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Local and global coincidence homology classes

Jean-Paul Brasselet*and Tatsuo Suwa†

Comme c'est curieux !
Comme c'est bizarre !
et **quelle coïncidence** *!*
J'ai pris le même train, Monsieur, moi aussi !
Eugène Ionesco
La cantatrice chauve.

Abstract

For two differentiable maps between two manifolds of possibly different dimensions, the local and global coincidence homology classes are introduced and studied by Bisi-Bracci-Izawa-Suwa (2016) in the framework of Čech-de Rham cohomology. We take up the problem from the combinatorial viewpoint and give some finer results, in particular for the local classes. As to the global class, we clarify the relation with the cohomology coincidence class as studied by Biasi-Libardi-Monis (2015). In fact they introduced such a class in the context of several maps and we also consider this case. In particular we define the local homology class and give some explicit expressions. These all together lead to a generalization of the classical Lefschetz coincidence point formula.

Keywords: Alexander duality; Thom class; intersection product with map; coincidence homology class; Lefschetz coincidence point formula.

Mathematics Subject Classification (2010) : 14C17, 37C25, 54H25, 55M05, 55M20, 57R20, 58C30.

1 Introduction

After Poincaré [9] and Brouwer [4], the Lefschetz fixed point theorem provided a new insight on the fixed point theory. S. Lefschetz [8] proved that, given a self map f of a compact oriented manifold M, the sum of indices at fixed points is equal to the alternating sum of the matrix traces of the linear maps induced by f on the homology of M with rational coefficients.

In fact, Lefschetz [7] provided the result in the context of coincidences. Given two maps f and g between compact oriented manifolds M and N of the same dimension, the coincidence points are defined as points $x \in M$ such that $f(x) = g(x)$. At these points, one defines the coincidence index and the Lefschetz result expresses the sum of indices in

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terms of alternating sum of suitable matrix traces (cf. Theorem 4.4 below). The Lefschetz fixed point formula is just the case $M = N$ and g is the identity map 1_M on M.

The result has been generalized in two ways. The first one is the case of manifolds with different dimensions. In fact it seems that Lefschetz himself possibly considered this case (cf. p. 28 in [11]). There have been a number of literatures on this (e.g., [10] and references therein). In [2], the global and local homology classes are introduced and some explicit formulas are given. The second one is the case of multi-coincidence. In [1], the Lefschetz class for several maps is introduced and studied.

In this paper, we will recall the main definitions and results concerning the Lefschetz coincidence indices and classes in the case of manifolds with same and possibly different dimensions. We study the global and local classes of [2] from combinatorial viewpoint and give some finer results. We also study the situation of multicoincidence, providing an alternative definition for the global class in homology. We further define local classes and give some explicit formulas.

The paper is organized as follows. In Section 2 we review, from the combinatorial viewpoint, the Poincaré and Alexander dualities, the Thom class and localized intersection products. After a brief description of the coincidence problem in Section 3, we recall in Section 4, the original Lefschetz coincidence point formula and sketch an outline of the proof. One of the key ingredients is that the local index is given as the local mapping degree of the difference of the maps.

We then take up the case of two maps between manifolds of possibly different dimensions in Section 5. We recall the definitions of the global and local coincidence classes (Definitions 5.5 and 5.9) as well as a general coincidence point theorem (Theorem 5.11). For the global class we prove that it correspnds to the cohomology coincidence class of [1] via the Poincaré duality (Theorem 5.7). The local class is representen by a cycle of the form $\sum c_{\boldsymbol{s}} \boldsymbol{s}$, where $\boldsymbol{s}$ runs through the simplices of an appropriate dimension in the coincidence set. Thus once we know the coefficient $c_{\boldsymbol{s}}$ for each simplex $\boldsymbol{s}$, we have an explicit expression for the local class. This can be done with the aid of the representation of the Thom in the Čech-de Rham cohomology. A general formula is given in Theorem 5.14 and, in some specific cases, it leads to the formulas as in Theorems 5.16, 5.18 and Corollary 5.19.

Finally we consider the case of several maps in Section 6. Since this case can be reduced to the case of two maps, everything done for two maps may be applied to obtain the results for this case (Theorems 6.5, 6.6 and Corollary 6.7).

2 Basic tools of algebraic topology

In the sequel, the homology is that of locally finite chains and the cohomology is that of cochains on finite chains, both with integral coefficients, unless otherwise stated. Also for a cycle C, its class in the homology of the ambient space is denoted by $[C]$, while the class in the homology of its support is simply denoted by C.

2.1 Poincaré and Alexander dualities

Combinatorial definitions of the dualities presented here can be found in [3], see also [13] and [14].

Let M be a connected oriented C^∞ manifold of dimension m. We take a triangulation K_0 of M and let K be the barycentric subdivision of K_0. We further take the barycentric subdivision K' of K. We take the second barycentric subdivision so that the star $S_{K'}(S)$ of a subcomplex S with respect to K', i.e., the union of simplices of K' intersecting with S, proper deformation retracts to S. Let K^* denote the cell decomposition dual to K. Thus for a simplex $\boldsymbol{s}$ of K its dual cell $\boldsymbol{s}^*$ is the union of simplices of K' that intersect with $\boldsymbol{s}$ only at the barycenter $b_{\boldsymbol{s}}$ of $\boldsymbol{s}$. We orient the simplices and cells so that, if $\boldsymbol{s}$ is an $(m-p)$-simplex of K and $\boldsymbol{s}^*$ its dual p-cell, the orientation of $\boldsymbol{s}^*$ followed by that of $\boldsymbol{s}$ gives the orientation of M. This orientation convention is that of [14].

Poincaré duality: We denote by $C^p_{K^*}(M)$ and $C^K_{m-p}(M)$ the groups of p-cochains in K^* and $(m-p)$-chains in K and define

$$P : C^p_{K^*}(M) \longrightarrow C^K_{m-p}(M) \qquad \text{by} \;\; u \mapsto \sum_{\boldsymbol{s}} \langle \boldsymbol{s}^*, u \rangle \boldsymbol{s}, \tag{2.1}$$

where $\langle\ ,\ \rangle$ denotes the Kronecker paring and the sum is taken over all $(m-p)$-simplices $\boldsymbol{s}$ of K. Then it induces the Poincaré duality isomorphism

$$P_M : H^p(M) \xrightarrow{\sim} H_{m-p}(M).$$

It is shown that P_M is given by the left cap product with the fundamental class of M, the class of the sum of all m-simplices in M. Thus a class c in $H^p(M)$ corresponds to the class γ in $H_{m-p}(M)$ such that

$$\langle M, c \smile a \rangle = \langle \gamma, a \rangle \qquad \text{for all } a \in H^{m-p}(M). \tag{2.2}$$

In the sequel P_M will simply be denoted by P if there is no fear of confusion.

Remark 2.3 In some literature such as [15], the Poincaré duality isomorphism is defined so that it is given by the right cap product with the fundamental class. If we denote this isomorphism by P'_M, we have

$$P'_M = (-1)^{p(m-p)} P_M.$$

Alexander duality: Let S be a K_0-subcomplex of M. We denote by $S_{K'}(S)$ and $O_{K'}(S)$ the star and the open star of S in K' and set

$$C^p_{K^*}(M, M \smallsetminus O_{K'}(S)) = \{\, u \in C^p_{K^*}(M) \mid \langle \boldsymbol{s}^*, u \rangle = 0 \text{ for } \boldsymbol{s} \not\subset S \,\}$$

Then (2.1) induces

$$C^p_{K^*}(M, M \smallsetminus O_{K'}(S)) \longrightarrow C^K_{m-p}(S)$$

and the Alexander isomorphism

$$A_{M,S} : H^p(M, M \smallsetminus S) \xrightarrow{\sim} H_{m-p}(S).$$

In the sequel $A_{M,S}$ is denoted by A if there is no fear of confusion. We have the following commutative diagram:

$$\begin{tikzcd} H^p(M, M \smallsetminus S) \arrow[r, "j^*"] \arrow[d, "\wr"', "A"] & H^p(M) \arrow[d, "\wr"', "P"] \\ H_{m-p}(S) \arrow[r, "i_*"] & H_{m-p}(M), \end{tikzcd} \tag{2.4}$$

where $i : S \to M$ and $j : (M, \emptyset) \to (M, S)$ are inclusions.

Thom homomorphism : Let X be an oriented pseudo-manifold of dimension d in M (cf. Definition 5.17 below). For a $(d-p)$-simplex $\boldsymbol{s}$ of K, $\boldsymbol{s}^* \cap X$ is a p-chain of K'. The homomorphism

$$P : C^p_{K'}(X) \longrightarrow C^K_{d-p}(X) \qquad \text{given by} \quad u \mapsto \sum_{\boldsymbol{s}} \langle \boldsymbol{s}^* \cap X, u \rangle \boldsymbol{s},$$

induces the Poincaré homomorphism

$$P_X : H^p(X) \longrightarrow H_{d-p}(X).$$

It is also given by the cap product with the fundamental class of X.

Setting $k = m - d$, the Thom homomorphism

$$T_{M,X} : H^p(X) \longrightarrow H^{p+k}(M, M \smallsetminus X)$$

is induced from the homomorphism

$$T : C^p_{K'}(X) \longrightarrow C^{p+k}_{K^*}(M, M \smallsetminus O_{K'}(X)) \qquad \text{given by} \quad \langle \boldsymbol{s}^*, T(u) \rangle = \langle \boldsymbol{s}^* \cap X, u \rangle.$$

Extending (2.4) for $S = X$, we have the following commutative pentahedron (square pyramid) where P_X and $T_{M,X}$ are isomorphisms if X is a manifold. The map $i_!$ is the Gysin map and the map a is defined as $a = P_M \circ j^* = i_* \circ A_{M,X}$.

$$\begin{tikzcd} & H^{p+k}(M, M \smallsetminus X) \arrow[dl, "j^*"'] \arrow[dr, dashed, "a"] \arrow[ddr, "A_{M,X} \simeq"] & \\ H^{p+k}(M) \arrow[rr, dashed, "P_M \simeq"'] & & H_{d-p}(M) \\ & H^p(X) \arrow[ul, "i_!"] \arrow[uul, "T_{M,X}\ (\simeq)"] \arrow[rr, "P_X\ (\simeq)"'] & & H_{d-p}(X). \arrow[ul, dashed, "i_*"] \end{tikzcd} \tag{2.5}$$

Definition 2.6 The *Thom class* Ψ_X of X is a class in $H^k(M, M \smallsetminus X)$ defined as

$$\Psi_X = T_{M,X}([1]), \qquad [1] \in H^0(X).$$

Thus it may also be written as

$$\Psi_X = (A_{M,X})^{-1} X. \tag{2.7}$$

Remark 2.8 If we use the convention that the duality homomorphisms are given by the right cap product (cf. Remark 2.3), then the Thom class Ψ'_X in this convention is given by

$$\Psi'_X = (-1)^{d(m-d)} \Psi_X.$$

2.2 Intersection product and localized intersection product

Let M be a connected oriented C^∞ manifold of dimension m, as before. For two classes $a \in H_r(M)$ and $b \in H_s(M)$, the *intersection product* $a \bullet b$ is defined by

$$a \bullet b = P(P^{-1}a \smile P^{-1}b) \qquad \text{in} \quad H_{r+s-m}(M), \tag{2.9}$$

where $\smile$ denotes the cup product. Then $a \bullet b$ is additive in a and b and we have

$$b \bullet a = (-1)^{(m-r)(m-s)} a \bullet b. \tag{2.10}$$

Suppose M is compact. In this case, if $r + s = m$, then $a \bullet b$ is in $H_0(M) = \mathbb{Z}$ and may be thought of as an integer.

Remark 2.11 The above intersection product remains the same if we use P'_M instead of P_M.

Let S_1 and S_2 be two K_0-subcomplexes of M and set $S = S_1 \cap S_2$. Let A_1, A_2 and A denote the Alexander isomorphisms for (M, S_1), (M, S_2) and (M, S), respectively. For two classes $a \in H_r(S_1)$ and $b \in H_s(S_2)$, the *intersection product* $(a \bullet b)_S$ *localized at* S is defined by

$$(a \bullet b)_S = A(A_1^{-1}a \smile A_2^{-1}b) \qquad \text{in} \quad H_{r+s-m}(S), \tag{2.12}$$

where $\smile$ denotes the cup product

$$H^{m-r}(M, M \smallsetminus S_1) \times H^{m-s}(M, M \smallsetminus S_2) \xrightarrow{\smile} H^{2m-r-s}(M, M \smallsetminus S).$$

Then $(a \bullet b)_S$ is additive in a and b and satisfies a relation similar to (2.10). Letting $i_1 : S_1 \hookrightarrow M$, $i_2 : S_2 \hookrightarrow M$ and $i : S \hookrightarrow M$ be the inclusions, from (2.4) we see that the definitions (2.9) and (2.12) are consistent in the sense that

$$(i_1)_* a \bullet (i_2)_* b = i_*(a \bullet b)_S.$$

In the above situation, suppose S has a finite number of connected components (S_λ). Then $H_{r+s-m}(S) = \bigoplus_\lambda H_{r+s-m}(S_\lambda)$ and we have the intersection product $(a \bullet b)_{S_\lambda}$ in $H_{r+s-m}(S_\lambda)$ for each λ. We have the "localization formula"

$$(i_1)_* a \bullet (i_2)_* b = \sum_\lambda (i_\lambda)_*(a \bullet b)_{S_\lambda} \qquad \text{in} \quad H_{r+s-m}(M),$$

where $i_\lambda : S_\lambda \hookrightarrow M$ denotes the inclusion. It is a formula of the same nature as the one given in Theorem 5.2 below. Suppose S is compact. Then, if $r+s=m$, $H_0(S_\lambda)=\mathbb{Z}$ for each λ and $(a\bullet b)_{S_\lambda}$ is an integer. In this case, the above formula is written as

$$(i_1)_*a\bullet(i_2)_*b=\sum_\lambda (a\bullet b)_{S_\lambda}. \tag{2.13}$$

The above dualities and intersection products may also be defined in the framework of Čech-de Rham cohomology (cf. [2], [12], [14]).

3 Coincidence problem

Let M and N denote connected oriented C^∞ manifolds of dimensions m and n, respectively. Let us consider two C^∞ maps $f:M\to N$ and $g:M\to N$.

Definition 3.1 The coincidence set $\mathrm{Coin}(f,g)$ is defined as

$$\mathrm{Coin}(f,g)=\{\,x\in M\mid f(x)=g(x)\,\}.$$

In the subsequent sections we introduce global and local invariants for the pair (f,g), study the relation among them and try to express the invariants explicitly. We then generalize the results to the case of several maps.

We denote by Γ_f and Γ_g the graphs of f and g in $M\times N$, respectively. They are both m-dimensional submanifolds of $M\times N$. We orient Γ_f so that the map $\tilde f: M\to\Gamma_f\subset M\times N$ given by $\tilde f(x)=(x,f(x))$ is an orientation preserving diffeomorphism, and similarly for Γ_g.

4 Lefschetz coincidence point formula - case of compact manifolds of the same dimension

In this section we review the original Lefschetz coincidence point formula in the case of C^∞ maps between manifolds of the same dimension, which will be generalized in various settings in the subsequent sections.

Let M and N denote compact connected oriented C^∞ manifolds of the same dimension m. Let us consider two maps $f:M\to N$ and $g:M\to N$.

Lefschetz number : In this paragraph, we consider homology and cohomology with $\mathbb{Q}$ coefficients and denote by f_p and f^p the homomorphisms induced by f on the p-th homology and p-th cohomology, respectively, and similarly for g_p and g^p. For each p, we have the commutative diagram :

$$\begin{tikzcd}
H^p(M,\mathbb{Q}) \arrow[r, "\sim", "P_M"'] & H_{m-p}(M,\mathbb{Q}) \arrow[r, "\sim", "K_M"'] \arrow[d, "f_{m-p}"] & H^{m-p}(M,\mathbb{Q})^* \arrow[d, "(f^{m-p})^*"] \\
H^p(N,\mathbb{Q}) \arrow[r, "\sim", "P_N"'] \arrow[u, "f^p"] & H_{m-p}(N,\mathbb{Q}) \arrow[r, "\sim", "K_N"'] & H^{m-p}(N,\mathbb{Q})^*,
\end{tikzcd}$$

where K_M and K_N denote the isomorphisms induced by the Kronecker pairing. Considering a similar diagram for g, we set

$$\tilde{g}^p = P_N^{-1} \circ g_{m-p} \circ P_M : H^p(M, \mathbb{Q}) \longrightarrow H^p(N, \mathbb{Q}).$$

Definition 4.1 The *Lefschetz number* $\mathrm{Lef}(f, g)$ of the pair (f, g) is defined by

$$\mathrm{Lef}(f, g) = \sum_{p=0}^{m} (-1)^p \cdot \mathrm{tr}(f^p \circ \tilde{g}^p).$$

We give a proof of the following for the sake of completeness:

Proposition 4.2 *We have*

$$\mathrm{Lef}(g, f) = (-1)^m \, \mathrm{Lef}(f, g).$$

Proof: First note that if we set

$$D_M^p = (K_M \circ P_M)^p : H^p(M, \mathbb{Q}) \xrightarrow{\sim} H^{m-p}(M, \mathbb{Q})^*,$$

then its transpose (dual map) is equal to D_M^{m-p}, i.e., $(D_M^p)^* = D_M^{m-p}$. We compute, omitting the superscripts for D,

$$\begin{aligned}(g^p \circ \tilde{f}^p)^* &= (g^p \circ (D_N^{-1}) \circ (f^{m-p})^* \circ D_M)^* = D_M \circ f^{m-p} \circ (D_N^{-1}) \circ (g^p)^* \\ &= D_M \circ f^{m-p} \circ (D_N^{-1}) \circ (g^p)^* \circ D_M \circ D_M^{-1} = D_M \circ f^{m-p} \circ \tilde{g}^{m-p} \circ D_M^{-1}.\end{aligned}$$

Thus

$$\mathrm{tr}(g^p \circ \tilde{f}^p) = \mathrm{tr}(f^{m-p} \circ \tilde{g}^{m-p}),$$

which proves the proposition. □

Remark 4.3 1. If we set

$$\tilde{g}_p = P_M \circ g^{m-p} \circ P_N^{-1} : H_p(N, \mathbb{Q}) \longrightarrow H_p(M, \mathbb{Q}),$$

then we may write

$$\mathrm{Lef}(f, g) = \sum_{p=0}^{m} (-1)^p \cdot \mathrm{tr}(\tilde{g}_p \circ f_p).$$

2. If we use P'_M and P'_N (cf. Remark 2.3) the above remains the same.

Local mapping degree: Let $f, g : M \to N$ be as above. Suppose p is an isolated point in $\mathrm{Coin}(f, g)$. Let U be a coordinate neighborhood around p with coordinates $x = (x_1, \ldots, x_m)$ in M and V a coordinate neighborhood around $f(p) = g(p)$ in N. Also let B be a closed ball around p in U such that $f(B) \subset V$ and $g(B) \subset V$. Thus we may consider the map $g - f : B \to \mathbb{R}^m$ whose image is the origin 0 in $\mathbb{R}^m$ only at p. The boundary ∂B is homeomorphic to the unit sphere S^{m-1} and we have the map

$$\gamma : \partial B \longrightarrow S^{m-1} \qquad \text{defined by} \quad \gamma(x) = \frac{g(x) - f(x)}{\|g(x) - f(x)\|}.$$

We denote the degree of this map by $\deg(g-f,p)$ and call it the *coincidence index* of (f,g) at p.

An isolated coincidence point p of the pair (f,g) is said to be *non-degenerate* if

$$\det(J_g(p)-J_f(p))\neq 0,$$

where $J_f(p)$ and $J_g(p)$ denote the Jacobian matrices of f and g at p. If p is a non-degenerate coincidence point, then we have

$$\deg(g-f,p)=\operatorname{sgn}\det(J_g(p)-J_f(p)).$$

With these, we have:

Theorem 4.4 (Lefschetz coincidence point formula) *Let M and N be compact connected oriented C^∞ manifolds of the same dimension and let $f,g\colon M\to N$ be C^∞ maps. Suppose* $\mathrm{Coin}(f,g)$ *consists of a finite number of isolated points. Then*

$$\mathrm{Lef}(f,g)=\sum_{p\in\mathrm{Coin}(f,g)}\deg(g-f,p).$$

In particular, $\mathrm{Lef}(f,g)$ *is an integer, which is zero if* $\mathrm{Coin}(f,g)=\emptyset$.

The above theorem applied to the case $N=M$ and $g=1_M$, the identity map of M, gives the Lefschetz fixed point formula for f.

Basically the proof of the above theorem consists of the following two parts. Note that, in the case under consideration, Γ_f and Γ_g are both m-cycles and the intersection product $[\Gamma_f]\bullet[\Gamma_g]$ is in $H_0(M\times N)=\mathbb{Z}$.

Part I. To show that $\mathrm{Lef}(f,g)=[\Gamma_f]\bullet[\Gamma_g]$.

Part II. To show that $[\Gamma_f]\bullet[\Gamma_g]$ localizes at the points of $\mathrm{Coin}(f,g)$ and the local contribution from each point is equal to the local mapping degree.

In some literature such as [15], the intersection number does not appear explicitly. Instead the following number is introduced. Let Δ_N denotes the diagonal in $N\times N$ and define $(f,g):M\to N\times N$ by $(f,g)(x)=(f(x),g(x))$. Consider the commutative diagram:

$$\begin{array}{ccccc}
H^m(N\times N, N\times N\smallsetminus\Delta_N) & \xrightarrow{j^*} & H^m(N\times N) & \xrightarrow{(f,g)^*} & H^m(M)\\
\wr\downarrow A'_{N\times N,\Delta_N} & & \wr\downarrow P'_{N\times N} & & \wr\downarrow P'_M\\
H_m(\Delta_N) & \xrightarrow{\quad i_*\quad} & H_m(N\times N) & & H_0(M),
\end{array}$$

The *Lefschetz class* of (f,g) is defined by (cf. Remarks 2.3 and 2.8)

$$L(f,g)=(f,g)^*\circ j^*(\Psi'_{\Delta_N})=(f,g)^*\circ(P'_{N\times N})^{-1}([\Delta_N])\qquad\text{in }\ H^m(M). \tag{4.5}$$

Then we have (cf. Theorem 5.7 below)

$$P'_M L(f,g)=[\Gamma_f]\bullet[\Gamma_g].$$

Part I is of purely global nature and can be done by direct computations taking bases of homology or cohomology of M and N ([15], also [2]). For this, it is not necessary to consider the Thom class Ψ'_{Δ_N}. In [15] it is used for Part II. See also [6] for a similar approach and the statement of the theorem as above. In [2], a simple direct proof for these is given using the expression of the Thom class of Γ_g in the framework of the Čech-de Rham cohomology.

5 Coincidence of two maps between manifolds of possibly different dimensions

In this section, we generalize the results in the previous section to the case of two maps between manifolds with possibly different dimensions.

5.1 Intersection product with a map

We recall the notion of intersection product with a map, see [2], [13] and [14], also [5] in the algebraic category.

Definition 5.1 Let W and M be oriented C^∞ manifolds of dimensions m' and m, respectively, and $F : M \to W$ a C^∞ map. We define the intersection product $M \bullet_F$ so that the first diagram below is commutative. Also, for a subcomplex $\tilde{S}$ of a triangulation of W, we set $S = F^{-1}(\tilde{S})$ and suppose S is a subcomplex of a triangulation of M. We then define the localized intersection product $(M \bullet_F)_S$ so that the second diagram is commutative:

$$\begin{tikzcd} H^p(W) \arrow[r, "\sim", "P"'] \arrow[d, "F^*"] & H_{m'-p}(W) \arrow[d, "M\cdot_F"] \\ H^p(M) \arrow[r, "\sim", "P"'] & H_{m-p}(M), \end{tikzcd} \qquad \begin{tikzcd} H^p(W, W \smallsetminus \tilde{S}) \arrow[r, "\sim", "A"'] \arrow[d, "F^*"] & H_{m'-p}(\tilde{S}) \arrow[d, "(M\cdot_F)_S"] \\ H^p(M, M \smallsetminus S) \arrow[r, "\sim", "A"'] & H_{m-p}(S). \end{tikzcd}$$

Note that, if M is a submanifold of W and if $F = \iota : M \hookrightarrow W$ is the inclusion, the products $M \bullet_\iota$ and $(M \bullet_\iota)_S$ coincide with $(M \bullet)_M$ and $(M \bullet)_S$, respectively, defined in Section 2 (cf. [2] Proposition 3.9, also [13] Section 7).

In the above situation, suppose S has only a finite number of connected components $(S_\lambda)_\lambda$. Then

$$H_{m-p}(S) = \bigoplus_\lambda H_{m-p}(S_\lambda)$$

and, for a class c in $H_{m-p}(S)$, the class $(M \bullet_F c)_S$ determines a class $(M \bullet_F c)_{S_\lambda}$ in $\breve{H}_{m-p}(S_\lambda)$ for each λ. We have the "residue theorem", which basically follows from (2.4):

Theorem 5.2 *Let C be an $(m'-p)$-cycle in W with support $\tilde{S} = |C|$. Suppose $S = F^{-1}\tilde{S}$ has a finite number of connected components $(S_\lambda)_\lambda$. Then we have*

$$M \bullet_F [C] = \sum_\lambda (i_\lambda)_*(M \bullet_F C)_{S_\lambda} \qquad \text{in} \quad H_{m-p}(M),$$

where $i_\lambda : S_\lambda \hookrightarrow M$ denotes the inclusion.

5.2 Global classes

5.2.1 The Lefschetz coincidence cohomology class.

We recall the Lefschetz coincidence cohomology class as considered in [1], which is a generalization of (4.5). See also [10] for related problems.

Let X be a topological space and N an oriented manifold of dimension n. Suppose we have two maps $f, g : X \to N$. Then we have the map $(f, g) : X \to N \times N$ and the diagram:

$$\begin{array}{ccccc} H^n(N \times N, N \times N \smallsetminus \Delta_N) & \xrightarrow{j^*} & H^n(N \times N) & \xrightarrow{(f,g)^*} & H^n(X) \\ \wr\downarrow A'_{N\times N, \Delta_N} & & \wr\downarrow P'_{N\times N} & & \\ H_n(\Delta_N) & \xrightarrow{\quad i_* \quad} & H_n(N \times N). & & \end{array}$$

Denoting by $\Psi'_{\Delta_N} = (A')^{-1}\Delta_N$ the Thom class of Δ_N, the *Lefschetz coincidence cohomology class* is defined as

$$L(f, g) = (j \circ (f, g))^*(\Psi'_{\Delta_N}) = (f, g)^*(P')^{-1}[\Delta_N] = (-1)^n (f, g)^* P^{-1}[\Delta_N] \qquad \text{in } \ H^n(X). \tag{5.3}$$

In [1] it is assumed that N is compact and the cohomology is with rational coefficients. Note that $L(g, f) = (-1)^{n^2} L(f, g) = (-1)^n L(f, g)$. Note also that as long as we consider the global class, it is not necessary to consider the Thom class of Δ_N.

In the case $X = M$ is an oriented manifold of dimension m, we have the commutative diagram

$$\begin{array}{ccc} H^n(N \times N) & \xrightarrow{(f,g)^*} & H^n(M) \\ \wr\downarrow P_{N\times N} & & \wr\downarrow P_M \\ H_n(N \times N) & \xrightarrow{M\cdot_{(f,g)}} & H_{m-n}(M) \end{array}$$

so that

$$P_M L(f, g) = (-1)^n M \bullet_{(f,g)} [\Delta_N]. \tag{5.4}$$

5.2.2 Global coincidence homology class

The global homology coincidence class is defined in [2] in the framework of Čech-de Rham cohomology. It can be done as well in the combinatorial context.

Let M and N be oriented manifolds of dimensions m and n, respectively. Suppose we have two maps $f, g : M \to N$. Let 1_M denote the identity map of M and consider the map $\tilde{f} = (1_M, f) : M \to M \times N$. Setting $W = M \times N$, we have the commutative diagram:

$$\begin{array}{ccc} H^n(W) & \xrightarrow[P_W]{\sim} & H_m(W) \\ \downarrow \tilde{f}^* & & \downarrow M\cdot_{\tilde{f}} \\ H^n(M) & \xrightarrow[P_M]{\sim} & H_{m-n}(M). \end{array}$$

Let us denote by Γ_g the graph of g in W, that defines an m-cycle whose homology class in $H_m(W)$ is denoted by $[\Gamma_g]$.

Definition 5.5 The *global coincidence homology class* $\Lambda(f,g)$ is defined by

$$\Lambda(f,g) = M \bullet_{\tilde{f}} [\Gamma_g] \qquad \text{in} \;\; H_{m-n}(M).$$

Note that $\tilde{f}$ induces an isomorphism $\tilde{f}_* : H_{m-n}(M) \xrightarrow{\sim} H_{m-n}(\Gamma_f)$ and $\Lambda(f,g)$ corresponds to $\Gamma_f \bullet [\Gamma_g]$ in $H_{m-n}(\Gamma_f)$, which is sent to $[\Gamma_f] \bullet [\Gamma_g]$ in $H_{m-n}(W)$ by the canonical homomorphism $H_{m-n}(\Gamma_f) \to H_{m-n}(W)$.

Remark 5.6 1. $\Lambda(g,f) = (-1)^{n^2}\Lambda(f,g) = (-1)^n \Lambda(f,g)$.

2. In particular, if $m = n$ and if M and N are compact, $\Lambda(f,g) = [\Gamma_f] \bullet [\Gamma_g]$ is the intersection number, which in turn coincides with the Lefschetz number $\mathrm{Lef}(f,g)$ defined in Section 4.

5.2.3 "Coincidence" of the two Lefschetz coincidence classes

Theorem 5.7 *If M and N are compact,*

$$P_M L(f,g) = (-1)^{n(m-n)} \Lambda(f,g), \quad \textit{i.e.,} \quad P'_M L(f,g) = \Lambda(f,g) \quad \textit{in} \;\; H_{m-n}(M,\mathbb{Q}).$$

Proof: Let us consider the Poincaré isomorphisms

$$P_{N\times N} : H^n(N\times N) \xrightarrow{\sim} H_n(N\times N) \quad \text{and} \quad P_{M\times N} : H^n(M\times N) \xrightarrow{\sim} H_m(N\times N)$$

and set $\eta_\Delta = P^{-1}_{N\times N}(\Delta_N)$ and $\eta_g = P^{-1}_{M\times N}(\Gamma_g)$. By (5.4), it suffices to prove

$$(g,f)^*\eta_\Delta = (-1)^{n(m-n)} \tilde{f}^*\eta_g \qquad \text{in} \;\; H^n(M,\mathbb{Q}). \tag{5.8}$$

Let $\{\mu_i^p\}_i$ be a basis of $H^p(M)$. We set $q = m-p$ and let $\{\check{\mu}_j^q\}_j$ be the basis of $H^q(M)$ dual to $\{\mu_i^p\}_i$:

$$\langle M, \mu_i^p \smile \check{\mu}_j^q \rangle = \delta_{ij}.$$

We also take a basis $\{\nu_k^p\}_k$ of $H^p(N)$ and the basis $\{\check{\nu}_\ell^r\}_\ell$ of $H^q(N)$ dual to $\{\nu_k^p\}_k$, $r = n-p$. By the Künneth formula, a basis of $H^n(M\times N)$ is given by

$$\left\{ \xi_{i,\ell}^{p,r} = \pi_1^*\mu_i^p \smile \pi_2^*\check{\nu}_\ell^r \right\}_{p+r=n},$$

where π_1 and π_2 are projections onto the first and second factors. Also a basis of the cohomology $H^n(N\times N)$ is given by

$$\left\{ \eta_{k,\ell}^{p,r} = \varpi_1^*\nu_k^p \smile \varpi_2^*\check{\nu}_\ell^r \right\}_{p+r=n},$$

where ϖ_1 and ϖ_2 are projections onto the first and second factors. By definition

$$\langle N\times N, \eta_\Delta \smile \varphi \rangle = \langle \Delta, \varphi \rangle.$$

We write

$$\eta_\Delta = \sum a_{k,\ell}^p \eta_{k,\ell}^{p,r}$$

Letting $\varphi = \varpi_1^* \check{\nu}_{k'}^{p'} \smile \varpi_2^* \nu_{\ell'}^{n-p'}$, we see that $a_{k,\ell}^p = (-1)^{n-p}\delta_{k,\ell}$ and $\eta_\Delta = \sum (-1)^{n-p}\eta_{k,k}^{p,r}$. Thus we have

$$(g,f)^*\eta_\Delta = \sum (-1)^{n-p} g^*\nu_k^p \smile f^*\check{\nu}_k^{n-p}.$$

Next, by definition

$$\langle M \times N, \eta_g \smile \psi\rangle = \langle \Gamma_g, \psi\rangle.$$

We write

$$\eta_g = \sum b_{i,\ell}^p \xi_{i,\ell}^{p,r}$$

By similar computations, letting $\psi = \pi_1^* \check{\mu}_{i'}^{p'} \smile \pi_2^* \nu_{\ell'}^{m-p'}$, we have

$$\tilde{f}^*\eta_g = \sum (-1)^\varepsilon g^*\nu_k^p \smile f^*\check{\nu}_k^{n-p},$$

where $\varepsilon = (m-p)p - (n-p)m$, and we have (5.8). □

5.3 Local classes

5.3.1 The local coincidence homology class

We recall the local coincidence homology class defined in [2] in our context.

Note that $\mathrm{Coin}(f,g) = \tilde{f}^{-1}(\Gamma_g)$, which will be simply denoted by C. We have the commutative diagram:

$$\begin{array}{ccc} H^n(W, W \smallsetminus \Gamma_g) & \xrightarrow[A]{\sim} & H_m(\Gamma_g) \\ \Big\downarrow \tilde{f}^* & & \Big\downarrow (M \cdot_{\tilde{f}} \ )_C \\ H^n(M, M \smallsetminus C) & \xrightarrow[A]{\sim} & H_{m-n}(C). \end{array}$$

Definition 5.9 ([2], Definition 4.2) The *local coincidence class* $\Lambda(f,g;C)$ of the pair (f,g) at C is defined to be the localized intersection class :

$$\Lambda(f,g;C) = (M \bullet_{\tilde{f}} \Gamma_g)_C \qquad \text{in} \quad H_{m-n}(C).$$

If we denote by Ψ_{Γ_g} the Thom class of Γ_g, we have

$$\Lambda(f,g;C) = A\tilde{f}^*\Psi_{\Gamma_g}. \tag{5.10}$$

Suppose $C = \mathrm{Coin}(f,g)$ has a finite number of connected components $(C_\lambda)_\lambda$. Then we have $H_{m-n}(C) = \oplus_\lambda H_{m-n}(C_\lambda)$ and accordingly we have the local coincidence class $\Lambda(f,g;C_\lambda)$ in $H_{m-n}(C_\lambda)$. We have a general coincidence point theorem:

Theorem 5.11 ([2] Theorem 4.4) *In the above situation*

$$\Lambda(f,g) = \sum_\lambda (\iota_\lambda)_* \Lambda(f,g;C_\lambda) \qquad \textit{in} \quad H_{m-n}(M),$$

where $\iota_\lambda : C_\lambda \hookrightarrow M$ denotes the inclusion.

In the sequel we give explicit expressions of the local classes. For this, we use (5.10) and a representation of the Thom class in the relative Čech-de Rham cohomology.

5.3.2 Čech-de Rham cohomology

Let M be a C^∞ manifold of dimension m. For an open set U of M, we denote by $A^p(U)$ the vector space of complex valued C^∞ p-forms on U. The cohomology of the complex $(A^*(M), d)$ is the de Rham cohomology $H^*_d(M)$. The Čech-de Rham cohomology may be defined for an arbitrary open covering of M, however here we only consider coverings consisting of two open sets.

Čech-de Rham cohomology : Let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of M. We set $U_{01} = U_0 \cap U_1$ and define the complex vector space $A^p(\mathcal{U})$ as

$$A^p(\mathcal{U}) = A^p(U_0) \oplus A^p(U_1) \oplus A^{p-1}(U_{01}).$$

An element σ in $A^p(\mathcal{U})$ is given by a triple $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ with σ_i a p-form on U_i, $i = 0, 1$, and σ_{01} a $(p-1)$-form on U_{01}. We define an operator $D : A^p(\mathcal{U}) \to A^{p+1}(\mathcal{U})$ by

$$D\sigma = (d\sigma_0, d\sigma_1, \sigma_1 - \sigma_0 - d\sigma_{01}).$$

Then we see that $D \circ D = 0$ so that we have a complex $(A^*(\mathcal{U}), D)$. The p-th *Čech-de Rham cohomology* of $\mathcal{U}$, denoted by $H^p_D(\mathcal{U})$, is the p-th cohomology of this complex. It is also abbreviated as ČdR cohomology. We denote the class of a cocycle σ by $[\sigma]$. It can be shown that the map $A^p(M) \to A^p(\mathcal{U})$ given by $\omega \mapsto (\omega|_{U_0}, \omega|_{U_1}, 0)$ induces an isomorphism

$$\alpha : H^p_d(M) \xrightarrow{\sim} H^p_D(\mathcal{U}). \tag{5.12}$$

Relative Čech-de Rham cohomology : Let S be a closed set in M. Letting $U_0 = M \smallsetminus S$ and U_1 a neighborhood of S in M, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. If we set

$$A^p(\mathcal{U}, U_0) = \{\, \sigma \in A^p(\mathcal{U}) \mid \sigma_0 = 0 \,\},$$

we see that $(A^*(\mathcal{U}, U_0), D)$ is a subcomplex of $(A^*(\mathcal{U}), D)$. We denote by $H^p_D(\mathcal{U}, U_0)$ the p-th cohomology of this complex.

Suppose S is a subcomplex relative to a C^∞ triangulation K_0 of M. Then we have a natural isomorphism :

$$H^p_D(\mathcal{U}, U_0) \simeq H^p(M, M \smallsetminus S; \mathbb{C}).$$

Let K, K' and K^* be as before. We may assume that U_1 contains the open star $O_{K'}(S)$ of S in K'. Let R_1 be an m-dimensional manifold with piecewise C^∞ boundary in $O_{K'}(S)$ containing S in its interior, for example we may take the star $S_{K''}(S)$ of S in the barycentric subdivision K'' of K' as R_1. We set $R_{01} = -\partial R_1$.

Theorem 5.13 ([13, 14]) *The Alexander isomorphism*

$$A : H^p_D(\mathcal{U}, U_0) \xrightarrow{\sim} H_{m-p}(S, \mathbb{C})$$

is induced from the homomorphism

$$A^p(\mathcal{U}, U_0) \longrightarrow C^K_{m-p}(S, \mathbb{C}) \quad \textit{given by } \sigma = (0, \sigma_1, \sigma_{01}) \mapsto \sum_{\boldsymbol{s}} \Big(\int_{\boldsymbol{s}^* \cap R_1} \sigma_1 + \int_{\boldsymbol{s}^* \cap R_{01}} \sigma_{01} \Big) \boldsymbol{s},$$

where $\boldsymbol{s}$ runs through the $(m-p)$-simplices of K in S.

5.3.3 Explicit expressions of local classes

If we use the Čech-de Rham cohomology, the local class is given as follows. Let $W_0 = W \smallsetminus \Gamma_g$ and W_1 a neighborhood of Γ_g and consider the covering $\mathcal{W} = \{W_0, W_1\}$ of W. Let $(0, \psi_1, \psi_{01})$ be a representative of the Thom class Ψ_{Γ_g} in $H^n_D(\mathcal{W}, W_0) \simeq H^n(W, W \smallsetminus \Gamma_g; \mathbb{C})$. Let $U_0 = M \smallsetminus C$, $C = \mathrm{Coin}(f, g)$, and U_1 a neighborhood of C such that $\tilde{f}(U_1) \subset W_1$. Let R_1 and R_{01} be as in Theorem 5.13. Then from Theorem 5.13 and (5.10), we have:

Theorem 5.14 *The class $\Lambda(f, g; C)$ in $H_{m-n}(C, \mathbb{C})$ is represented by the cycle*

$$\sum_{\boldsymbol{s}} c_{\boldsymbol{s}} \boldsymbol{s}, \qquad c_{\boldsymbol{s}} = \int_{\boldsymbol{s}^* \cap R_1} \tilde{f}^* \psi_1 + \int_{\boldsymbol{s}^* \cap R_{01}} \tilde{f}^* \psi_{01},$$

where $\boldsymbol{s}$ runs through the $(m-n)$-simplices of K in C.

If C has a finite number of connected components (C_λ), we take U_1 (and K_0) so that $U_1 = \bigcup_\lambda U_\lambda$, where $U_\lambda \supset O_{K'}(C_\lambda)$, for each λ, and $U_\lambda \cap U_\mu = \emptyset$, if $\lambda \neq \mu$. Then the class $\Lambda(f, g; C_\lambda)$ in $H_{m-n}(C_\lambda, \mathbb{C})$ is represented by the above cycle with $\boldsymbol{s}$ running through the $(m-n)$-simplices of K in C_λ.

(a) The case $m = n$ and C_λ is compact: In this case, the local class $\Lambda(f, g; C_\lambda)$ is in $H_0(C_\lambda) = \mathbb{Z}$ so that it is an integer. By Theorem 5.14 and the above remark, it is given by

$$\Lambda(f, g; C_\lambda) = \int_{R_\lambda} \tilde{f}^* \psi_1 + \int_{R_{0\lambda}} \tilde{f}^* \psi_{01}, \tag{5.15}$$

where $R_\lambda = R_1 \cap U_\lambda$ and $R_{0\lambda} = -\partial R_\lambda$.

Now suppose $C_\lambda = \{a\}$ is an isolated point. A short proof of the following formula using the Thom class in the Čech-de Rham cohomology is given in [2]:

Theorem 5.16 *We have:*

$$\Lambda(f, g; a) = \deg(g - f, a).$$

From Theorems 5.11 and 5.16, we recover Theorem 4.4.

(b) The case C_λ is a pseudo-manifold:

Definition 5.17 A *pseudo-manifold* X of dimension d in M is a subcomplex of M with respect to a triangulation of M satisfying the following conditions:

(1) Every simplex in X is a face of some d-simplex in X,

(2) Every $(d-1)$-simplex is the face of exactly two d-simplices,

(3) The d-simplices in X can be oriented so that, if $\boldsymbol{s}$ is a $(d-1)$-simplex in X and if $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are the two simplices that contain $\boldsymbol{s}$ in their boundary, then the prescribed orientations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ induce opposite orientations of $\boldsymbol{s}$.

A pseudo-manifold X is said to be *oriented*, once orientations of d-simplices in X satisfying (3) above are fixed. We say that X is *irreducible* if $X \smallsetminus X^{d-2}$ is connected, where X^{d-2} denotes the $(d-2)$-skeleton of X. Then we have a decomposition into irreducible components:

$$X = \bigcup_i X_i.$$

If X is oriented, each X_i carries a fundamental cycle, the union of d-simplices in X_i and it defines a class in $H_d(X)$. In fact $H_d(X)$ is generated by these classes.

Let C_λ be a connected component of $C = \mathrm{Coin}(f, g)$ as above and suppose it is an oriented pseudo-manifold of dimension $m - n$. If $\boldsymbol{s}$ is an $(m-n)$-simplex in C_λ, $\boldsymbol{s}^*$ is an n-cell such that $\boldsymbol{s}^* \cap C_\lambda = \{b_{\boldsymbol{s}}\}$, Thus $\deg(g-f)|_{\boldsymbol{s}^*}$ makes sense.

Theorem 5.18 *In the above situation, the local class $\Lambda(f, g; C_\lambda)$ in $H_{m-n}(C_\lambda)$ is represented by the cycle*

$$\sum_{\boldsymbol{s}} \deg(g-f)|_{\boldsymbol{s}^*} \cdot \boldsymbol{s},$$

where $\boldsymbol{s}$ runs through the $(m-n)$-simplices of K in C_λ.

Proof: This follows from (5.10) and Theorems 5.14, 5.16. □

Corollary 5.19 *In the above situation, if $C_\lambda = \bigcup_i C_{\lambda,i}$ is the irreducible decomposition,*

$$\Lambda(f, g; C_\lambda) = \sum_i \deg(g_{\lambda,i} - f_{\lambda,i}) \cdot C_{\lambda,i} \qquad \text{in} \quad H_{m-n}(C_\lambda),$$

where $g_{\lambda,i} - f_{\lambda,i}$ is the restriction of $g - f$ to a small ball of dimension n transverse to $C_{\lambda,i}$ at a non-singular point of $C_{\lambda,i}$.

In the case C_λ is a manifold, the above reduces to Proposition 4.8 in [2]. Note that we do not need the compactness of M or C_λ.

6 Coincidence of several maps

6.1 Global cohomology class

Let X be a topological space and N a compact connected oriented n-dimensional manifold. In [1] the authors consider p maps $f_1, \ldots, f_p : X \to N$, $p \geq 2$, and define a Lefschetz class $L(f_1, \ldots, f_p) \in H^{(p-1)n}(X, \mathbb{Q})$. They prove that $L(f_1, \ldots, f_p) \neq 0$ implies that one has "multi-coincidence" $f_1(x) = \cdots = f_p(x)$ for some $x \in X$.

We recall the definition of the class $L(f_1, \ldots, f_p)$ and give explicit local contributions subsequently. Again, it is not necessary to consider the Thom class of Δ_N or to assume the compactness of N. Also, it is defined in $H^{(p-1)n}(X, \mathbb{Z})$.

Definition 6.1 The *Lefschetz coincidence cohomology class* of $f_1, \ldots, f_p$ is defined as

$$L(f_1, \ldots, f_p) = L(f_1, f_2) \smile \cdots \smile L(f_{p-1}, f_p) \qquad \text{in} \quad H^{(p-1)n}(X).$$

6.2 Global homology classes

Suppose $X = M$ is an oriented manifold of dimension m.

Definition 6.2 We define the *global coincidence homology* class as

$$\Lambda(f_1, \dots, f_p) = \Lambda(f_1, f_2) \bullet \cdots \bullet \Lambda(f_{p-1}, f_p) \qquad \text{in} \quad H_{m-(p-1)n}(M).$$

From Theorem 5.7, we have:

Theorem 6.3 *We have the equality*:

$$\Lambda(f_1, \dots, f_p) = P'_M L(f_1, \dots, f_p).$$

6.3 Local homology classes

Let us consider the case of p maps $f_1, \dots, f_p : M \to N$. We set $W = M \times N$. We denote by $C_{i,i+1}$ the coincidence set

$$C_{i,i+1} = \mathrm{Coin}(f_i, f_{i+1}) \qquad \text{for } i = 1, \cdots, p-1.$$

Then we have

$$\bigcap_{i=1}^{p-1} C_{i,i+1} = \mathrm{Coin}(f_1, \cdots, f_p),$$

which we denote by C.

For each i, $1 \le i \le p-1$, we have a commutative diagram:

$$\begin{array}{ccc} H^n(W, W \smallsetminus \Gamma_{f_{i+1}}) & \xrightarrow[A]{\sim} & H_m(\Gamma_{f_{i+1}}) \\ \Big\downarrow \tilde{f}_i^* & & \Big\downarrow (M \cdot_{\tilde{f}_i})_{C_{i,i+1}} \\ H^n(M, M \smallsetminus C_{i,i+1}) & \xrightarrow[A]{\sim} & H_{m-n}(C_{i,i+1}) \end{array}$$

and the class

$$\Lambda(f_i, f_{i+1}; C_{i,i+1}) = (M \bullet_{\tilde{f}_i} \Gamma_{f_{i+1}})_{C_{i,i+1}} \qquad \text{in} \quad H_{m-n}(C_{i,i+1}).$$

Definition 6.4 We define the *local coincidence class* as

$$\Lambda(f_1, \dots, f_p; C) = (\Lambda(f_1, f_2; C_{1,2}) \bullet \cdots \bullet \Lambda(f_{p-1}, f_p; C_{p-1,p}))_C \qquad \text{in} \quad H_{m-(p-1)n}(C).$$

Suppose $C = \mathrm{Coin}(f_1, \dots, f_p)$ has a finite number of connected components $(C_\lambda)_\lambda$. Then we have the local coincidence class $\Lambda(f_1, \dots, f_p; C_\lambda)$ in $H_{m-n}(C_\lambda)$ and we have a general coincidence point theorem as in Section 5:

Theorem 6.5 *In the above situation, we have*

$$\Lambda(f_1, \dots, f_p) = \sum_\lambda (\iota_\lambda)_* \Lambda(f_1, \dots, f_p; C_\lambda) \qquad \textit{in} \quad H_{m-(p-1)n}(M),$$

where $\iota_\lambda : C_\lambda \hookrightarrow M$ *denotes the inclusion.*

From Corollary 5.19, we have an explicit expression of the local coincidence class. For simplicity we assume that each $C_{i,i+1}$ is an irreducible oriented pseudo-manifold of dimension $m-n$. Let $B_{i,i+1}$ be a small ball of dimension n in M transverse to $C_{i,i+1}$ at a non-singular point of $C_{i,i+1}$. Then we have

Theorem 6.6 *In the above situation, we have the following formula in* $H_{m-(p-1)n}(C)$:

$$\Lambda(f_1,\dots,f_p;C)=\deg((f_2-f_1)|_{B_{1,2}})\cdots\deg((f_p-f_{p-1})|_{B_{p-1,p}})\cdot(C_{1,2}\bullet\cdots\bullet C_{p-1,p})_C.$$

If $m=(p-1)n$ and if the $C_{i,i+1}$'s intersect transversely, C consists of isolated points and we have:

Corollary 6.7 *In the above situation, for a point x in C,*

$$\Lambda(f_1,\dots,f_p;x)=\deg((f_2-f_1)|_{B_{1,2}})\cdots\deg((f_p-f_{p-1})|_{B_{p-1,p}}).$$

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