and the condition (6.25) yields

$$\lambda_2 - \tilde{\lambda}_2 \leq \frac{1}{1+d}\left(2\rho^* - \delta\right).$$

It follows that we can find a positive $\rho^* < \delta/2$ to obtain the claimed inequality $\lambda_2 < \tilde{\lambda}_2$.

In addition, if $d = 1$ and if the non resonance condition (6.5) is satisfied for instance for $i = 2$, then we see that we can take $\rho^* > 0$ small enough so that (6.25) leads to an impossibility. This means that the case that we considered, namely, $\gamma = (+, \lambda_1, \lambda_2)$ and $\tilde{\gamma} = (-, \lambda_1, \tilde{\lambda}_2)$, never occurs under this condition. □

Let us now finally prove the desired result.

Proof of Theorem 6.10. 1) Let us first prove that the graph is a forest. By definition, we have to show that it has no $\hat{\rho}$-cycle. Assume by contradiction that there exists a $\hat{\rho}$-cycle:

$$\gamma_0 \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_1 \overset{\hat{\rho}}{\Longleftrightarrow} \cdots \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_{n-1} \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_0,$$

where, by definition, $n \geq 3$ and $\gamma_i \neq \gamma_j$ for $i \neq j$. Let $\gamma_{\min} \in \{\gamma_0, \dots, \gamma_{n-1}\}$ be such that

$$|\sigma(\gamma_{\min})| = \min_{i \in \{0,\dots,n-1\}} |\sigma(\gamma_i)|.$$

We distinguish two cases.

- Case 1 : $|\sigma(\gamma_{\min})| < \sigma^*$
  We claim that in this case, for $\hat{\rho}$ well chosen, we have a 0-cycle, that is

$$\sigma(\gamma_i) = \sigma(\gamma_{\min}), \quad \forall i \in \{0, \dots, n-1\}. \tag{6.27}$$

  This will establish a contradiction with point 2 of Lemma 5.3. If (6.27) is not true, we can take the smallest index $i_0 \in \{0, \dots, n-2\}$ such that

$$\begin{cases} \sigma(\gamma_{i_0}) = \sigma(\gamma_{\min}), \\ \sigma(\gamma_{i_0+1}) \neq \sigma(\gamma_{\min}). \end{cases} \tag{6.28}$$

  In particular we have

$$|\sigma(\gamma_{i_0})| < \sigma^*. \tag{6.29}$$

  Moreover, by definition of $\overset{\hat{\rho}}{\Longleftrightarrow}$, we have

$$|\sigma(\gamma_{i_0+1}) - \sigma(\gamma_{i_0})| \leq \hat{\rho}, \tag{6.30}$$

  and, taking $\hat{\rho} \leq \sigma^*$, we deduce that

$$|\sigma(\gamma_{i_0+1})| < 2\sigma^*. \tag{6.31}$$

  Therefore, $|\sigma(\gamma_{i_0}) - \sigma(\gamma_{i_0+1})| \geq \delta^*$, where

$$\delta^* = \inf \left\{ |\sigma(\gamma) - \sigma(\tilde{\gamma})|, \quad \text{s.t.} \quad \gamma, \tilde{\gamma} \in \Gamma \setminus \widehat{\Gamma}^+, \quad |\sigma(\gamma)| < \sigma^*, \quad |\sigma(\tilde{\gamma})| < 2\sigma^*, \quad \sigma(\gamma) \neq \sigma(\tilde{\gamma}) \right\}.$$

  Since this quantity is the minimum of a finite number of positive values, it satisfies $\delta^* > 0$. Therefore we can impose on the parameter $\hat{\rho}$ the additional condition $\hat{\rho} < \delta^*$ to obtain a contradiction with (6.30). This establishes (6.27).
- Case 2 : $|\sigma(\gamma_{\min})| \geq \sigma^*$
  By item (i) of Lemma 6.11, we have

$$s(\gamma_{i+1}) = -s(\gamma_i), \quad \forall i \in \{0, \dots, n-1\}, \tag{6.32}$$

  where we introduced $\gamma_n = \gamma_0$ for convenience. Note in particular that n is even.

– Let us now show that the kinds of the arrows in the $\hat{\rho}$-cycle necessarily alternate, that is

$$\begin{cases} \gamma_i \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_{i+1} & \Longrightarrow \quad \gamma_{i+1} \xleftrightarrow[\lambda_1]{\hat{\rho}} \gamma_{i+2}, \\ \gamma_i \xleftrightarrow[\lambda_1]{\hat{\rho}} \gamma_{i+1} & \Longrightarrow \quad \gamma_{i+1} \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_{i+2}. \end{cases} \tag{6.33}$$

We will show the first of these properties, the proof of the other being similar.
Assume, by contradiction, that we have

$$\gamma_i \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_{i+1} \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_{i+2}.$$

Since $\gamma_i \neq \gamma_{i+2}$, we deduce that $\gamma_i \xleftrightarrow[\lambda_2]{2\hat{\rho}} \gamma_{i+2}$ and, choosing $2\hat{\rho} \leq \rho^*$, we can use again item (i) of Lemma 6.11 to deduce that $s(\gamma_i) \neq s(\gamma_{i+2})$, which is a contradiction with (6.32).

– Let us assume for instance that $s(\gamma_0) = +$ and $\gamma_0 \xleftrightarrow[\lambda_2]{\hat{\rho}} \gamma_1$, the other cases being similar. Recalling (6.32), and since $\hat{\rho} \leq \rho^*$, by item (ii) of Lemma 6.11, we thus have

$$\lambda_1(\gamma_0) < \lambda_1(\gamma_1).$$

By using (6.33) we see that the second arrow in the cycle is $\gamma_1 \xleftrightarrow[\lambda_1]{\hat{\rho}} \gamma_2$, which gives $\lambda_1(\gamma_1) = \lambda_1(\gamma_2)$. By induction, we eventually obtain

$$\lambda_1(\gamma_0) < \lambda_1(\gamma_1) = \lambda_1(\gamma_2) < \ldots < \lambda_1(\gamma_{n-1}) = \lambda_1(\gamma_n).$$

This is impossible since, by definition of the cycle we have $\gamma_n = \gamma_0$.
This shows that there is no $\hat{\rho}$-cycle in the case $|\sigma(\gamma_{\min})| \geq \sigma^*$ either.

2) Let us now prove the estimates (6.22) and (6.23). We consider a path:

$$\gamma_0 \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_1 \overset{\hat{\rho}}{\Longleftrightarrow} \cdots \overset{\hat{\rho}}{\Longleftrightarrow} \gamma_n,$$

where $n \geq 1$ and $\gamma_i \neq \gamma_j$ for $i \neq j$ and we define

$$m = \min_{0 \leq i \leq n} |\sigma(\gamma_i)|.$$

Let $i_0 \in \{0, \ldots, n\}$ such that $|\sigma(i_0)| = m$. By definition of $\overset{\hat{\rho}}{\Longleftrightarrow}$ we see that, for any $i \in \{0, \ldots, n\}$, we have

$$|\sigma(\gamma_i) - \sigma(\gamma_{i_0})| \leq \hat{\rho}|i - i_0| \leq \hat{\rho} n,$$

and therefore

$$|\sigma(\gamma_i)| \leq m + \hat{\rho} n. \tag{6.34}$$

Since by definition all the elements in the path are distinct, we have

$$N(m + \hat{\rho} n) \geq n,$$

where N is the counting function defined in (4.15). Using (4.16), we get

$$n \leq \kappa_0(m + \hat{\rho} n),$$

and, choosing $\hat{\rho} \leq 1/(2\kappa_0)$, we obtain the first estimate (6.22):

$$n \leq 2\kappa_0 m.$$

Plugging this inequality into (6.34), we obtain the second estimate (6.23).

□

6.6. **More general control operators**

In this section we extend the result of Theorem 3.2 to more general control operators thanks to the so-called Lebeau-Robbiano method.

More precisely, we prove Theorem 3.3, which is concerned with control operators $B \in \mathcal{L}(U, D(A^*)')$ whose adjoint is of the following form on $\mathbb{C}^2 \otimes D(A_1) \otimes D(A_2)$:

$$B^* = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes L_1^* \otimes B_2^* \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B_1^* \otimes L_2^* \end{pmatrix},$$

for some operators $L_i^* \in \mathcal{L}(H_i)$ $(i = 1, 2)$ subject to the Lebeau-Robbiano type estimates (3.3).

This case is much harder to handle than the previous case of control operator B_{ref} (corresponding to $L_1 = L_2 = \mathrm{Id}$) because we lose some important orthogonal properties. For this operator B, the proof given in Section 6.1 cannot be simply adapted because the expansions of the controls u_i into Fourier series no longer seem usable.

However, thanks to the estimate of the control cost previously obtained for the system $(-A, B_{\mathrm{ref}})$, we can use the Lebeau-Robbiano method to deal with this more general case.

We recall that the purpose of the Lebeau-Robbiano method is precisely to allow the change of the control operator for null controllable systems under some conditions. More precisely,

Theorem 6.12 (Lebeau-Robbiano method)**.** *Let $\mathcal{B}_{\mathrm{ref}} \in \mathcal{L}(U, D(\mathcal{A}^*)')$. Assume that there exists a family of operators $\{P_\mu\}_{\mu>0} \subset \mathcal{L}(D(\mathcal{A}^*))$ such that*

$$e^{-t\mathcal{A}^*}(\operatorname{Ran} P_\mu) \subset \operatorname{Ran} P_\mu, \quad \forall t \geq 0, \forall \mu > 0, \qquad \sup_{\mu>0} \|P_\mu\|_{\mathcal{L}(D(\mathcal{A}^*))} < +\infty, \tag{6.35}$$

and that satisfies the following three key properties for some $C_1, C_2 > 0$:

(i) ***Controllability of the reference system:*** *The system $(-\mathcal{A}, \mathcal{B}_{\mathrm{ref}})$ is null controllable at any time $T > 0$, with control cost satisfying, for some $p_0 \in (0, 1)$,*

$$\mathrm{cost}_T(-\mathcal{A}, \mathcal{B}_{\mathrm{ref}}) \leq C_1 \exp\left(\frac{C_1}{T^{\frac{p_0}{1-p_0}}}\right), \quad \forall T > 0.$$

(ii) ***A relative observability property*** *(of the operator $\mathcal{B}^*$ with respect to the reference operator $\mathcal{B}_{\mathrm{ref}}^*$): there exists $\eta \in [0, 1)$ such that*

$$\|\mathcal{B}_{\mathrm{ref}}^* z\|_U \leq C_1 e^{C_1 \mu^\eta} \|\mathcal{B}^* z\|_U, \quad \forall \mu > 0, \quad \forall z \in \operatorname{Ran} P_\mu, \tag{6.36}$$

(iii) ***A dissipation property:***

$$\left\| e^{-t\mathcal{A}^*} z \right\|_{D(\mathcal{A}^*)} \leq C_1 e^{-C_2 \mu t} \|z\|_{D(\mathcal{A}^*)}, \quad \forall t \geq 0, \quad \forall \mu > 0, \quad \forall z \in \operatorname{Ran}(\mathrm{Id} - P_\mu). \tag{6.37}$$

Then, the system $(-\mathcal{A},\mathcal{B})$ is null controllable in time T for every $T>0$, with control cost satisfying, for some $C>0$,

$$\mathrm{cost}_T(-\mathcal{A},\mathcal{B})\leq C\exp\left(\frac{C}{T^{\frac{p}{1-p}}}\right),\quad \forall T>0,$$

with $p=\max(p_0,\eta)$.

Theorem 6.12 is a simple adaptation of the abstract Lebeau-Robbiano method established in [30, Theorem 2.2] (see pp. 1469-1470). The main difference is that we consider here operators P_μ which are not necessarily orthogonal projections, which is an important feature when considering systems of PDEs since they are not self-adjoint in general. However, the last property in (6.35) is enough to make the same proof work.

Let us now turn to the proof of the desired result:

Proof of Theorem 3.3. We simply show that we are in the configuration of Theorem 6.12 with $\mathcal{A}=A$, $\mathcal{B}_0=B_{\mathrm{ref}}$, $\mathcal{B}=B$ and

$$P_\mu z=\sum_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+\\ |\sigma(\gamma)|\leq\mu\\ 0\leq k\leq k_\mu}}\left\langle z,\Phi_\gamma^{k,*}\right\rangle_{D(A^*)}\tilde{\Phi}_\gamma^k,$$

where $\left\{\Phi_\gamma^{k,*}\right\}_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+\\ 0\leq k\leq k_\gamma}}$ is the biorthogonal family, in $D(A^*)$, of the Riesz basis $\left\{\tilde{\Phi}_\gamma^k\right\}_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+\\ 0\leq k\leq k_\gamma}}$, where $\tilde{\Phi}_\gamma^k=\Phi_\gamma^k/\left\|\Phi_\gamma^k\right\|_{D(A^*)}$ (see Proposition 4.9).

- Note that P_μ is not an orthogonal projection since we only have a Riesz basis and not necessarily a Hilbert basis ($P_\mu^*\neq P_\mu$, unless $M^*=M$). However, this family of projections clearly satisfies the conditions in (6.35).
- The controllability of the reference system was established in Theorem 3.2 for any $p_0>\max\{\theta_1,\theta_2,\nu\}$.
- The relative observability property (6.36) holds with $\eta=\max(\eta_1,\eta_2)$ thanks to our assumption (3.3) on the operators L_i. Note that this has to be checked only on linear combinations of eigenfunctions of A^* since the observation operators B^* and B_{ref}^* do not see the generalized eigenfunctions $\tilde{f}_\gamma$: $B^*\Phi_\gamma^1=B_{\mathrm{ref}}^*\Phi_\gamma^1=0$ for $\gamma\in\widehat{\Gamma}^+$ (see (4.14)).
- Finally, the dissipation property (6.37) is easy to check because, from the Riesz basis property, the semigroup of $-A^*$ is explicitely given for every $t\geq 0$ and $z\in\mathrm{Ran}\,(\mathrm{Id}-P_\mu)$ by

$$e^{-tA^*}z=\sum_{\substack{\gamma\in\Gamma\setminus\widehat{\Gamma}^+\\ |\sigma(\gamma)|>\mu}}e^{-\sigma(\gamma)t}\left\langle z,\Phi_\gamma^{0,*}\right\rangle_{D(A^*)}\tilde{\Phi}_\gamma^0+\sum_{\substack{\gamma\in\widehat{\Gamma}^+\\ |\sigma(\gamma)|>\mu}}e^{-\sigma(\gamma)t}\left\langle z,\Phi_\gamma^{1,*}\right\rangle_{D(A^*)}\left(\tilde{\Phi}_\gamma^1-t(A^*-\sigma(\gamma))\tilde{\Phi}_\gamma^1\right).$$

□

7. Application to the boundary null controllability of coupled linear parabolic systems on cartesian geometries

As mentioned in the introduction, our main motivation for the abstract results proved in this paper is their application to actual multi-dimensional boundary null controllability issues for coupled parabolic systems. A typical such system to which our general analysis applies is the following 2D two-component system controlled

from the boundary by only one control:

$$\begin{cases} \dfrac{\partial y_1}{\partial t} - d\ \mathrm{div}\,(K(x)\nabla y_1) = m_{11}y_1 + m_{12}y_2 & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial y_2}{\partial t} - \quad \mathrm{div}\,(K(x)\nabla y_2) = m_{21}y_1 + m_{22}y_2 & \text{in } (0,T)\times\Omega, \\ y_1 = 1_\gamma u, \quad y_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0 & \text{in } \Omega, \end{cases} \tag{7.1}$$

where the domain $\Omega \subset \mathbb{R}^2$ is the rectangle defined in (1.2) (see Figure 1) and the diffusion tensor has the following form

$$K(x) = \begin{pmatrix} k_1(x_1) & 0 \\ 0 & k_2(x_2) \end{pmatrix},$$

with $k_i \in W^{1,\infty}(0, X_i)$, $\inf_{(0,X_i)} k_i > 0$ and $d > 0$ is a parameter accounting for the ratio of diffusion between the two components in the system.

The first equation is controlled from the boundary on a non empty relative open subset γ of $\partial\Omega$. On the other hand, the second equation has no control, but it is coupled to the first equation via a constant internal coupling term, so that it is indirectly controlled, as soon as $m_{21} \neq 0$.

We recall that the case $d = 1$ and $K(x) = \mathrm{Id}$ was studied in the literature:

- The approximate controllability of the system (7.1) was studied in [31] when the underlying operator is the Laplacian.

  Notably, it was established in [31, Theorem 2.15] that this system is approximately controllable in time T for every $T > 0$ if the Kalman condition $m_{21} \neq 0$ holds and if γ satisfies the geometric condition (7.4), which was introduced in this very paper.

  On the other hand, when γ intersects only one face of the boundary $\partial\Omega$, say for instance $\gamma = \omega_1 \times \{0\}$, then it was shown in [31, Theorem 2.14] that the approximate controllability of the system is equivalent to the approximate controllability of the 1D reduced system

$$\begin{cases} \dfrac{\partial y_1}{\partial t} - \dfrac{\partial^2 y_1}{\partial x^2} = m_{11}y_1 + m_{12}y_2, & \text{in } (0,T)\times(0,X_1), \\ \dfrac{\partial y_2}{\partial t} - \dfrac{\partial^2 y_2}{\partial x^2} = m_{21}y_1 + m_{22}y_2, & \text{in } (0,T)\times(0,X_1), \\ y_1 = 1_{\{0\}}u, \quad y_2 = 0, & \text{on } (0,T)\times\{0, X_1\}, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \text{in } (0,X_1). \end{cases} \tag{7.2}$$

- This second result of [31] was then extended to the null controllability property in [9, Theorem 1.2].
- As already mentioned, the approximate and null controllability of the one-dimensional system (7.2) were studied in the seminal work [23]. More precisely, it was shown in [23, Theorem 1.1] (resp. [23, Theorem 5.2]) that this system is null (resp. approximately) controllable in time T if, and only if, the Kalman condition $m_{21} \neq 0$ holds and we have the following "non resonance" condition:

$$\left(\lambda + \theta = \tilde\lambda + \tilde\theta \quad \Longrightarrow \quad \lambda = \tilde\lambda \text{ and } \theta = \tilde\theta\right), \quad \forall \lambda, \tilde\lambda \in \sigma\left(-\frac{\partial^2}{\partial x^2}\right), \forall \theta, \tilde\theta \in \sigma(M^*). \tag{7.3}$$

Remark 7.1. In the case of the Laplace operator considered in that reference, the non resonance condition (7.3) is in fact equivalent to our condition (6.5).

Note that, if M has only one eigenvalue, then this condition (7.3) is automatically satisfied.

However, in the case where M has two distinct eigenvalues, the Kalman condition is not sufficient to ensure null controllability and it is needed to assume that (7.3) holds.

On the other hand, Theorem 3.3 leads to the following new result.

Theorem 7.2. *Assume that the Kalman condition $m_{21} \neq 0$ holds and that there exist two non empty open subsets $\omega_1 \subset (0, X_1)$ and $\omega_2 \subset (0, X_2)$, such that (see e.g. Figure 1)*

$$(\omega_1 \times \{0\}) \cup (\{0\} \times \omega_2) \subset \gamma. \tag{7.4}$$

Then, for any $y^0 \in H^{-1}(\Omega)^2$ the system (7.1) is null controllable in time T for every $T > 0$, with the estimate

$$\|u\|_{L^2((0,T)\times\partial\Omega)} \leq C \exp\left(\frac{C}{T}\right) \|y^0\|_{H^{-1}(\Omega)^2}.$$

To the best of our knowledge, Theorem 7.2 is the first and only result concerning the controllability properties of the two-dimensional system (7.1) for any value of the ratio of diffusions $d > 0$.

Remark 7.3. Combined with the results of the literature recalled just before the statement of Theorem 7.2, we see that this result shows a very strong influence of the geometry of the control domain: there are some two-dimensional systems (7.1) which are null controllable in arbitrary small time if two perpendicular faces of the boundary are controlled, whereas they are not even approximately controllable if the control acts on only one face. An explicit example of such systems is

$$\begin{cases} \dfrac{\partial y_1}{\partial t} - \Delta y_1 = 0 & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial y_2}{\partial t} - \Delta y_2 = y_1 + 3y_2 & \text{in } (0,T)\times\Omega, \end{cases}$$

posed on the square domain $\Omega = (0,\pi)^2$, for which we can check that the non resonance condition (7.3) fails for both $i = 1$ and $i = 2$.

Proof of Theorem 7.2. This result will be a straightforward consequence of Theorem 3.3 once we will have checked that we are under the framework of Section 2.2. We observe also that it is enough to consider the case

$$\gamma = (\omega_1 \times \{0\}) \cup (\{0\} \times \omega_2). \tag{7.5}$$

Indeed, the system (7.1) corresponds to the abstract control system (2.1) with the following functional framework

- The state space is
$$H = L^2(\Omega)^2 = \mathbb{C}^2 \,\widehat{\otimes}\, L^2(0,X_1) \,\widehat{\otimes}\, L^2(0,X_2).$$
- The operator A is the closure of the operator
$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) - M \otimes \mathrm{Id} \otimes \mathrm{Id},$$
with domain $\mathbb{C}^2 \otimes D(A_1) \otimes D(A_2)$, where, for $i = 1, 2$, A_i is the one-dimensional and scalar positive Dirichlet Laplacian on the space $H_i = L^2(0, X_i)$:
$$A_i = -\frac{\partial}{\partial x_i}\left(k_i(x_i)\frac{\partial}{\partial x_i}\,\cdot\,\right), \tag{7.6}$$

with domain $D(A_i) = H^2(0, X_i) \cap H_0^1(0, X_i)$.

- For γ as in (7.5), the control space can be taken as

$$U = L^2(0, X_1) \times L^2(0, X_2),$$

where for each $(u_1, u_2) \in U$ a control u for (7.1) is simply

$$u(x) = \begin{cases} u_1(x_1), & \text{if } x_1 \in \omega_1 \text{ and } x = (x_1, 0) \\ u_2(x_2), & \text{if } x_2 \in \omega_2 \text{ and } x = (0, x_2), \\ 0, & \text{otherwise.} \end{cases}$$

- For $i = 1, 2$, we introduce $B_i^* \in \mathcal{L}(D(A_i), \mathbb{C})$ to be the one-dimensional scalar operator

$$B_i^* z = -k_i(0) \frac{\partial z}{\partial x}(0),$$

and $L_i^* \in \mathcal{L}(H_i)$ is simply given by

$$L_i^* z = 1_{\omega_i} z.$$

Then, it is easily checked that the control operator B defined as in (2.17) is such that, for any $u \in U$ and any $\in D(A^*)$, we have

$$\langle Bu, z \rangle_{D(A^*)', D(A^*)} = -\int_{\omega_1} u_1(x_1) k_2(0) \overline{\frac{\partial z_1}{\partial x_2}}(x_1, 0)\, dx_1 - \int_{\omega_2} u_2(x_2) k_1(0) \overline{\frac{\partial z_1}{\partial x_1}}(0, x_2)\, dx_2.$$

Let us now check the assumptions of Section 2.2.

- We recall that A_i is a positive self-adjoint operator with compact resolvent and that its spectrum satisfies
  - the gap condition
$$\mathrm{Gap}(\Lambda_i) > 0,$$
  - the counting function estimate
$$\exists C_i > 0, \quad N_{\Lambda_i}(r) \leq C_i \sqrt{r}, \quad \forall r > 0.$$

  Those are very classical results; a self-contained proof is for instance given in [3, Section 2].

  Note in particular that, since $\theta_1 = \theta_2 = 1/2$, the condition $\theta_1 + \theta_2 \leq 1$ is fulfilled.
- It is clear that $(-A_i, B_i)$ satisfies the Fattorini-Hautus test and that the eigenfunction ϕ_{i,λ_i} of A_i associated with the eigenvalue $\lambda_i \in \Lambda_i$ which is such that $B_i^* \phi_{i,\lambda_i} = 1$, that is such that

$$-k_i(0) \phi_{i,\lambda_i}'(0) = 1,$$

satisfies

$$\|\phi_{i,\lambda_i}\|_{L^2(0,X_i)} \leq \frac{C}{\sqrt{\lambda_i}}.$$

This is also given in [3, Theorem 1.1].

This estimate implies that the upper bound (2.15) holds with

$$\nu_i = 0.$$

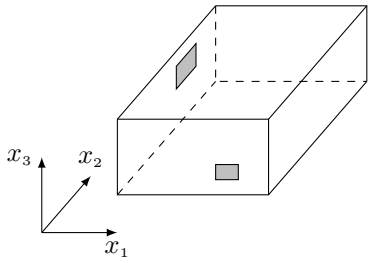

FIGURE 3. The geometry of the boundary control problem in 3D. The control domain is the part of the boundary which is represented in gray.

- The property (3.3) concerning the operators L_i holds with

$$\eta_i = \frac{1}{2}.$$

This is nothing but the one-dimensional Lebeau-Robbiano spectral inequality [27, Theorem 3] (or Turán's inequality, see [36, Corollary 3.3]).

All the assumptions of Section 2.2 are fulfilled, so that Theorem 3.3 can be applied and shows that the system (7.1) is null controllable in time T for every $T > 0$.

The result is first obtained in the space $D(A^*)' = (H^2(\Omega)^2 \cap H_0^1(\Omega)^2)'$. In addition, note that

$$V' = H^{-1}(\Omega)^2$$

is an admissible subspace for our system (7.1) (see Definition 2.4). This is a direct consequence of the following well-known elliptic regularity estimate satisfied by the solution z to the corresponding adjoint system (2.3) in any time $T > 0$: there exists $C > 0$ such that

$$\begin{aligned}\int_0^T \int_{\partial\Omega} \left\| \frac{\partial z}{\partial n}(t,\sigma) \right\|_{\mathbb{C}^2}^2 d\sigma dt &\leq C \int_0^T \|z(t,\cdot)\|_{H^2(\Omega)^2}^2 \, dt \\ &\leq C \int_0^T \|Az(t,\cdot)\|_{L^2(\Omega)^2}^2 \, dt \\ &\leq C \left\|z^T\right\|_{H_0^1(\Omega)^2}^2, \quad \forall z^T \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2,\end{aligned}$$

where $\partial/\partial n$ denotes the normal derivative.

□

By similar considerations, Theorem 3.4 proves the null controllability at any time T of the system (7.1) posed on a 3D parallelepiped, with a diffusion tensor of the form

$$K(x) = \begin{pmatrix} k_1(x_1) & 0 & 0 \\ 0 & k_2(x_2) & 0 \\ 0 & 0 & k_3(x_3) \end{pmatrix},$$

and with a boundary control supported on two non parallel faces as shown for instance in Figure 3.

Appendix A. Properties of A and B

A.1. **Tensor products**

Let us briefly recall some basic facts about tensor products. More material can be found for instance in [34, Chapters II.4 and VIII.10] and [33, Chapter XIII.9].

For $\varphi_1 \in H_1$ and $\varphi_2 \in H_2$, we denote by $\varphi_1 \otimes \varphi_2 : H_1 \times H_2 \longrightarrow \mathbb{C}$ the pure tensor product of φ_1 with φ_2, that is the continuous bilinear form defined, for every $(h_1, h_2) \in H_1 \times H_2$, by

$$(\varphi_1 \otimes \varphi_2)(h_1, h_2) = \langle \varphi_1, h_1 \rangle_{H_1} \langle \varphi_2, h_2 \rangle_{H_2}.$$

Then, the so-called algebraic tensor space is

$$H_1 \otimes H_2 = \text{span}\left\{\varphi_1 \otimes \varphi_2, \quad \text{s.t.} \quad \varphi_1 \in H_1, \quad \varphi_2 \in H_2\right\}.$$

We will denote by $E_1 \otimes E_2 = \text{span}\left\{\varphi_1 \otimes \varphi_2, \quad \varphi_1 \in E_1, \quad \varphi_2 \in E_2\right\}$ for any subspaces $E_1 \subset H_1$ and $E_2 \subset H_2$. On the vector space $H_1 \otimes H_2$ we introduce the following inner product, first defined on pure tensor products by

$$\langle \varphi_1 \otimes \varphi_2, \widetilde{\varphi}_1 \otimes \widetilde{\varphi}_2 \rangle_{H_1 \otimes H_2} = \langle \varphi_1, \widetilde{\varphi}_1 \rangle_{H_1} \langle \varphi_2, \widetilde{\varphi}_2 \rangle_{H_2}, \tag{A.1}$$

and then extended by linearity to all of $H_1 \otimes H_2$. It can be checked that this inner product is well defined (i.e. that two different writing of an element in $H_1 \otimes H_2$ yields the same value in the computation of the inner product). This makes $H_1 \otimes H_2$ a pre-Hilbert space but this space is in general not complete (because H_1 and H_2 are infinite dimensional). This motivates the introduction of its completion with respect to this inner product, which will be denoted by

$$H_1 \,\widehat{\otimes}\, H_2.$$

Let now $\tilde{H}_1$, $\tilde{H}_2$ be two complex Hilbert spaces and $A_1 : H_1 \to \tilde{H}_1$, $A_2 : H_2 \to \tilde{H}_2$ be two bounded linear operators. There exists a unique bounded linear operator from $H_1 \,\widehat{\otimes}\, H_2$ into $\tilde{H}_1 \,\widehat{\otimes}\, \tilde{H}_2$, denoted by $A_1 \otimes A_2$, that satisfies

$$(A_1 \otimes A_2)(\phi_1 \otimes \phi_2) = (A_1\phi_1) \otimes (A_2\phi_2), \quad \forall \phi_1 \in H_1, \phi_2 \in H_2.$$

Moreover, we have

$$\|A_1 \otimes A_2\|_{\mathcal{L}(H_1 \widehat{\otimes} H_2, \tilde{H}_1 \widehat{\otimes} \tilde{H}_2)} = \|A_1\|_{\mathcal{L}(H_1, \tilde{H}_1)} \|A_2\|_{\mathcal{L}(H_2, \tilde{H}_2)}.$$

We refer for instance to [34, Proposition VIII.10, p.299].

A.2. **Proofs of the properties of A**

Let us start by showing that our operator A is a bounded perturbation of a self-adjoint operator with compact resolvent.

Proof of Proposition 2.5. Let us denote by A_{00} the operator (2.10). First, it is clear that $D(A_{00})$ is dense in H and that the operator A_{00} is symmetric. In particular, it is closable. Moreover, its closure is self-adjoint if, and only if, both $\text{Ran}\,(A_{00} + i)$ and $\text{Ran}\,(A_{00} - i)$ are dense in H (see e.g. [34, Corollary VIII.2, p.257]). This clearly holds here since, in fact, any $f \in H$ can be written as

$$f = \sum_{\substack{\lambda_1 \in \Lambda_1 \\ \lambda_2 \in \Lambda_2}} \left(\frac{A_{00} \pm i}{d(\lambda_1 + \lambda_2) \pm i} \left((P^1_{\lambda_1,\lambda_2} f) \otimes \phi_{1,\lambda_1} \otimes \phi_{2,\lambda_2} \right) + \frac{A_{00} \pm i}{\lambda_1 + \lambda_2 \pm i} \left((P^2_{\lambda_1,\lambda_2} f) \otimes \phi_{1,\lambda_1} \otimes \phi_{2,\lambda_2} \right) \right),$$

for some finite dimensional operators $P^1_{\lambda_1,\lambda_2}, P^2_{\lambda_1,\lambda_2} \in \mathcal{L}(H, \mathbb{C}^2)$ (recall that $\Lambda_1, \Lambda_2 \subset \mathbb{R}$). This also shows that the closure of $A_{00} \pm i$, and thus A_0, has a compact resolvent. □

Let us now prove the claimed spectral properties of A. We recall that a family in a Hilbert space is a Riesz basis if it is the image of an orthonormal basis through an invertible bounded linear operator. We refer for instance to [38, Section 1.8] for material on Riesz basis.

Proof of Proposition 4.9. 1) Let $\hat{\sigma} \in \mathbb{C}$ be fixed and let $\Phi \in D(A^*)$. Writing

$$\Phi = \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} v_{\lambda_1,\lambda_2} \otimes \phi_{1,\lambda_1} \otimes \phi_{2,\lambda_2}, \tag{A.2}$$

for some vectors $v_{\lambda_1,\lambda_2} \in \mathbb{C}^2$, a computation shows that $\Phi \in \ker(\hat{\sigma} - A^*)$ if, and only if,

$$v_{\lambda_1,\lambda_2} \in \ker G_\lambda, \quad G_\lambda = \begin{pmatrix} d\lambda - m_{11} - \hat{\sigma} & -m_{21} \\ -m_{12} & \lambda - m_{22} - \hat{\sigma} \end{pmatrix},$$

for every $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, where we use our standard notation $\lambda = \lambda_1 + \lambda_2$. Therefore, $\hat{\sigma}$ is an eigenvalue of A^* if, and only if, $\ker G_\lambda \neq \{0\}$ for some $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. Since $m_{21} \neq 0$ by assumption, this is equivalent to

$$\hat{\sigma} = d\lambda - m_{11} - m_{21} r, \tag{A.3}$$

where r is a root of

$$m_{21} r^2 + ((1-d)\lambda + m_{11} - m_{22})\, r - m_{12} = 0.$$

In addition, the discriminant of this equation is exactly Δ_λ defined in (4.1) and its two complex roots are

$$\frac{-((1-d)\lambda + m_{11} - m_{22}) - \sqrt{\Delta_\lambda}}{2m_{21}}, \quad \frac{-((1-d)\lambda + m_{11} - m_{22}) + \sqrt{\Delta_\lambda}}{2m_{21}}. \tag{A.4}$$

It is then clear that, given an eigenvalue $\hat{\sigma}$, there is only a finite number of $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ that satisfy (A.3), so that the series in (A.2) is in fact over a finite set. Moreover, the first component of each $v_{\lambda_1,\lambda_2} \in \ker G_\lambda \setminus \{0\}$ is necessarily nonzero since $m_{21} \neq 0$. This shows that Φ can be written as a linear combination of the Φ^0_γ defined in (4.13) (a simple computation shows that the expression of the eigenvalues coincides with the one in (4.5), depending on the sign $s(\gamma)$).

The proof of the statement concerning the generalized eigenspaces simply relies on computations and will be omitted.

2) The proof of the estimates in (v) is a straightforward computation

$$\begin{aligned}
\left\|\Phi^0_\gamma\right\|_{D(A^*)} &= (1 + |\sigma(\gamma)|)\left\|\Phi^0_\gamma\right\|_H \quad \text{(since } \Phi^0_\gamma \text{ is an eigenfunction)} \\
&= (1 + |\sigma(\gamma)|)\sqrt{1 + |r_\gamma|^2}\left\|\phi_{1,\lambda_1(\gamma)}\right\|_{H_1}\left\|\phi_{2,\lambda_2(\gamma)}\right\|_{H_2} \quad \text{(recall (4.13))} \\
&\leq C_1(1 + |\sigma(\gamma)|)\sqrt{1 + |r_\gamma|^2} e^{2C_1\lambda(\gamma)^\nu} \quad \text{(by (2.15))} \\
&\leq C_2\,|\sigma(\gamma)|^2\, e^{C_2|\sigma(\gamma)|^\nu} \quad \text{(by (4.9) and (4.12)),}
\end{aligned}$$

where $C_1, C_2 > 0$ do not depend on γ. The reasoning is similar for the estimate of the norm of Φ^1_γ.

3) Let us now show that the family $\left\{\Phi^k_\gamma\right\}_{\substack{\gamma\in\Gamma \\ 0\leq k\leq k_\gamma}}$ is complete in H. Let then $z \in H$ be such that

$$\left\langle z, \Phi^k_\gamma \right\rangle_H = 0, \quad \forall \gamma \in \Gamma, \quad \forall 0 \leq k \leq k_\gamma. \tag{A.5}$$

and let us show that necessarily $z = 0$. Since H is by definition the completion of $\mathbb{C}^2 \otimes H_1 \otimes H_2$, it is equivalent to prove that (A.5) implies that

$$\langle z, h \rangle_H = 0, \quad \forall h \in \mathbb{C}^2 \otimes H_1 \otimes H_2,$$

and since $\{\phi_{i,\lambda_i}\}_{\lambda_i \in \Lambda_i}$ is complete in H_i $(i = 1, 2)$, this is equivalent by linearity to

$$\left\langle z, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \phi_{1,\lambda_1} \otimes \phi_{2,\lambda_2} \right\rangle_H = 0, \quad \forall \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2, \forall \lambda_1 \in \Lambda_1, \forall \lambda_2 \in \Lambda_2. \tag{A.6}$$

Let $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ be given. We define $\gamma^+ = (+, \lambda_1, \lambda_2)$ and $\gamma^- = (-, \lambda_1, \lambda_2)$.

- If $\gamma^+ \notin \widehat{\Gamma}$ (we automatically have $\gamma^- \notin \widehat{\Gamma}$), then the two vectors of $\mathbb{C}^2$

$$\begin{pmatrix} 1 \\ r_{\gamma^+} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ r_{\gamma^-} \end{pmatrix}$$

  are linearly independent. Therefore the conditions $\left\langle z, \Phi^0_{\gamma^+} \right\rangle_H = \left\langle z, \Phi^0_{\gamma^-} \right\rangle_H = 0$ from (A.5) imply (A.6).
- If $\gamma^+ \in \widehat{\Gamma}$, then the two vectors of $\mathbb{C}^2$

$$\begin{pmatrix} 1 \\ r_{\gamma^+} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -\frac{1}{m_{21}} \end{pmatrix},$$

  are linearly independent. Therefore the conditions $\left\langle z, \Phi^0_{\gamma^+} \right\rangle_H = \left\langle z, \Phi^1_{\gamma^+} \right\rangle_H = 0$ from (A.5) imply (A.6).

4) Let $\tilde{\Phi}^k_\gamma = \Phi^k_\gamma / \left\|\Phi^k_\gamma\right\|_H$. Proving that the family F is a Riesz basis of H is equivalent to show that (see e.g. [38, Theorem 1.9]) there exist $m, M > 0$ such that, for every scalars $(\alpha^k_\gamma)_{\substack{\gamma \in \Gamma \setminus \widehat{\Gamma}^+ \\ 0 \le k \le k_\gamma}} \subset \mathbb{C}$, cofinitely many of them being equal to 0, we have

$$m \sum_{\substack{\gamma \in \Gamma \setminus \widehat{\Gamma}^+ \\ 0 \le k \le k_\gamma}} \left|\alpha^k_\gamma\right|^2 \le \left\| \sum_{\substack{\gamma \in \Gamma \setminus \widehat{\Gamma}^+ \\ 0 \le k \le k_\gamma}} \alpha^k_\gamma \tilde{\Phi}^k_\gamma \right\|^2_H \le M \sum_{\substack{\gamma \in \Gamma \setminus \widehat{\Gamma}^+ \\ 0 \le k \le k_\gamma}} \left|\alpha^k_\gamma\right|^2. \tag{A.7}$$

For every $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ we set $\gamma^+ = (+, \lambda_1, \lambda_2)$ and $\gamma^- = (-, \lambda_1, \lambda_2)$, and

$$T_{\lambda_1,\lambda_2} = \begin{cases} \alpha^0_{\gamma^-} \tilde{\Phi}^0_{\gamma^-} + \alpha^0_{\gamma^+} \tilde{\Phi}^0_{\gamma^+}, & \text{if } \gamma^+ \notin \widehat{\Gamma}^+, \\ \alpha^0_{\gamma^-} \tilde{\Phi}^0_{\gamma^-} + \alpha^1_{\gamma^-} \tilde{\Phi}^1_{\gamma^-}, & \text{if } \gamma^+ \in \widehat{\Gamma}^+. \end{cases}$$

The sum we have to estimate simply reads $\sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} T_{\lambda_1,\lambda_2}$. Since it is easily seen from the definitions of the (generalized) eigenfunctions that all those terms are pairwise orthogonal, we are finally led to find two numbers $m, M > 0$, independent of λ_1, λ_2 and of the coefficients $\alpha^\bullet_\bullet$ such that (A.7) hold for each term T_{λ_1,λ_2}.

Let us consider such a pair $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$.

- Case 1 : $\lambda \in \widehat{\Lambda}$.
  We have $T_{\lambda_1,\lambda_2} = \alpha^0_{\gamma^-}\tilde{\Phi}^0_{\gamma^-} + \alpha^1_{\gamma^-}\tilde{\Phi}^1_{\gamma^-}$. The normalization condition immediately gives

$$\|T_{\lambda_1,\lambda_2}\|_H^2 \leq 2(|\alpha^0_{\gamma^-}|^2 + |\alpha^1_{\gamma^-}|^2).$$

  For the lower bound, we use the estimate (proved at the end of this section)

$$\left|\alpha\frac{1}{\sqrt{1+|a|^2}}\begin{pmatrix}1\\a\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \geq \frac{1}{\sqrt{2}}(|\alpha|^2+|\beta|^2)\frac{1}{1+|a|^2}, \quad a,b,\alpha,\beta \in \mathbb{C}, \tag{A.8}$$

  to deduce

$$\|T_{\lambda_1,\lambda_2}\|_H^2 \geq \frac{1}{\sqrt{2}}(|\alpha^0_{\gamma^-}|^2 + |\alpha^1_{\gamma^-}|^2)\frac{1}{1+|r_{\gamma^+}|^2}.$$

  If $\widehat{\Lambda}$ is finite, we immediately deduce a lower bound for the terms of this kind. Otherwise, by Remark 4.2, we necessarily are in the case $d = 1$, $\Delta = 0$, in which case it is easily seen from (A.4) that

$$r_{\gamma^+} = \frac{m_{22} - m_{11}}{2m_{21}},$$

  which is obviously a bounded quantity. The claim is proved.
- Case 2 : $\lambda \notin \widehat{\Lambda}$.
  We know that $\gamma^+ = (+, \lambda_1, \lambda_2) \notin \widehat{\Gamma}^+$ and therefore we have $T_{\lambda_1,\lambda_2} = \alpha^0_{\gamma^-}\tilde{\Phi}^0_{\gamma^-} + \alpha^0_{\gamma^+}\tilde{\Phi}^0_{\gamma^+}$.
  Here also the normalization condition gives the upper bound

$$\|T_{\lambda_1,\lambda_2}\|_H^2 \leq 2(|\alpha^0_{\gamma^-}|^2 + |\alpha^0_{\gamma^+}|^2).$$

  For the lower bound, we use the estimate (proved at the end of this section)

$$\left|\alpha\frac{1}{\sqrt{1+|a|^2}}\begin{pmatrix}1\\a\end{pmatrix} + \beta\frac{1}{\sqrt{1+|b|^2}}\begin{pmatrix}1\\b\end{pmatrix}\right|^2 \geq \frac{1}{\sqrt{2}}(|\alpha|^2+|\beta|^2)\frac{|a-b|^2}{(1+|a|^2)(1+|b|^2)}, \quad a,b,\alpha,\beta \in \mathbb{C}, \tag{A.9}$$

  to get

$$\|T_{\lambda_1,\lambda_2}\|_H^2 \geq \frac{1}{\sqrt{2}}(|\alpha^0_{\gamma^-}|^2 + |\alpha^0_{\gamma^+}|^2)\frac{|r_{\gamma^+} - r_{\gamma^-}|^2}{(1+|r_{\gamma^+}|^2)(1+|r_{\gamma^-}|^2)}.$$

  By the definition of r_γ in (4.13), and the expressions (4.5), (4.8), we have

$$|r_{\gamma^+} - r_{\gamma^-}|^2 = \frac{|\Delta_\lambda|}{|m_{21}|^2} \neq 0.$$

  – If $d = 1$, then $\Delta_\lambda = \Delta$ and we obtain from (A.4) that

$$r_\gamma = \frac{2s(\gamma)\sqrt{\Delta} + m_{11} - m_{22}}{-2m_{21}}, \quad \forall \gamma \in \Gamma,$$

  and in particular the values of r_{γ^+} and r_{γ^-} only depend on M, which proves the claim.

– In the case $d \neq 1$, we write

$$|r_{\gamma^+} r_{\gamma^-}| = \frac{|(\max(d,1)-1)\lambda + \sigma^+ + m_{11} + \varepsilon_\lambda| \; |(\min(d,1)-1)\lambda + \sigma^- + m_{11} - \varepsilon_\lambda|}{|m_{21}|^2}.$$

It follows from (4.1) that

$$|r_{\gamma^+} - r_{\gamma^-}|^2 \underset{\lambda \to +\infty}{\sim} C_1 (d-1)^2 \lambda^2,$$

$$|r_{\gamma^+} r_{\gamma^-}|^2 \underset{\lambda \to +\infty}{\sim} C_2 (d-1)^2 \lambda^2,$$

for some $C_1, C_2 > 0$ that do not depend on γ^+, γ^-, and thus

$$|r_{\gamma^+}|^2 + |r_{\gamma^-}|^2 \underset{\lambda \to +\infty}{\sim} C_1 (d-1)^2 \lambda^2.$$

Therefore, the quantity

$$\frac{|r_{\gamma^+} - r_{\gamma^-}|^2}{(1+|r_{\gamma^+}|^2)(1+|r_{\gamma^-}|^2)},$$

has a positive limit when $\lambda \to +\infty$, which concludes the proof.

5) Since $A^* \Phi^0_\gamma = \sigma(\gamma) \Phi^0_\gamma$ and $A^* \Phi^1_\gamma = \sigma(\gamma) \Phi^1_\gamma + \Phi^0_\gamma$ if $\gamma \in \widehat{\Gamma}$, it is not difficult to deduce from what precedes that the same family normalized in $D(A^*)$ is also a Riesz basis of $D(A^*)$.

6) We have seen in step 1) that $\sigma(\Gamma) \subset \sigma(A^*)$. Let us now prove the reverse inclusion.

Let then $\mu \in \mathbb{C} \backslash \sigma(\Gamma)$ and $f \in H$. Let $\left\{ \Phi^{k,*}_\gamma \right\}_{\substack{\gamma \in \Gamma \backslash \widehat{\Gamma}^+ \\ 0 \leq k \leq k_\gamma}}$ be the biorthogonal family in H to the Riesz basis $\left\{ \tilde{\Phi}^k_\gamma \right\}_{\substack{\gamma \in \Gamma \backslash \widehat{\Gamma}^+ \\ 0 \leq k \leq k_\gamma}}$ (see e.g. [38, Theorem 1.9] for its existence). We set

$$z = \sum_{\gamma \in \Gamma \backslash \widehat{\Gamma}^+} \frac{\left\langle f, \Phi^{0,*}_\gamma \right\rangle_H}{\mu - \sigma(\gamma)} \tilde{\Phi}^0_\gamma + \sum_{\gamma \in \widehat{\Gamma}^-} \frac{\left\langle f, \Phi^{1,*}_\gamma \right\rangle_H}{\mu - \sigma(\gamma)} \left(\tilde{\Phi}^1_\gamma - \frac{\left\| \Phi^0_\gamma \right\|_H}{\left\| \Phi^1_\gamma \right\|_H} \tilde{\Phi}^0_\gamma \right). \tag{A.10}$$

Note that, $\operatorname{dist}(\mu, \sigma(\Gamma)) > 0$ as $\sigma(\Gamma)$ is obviously closed and $\mu \notin \sigma(\Gamma)$, and additionally for any $\gamma \in \widehat{\Gamma}^-$, we have

$$\frac{\left\| \Phi^0_\gamma \right\|_H}{\left\| \Phi^1_\gamma \right\|_H} = |m_{21}| \sqrt{1 + |r_\gamma|^2} \leq C(1 + |\sigma(\gamma)|).$$

Therefore, both sums in (A.10) are absolutely convergent in H, and using the closedness of A^*, we can check that $z \in D(A^*)$ with $(\mu - A^*) z = f$, which proves the claim.

7) It remains to prove the asymptotic property (4.16) of the counting function N. Note that it is enough to consider $r > 1$ since $N(r) = 0$ otherwise, still thanks to (4.9). Fist of all, we obviously have

$$N(r) \leq N_+(r) + N_-(r),$$

where

$$N_\pm(r) = \text{ card } \left\{ \gamma \in \Gamma^\pm, \quad \text{s.t.} \quad |\sigma(\gamma)| < r \right\}.$$

Let us for instance estimate N_+. Let $\gamma \in \Gamma^+$. From the formula (4.8), we get

$$|\lambda(\gamma)| \leq \frac{1}{\max(d,1)} \left(|\sigma(\gamma)| + \left|\sigma^+\right| + \left|\varepsilon_{\lambda(\gamma)}\right| \right) \leq \frac{|\sigma(\gamma)|}{\max(d,1)} \left(1 + \left|\sigma^+\right| + \left|\varepsilon_{\lambda(\gamma)}\right| \right),$$

by using (4.9). It follows that the condition $|\sigma(\gamma)| < r$ implies

$$|\lambda(\gamma)| < Cr, \quad C = \frac{1}{\max(d,1)} \left(1 + \left|\sigma^+\right| + \sup_{\lambda \geq 0} |\varepsilon_\lambda| \right).$$

It follows that

$$N_+(r) \leq \text{ card } \{(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2, \quad \text{s.t.} \quad |\lambda_1 + \lambda_2| < Cr\},$$

and since $\lambda_1, \lambda_2 \geq 0$ this gives

$$N_+(r) \leq N_{\Lambda_1}(Cr) N_{\Lambda_2}(Cr),$$

with the same estimate for N_-. It then follows from the asymptotics (2.8) of N_{Λ_1} and N_{Λ_2} that

$$N(r) \leq \kappa_0 r^\theta, \quad \forall r > 1, \tag{A.11}$$

where

$$\kappa_0 = 2\kappa_1 \kappa_2 C^\theta, \qquad \theta = \theta_1 + \theta_2.$$

Since $\theta \leq 1$ by assumption (2.9) and $r > 1$, this yields the desired asymptotic (4.16).

□

It remains to prove the technical lemma we used during the proof.

Proof of the estimate (A.9). Let

$$x = \frac{1}{\sqrt{1+|a|^2}} \begin{pmatrix} 1 \\ a \end{pmatrix}, \quad y = \frac{1}{\sqrt{1+|b|^2}} \begin{pmatrix} 1 \\ b \end{pmatrix}.$$

We have

$$|\alpha x + \beta y|^2 = \left\langle G \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\rangle,$$

where G is the Gram matrix of x and y

$$G = \begin{pmatrix} 1 & \frac{1+a\bar{b}}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}} \\ \frac{1+\bar{a}b}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}} & 1 \end{pmatrix}.$$

Since all the entries in G have a modulus less than 1, we have

$$\left\langle G \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\rangle \geq \frac{|\det(G)|}{\sqrt{2}} \left\| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\|^2_{\mathbb{C}^2}.$$

A straightforward computation shows that

$$\det(G) = \frac{|a-b|^2}{(1+|a|^2)(1+|b|^2)},$$

which concludes the proof. □

A.3. **Construction of** B

We conclude this appendix with the proof of the proposition defining our control operator B.

Proof of Proposition 2.7. We will denote by B' the operator in the right-hand side of (2.17) with domain

$$D_0 = \mathbb{C}^2 \otimes D(A_1) \otimes D(A_2).$$

Note that D_0 is dense in $D(A_0)$ by the very definition of the domain of the closure (recall that A_0 is the closure of (2.10)) and $D(A^*) = D(A) = D(A_0)$, therefore D_0 is dense in $D(A^*)$.

Consequently, the claim is equivalent to show that there exists $C > 0$ such that

$$\|B'z\|_U \leq C\left(\|A^*z\|_H + \|z\|_H\right), \quad \forall z \in D_0.$$

Below, we denote by C a positive number that may change from line to line but that does not depend on z. Still by a density argument, it is equivalent to prove such an estimate for any $z \in D_0$ of the form

$$z = \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} \begin{pmatrix} \alpha_{\lambda_1,\lambda_2} \\ \beta_{\lambda_1,\lambda_2} \end{pmatrix} \otimes \phi_{1,\lambda_1} \otimes \phi_{2,\lambda_2},$$

with $(\alpha_{\lambda_1,\lambda_2})_{\substack{\lambda_1\in\Lambda_1\\ \lambda_2\in\Lambda_2}}, (\beta_{\lambda_1,\lambda_2})_{\substack{\lambda_1\in\Lambda_1\\ \lambda_2\in\Lambda_2}} \subset \mathbb{C}$, cofinitely many of them being equal to 0. By definition of B', we have

$$B'z = \begin{pmatrix} L_1^* \left(\displaystyle\sum_{\lambda_1 \in \Lambda_1} \left(\sum_{\lambda_2 \in \Lambda_2} \alpha_{\lambda_1,\lambda_2} B_2^* \phi_{2,\lambda_2} \right) \phi_{1,\lambda_1} \right) \\ L_2^* \left(\displaystyle\sum_{\lambda_2 \in \Lambda_2} \left(\sum_{\lambda_1 \in \Lambda_1} \alpha_{\lambda_1,\lambda_2} B_1^* \phi_{1,\lambda_1} \right) \phi_{2,\lambda_2} \right) \end{pmatrix}.$$

Since $L_i^* \in \mathcal{L}(H_i)$, we have

$$\|B'z\|_U^2 \leq C \left(\left\| \sum_{\lambda_1 \in \Lambda_1} \left(\sum_{\lambda_2 \in \Lambda_2} \alpha_{\lambda_1,\lambda_2} B_2^* \phi_{2,\lambda_2} \right) \phi_{1,\lambda_1} \right\|_{H_1}^2 + \left\| \sum_{\lambda_2 \in \Lambda_2} \left(\sum_{\lambda_1 \in \Lambda_1} \alpha_{\lambda_1,\lambda_2} B_1^* \phi_{1,\lambda_1} \right) \phi_{2,\lambda_2} \right\|_{H_2}^2 \right).$$

Since the family $(\phi_{i,\lambda_i})_{\lambda_i \in \Lambda_i}$ is orthogonal in H_i we have

$$\|B'z\|_U^2 = \sum_{\lambda_1 \in \Lambda_1} \left| B_2^* \sum_{\lambda_2 \in \Lambda_2} \alpha_{\lambda_1,\lambda_2} \phi_{2,\lambda_2} \right|^2 \|\phi_{1,\lambda_1}\|_{H_1}^2 + \sum_{\lambda_2 \in \Lambda_2} \left| B_1^* \sum_{\lambda_1 \in \Lambda_1} \alpha_{\lambda_1,\lambda_2} \phi_{1,\lambda_1} \right|^2 \|\phi_{2,\lambda_2}\|_{H_2}^2.$$

Using that $B_i^* \in L(D(A_i), \mathbb{C})$, we deduce that

$$\|B'z\|_U^2 \leq C \sum_{\lambda_1 \in \Lambda_1} \left\| A_2 \sum_{\lambda_2 \in \Lambda_2} \alpha_{\lambda_1,\lambda_2} \phi_{2,\lambda_2} \right\|_{H_2}^2 \|\phi_{1,\lambda_1}\|_{H_1}^2 + C \sum_{\lambda_2 \in \Lambda_2} \left\| A_1 \sum_{\lambda_1 \in \Lambda_1} \alpha_{\lambda_1,\lambda_2} \phi_{1,\lambda_1} \right\|_{H_1}^2 \|\phi_{2,\lambda_2}\|_{H_2}^2 ,$$

and still by orthogonality of the family of eigenfunctions we get

$$\begin{aligned}
\|B'z\|_U^2 &\leq C \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} |\alpha_{\lambda_1,\lambda_2}|^2 (\lambda_1^2 + \lambda_2^2) \|\phi_{1,\lambda_1}\|_{H_1}^2 \|\phi_{2,\lambda_2}\|_{H_2}^2 \\
&\leq C \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} |\alpha_{\lambda_1,\lambda_2}|^2 (\lambda_1 + \lambda_2)^2 \|\phi_{1,\lambda_1}\|_{H_1}^2 \|\phi_{2,\lambda_2}\|_{H_2}^2 \\
&= C \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) z \right\|_H^2 .
\end{aligned}$$

It follows that

$$\begin{aligned}
\|B'z\|_U^2 &\leq C \left\| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2) z \right\|_H^2 \\
&= C \|A^* z + (M^* \otimes \mathrm{Id} \otimes \mathrm{Id}) z\|_H^2 \\
&\leq C (\|A^* z\|_H + \|z\|_H)^2 .
\end{aligned}$$

The proof is complete.

□

Appendix B. Basic elements from graph theory

We recall here some very basic definitions and one result coming from graph theory. We refer for instance to [37] for the details.

Definition B.1.

- A (simple, undirected) graph is a pair of two sets $(\mathcal{N}, \mathcal{E})$ with

$$\mathcal{E} \subset \{\{\gamma, \tilde{\gamma}\}, \quad \text{s.t.} \quad \gamma, \tilde{\gamma} \in \mathcal{N}, \quad \gamma \neq \tilde{\gamma}\} .$$

  The elements of $\mathcal{N}$ are called the nodes and the elements of $\mathcal{E}$ are called the egdes.
- Let $\gamma, \tilde{\gamma} \in \mathcal{N}$. A path from γ to $\tilde{\gamma}$ is a finite sequence of distinct edges of the form

$$(\{\gamma_0, \gamma_1\}, \ldots, \{\gamma_{n-1}, \gamma_n\}) ,$$

  where

$$\begin{gathered}
n \geq 1, \\
\{\gamma_i, \gamma_{i+1}\} \in \mathcal{E}, \quad \forall i \in \{0, \ldots, n-1\}, \\
\gamma_0 = \gamma, \quad \gamma_n = \tilde{\gamma}, \\
\gamma_i \neq \gamma_j, \quad \forall i, j \in \{0, \ldots, n\}, \quad i \neq j.
\end{gathered}$$

The integer n is called the length of the path.

A graph $(\mathcal{N}, \mathcal{E})$ is connected if for every $\gamma, \tilde{\gamma} \in \mathcal{N}$ there exists a path from γ to $\tilde{\gamma}$.

- A cycle is a finite sequence of edges of the form

$$\left(\{\gamma_0, \gamma_1\}, \{\gamma_1, \gamma_2\}, \dots, \{\gamma_{n-2}, \gamma_{n-1}\}, \{\gamma_{n-1}, \gamma_0\}\right),$$

where

$$\begin{gathered} n \geq 3, \\ \{\gamma_i, \gamma_{i+1}\} \in \mathcal{E}, \quad \forall i \in \{0, \dots, n-2\}, \\ \gamma_i \neq \gamma_j, \quad \forall i, j \in \{0, \dots, n-1\}, \quad i \neq j. \end{gathered}$$

A graph with no cycles is called a forest. A connected graph with no cycles is called a tree.

The only result that we need is the following classical and simple one (see e.g. [37, Remark 1.2.7]):

Proposition B.2. *The relation "$\gamma = \tilde{\gamma}$ or there exists a path from γ to $\tilde{\gamma}$" is an equivalence relation over $\mathcal{N} \times \mathcal{N}$.*

It follows that any graph can be partitioned into connected subgraphs. Indeed, for $\gamma \in \mathcal{N}$, let us denote by $\mathcal{N}_\gamma$ its equivalence class. Thus, we have the natural partition of $\mathcal{N}$:

$$\mathcal{N} = \bigcup_{\gamma \in \mathcal{N}} \mathcal{N}_\gamma,$$

and each subgraph $(\mathcal{N}_\gamma, \mathcal{E}_\gamma)$ is connected, where $\mathcal{E}_\gamma = \left\{\{\tilde{\gamma}, \tilde{\tilde{\gamma}}\} \in \mathcal{E}, \quad \text{s.t.} \quad \tilde{\gamma}, \tilde{\tilde{\gamma}} \in \mathcal{N}_\gamma\right\}$. In particular, a forest is partitioned into trees (the union of two graphs $(\mathcal{N}_1, \mathcal{E}_1)$ and $(\mathcal{N}_2, \mathcal{E}_2)$ is by definition the graph $(\mathcal{N}_1 \cup \mathcal{N}_2, \mathcal{E}_2 \cup \mathcal{E}_2)$; the intersection of graphs is defined similarly).

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